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Efficient Solvability of the Weighted Vertex Coloring Problem for Some Two Hereditary Graph Classes

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Abstract—The weighted vertex coloring problem for a given weighted graph is to minimize the number of colors so that for each vertex the number of the colors that are assigned to this vertex is equal to its weight and the assigned sets of vertices are disjoint for any adjacent vertices. For all but four hereditary classes that are defined by two connected 5-vertex induced prohibitions, the computational complexity is known of the weighted vertex coloring problem with unit weights. For four of the six pairwise intersections of these four classes, the solvability was proved earlier of the weighted vertex coloring problem in time polynomial in the sum of the vertex weights. Here we justify this fact for the remaining two intersections.

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INTRODUCTION

We consider only *simple graphs*, i.e., the undirected unlabeled graphs without loops and multiple edges. A set of graphs closed under isomorphism and vertex removal is called a *hereditary graph class*. Every hereditary graph class $\mathcal{X}$ can be defined by the set of forbidden induced subgraphs $\mathcal{Y}$, and this is written as follows: $\mathcal{X} = \mathrm{Free}(\mathcal{Y})$. The graphs of the class $\mathcal{X}$ are also called $\mathcal{Y}$*-free*. If $\mathcal{Y} = \{H\}$ then the graphs in $\mathcal{X}$ will be called H*-free*.

Let $G = (V, E)$ be a graph and let $w\colon V \to \mathbb{N} \cup \{0\}$ be a weight function. The pair (G, w) is called a *weighted graph*. A *vertex coloring* of some (G, w) is an arbitrary mapping $c\colon V \to 2^{\mathbb{N}}$ for which $|c(v)| = w(v)$ for every vertex $v \in V$ and $c(u) \cap c(v) = \varnothing$ for every edge $uv \in E$. Elements of the set $\bigcup_{v\in V}\{c(v)\}$ are called *colors*. It is assumed that the vertices of zero weight are not colored and hence can be removed from G. The use of zero weights is justified by the fact that we propose some reduction of weighted graphs that consists in removing some special vertices and reducing the weights of some other vertices with the preservation of nonnegativity of their weights. This makes it possible to control the weighted chromatic number.

The least number of colors in the vertex colorings of (G, w) is called the *weighted chromatic number* of (G, w) and denoted by $\chi_w(G)$. The *weighted vertex coloring problem* (further, briefly, *Problem* WVC) for a given weighted graph (G, w) and a number k consists in determining whether $\chi_w(G) \le k$ or not. The nonweighted version of Problem WVC (i.e., the version with unit vertex weights) is called the *vertex coloring problem* (*Problem* VC). Problems VC and WVC are classical NP-complete problems on graphs [1].

As usual, P_n, C_n, and O_n denote a simple path, a simple cycle, and the empty graph on n vertices; and $K_{p,q}$ is a complete bipartite graph with p vertices in one part and q vertices in the other. Designate as $K_{2,3}^{+}$ the graph obtained by adding to $K_{2,3}$ an edge incident to vertices of degree 3 in $K_{2,3}$. The graph W_4 is obtained from a cycle with 4 vertices by adding a new vertex and all edges incident to the added vertex

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and the vertices of the cycle. A graph *butterfly* is the result of identifying some two vertices belonging to two triangles.

Problem VC is polynomially solvable for the class Free($\{H\}$) if H is an induced subgraph in P_4 or in the graph $P_3 + P_1$ (i.e, in the disjoint union in the graphs P_3 and P_1); otherwise, Problem VC is NP-complete in this class [2]. However, no complete complexity classification of Problem VC exists even in the case of a pair of forbidden induced subgraphs. Moreover, for all hereditary classes but three defined by prohibitions with at most 4 vertices each the computational status of Problem VC is known [3]. Some recent results on the complexity of Problem VC in hereditary classes defined by prohibitions of small size are presented in [4–15].

In [9–14], the question was considered of the computational complexity of Problem VC for two connected 5-vertex forbidden induced subgraphs. At present, the complexity status of Problem VC is known for all sets of prohibitions of this kind but the following four:

- $\{K_{1,3},\ \text{butterfly}\}$,
- $\{P_5, H\}$, where $H \in \{K_{2,3}, K_{2,3}^{+}, W_4\}$.

