*ISSN 0037–4466, Siberian Mathematical Journal, 2023, Vol. 64, No. 4, pp. 807–818.* 

# KNOT AS A COMPLETE INVARIANT OF THE DIFFEOMORPHISM OF SURFACES WITH THREE PERIODIC ORBITS

**D. A. Baranov, E. S. Kosolapov, and O. V. Pochinka** UDC 517.938.5

**Abstract:** It is known that Morse–Smale diffeomorphisms with two hyperbolic periodic orbits exist only on the sphere and they are all topologically conjugate to each other. However, if we allow three orbits to exist then the range of manifolds admitting them widens considerably. In particular, the surfaces of arbitrary genus admit such orientation-preserving diffeomorphisms. In this article we find a complete invariant for the topological conjugacy of Morse–Smale diffeomorphisms with three periodic orbits. The invariant is completely determined by the homotopy type (a pair of coprime numbers) of the torus knot which is the space of orbits of an unstable saddle separatrix in the space of orbits of the sink basin. We use the result to calculate the exact number of the topological conjugacy classes of diffeomorphisms under consideration on a given surface as well as to relate the genus of the surface to the homotopy type of the knot.

**DOI:** 10.1134/S0037446623040031

**Keywords:** knot, surface, gradient-like diffeomorphism

## 1. Introduction and Statements

Consider a closed orientable surface $S_p$ of genus $p \geq 0$ which is equipped with some metric $d$. Two homeomorphisms $f, f' : S_p \to S_p$ are *topologically conjugate* whenever there exists an orientation-preserving homeomorphism $h : S_p \to S_p$ with $f' = h \circ f \circ h^{-1}$.

A point $x \in S_p$ is *wandering* for a given homeomorphism $f$ whenever there exists an open neighborhood $U_x$ of $x$ such that $f^n(U_x) \cap U_x = \varnothing$ for all $n \in \mathbb{N}$. Otherwise, the point is *nonwandering*. The set of nonwandering points for $f$ is referred to as the *nonwandering set* and denoted by $\Omega_f$. If $\Omega_f$ is a finite set then each $r \in \Omega_f$ is periodic with some period $m_r \in \mathbb{N}$.

If $f$ is a diffeomorphism then $r \in \Omega_f$ is *hyperbolic* whenever all eigenvalues of the Jacobi matrix $\left(\frac{\partial f^{m_r}}{\partial x}\right)\big|_r$ have absolute values distinct from 1. If the absolute values of all eigenvalues are less or greater than 1 then the point $r$ is a *sink* or a *source*. Sinks and sources are called *nodes*. If a hyperbolic periodic point is not a node then it is a *saddle point*.

Given a hyperbolic periodic point $r$ of a diffeomorphism $f$, denote by $q_r$ the number of the eigenvalues of the Jacobi matrix $\left(\frac{\partial f^{m_r}}{\partial x}\right)\big|_r$ whose absolute values are greater than 1. The hyperbolic structure of a periodic point $r$ implies the existence of the *stable* manifold

$$W_r^s = \{x \in S_p : \lim_{k\to+\infty} d(f^{k\cdot m_r}(x), r) = 0\}$$

and the *unstable* manifold

$$W_r^u = \{x \in S_p : \lim_{k\to+\infty} d(f^{-k\cdot m_r}(x), r) = 0\}$$

which are smooth embeddings of $\mathbb{R}^{2-q_r}$ and $\mathbb{R}^{q_r}$ respectively.

This research was supported by the Russian Science Foundation (Grant no. 22–11–00027), with exception of the numerical calculations of the number of periodic mappings on a given surfaces. The calculations were supported by the Laboratory of Dynamical Systems and Applications of the National Research University Higher School of Economics. The laboratory was arranged as part of a megagrant of the Ministry of Science and Higher Education of the Russian Federation (Project 075–15–2022–1101).

Original article submitted March 30, 2022; revised July 14, 2022; accepted August 15, 2023.

Both stable and unstable manifolds are called *invariant manifolds*. Each connected component of $W^u_r \setminus r$ or $W^s_r \setminus r$ is called an *unstable* or a *stable separatrix*. A diffeomorphism $f : S_p \to S_p$ is a *Morse–Smale diffeomorphism* whenever $\Omega_f$ is finite and hyperbolic, while the invariant manifolds of periodic points meet transversely. If the invariant manifolds of distinct saddle points are disjoint then the Morse–Smale diffeomorphism $f : S_p \to S_p$ is *gradient-like*.

The *periodic data* of the periodic orbit $\mathcal{O}_r$ of a periodic point $r$ is the tuple $(m_r, q_r, \nu_r)$, where $m_r$ is the period of $r$, while $q_r = \dim\ W^u_r$, and $\nu_r$ is the orientation type of $r$; i.e., $\nu_r = +1$ or $\nu_r = -1$ according as $f^{m_r}|_{W^u_r}$ preserves or changes orientation. For each orientation-preserving diffeomorphism the orientation type of all nodes is $+1$, while the orientation type of saddle points can be either $+1$ or $-1$.

Denote by $G$ the set of orientation-preserving Morse–Smale diffeomorphisms $f : S_p \to S_p$ whose nonwandering set consists precisely of three periodic orbits.

**Proposition 1.1** [1, Theorems 2.1 and 2.2]. *The nonwandering set of each diffeomorphism $f \in G$ consists of a sink orbit $\mathcal{O}_\omega$, a source orbit $\mathcal{O}_\alpha$, and a saddle orbit $\mathcal{O}_\sigma$. Furthermore, the saddle orbit has the negative orientation type, while at least one of the nodal orbits of the diffeomorphism has period* 1.

To make this article self-contained, we prove Proposition 1.1 in Section 3. Henceforth, assume for definiteness that the sink $\omega$ is fixed. Its hyperbolicity implies that the diffeomorphism $f|_{W^s_\omega}$ is topologically conjugate using a homeomorphism $\psi_f : W^s_\omega \to \mathbb{R}^2$ to the linear diffeomorphism $A : \mathbb{R}^2 \to \mathbb{R}^2$ described as

$$A(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_2}{2}\right);$$

see [2, Proposition 2.5] for instance. Put $\mathbb{T}^2 = (\mathbb{R}^2 \setminus (0,0))/A$ and agree to denote the natural projection by $p : \mathbb{R}^2 \setminus (0,0) \to \mathbb{T}^2$. Introduce generators on the torus as follows: Refer as a *parallel* $L$ on $\mathbb{T}^2$ to the image of

$$\mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\},$$

i.e., $L = p(\mathbb{S}^1)$ oriented counterclockwise; every parallel has homotopy type $\langle 1, 0\rangle$. Refer as a *meridian* $M$ to the image of the positive semiaxis $Ox_1^+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 = 0\}$ of the axis $Ox_1$, i.e., $M = p(Ox_1^+)$, oriented in the direction of the decreasing of $x_1$; the meridian has homotopy type $\langle 0, 1\rangle$; see Fig. 1.