The complexity status of Problem VC has not been clarified for each of these four classes. In [11], we proved that Problem VC is polynomially solvable for the class of $\{P_5, K_{1,3}\}$-free graphs. Consequently, the intersection of the class Free($\{K_{1,3}, \text{butterfly}\}$) with each of the remaining three classes under consideration gives an example of the polynomial solvability of Problem VC. It is not hard to verify that this remains valid for Problem WVC and polynomial solvability with respect to the sum of the vertex weights; this fact is proved in the present article. In [15], we considered the class Free($\{P_5, K_{2,3}, W_4\}$) and proved the solvability of Problem WVC in time polynomial of the sum of the vertex weights. In the present article, we consider the classes Free($\{P_5, K_{2,3}, K_{2,3}^{+}\}$) and Free($\{P_5, K_{2,3}^{+}, W_4\}$) and prove the polynomial-time solvability of Problem WVC for their graphs. These results imply that Problem VC is polynomially solvable in each of the above-mentioned three classes. This makes sure that Problem VC is polynomially solvable for each of the classes

$$\text{Free}(\{P_5, K_{2,3}\}), \qquad \text{Free}(\{P_5, K_{2,3}^{+}\}), \qquad \text{Free}(\{P_5, W_4\}).$$

The authors hope that their result will be useful in constructing polynomial algorithms for solving Problem VC in these classes.

1. NOTATIONS

Let v be a vertex in a graph. For every $k \geq 0$ let $N_k(v)$ be the set of vertices in a graph situated at distance exactly k from v. Clearly, $N_0(v) = \{v\}$ and $N_1(v) = N(v)$ is the neighborhood of v. For vertices $v_1, \dots, v_k$ in some graph and some subset V' of its vertex set, we adopt the notations

$$N_{V'}^{\cap}(v_1, v_2, \dots, v_k) = N(v_1) \cap N(v_2) \cap \dots \cap N(v_k) \cap V',$$

$$N_{V'}^{\cup}(v_1, v_2, \dots, v_k) = (N(v_1) \cup N(v_2) \cup \dots \cup N(v_k)) \cap V',$$

$$N_{V'}^{-}(v_1, v_2) = (N(v_1) \setminus N(v_2)) \cap V'.$$

If $k = 1$ then we write $N_{V'}(v_1)$ instead of $N_{V'}^{\cap}(v_1) = N_{V'}^{\cup}(v_1)$, and $N_{V'}^{-}(v_1)$ means the set $V' \setminus N(v_1)$. If V' coincides with vertex set of the graph then we write $N^{-}(v_1, v_2)$ instead of $N_{V'}^{-}(v_1, v_2)$.

Let G be a graph and let $A \subseteq V(G)$ and $B \subseteq V(G)$. The symbol $\overline{G}$ designates the complementary graph to G. Then $G(A)$ is a subgraph in G induced by a set of vertices A, and $G \setminus A$ is the result of removing all elements of A from G. A subset A is *completely adjacent* to a subset B if each vertex in A is adjacent to each vertex in B. A subset A is *completely nonadjacent* to a subset B if no vertex in A is adjacent to no vertex in B. We assume that the empty set of vertices is both completely adjacent and completely nonadjacent to any set of vertices.

An *independent subset* in a graph is a subset of its pairwise nonadjacent vertices. Each subset of pairwise adjacent vertices in a graph is a *clique*.

2. SOME ALGORITHMIC PROVISIONS FOR EFFICIENT ALGORITHMS FOR SOLVING PROBLEM WVC

Let $G = (V, E)$ be a graph. A subset $M \subseteq V$ is called a *module* of G if each vertex in $V \setminus M$ is either adjacent to all elements of M or to none of them. A module of the graph is called *trivial* if it contains only one vertex of the graph or all its vertices; otherwise, it is called *nontrivial*. A *separating clique of a graph* is its clique whose removal increases the number of its connected components. A connected graph is called *atomic* if it contains neither nontrivial modules no separating cliques. The following result is rather well known (for example, see Lemma 1 in [15]):

Lemma 1. *For every hereditary graph class, Problem* WVC *is reduced to the same problem for its atomic graphs in polynomial time.*

Given $v \in V$, refer to $V \setminus N(v)$ as the *antineighborhood of* v and denote it by $\overline{N(v)}$.

Lemma 2. *Let* (G, w) *be a weighted graph containing a vertex* v *such that* $\overline{N(v)} = \{v, v_1, \dots, v_k\}$ *is an independent set. Then*

$$\chi_w(G) = \chi_{w'}(G \setminus \{v\}) + w(v),$$

where $w'(u) = w(u)$ *for each vertex* $u \notin \overline{N(v)}$ *and* $w'(u) = \max(w(u) - w(v), 0)$ *for each vertex* $u \neq v$ *belonging to* $\overline{N(v)}$.

Proof. Since $\overline{N(v)}$ is independent, each color used for v can also be used for every vertex in $\overline{N(v)} \setminus \{v\}$ with the preservation of the admissibility of the coloring and the total number of colors used therein. Thus, it suffices to consider the colorings of (G, w) in which, for every vertex $u \in \overline{N(v)}$, some of the $\min(w(v), w(u))$ colors for u coincide with some of the $\min(w(v), w(u))$ colors for v. Removing v from G and decreasing $w(u)$ by $\min(w(v), w(u))$ for each $u \in \overline{N(v)} \setminus \{v\}$, we obtain a weighted graph $(G \setminus \{v\}, w')$ that can be colored with $\chi_w(G) - w(v)$ colors. Therefore,

$$\chi_w(G) \geq \chi_{w'}(G \setminus \{v\}) + w(v).$$

On the other hand, each coloring of $(G \setminus \{v\}, w')$ can be complemented to a coloring of (G, w) by using new $w(v)$ colors for coloring the vertex v and adding any new $w(u) - w'(u)$ colors for coloring each vertex $u \in \overline{N(v)} \setminus \{v\}$. Consequently,

$$\chi_w(G) \leq \chi_{w'}(G \setminus \{v\}) + w(v).$$

Lemma 2 is proved. □

Call an atomic graph *irreducible* if the antineighborhood of each its vertex is not an independent set.

Lemmas 1 and 2 imply the following

Lemma 3. *For every hereditary graph class, Problem* WVC *is reduced to the same problem for its irreducible graphs in polynomial time.*

Some exhaustion analog of Lemma 2 for the case when $\overline{N(v)} \setminus \{v\}$ is a clique is presented in the following obvious assertion:

Lemma 4. *Let* (G, w) *be a weighted graph containing a vertex* v *such that*

$$\overline{N(v)} \setminus \{v\} = \{v_1, \dots, v_k\}$$

is a clique. Let Ω *be the family of arrangements of weights* w' *to the vertices of* $G \setminus \{v\}$ *such that*

- $w'(u) = w(u)$ *for each vertex* $u \notin \overline{N(v)}$,
- *for some nonnegative integers* $w_1, w_2, \dots, w_k$ *whose sum is equal to* $w(v)$, *we have* $w'(v_i) = \max(w(v_i) - w_i, 0)$ *for each* $1 \leq i \leq k$.

Then

$$\chi_w(G) = \min_{w' \in \Omega} \chi_{w'}(G \setminus \{v\}) + w(v).$$

Proof. Consider an arbitrary vertex coloring c of (G, w). Given $1 \leq i \leq k$, denote by w_i the number of common colors for the vertices v and v_i. We may assume that $w_1 + w_2 + \dots + w_k = w(v)$; otherwise, the colors of

$$c(v) \setminus \bigcup_{i=1}^{k} c(v_i)$$

can be used for coloring $v_1, v_2, \dots, v_k$ without loss of optimality for c. Remove v from G; then, for each $1 \leq i \leq k$, replace the weight of v_i by $\max(w(v_i) - w_i, 0)$ and obtain some element $w^* \in \Omega$. Clearly,

$$\chi_w(G) = \chi_{w^*}(G \setminus \{v\}) + w(v) \geq \min_{w' \in \Omega} \chi_{w'}(G \setminus \{v\}) + w(v).$$