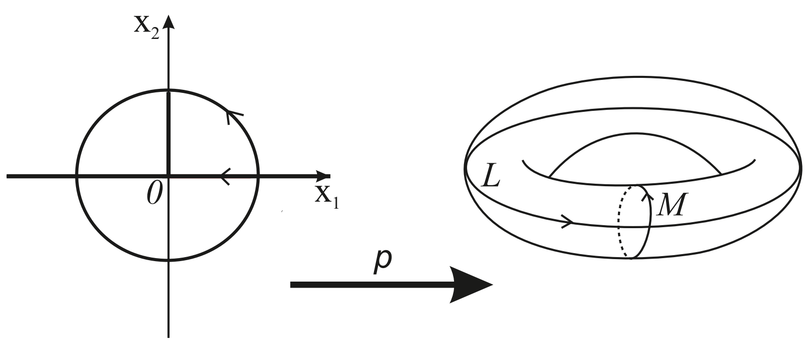


**Fig. 1.** The parallel $L$ and meridian $M$ on a torus.

Put $p_f = p\psi_f : V_f \to \mathbb{T}^2$ and $\gamma_f = p_f\big(W^u_{\mathcal{O}_\sigma}\big)$. According to [2], $\gamma_f$ on $\mathbb{T}^2$ is an essential knot of homotopy type $\langle \lambda_f, \mu_f\rangle$, where $\mu_f > 0$ and $\gcd(\lambda_f, \mu_f) = 1$. The homotopy type of $\gamma_f$ depends on the choice of $\psi_f$ so that if $(\widetilde{\lambda}_f, \widetilde{\mu}_f)$ is the homotopy type of $\gamma_f$ for some homeomorphism $\widetilde{\psi}_f$ distinct from $\psi_f$ then $\widetilde{\mu}_f = \mu_f$ and $\widetilde{\lambda}_f \equiv \lambda_f \pmod{\mu_f}$. Therefore, without loss of generality we may choose $\psi_f$ so that $\gamma_f$ has homotopy type

$$\langle \lambda_f, \mu_f\rangle : \mu_f > 0, \quad \gcd(\lambda_f, \mu_f) = 1, \quad 0 \le \lambda_f < \mu_f. \qquad (*)$$

The main result of this article is a proof of the following theorems:

**Theorem 1.** *The topological conjugacy class of a diffeomorphism $f \in G$ is uniquely determined by the homotopy type $\langle \lambda_f, \mu_f \rangle$ of the knot $\gamma_f$; i.e., two diffeomorphisms $f, f' \in G$ are topologically conjugate if and only if $\lambda_f = \lambda_{f'}$ and $\mu_f = \mu_{f'}$.*

**Theorem 2.** *On a surface $S_p$ of genus $p \geq 0$ a diffeomorphism $f \in G$ with the knot $\gamma_f$ of homotopy type $\langle \lambda_f, \mu_f \rangle$ exists if and only if*

$$\mu_f = 4p \text{ or } \mu_f = 4p + 2.$$

*Furthermore, the number $N_p$ of the topological conjugacy classes of diffeomorphisms $f \in G$ on the surface $S_p$ can be calculated as*

$$N_p = \varphi(4p) + \varphi(4p + 2),$$

*where $\varphi(n)$ is Euler's totient function counting the positive integers coprime to $n$ and not exceeding $n$.*

## 2. Periodic Homeomorphisms of a Surface

A homeomorphism $\varphi : S_p \to S_p$ is *periodic* whenever there exists $n \in \mathbb{N}$ such that $\varphi^n = \mathrm{id}$. The least of these $n$ is the *period* of $\varphi$. A point $x_0$ is called a *point of smaller period* $n_0 < n$ of a homeomorphism $\varphi$ whenever $\varphi^{n_0}(x_0) = x_0$.

Henceforth we consider orientation-preserving periodic homeomorphisms. According to Nielsen's results [3], see also [4], for every such homeomorphism $\varphi : S_p \to S_p$ the set of points of smaller period is finite, while the orbit space of the action of $\varphi$ on $S_p$ is a sphere with $g$ handles (a modular surface). In a neighborhood of a point $x_0$ of smaller period $n_0$ the mapping $f^{n_0}$ is conjugate to the rotation by some rational angle $2\pi \frac{\delta_0}{\lambda_0}$, where $\lambda_0 = \frac{n}{n_0}$.

Denote by $X_i$, for $i = 1, \dots, k$, the orbits of points of smaller period; and by $n_i$, their periods. Put

$$\lambda_i = \frac{n}{n_i}.$$

Denote by $\frac{\delta_i}{\lambda_i}$ the corresponding winding number and define $d_i$ by the condition $d_i \delta_i \equiv 1 \pmod{\lambda_i}$.

The tuple $(n, p, g, n_1, \dots, n_k, d_1, \dots, d_k)$ of parameters of a periodic homeomorphism $\varphi$ is the *complete characteristic* of $\varphi$.

**Proposition 2.1** [3]. *Two periodic homeomorphisms are topologically conjugate if and only if they share complete characteristics up to reindexing.*

**Proposition 2.2** [3]. *The complete characteristic $(n, p, g, n_1, \dots, n_k, d_1, \dots, d_k)$ is realized by some periodic homeomorphism $\varphi : S_p \to S_p$ if and only if the following are met:*

- $2p + \sum_{i=1}^{k} n_i - 2 = n(2g + k - 2)$,
- $\sum_{i=1}^{k} d_i n_i \equiv 0 \pmod{n}$,
- *if $g = 0$ then* $\gcd(d_1 n_1, \dots, d_k n_k, n) = 1$.

**Proposition 2.3** [4, 5]. *Given a periodic homeomorphism $\varphi$ with complete characteristic*

$$(n, p, n_1, \dots, n_k, d_1, \dots, d_k),$$

*the following hold:*

(1) $g \leq p$;

(2) $k \leq 2(p + 1)$;

(3) $n \leq 4p + 2$.

These inequalities imply immediately that finding all periodic homeomorphisms on a surface with a fixed number of handles is an algorithmic problem. The following lemma yields some algorithmic criterion for the realizability of a characteristic by a periodic homeomorphism.

**Lemma 1** (algorithmic criterion). *The tuple $(n, p, g, n_1, \dots, n_k, d_1, \dots, d_k)$ is the complete characteristic of a periodic mapping $\varphi$ if and only if the following are met, with $\lambda_i = \frac{n}{n_i}$ and $\lambda = \operatorname{lcm}(\lambda_1, \dots, \lambda_k)$:*

*In the case $g = 0$:*

*(1) $\sum_{i=1}^{k} \frac{d_i}{\lambda_i} \in \{1, \dots, k-1\}$;*

*(2) $n = \lambda$ and $p = \frac{\lambda - \sum_{i=1}^{k} \frac{\lambda}{\lambda_i}}{2} + 1$;*

*(3) $\gcd(d_1, \dots, d_k, n) = 1$.*

*In the case $g \neq 0$:*

*(1) $\sum_{i=1}^{k} \frac{d_i}{\lambda_i} \in \{1, \dots, k-1\}$;*

*(2) $n = \tau\lambda$, $\tau \in \mathbb{N}$, and $p = \frac{\lambda(2g+k-2) - \sum_{i=1}^{k} \frac{\lambda}{\lambda_i}}{2}\tau + 1$.*