Now, consider an optimal vertex coloring of $(G \setminus \{v\}, w^{**})$, where

$$w^{**} = \arg\min_{w' \in \Omega} \chi_{w'}(G \setminus \{v\}),$$

and the corresponding nonnegative integers $w_1, w_2, \dots, w_k$ whose sum is equal to $w(v)$. This coloring gives a partial vertex coloring of (G, w). For each $1 \leq i \leq k$, replace the vertex v_i by $w(v_i)$ by using w_i new colors. We assume that for distinct v_i and v_j the sets of their new colors are disjoint. Use new $w(v)$ colors for coloring the vertex v. Thus,

$$\chi_w(G) \leq \chi_{w^{**}}(G \setminus \{v\}) + w(v) = \min_{w' \in \Omega} \chi_{w'}(G \setminus \{v\}) + w(v).$$

Hence,

$$\chi_w(G) = \min_{w' \in \Omega} \chi_{w'}(G \setminus \{v\}) + w(v).$$

Lemma 4 is proved. □

The number of solutions to the equation $w_1 + w_2 + \dots + w_k = w(v)$ in nonnegative integers is equal to $\binom{k}{w(v)+k-1}$. Therefore, Problem WVC for the pair (G, w) is reduced to $\binom{k}{w(v)+k-1}$ Problems WVC, each on the graph $G \setminus \{v\}$; moreover, passage to each of these problem is carried out in polynomial time.

Given a weighted graph $(G' = (V', E'), w')$, where $V' = \{v'_1, \dots, v'_n\}$, define the operation of *weight unitizing*. The result of this operation is a graph $G_{w'}$ with vertex set partitions into cliques $Q_1, \dots, Q_n$, where $|Q_i| = w'_i$ for each $1 \leq i \leq n$. For every i and j, the clique Q_i is *completely adjacent* with Q_j (respectively, *completely nonadjacent*) if and only if $v'_i v'_j \in E'$ (respectively, $v'_i v'_j \notin E'$).

We say that the operation of weight unitizing *preserves a graph class* $\mathcal{X}$ if, for every graph $G' \in \mathcal{X}$ and weight function w', we have $G_{w'} \in \mathcal{X}$. For example, it is not hard to check that the class $\mathcal{X} = \mathrm{Free}(\{P_5, K_{1,3}\})$ is preserved under weight unitizing. Obviously, we have

Lemma 5. *For every graph class preserved under weight unitizing, Problem* WVC *is reduced to Problem* VC *in time polynomial in the sum of the weights.*

In [14, Lemma 11], we proved

Lemma 6. *For every fixed $C > 0$, there exists $C' > 0$ such that Problem* WVC *for a graph $(G = (V, E), w)$ with $|V| \leq C$ is solvable in time*

$$O\Big(\Big(\sum_{v \in V} w(v)\Big)^{C'}\Big).$$

In [15] (see Lemma 10 in [14]), we proved

Lemma 7. *For every O_3-free graph $(G = (V, E), w)$, Problem* WVC *is solvable in time*

$$O\Big(\Big(\sum_{v \in V} w(v)\Big)^{3}\Big).$$

A graph is called a *Berge graph* if it belongs to the class

$$\text{Free}\left(\{C_{2i+1} \mid i \geq 1\} \cup \{\overline{C}_{2i+1} \mid i \geq 1\}\right).$$

A graph is called *perfect* if its chromatic and clique numbers (i.e., the weighted chromatic number for the unit collection of weights and the size of a greatest clique) are equal and this holds for each of its induced subgraph. It was proved in [16] that a graph is perfect if and only if this is a Berge graph.

The following assertion is known (see [17]):

Lemma 8. *Problem* WVC *is polynomially solvable for perfect graphs.*

3. EFFICIENT SOLVABILITY OF PROBLEM WVC FOR $\{P_5, K_{2,3}, K_{2,3}^+\}$-FREE GRAPHS

Describe the general scheme of our algorithm for $\{P_5, K_{2,3}, K_{2,3}^+\}$-free graphs. By Lemma 3, we can consider only irreducible graphs of the class $\text{Free}(\{P_5, K_{2,3}, K_{2,3}^+\})$.