Proof. Condition (1) follows from the first claim of Proposition 2.2. Namely,

$$d_1 n_1 + \dots + d_k n_k \equiv 0 \pmod{n} \Leftrightarrow \frac{d_1 n}{\lambda_1} + \dots + \frac{d_k n}{\lambda_k} \equiv 0 \pmod{n}.$$

Since $0 < d_i < \lambda_i$, we see that $0 < \frac{d_i n}{\lambda_i} < n$. This implies that

$$\frac{d_1 n}{\lambda_1} + \dots + \frac{d_k n}{\lambda_k} \in \{n, 2n, \dots, (k-1)n\} \Rightarrow \frac{d_1}{\lambda_1} + \dots + \frac{d_k}{\lambda_k} \in \{1, 2, \dots, k-1\}.$$

Condition 2 follows from the first claim in Proposition 2.2. Indeed, by definition $n = \lambda_i n_i$ for $i = 1, \dots, k$, which implies that $n = \tau\lambda$ for some $\tau \in \mathbb{N}$. Then from the equation on the Euler characteristics we express the genus $p$ of the original surface:

$$p = \frac{\lambda(2g + k - 2) - \sum\limits_{i=1}^{k} \frac{\lambda}{\lambda_i}}{2}\tau + 1.$$

Thus, the proof of Lemma 1 is complete in the case $g \neq 0$.

Verify that $\tau = 1$ in the case $g = 0$. Indeed, suppose that this is false and $n = \tau\lambda$, where $\tau > 1$. Then

$$\gcd(d_1 n_1, \dots, d_k n_k, n) = \gcd\left(\frac{d_1 n}{\lambda_1}, \dots, \frac{d_k n}{\lambda_k}, n\right) = \gcd\left(\tau\frac{\lambda}{\lambda_1} d_1, \dots, \tau\frac{\lambda}{\lambda_k} d_k, \tau\lambda\right) \geq \tau > 1.$$

This contradicts condition (3) of Proposition 2.2.

Therefore, if $g = 0$ then $n = \lambda = \operatorname{lcm}(\lambda_1, \dots, \lambda_k)$. Since condition (3) of Proposition 2.2 yields

$$\gcd\left(\frac{n}{\lambda_1} d_1, \dots, \frac{n}{\lambda_k} d_k, n\right) = 1,$$

we infer that

$$\gcd\left(\frac{n}{\lambda_1}, \dots, \frac{n}{\lambda_k}\right) = 1.$$

This implies that

$$\gcd\left(\frac{n}{\lambda_1} d_1, \dots, \frac{n}{\lambda_k} d_k\right) = \gcd(d_1, \dots, d_k).$$

Then

$$\gcd(d_1, \dots, d_k, n) = \gcd\left(\frac{n}{\lambda_1} d_1, \dots, \frac{n}{\lambda_k} d_k, n\right) = 1,$$

and the proof of Lemma 1 is complete. $\square$

Next we give some corollaries of the algorithmic criterion.

**Corollary 2.1.** *There exists no periodic homeomorphism with exactly one point of a smaller period.*

**Corollary 2.2.** *Every periodic homeomorphism with two points of a smaller period for $g \neq 0$ has complete characteristic of the form*

$$(n = \tau\lambda,\ p = \tau(2g-1)+1,\ n_1 = n_2 = \lambda,\ d_1 + d_2 = \lambda).$$

*In the case $g = 0$ every periodic homeomorphism is conjugate to a rational-angle rotation of the sphere about its polar axis.*

## 3. Dynamics of a Class $G$ Diffeomorphism

**3.1. A linearizing neighborhood of a saddle point.** Consider an orientation-preserving Morse–Smale diffeomorphism $f : S_p \to S_p$ and a saddle periodic point $\sigma$ of $f$ of period $m_\sigma$ and orientation type $\nu_\sigma$. Denote by $a_{\nu_\sigma} : \mathbb{R}^2 \to \mathbb{R}^2$ the diffeomorphism defined as

$$a_{\nu_\sigma}(x, y) = \left(\nu_\sigma \cdot \frac{x}{2}, \nu_\sigma \cdot 2y\right).$$

The diffeomorphism $a_{\nu_\sigma} : \mathbb{R}^2 \to \mathbb{R}^2$ has the unique fixed saddle point at the origin $O$ with the stable manifold $W^s_O = Ox_1$ and unstable manifold $W^u_O = Ox_2$. For $t \in (0, 1]$ put

$$\mathscr{N}^t = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 x_2| \leq t\}, \quad \mathscr{N} = \mathscr{N}^1.$$

Define the pair of transversal foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ in the neighborhood $\mathscr{N}$ as follows:

$$\mathscr{F}^u = \bigcup_{c_2 \in Ox_2} \{(x_1, x_2) \in \mathscr{N} : x_2 = c_2\},$$

$$\mathscr{F}^s = \bigcup_{c_1 \in Ox_1} \{(x_1, x_2) \in \mathscr{N} : x_1 = c_1\}.$$

Observe that $\mathscr{N}$ is invariant under the diffeomorphism $a_\nu$ that carries the leaves of the foliation $\mathscr{F}^u$ or $\mathscr{F}^s$ into leaves of the same foliation.

Refer to a neighborhood $N_\sigma$ of a saddle point $\sigma$ as *linearizing* whenever there is a homeomorphism $h_\sigma : N_\sigma \to \mathscr{N}$ conjugating the diffeomorphism $f^{m_\sigma}|_{N_\sigma}$ with the canonical diffeomorphism $a_{\nu_\sigma}|_{\mathscr{N}}$.

By way of the homeomorphism $h_\sigma^{-1}$ the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ induce the $f^{m_\sigma}$-invariant foliations $F^u_\sigma$ and $F^s_\sigma$ on the linearizing neighborhood of $N_\sigma$; see Fig. 2.

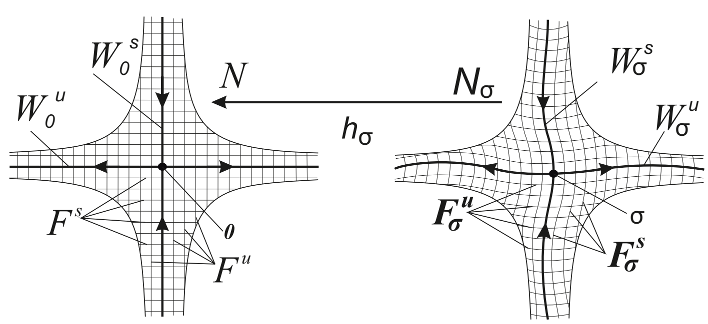


**Fig. 2.** A linearizing neighborhood of a saddle point $\sigma$.

**Proposition 3.1** [2, Theorem 2.2]. *Each saddle point of an orientation-preserving Morse–Smale diffeomorphism $f : S_p \to S_p$ has a linearizing neighborhood.*

Put $\mathscr{N}^u = \mathscr{N} \setminus Ox_1$ and denote by $\widehat{\mathscr{N}}^u_{\nu_\sigma} = \mathscr{N}^u/a_{\nu_\sigma}$ the orbit space of the action of the group $\{a^n_{\nu_\sigma}, n \in \mathbb{Z}\}$ on $\mathscr{N}^u$. Furthermore, denote the natural projection by $p_{\widehat{\mathscr{N}}^u_{\nu_\sigma}} : \mathscr{N}^u \to \widehat{\mathscr{N}}^u_{\nu_\sigma}$. The fundamental domain[1)] of the action of the group $\{a^n_{\nu_\sigma}, n \in \mathbb{Z}\}$ on $\mathscr{N}^u$ in the case $\nu_\sigma = +1$ consists of two disjoint curvilinear trapezoids, each of which has equivalent points on the horizontal boundary segments; while in the case $\nu_\sigma = -1$ we can choose the fundamental domain as one curvilinear trapezoid (with equivalent points on the horizontal boundary segment).

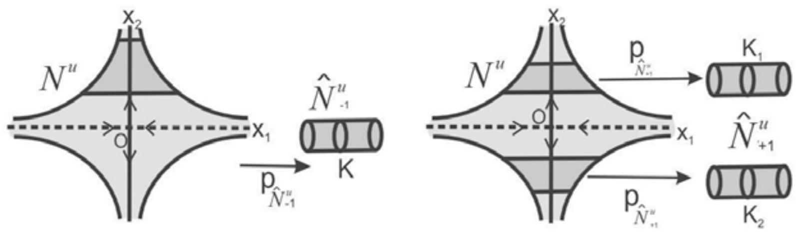


**Fig. 3.** The orbit space of $\widehat{\mathscr{N}}^u_{\nu_\sigma}$.

Fig. 3 with these trapezoids filled shows how we obtain from them the manifold $\widehat{\mathscr{N}}^u_{\nu_\sigma}$ depending on the choice of $\nu_\sigma$ by identifying the boundaries via the diffeomorphism $a_{\nu_\sigma}$.

**Proposition 3.2** [6, Proposition 5]. *The manifold $\widehat{\mathscr{N}}^u_{\nu_\sigma}$ has the following topological type depending on $\nu_\sigma$:*

- *the space $\widehat{\mathscr{N}}^u_{-1}$ is homeomorphic to a two-dimensional annulus $K$;*
- *the space $\widehat{\mathscr{N}}^u_{+1}$ is homeomorphic to a pair $K_1$, $K_2$ of two-dimensional annuli.*

A similar statement holds for $\mathscr{N}^s = \mathscr{N} \setminus Ox_2$. Moreover, the mapping

$$\widehat{\psi}_\sigma = p_{\widehat{\mathscr{N}}^s_{\nu_\sigma}} p^{-1}_{\widehat{\mathscr{N}}^u_{\nu_\sigma}} : \partial\widehat{\mathscr{N}}^u_{\nu_\sigma} \to \partial\widehat{\mathscr{N}}^s_{\nu_\sigma}$$

is well-defined, and $\widehat{\psi}_\sigma$ is called a *surgery mapping* (Fig. 4).

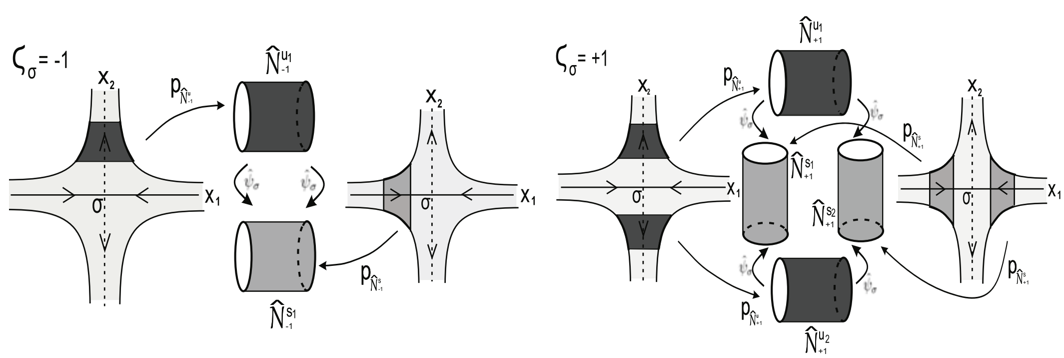


**Fig. 4**

Put $N^t_\sigma = h^{-1}_\sigma(\mathscr{N}^t)$, $N^{ut}_\sigma = N^t_\sigma \setminus W^s_\sigma$, $N^{st}_\sigma = N^t_\sigma \setminus W^u_\sigma$, and $\widehat{N}^{ut}_\sigma = N^{ut}/f$, $\widehat{N}^{st}_\sigma = N^{st}/f$, as well as $N^u_\sigma = N^{u1}_\sigma$, $N^s_\sigma = N^{s1}_\sigma$, $\widehat{N}^u_\sigma = \widehat{N}^{u1}_\sigma$, and $\widehat{N}^s_\sigma = \widehat{N}^{s1}_\sigma$.

[1)] Refer as a *fundamental domain* for an action of a group $G$ on a topological space $X$ to a closed set $D_G \subset X$ such that there exists a set $\widetilde{D}_G$ with the following properties: (1) $\mathrm{cl}(\widetilde{D}_G) = D_G$; (2) $g(\widetilde{D}_G) \cap \widetilde{D}_G = \varnothing$ for all $g \in G$ distinct from the neutral element of $G$; and (3) $\bigcup_{g\in G} g(\widetilde{D}_G) = X$.

### 3.2. The type of periodic orbits of a diffeomorphism $f \in G$.

**Lemma 2.** *The nonwandering set of an arbitrary diffeomorphism $f \in G$ consists of one sink orbit $\mathscr{O}_\omega$, one source orbit $\mathscr{O}_\alpha$, and one saddle orbit $\mathscr{O}_\sigma$; moreover, the saddle orbit has negative orientation type.*

Proof. Since $f$ is a Morse–Smale diffeomorphism, we infer that

$$S_p = \bigcup_{r \in \Omega_f} W^u_r = \bigcup_{r \in \Omega_f} W^s_r; \tag{$*$}$$

see [7, Theorem 2.3] or [2, Theorem 2.1] for instance. This implies that the diffeomorphism $f$ yields at least one sink orbit $\mathscr{O}_\omega$ and at least one source orbit $\mathscr{O}_\alpha$. According to [2, Corollary 2.2], the nonwandering set of a Morse–Smale diffeomorphism lacking saddle points consists of two fixed points. Then the third orbit of the diffeomorphism $f$ is a saddle orbit $\mathscr{O}_\sigma$. Let us verify that the saddle point $\sigma$ has negative orientation type.