In this section, we prove that each of these graphs $G = (V, E)$ either has at most 10 vertices or has at most four induced subgraphs each of which is isomorphic to $K_{1,3}$. In the first case, we apply to (G, w) some algorithm polynomial in the sum of the weights that exists by Lemma 6. In the case when $|V| \geq 11$ and there is a vertex v with $G(N(v)) \notin \text{Free}(\{O_3\})$, we apply the elimination from Lemma 4 to v, which turns out to be polynomial since $N(v) \setminus \{v\}$ is a clique on at most three vertices. Therefore, Problem WVC for $\{P_5, K_{2,3}, K_{2,3}^+\}$-free graphs is reduced to the same problem for $\{P_5, K_{1,3}\}$-free graphs in time polynomials in the sum of the weights. Problem WVC is solvable in polynomial time in the sum of the weights of the vertices for $\{P_5, K_{1,3}\}$-free graphs because Problem VC is polynomially solvable in $\text{Free}(\{P_5, K_{1,3}\})$ (see [11]) and the operation of weight unitizing preserves the class $\text{Free}(\{P_5, K_{1,3}\})$.

Let $G = (V, E)$ be an irreducible $\{P_5, K_{2,3}, K_{2,3}^+\}$-free graph with a vertex v such that $V' = N(v)$ contains an independent set $I = \{a, b, c\}$. Redenote the set $N_2(v)$ by V''.

Lemma 9. $N_3(v)$ *is empty.*

Proof. Suppose to the contrary that there exists $x \in N_3(v)$ adjacent to a vertex $y \in V''$. Note that $I \not\subseteq N(y)$ since otherwise I, v, and y would induce $K_{2,3}$.

Then $N_{V'}^-(y)$ is completely adjacent to $N_{V'}(y)$; otherwise, a vertex in $N_{V'}^-(y)$ and a vertex in $N_{V'}(y)$ nonadjacent to it together with v, x, and y induce P_5. Therefore, $N_I(y) \neq \varnothing$; otherwise, I, v, and an arbitrary vertex in $N_{V'}(y)$ induce $K_{2,3}^+$. It follows that $I \not\subseteq N(y)$ and $N_I(y) \neq \varnothing$; i.e., I cannot be independent; a contradiction. Hence, $N_3(v) = \varnothing$. Lemma 9 is proved. ◻

Lemma 10. *Given adjacent vertices $x, y \in V''$, we have either $N_{V'}^{\cup}(x, y) = V'$ or $N_{V'}(x) = N_{V'}(y)$.*

Proof. Suppose the contrary. Redenote $V' \setminus N_{V'}^{\cup}(x, y)$ by N'; while $N_{V'}^-(x, y) \cup N_{V'}^-(y, x)$, by N''; and $N_{V'}^{\cap}(x, y)$, by N'''.

By assumption, $N' \neq \varnothing$ and $N'' \neq \varnothing$. Then N' is completely adjacent to N'' since otherwise x, y, and v together with an element of N' and a nonadjacent element in N'' would induce P_5. Owing to this and the fact that I is an independent set and $G \in \text{Free}(\{K_{2,3}, K_{2,3}^+\})$, we have $I \not\subseteq N(x)$, $I \not\subseteq N(y)$ and $I \not\subseteq N'$, $I \not\subseteq N''$. Consequently, either $I \cap N' \neq \varnothing$ or each of the sets $N_{V'}^-(x, y)$, $N_{V'}^-(y, x)$, and N''' contains one element of I.

Suppose that $I \cap N' \neq \varnothing$. We can assume that $a \in I \cap N'$ and $u \in N_{V'}^-(x, y)$. Then, obviously, $ua \in E$, $b, c \in N'''$, and u is adjacent at least to one of the vertices b or c. The vertex u cannot be simultaneously adjacent to b and c since otherwise a, b, c, v, and u would induce $K_{2,3}^+$; therefore, we can assume that $ub \in E$ and $uc \notin E$. Consequently, a, u, b, y, and c induce P_5. Hence, $I \cap N' = \varnothing$.