Put

$$V_\omega = W^s_{\mathscr{O}_\omega} \setminus \mathscr{O}_\omega.$$

Denote by

$$\widehat{V}_\omega = V_\omega / f$$

the orbit space of the action of the group $F = \{f^k,\, k \in \mathbb{Z}\}$ on $V_\omega$; and by

$$p_{_\omega} : V_\omega \to \widehat{V}_\omega,$$

the natural projection. By [2, Proposition 2.5, p. 35], $\widehat{V}_\omega$ is diffeomorphic to the two-dimensional torus, while the natural projection

$$p_{_\omega} : V_\omega \to \widehat{V}_\omega$$

is a covering. Put $N^u_\sigma = N_{\mathscr{O}_\sigma} \setminus W^s_{\mathscr{O}_\sigma}$. By $(*)$, we have $N^u_\sigma \subset V_\omega$. Put $\widehat{N}^u_\sigma = p_{_\omega}(N^u_\sigma)$. By Proposition 3.2, $\widehat{N}^u_\sigma$ consists of one or two annuli when $\nu_\sigma = -1$ or $\nu_\sigma = +1$ that are not contractible on the torus $\widehat{V}_\omega$. The similar claim is valid for $\widehat{N}^s_\sigma$ that is the projection of $N^s_\sigma = N_{\mathscr{O}_\sigma} \setminus W^s_{\mathscr{O}_\sigma}$ onto $\widehat{V}_\alpha = V_\alpha / f$, where $V_\alpha = W^u_{\mathscr{O}_\alpha} \setminus \mathscr{O}_\alpha$ and $p_{_\alpha} : V_\alpha \to \widehat{V}_\alpha$ is the natural projection.

On the other hand, $(*)$ implies that $V_\alpha = V_\omega \setminus N^u_\sigma \cup N^s_\sigma$. Then

$$\widehat{V}_\alpha = \widehat{V}_\omega \setminus \widehat{N}^u_\sigma \cup \widehat{N}^s_\sigma.$$

Therefore, in order to obtain $\widehat{V}_\alpha$, we must cut out the annuli $\widehat{N}^u_\sigma$ from $\widehat{V}_\omega$ and glue the annuli $\widehat{N}^s_\sigma$ to the boundary of the resulting set via the surgery mapping.

If $\nu_\sigma = -1$ then each of the sets $\widehat{N}^u_\sigma$ and $\widehat{N}^s_\sigma$ consists of a sole annulus not contractible on $\widehat{V}_\omega$ and $\widehat{V}_\alpha$ respectively. Then $\widehat{V}_\omega \setminus \widehat{N}^u_\sigma$ amounts to an annulus. Thus, attaching $\widehat{N}^s_\sigma$ to its boundary, we again obtain a sole torus and, consequently, this case is possible.

If $\nu_\sigma = +1$ then each of the sets $\widehat{N}^u_\sigma$ and $\widehat{N}^s_\sigma$ consists of pairs of annuli not contractible on the tori $\widehat{V}_\omega$ and $\widehat{V}_\alpha$ respectively. Then $\widehat{V}_\omega \setminus \widehat{N}^u_\sigma$ amounts to two annuli. Thus, attaching to their boundaries the annuli $\widehat{N}^s_\sigma$ via the surgery mapping, we obtain two tori (each pair of annuli determines a separate torus); consequently, this case is impossible. $\square$

### 3.3. Periodic data of a diffeomorphism $f \in G$.

**Proposition 3.3** [2, Theorems 3.1 and 3.3]. *Every orientation-preserving gradient-like diffeomorphism $f : S_p \to S_p$ can be expressed as the composition $f = \varphi \circ \xi^1$, where $\xi^1$ is the translation by unit time along the trajectories of the gradient flow $\xi^t$ of some Morse function,*[2)] *while $\varphi$ is a periodic homeomorphism. Furthermore,*

[2)] A $C^2$-smooth function with nondegenerate critical points.

• *the points of smaller period of $\varphi$ are also periodic points of the diffeomorphism $f$ and, moreover, their periods coincide;*

• *the period of the separatrix of an arbitrary saddle point of the diffeomorphism $f$ coincides with the period of the homeomorphism $\varphi$.*

**Lemma 3.** *If $f = \varphi \circ \xi^1 \in G$ then the following hold:*

(1) *$\varphi$ has either two or three orbits of smaller period;*

(2) *If $\varphi$ has two orbits of smaller period then $\varphi$ has complete characteristic ($n = 2, g = 0, p = 0, n_1 = n_2 = 1$, and $d_1 = d_2 = 1$) and is topologically conjugate to the 180-degree rotation of the sphere about its polar axis;*

(3) *if the mapping $\varphi$ has exactly three points of smaller period then it has one of the following complete characteristics:*

(i) ($n = 4p, g = 0, p > 0, n_1 = 2p, n_2 = 1, n_3 = 1, d_1 = 1, d_2, d_3 = 2p - d_2$), $0 < d_2 < 2p$, *and* $\gcd(d_2, 2p) = 1$;

(ii) ($n = 4p, g = 0, p > 0, n_1 = 2p, n_2 = 1, n_3 = 1, d_1 = 1, d_2, d_3 = 6p - d_2$), $2p < d_2 < 4p$, *and* $\gcd(d_2, 2p) = 1$;

(iii) ($n = 4p+2, g = 0, p > 0, n_1 = 2p+1, n_2 = 2, n_3 = 1, d_1 = 1, d_2, d_3 = 2p+1-2d_2$), $0 < d_2 \le p$, *and* $\gcd(d_2, 2p+1) = 1$;

(iv) ($n = 4p+2, g = 0, p > 0, n_1 = 2p+1, n_2 = 2, n_3 = 1, d_1 = 1, d_2, d_3 = 6p+3-2d_2$), $p < d_2 \le 2p$, *and* $\gcd(d_2, 2p+1) = 1$.

Proof. By Lemma 2 the nonwandering set of the diffeomorphism $f \in G$ consists of the three periodic orbits: the sink orbit $\mathscr{O}_\omega$, the source orbit $\mathscr{O}_\alpha$, and the saddle orbit $\mathscr{O}_\sigma$. Denote their periods by $m_\omega$, $m_\alpha$, and $m_\sigma$.

Since $f = \varphi \circ \xi^1$ and the flow $\xi^t$ is generated by a Morse function, the Morse equalities (see [4] for instance) yield

$$m_\omega + m_\alpha - m_\sigma = 2 - 2p. \tag{$**$}$$

Denote the period of the homeomorphism $\varphi$ by $n$. According to Lemma 2, the saddle orbit of $f$ has negative orientation type. By Proposition 3.3, the period of the saddle separatrix equals $n$, which implies that $m_\sigma = \frac{n}{2}$.

Let us establish all claims of the lemma.

(1): If the periodic mapping $\varphi$ had more than three points of a smaller period then by Proposition 3.3 the mapping $f$ would have more than three periodic points, which contradicts the condition on the class $G$. Since the period of the saddle point equals $\frac{n}{2} < n$, the mapping $\varphi$ must have at least one point of a smaller period. By Corollary 2.1, there cannot be exactly one point of a smaller period.

(2): According to the Morse equality ($**$) in the case of two points of a smaller period of $\varphi$ we have $-\frac{n}{2} + \frac{n}{\lambda_2} + n = 2 - 2p$. Insert this equality into the first equality of Lemma 1:

$$\frac{n}{2} + \frac{n}{\lambda_2} + \frac{n}{2} - \frac{n}{\lambda_2} - n = n \cdot 2g.$$

Hence, $n - n = n \cdot 2g$. Therefore, $g = 0$. Corollary 2.2 implies that the periodic homeomorphism $\varphi$ is a rational-angle rotation of the sphere. It is clear that in our case this angle equals 180 degrees. Indeed, the period of the saddle point equals 1, but on the other hand, its period equals $\frac{n}{2}$, and so the period of $\varphi$ equals 2.

(3): Suppose that the periodic homeomorphism $\varphi$ corresponding to the diffeomorphism $f$ has the complete characteristic ($n, p, g, n_i, d_i$, for $1 \le i \le 3$). The second claim of Proposition 2.3 yields $p > 0$. By hypothesis and the property above, we find that $n_1 = \frac{n}{2}$. The Morse equality ($**$) shows that

$$-\frac{n}{2} + n_2 + n_3 = -\frac{n}{2} + \frac{n}{\lambda_2} + \frac{n}{\lambda_3} = 2 - 2p.$$

From the first equality of Lemma 1 we obtain $n = n \cdot (2g + 1) \Rightarrow g = 0$.

Put $n_2 = \frac{n}{\lambda_2}$ and $n_3 = \frac{n}{\lambda_3}$. Since $g = 0$, by Lemma 1 we infer that $n = \mathrm{lcm}(2, \lambda_2, \lambda_3)$. According to the second condition of Lemma 1, we see that $\frac{1}{2} + \frac{d_2}{\lambda_2} + \frac{d_3}{\lambda_3} = z$, where $z$ in either 1 or 2. Collecting all terms except the last one in the right-hand side, we obtain some fraction with denominator $\lambda_3$. Since the fractions on the two sides are equal and $d_i$ is coprime to $\lambda_i$, the equality is possible if and only if $\lambda_3$ divides $2\lambda_2$. Similarly we can show that $\lambda_2$ divides $2\lambda_3$. Consequently,

$$2\lambda_2 = t_1\lambda_3, 2\lambda_3 = t_2\lambda_2 \Rightarrow 4\lambda_2 = t_1 t_2 \lambda_2 \Rightarrow t_1 t_2 = 4.$$

Hence, up to reindexing we obtain just two cases: either (a) $\lambda_2 = \lambda_3$ or (b) $\lambda_2 = 2\lambda_3$.

In case (b) we have $n = \mathrm{lcm}(2, \lambda_3, 2\lambda_3) = 2\lambda_3 \Rightarrow n_2 = 2$, $n_3 = 1$. Inserting the available values of $n$, $d_1$, $n_1$, $n_2$, and $n_3$ into the second equality of Lemma 1, we obtain the complete characteristic $(n = 4k+2, g = 0, p = k, n_1 = 2k+1, n_2 = 2, n_3 = 1, d_1 = 1, d_2, d_3)$, where $k \in \mathbb{N}$ with $\gcd(d_2, 2k+1) = \gcd(d_3, 4k+2) = 1$, and $2d_2 + d_3 = 2k+1$ or $2d_2 + d_3 = 3(2k+1)$. It is clear that in the first case $2d_2 = 2k+1-d_3 \leq 2k$, whence $d_2 \leq k$, while in the second case $2d_2 = 6k+3-d_3 > 6k+3-(4k+2)$, whence $d_2 > k$. The equality $\gcd(d_2, 2k+1) = 1$ implies that $\gcd(2d_2, 2k+1) = 1$. Indeed, if the numbers $d_2$ and $2k+1$ are coprime then $\gcd(2d_2, 4k+2) = 2$, and so $\gcd(2d_2, 2k+1) = 1$ as required. Since either $d_3 = 2k+1-2d_2$ or $d_3 = 3(2k+1)-2d_2$, it follows that $\gcd(d_3, 2k+1) = \gcd(2d_2, 2k+1) = 1$. Considering that $d_3$ is odd, we obtain $\gcd(d_3, 4k+2) = 1$.

In case (a) we should consider two subcases: (a1) $\lambda_3$ is even; (a2) $\lambda_3$ is odd.

In case (a1) we have $n = \mathrm{lcm}(2, \lambda_3) = \lambda_3$. Hence, $n_2 = n_3 = 1$. Inserting the available values of $n$, $d_1$, $n_1$, $n_2$, and $n_3$ into the second equality of Lemma 1, we obtain the complete characteristic $(n = 4k$, $g = 0$, $p = k$, $n_1 = 2k$, $n_2 = 1$, $n_3 = 1$, $d_1 = 1$, $d_2, d_3)$, where $k \in \mathbb{N}$ with $\gcd(d_2, 4k) = \gcd(d_3, 4k) = 1$ and $d_2 + d_3 = 2k$ or $d_2 + d_3 = 6k$. It is clear that in the first case $d_2 < 2k$, while in the second $2k < d_2 < 4k$. It is also clear that $\gcd(d_2, 4k) = 1$ is equivalent to $\gcd(d_2, 2k) = 1$. Indeed, if $d_2$ and $4k$ are coprime then $d_2$ and $2k$ are coprime too. However, if $d_2$ and $2k$ are coprime then this implies that $d_2$ is necessarily odd, and so $d_2$ is coprime to $2 \cdot 2k = 4k$. Since $d_3 = \{2, 6\}k - d_2$, it follows that $d_3$ has the same residue modulo $2k$ as $d_2$. Hence, the condition $\gcd(d_2, 2k) = \gcd(d_3, 2k) = 1$ simplifies to $\gcd(d_2, 2k) = 1$.

In case (a2) we have $n = \mathrm{lcm}(2, \lambda_3) = 2\lambda_3$. Hence, $n_2 = n_3 = 2$. The Morse equality yields $\frac{-2\lambda}{2} + \frac{2\lambda}{\lambda} + \frac{2\lambda}{\lambda} = 2 - 2p$. Therefore, $\lambda = 2p + 2$, which is a contradiction because $\lambda$ was assumed odd. $\square$

Fig. 5 and 6 depict the results of numerical calculations of the number of periodic homeomorphisms, i.e., the number of periodic homeomorphisms for the given genus of the surface.

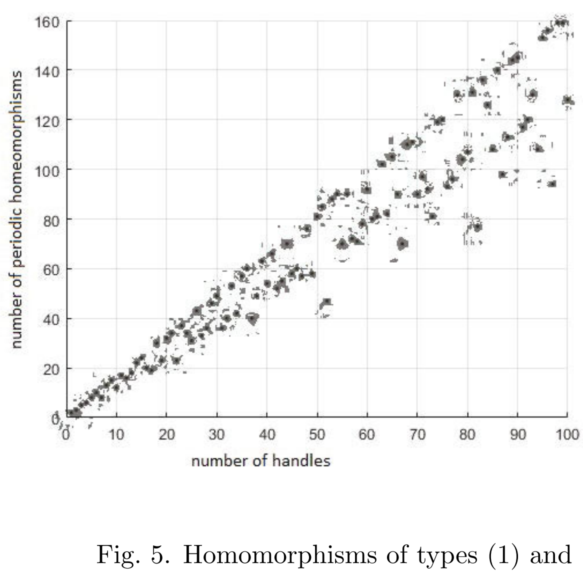


Fig. 5. Homomorphisms of types (1) and (2).