Suppose that each of the sets $N^{-}_{V'}(x,y)$, $N^{-}_{V'}(y,x)$, and N''' contains one element from I. We may assume that

$$a \in N^{-}_{V'}(x,y), \qquad b \in N^{-}_{V'}(y,x), \qquad c \in N''', \qquad w \in N'.$$

Clearly, $wa \in E$ and $wb \in E$. Since $G \in \mathrm{Free}(\{K^{+}_{2,3}\})$, we have $wc \notin E$. Then b, w, a, x, and c induce P_5; a contradiction. Hence, the initial assumption fails.

Lemma 10 is proved. □

Lemma 11. *Given adjacent vertices $x, y \in V''$, we have $N^{\cup}_{V'}(x,y) = V'$.*

Proof. Suppose the contrary. Then $N_{V'}(x) = N_{V'}(y)$ and $N_{V'}(x) \subset V'$ by Lemma 10. Denote by V^* the set of the vertices $z \in V''$ such that $N_{V'}(z) = N_{V'}(x)$. Designate as V^{**} the set of the vertices of the connected component of the subgraph $G(V^*)$ containing x and y. Prove that V^{**} is a nontrivial module in G.

Assume that V^{**} is not a module in G. Then there exist adjacent vertices $v_1 \in V^{**}$ and $v_2 \in V^{**}$ and a vertex $v_3 \in V'' \setminus V^{**}$ such that $v_1v_3 \notin E$ and $v_2v_3 \in E$. Clearly, $v_3 \notin V^*$. Consequently, $N^{\cup}_{V'}(v_2, v_3) = V'$ by Lemma 10. And so, there exists a vertex $u \in V'$ adjacent to v_3 and nonadjacent to v_2. Then v_1, v_2, v_3, u, and v induce P_5. Thus, V^{**} is a nontrivial module of G. Therefore, the initial assumption fails. Lemma 11 is proved. □

Lemma 12. *If $V' = I$ then $|V| \leq 10$.*

Proof. Since G is $K_{2,3}$-free, none of the vertices of V'' is adjacent to all vertices in the set $V' = I$. Suppose that $z' \in N^{\cap}(a,b) \setminus \{v\}$ and $z'' \in V'' \setminus N^{\cap}(a,b)$. If $z''c \in E$ then $z'z'' \in E$ since otherwise z'', c, v, and a or b and z' would induce P_5. If $z''c \notin E$ then z'' must be adjacent to a or b but not to both of them. Then $z'z'' \notin E$ since otherwise c, v, and a or b, z', and z'' would induce P_5. Consequently, $N^{\cap}(a,b) \setminus \{v\}$ is a module in G; therefore, it contains at most one vertex. Likewise, each of the sets $N^{\cap}(a,c) \setminus \{v\}$ and $N^{\cap}(b,c) \setminus \{v\}$ contains at most one vertex.

Suppose that $N(z') \cap I = \{a\}$, $z'' \in V''$, and $N(z'') \cap I \neq \{a\}$. We showed above that if $z'' \in N^{\cap}(a,b) \cup N^{\cap}(a,c)$ then $z'z'' \notin E$. If z'' is nonadjacent to a then $z'z'' \in E$ since otherwise z', a, v, and b or c and z'' would induce P_5. Consequently, $N(a) \setminus N^{\cup}(b,c)$ is a module in G and contains at most one vertex. Analogously,

$$|N(b) \setminus N^{\cup}(a,c)| \leq 1, \qquad |N(c) \setminus N^{\cup}(a,b)| \leq 1.$$

Hence, $|V| \leq 10$.

Lemma 12 is proved. □

Henceforth we assume that v has a neighborhood of the greatest size among all vertices of G containing three pairwise nonadjacent vertices and $|V'| \geq 4$.