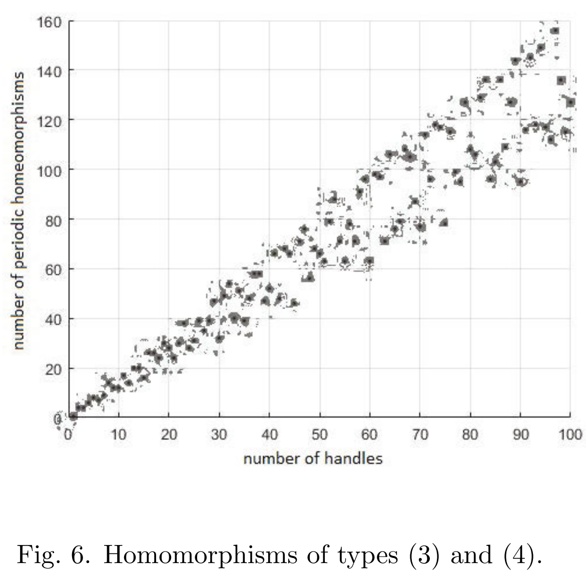


Fig. 6. Homomorphisms of types (3) and (4).

## 4. Classification of Class $G$ Diffeomorphisms

In this section we prove Theorem 1. Namely, we verify that two diffeomorphisms $f, f' \in G$ are topologically conjugate if and only if $\lambda_f = \lambda_{f'}$ and $\mu_f = \mu_{f'}$.

Necessity: If two diffeomorphisms $f$ and $f'$ are topologically conjugate then there exists a homeomorphism $h$ with $f' = h \circ f \circ h^{-1}$. Since the conjugating homeomorphism carries the invariant manifolds of periodic points into their analogs preserving stability and period, we infer that $h(W^s_\omega) = W^s_{\omega'}$ and $h(W^u_{\mathscr{O}_\sigma}) = W^u_{\mathscr{O}_\sigma}$. Put

$$\widehat{h} = p_f \circ h \circ p_f^{-1} : \mathbb{T}^2 \to \mathbb{T}^2.$$

Then $\widehat{h}(\gamma_f) = \gamma_{f'}$. Since $h$ carries the disk $d = \psi_f^{-1}(\mathbb{D}^2)$ to the disk $h(d)$ so that $\psi_{f'}(h(d))$ contains the origin, the knot $\widehat{h}(L)$ has the homotopy type $\langle 1, 0\rangle$; see [8] for instance. Then the induced isomorphism $\widehat{h}_*$ is determined by the matrix $\widehat{h}_* = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. Thus,

$$\widehat{h}_*(\langle \lambda_f, \mu_f\rangle) = (\lambda_f, \mu_f) \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \langle \lambda_f + c\mu_f, \mu_f\rangle.$$

On the other hand,

$$\widehat{h}_*(\langle \lambda_f, \mu_f\rangle) = \langle \lambda_{f'}, \mu_{f'}\rangle.$$

Hence, we find that $\mu_f = \mu_{f'}$ and $\lambda_f + c\mu_f = \lambda_{f'}$ and, consequently, $\lambda_{f'} = \lambda_f + c\mu_{f'}$. Condition $(*)$ implies that $\lambda_{f'} < \mu_{f'}$, but this is possible only for $c = 0$. Thus, $\lambda_f = \lambda_{f'}$ and $\mu_f = \mu_{f'}$.

Sufficiency: Suppose that $\lambda_f = \lambda_{f'}$ and $\mu_f = \mu_{f'}$. Construct a homeomorphism $h$ conjugating $f$ and $f'$ step-by-step.

Step 1. Construction of a homeomorphism $h_\omega : W^s_\omega \to W^s_{\omega'}$. According to [8], there exists a homeomorphism $\widehat{h} : \mathbb{T}^2 \to \mathbb{T}^2$ isotopic to the identity mapping with $\widehat{h}(\gamma_f) = \gamma_{f'}$. In this case the induced isomorphism $\widehat{h}_*$ is the identity mapping and, consequently (see [9] for instance), it lifts to a homeomorphism $h_\omega : W^s_\omega \backslash \omega \to W^s_{\omega'} \backslash \omega'$, (i.e., $p_{f'} \circ h_\omega = \widehat{h} \circ p_f$) which conjugates $f$ with $f'$ (i.e., $f' \circ h_\omega = h_\omega \circ f$) and satisfies $h_\omega(W^u_{\mathscr{O}_\sigma}) = W^u_{\mathscr{O}_{\sigma'}}$. Extend $h_\omega$ to $W^s_\omega$ by putting $h_\omega(\omega) = \omega'$.

Step 2. Modification of the homeomorphism $h_\omega$ in a neighborhood of $W^s_{\mathscr{O}_\sigma}$. By Lemma 2, the saddle orbit $\mathscr{O}_\sigma$ has negative orientation type $\nu_\sigma = -1$, which implies that the period of the saddle separatrix equals the number of all separatrices, and is therefore even. On the other hand, the knot $\gamma_f$ is the orbit space of saddle separatrices, and so the period of saddle separatrix equals $\mu_f$, while the period of the saddle point is $m_\sigma = \frac{\mu_f}{2}$. Furthermore, if $x$ belongs to the unstable separatrix $\ell^1_\sigma$ of an saddle $\sigma$ then the point $f^{m_\sigma}(x)$ belongs to another unstable separatrix $\ell^2_\sigma$ of the same saddle $\sigma$. Similar claims hold for the saddle $\sigma'$. Since $\mu_f = \mu_{f'}$, $f' \circ h_\omega = h_\omega \circ f$, and $h_\omega(W^u_{\mathscr{O}_\sigma}) = W^u_{\mathscr{O}_{\sigma'}}$, it follows that

$$h_\omega(W^u_\sigma \setminus \sigma) = W^u_{\sigma'} \setminus \sigma',$$

enabling us to extend the homeomorphism $h_\omega$ uniquely to the saddle orbit $\mathscr{O}_\sigma$.

Take a linearizing neighborhood $N_\sigma$ of the saddle $\sigma$. Choose $t_1 \in (0, 1]$ so that $h_\omega(N^{ut_1}_\sigma) \subset N^u_{\sigma'}$. Since $m_\sigma = m_{\sigma'}$ and $\nu_\sigma = \nu_{\sigma'}$, we can verify directly that

$$\widetilde{h}_\omega = \mu_{\sigma'} h_\omega \mu_\sigma^{-1}\big|_{\mathscr{N}^{ut_1}} : \mathscr{N}^{ut_1}_1 \to \mathscr{N}^u$$

is a topological embedding commuting with the diffeomorphism $a_{-1}$; i.e., $a_{-1}\widetilde{h}_\omega = \widetilde{h}_\omega a_{-1}$. Given $\kappa \in \{-1, +1\}$, define the topological embedding $\widetilde{\psi}_\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\widetilde{\psi}_\sigma(x_1, x_2) = (\widetilde{h}_\omega(x_1), \kappa \cdot x_2).$$

Choose $t_2 \in (0,1)$ so that $\widetilde{\psi}_\sigma(\mathscr{N}^{ut_2}) \subset \widetilde{h}_\omega(\mathscr{N}^{ut_1})$. Since

$$\widetilde{h}_\omega^{-1}\widetilde{\psi}_\sigma|_{Ox_1\setminus O} : Ox_1 \setminus O \to Ox_1 \setminus O$$

is the identity, assume without loss of generality that $\kappa$ is chosen so that the topological embedding $\theta_\sigma = \widetilde{h}_\omega^{-1}\widetilde{\psi}_\sigma|_{\mathscr{N}^{ut_2}} : \mathscr{N}^{ut_2} \to \mathscr{N}^u$ is orientation-preserving.

Put

$$\widehat{\theta}_\sigma = p_{\widehat{\mathscr{N}}^u_{-1}}\theta_\sigma(p_{\widehat{\mathscr{N}}^u_{-1}}|_{\mathscr{N}^{ut_2}_{-1}})^{-1} : \widehat{\mathscr{N}}^{ut_2}_{-1} \to \widehat{\mathscr{N}}^u_{-1},$$

$$K_\sigma = \widehat{\mathscr{N}}^u_{-1} \setminus \operatorname{int}\ \widehat{\mathscr{N}}^{ut_2}_{-1}, \quad Q_\sigma = \widehat{\mathscr{N}}^u_{-1} \setminus \operatorname{int}\widehat{\theta}_\sigma(\widehat{\mathscr{N}}^{ut_2}_{-1}).$$

By construction, each connected component of the sets $K_\sigma$ and $Q_\sigma$ is an annulus. According to [8], there is a homeomorphism $\widehat{\Theta}_\sigma : \widehat{\mathscr{N}}^u_{-1} \to \widehat{\mathscr{N}}^u_{-1}$ coinciding with $\widehat{\theta}_\sigma$ on $\widehat{\mathscr{N}}^{ut_2}_{-1}$ and identical on $\partial\widehat{\mathscr{N}}^u_{-1}$. Denote by $\widetilde{\Theta}_\sigma : \mathscr{N}^u \to \mathscr{N}^u$ a lift of the homeomorphism $\widehat{\Theta}_\sigma$, which is the identity mapping on $\partial\mathscr{N}$. Define the homeomorphism $\Theta_\sigma : N_\sigma \to h_\omega(N_\sigma)$ as

$$\Theta_\sigma(x) = \begin{cases} h_\omega(\mu_\sigma^{-1}(\widetilde{\Theta}_\sigma(\mu_\sigma(x)))) & \text{for } x \in N^u_\sigma, \\ \mu_{\sigma'}^{-1}(\widetilde{\psi}_\sigma(\mu_\sigma(x))) & \text{for } x \in W^s_\sigma. \end{cases}$$

Define the homeomorphism $\Theta : N_{\mathscr{O}_\sigma} \to h_\omega(N_{\mathscr{O}_\sigma})$ as

$$\Theta(x) = {f'}^k(\Theta_\sigma(f^{-k}(x))),$$

where $k \in \mathbb{Z}$ is chosen so that $f^{-k}(x) \in N_\sigma$.

Step 3. Define the homeomorphism $h : S_p \setminus \mathscr{O}_\alpha \to S_p \setminus \mathscr{O}_{\alpha'}$ as

$$h(x) = \begin{cases} h_\omega(x) & \text{for } x \in S_p \setminus (N_{\mathscr{O}_\sigma} \cup \mathscr{O}_\alpha), \\ \Theta(x) & \text{for } x \in N_{\mathscr{O}_\sigma} \end{cases}$$

and extend $h$ by continuity to $\mathscr{O}_\alpha$, assigning to $\alpha \in \mathscr{O}_\alpha$ the point $\alpha' \in \mathscr{O}_{\alpha'}$ so that $h(W^u_\alpha \setminus (\alpha)) = W^u_{\alpha'} \setminus \alpha'$. Then $h$ is the required homeomorphism. □

## 5. Connection between the Genus of the Supporting Surface and the Homotopy Type of the Knot $\gamma_f$

In this section we prove Theorem 2.

Proof. The diffeomorphism $f$ has homotopy type $(\lambda_f, \mu_f)$. Then by Proposition 3.3 the period $n$ of $f$ equals $\mu_f$. Using Lemma 3, we see that either $n = 4p$ or $n = 4p+2$, i.e., either $\mu_f = 4p$ or $\mu_f = 4p+2$. This implies that

$$p = \frac{\mu_f}{4} \text{ if } \mu_f = 0 \pmod 4, \quad p = \frac{\mu_f - 2}{4} \text{ if } \mu_f = 2 \pmod 4.$$

Since the number of gradient-like diffeomorphisms on the surface equals the number of periodic homeomorphisms [2, Theorem 3.2], invoking Lemma 3 again, we conclude that we can calculate their number $N_p$ as

$$N_p = \varphi(4p) + \varphi(4p+2),$$

where $\varphi(n)$ is Euler's totient function. □

## References


1. Medvedev T., Nozdrinova E., and Pochinka O., "On periodic data of diffeomorphisms with one saddle orbit," Topology Proc., vol. 54, 49–68 (2019).
2. Grines V., Medvedev T., and Pochinka O., *Dynamical Systems on 2- and 3-Manifolds*, Springer, Switzerland (2016).
3. Nielsen J., *Die Struktur periodischer Transformationen von Flächen.* Mat.-Fys. Medd. Danske Vid. Selsk,. Bd 15, Levin & Munksgaard, Copenhagen (1937).
4. Baranov D.A. and Pochinka O.V., "Classification of periodic transformations of an orientable surface of genus two," Zhurnal SVMO, vol. 23, no. 2, 147–158 (2021).
5. Wang Sh., "Maximum orders of periodic maps on closed surfaces," Topology Appl., vol. 41, 255–262 (1991).
6. Pochinka O.V., Grines V.Z., and Kapkaeva S.H., "A three-colour graph as a complete topological invariant for gradient-like diffeomorphisms of surfaces," Sb. Math., vol. 205, no. 10, 1387–1412 (2014).
7. Smale S., "Differentiable dynamical systems," Bull. Amer. Math. Soc., vol. 73, no. 6, 747–817 (1967).
8. Rolfsen D., *Knots and Links*, Amer. Math. Soc., Providence (2003).
9. Kosniowski C., *A First Course in Algebraic Topology*, Cambridge University, Cambridge (1980).



D. A. Baranov
Higher School of Economics, Nizhnii Novgorod, Russia
*E-mail address*: denbaranov0066@gmail.com

E. S. Kosolapov
St. Petersburg State University, St. Petersburg, Russia
*E-mail address*: egor-kosolapov@bk.ru

O. V. Pochinka
Higher School of Economics, Nizhnii Novgorod, Russia
https://orcid.org/0000-0002-6587-5305
*E-mail address*: olga-pochinka@yandex.ru