Lemma 13. *The set V'' is a clique with two or three vertices and the vertex of degree 3 of each induced subgraph $K_{1,3}$ in G belongs to the set $V'' \cup \{v\}$.*

Proof. Let x and y be arbitrary adjacent vertices in V''. Such vertices exist since $\overline{N(v)}$ must not be an independent set.

Put

$$N_1 = N^{-}_{V'}(x,y), \qquad N_2 = N^{-}_{V'}(y,x), \qquad N_3 = N^{\cap}_{V'}(x,y).$$

By Lemma 11, $N^{\cup}_{V'}(x,y) = V'$. So, from the independence of I and the $K_{2,3}$-freeness of G it follows that $I \nsubseteq N(x)$, $I \nsubseteq N(y)$ and $N_1 \neq \varnothing$, $N_2 \neq \varnothing$. Since $G \in \mathrm{Free}(\{K_{2,3}, K^{+}_{2,3}\})$, we have $|N^{\cap}_{V'}(x', y')| \leq 1$ for every nonadjacent vertices $x' \in V''$ and $y' \in V''$.

Show that there is no vertex in V'' adjacent to only one of the vertices x and y. Suppose that some $z \in V''$ is adjacent to x and not adjacent to y. To avoid the appearance of a subgraph P_5 induced by z, x,

y, a vertex from N_2, and v, it is necessary that $\{z\}$ be completely adjacent to N_2. Therefore, N_2 contains exactly one element a' and $\{z\}$ is completely nonadjacent to $(N^{\cap}_{V'}(y,x) \setminus \{a'\})$. If $\{z\}$ is not completely adjacent to N_1 then $N(x)$ contains three pairwise nonadjacent vertices (y, z, and some element from N_1); moreover, $|N(x)| > |V'|$. Clearly, $I \nsubseteq N(z)$. Hence, since $I \nsubseteq N(x)$ and $I \nsubseteq N(y)$, we can assume that $a' = a$, $b \in N_3$, and $c \in N_1$. Consequently, G contains the graph P_5 induced by c, z, a, y, and b. Therefore, our assumption of the existence of a vertex z fails. Thus, $G(V'')$ is a P_3-free graph, i.e., a disjoint sum of complete graphs.

Suppose now that z is a vertex from V'' adjacent neither to x nor to y. Consider the two cases: $N(z) \cap (N_1 \cup N_2) = \varnothing$ and $N(z) \cap (N_1 \cup N_2) \neq \varnothing$.

Assume that $N(z) \cap (N_1 \cup N_2) = \varnothing$. Then $N(z) \cap N_3 \neq \varnothing$. Clearly, $|N(z) \cap N_3| = 1$. Since $N(z) \cap N_3$ is not a separating clique of G, there exists a vertex $z' \in V''$ adjacent to z. By Lemma 11, $N^{\cup}_{V'}(z,z') = V'$. Consequently, $N(z') \cap (N_1 \cup N_2) \neq \varnothing$. The vertex z' is adjacent neither to x nor to y because otherwise, x, y, z, and z' would form a clique, which is impossible. Replacing z' by z, we can consider only the case when $N(z) \cap (N_1 \cup N_2) \neq \varnothing$.

Suppose that $N(z) \cap N_2 \neq \varnothing$; for $N(z) \cap N_1 \neq \varnothing$ the argument is similar. Clearly, $|N^{\cap}_{V'}(x,z)| \leq 1$ and $|N^{\cap}_{V'}(y,z)| \leq 1$; therefore, $N(z) \cap N_2 = \{a'\}$. We can assume that $N(z) \cap N_1 = \{b'\}$ since otherwise there must exist a vertex $z'' \in V''$ adjacent to z because $\{a'\}$ is not a separating clique. Then $N^{\cup}_{V'}(z,z'') = V'$ by Lemma 11; therefore, $V' \setminus \{a'\} \subseteq N(z'')$. Clearly, $z''x \notin E$ and $z''y \notin E$. Since $|V'| \geq 4$, either $|N^{\cap}_{V'}(z'',x)| \geq 2$ or $|N^{\cap}_{V'}(z'',y)| \geq 2$. Consequently, G contains either an induced $K_{2,3}$ or an induced $K^+_{2,3}$. Hence, we can assume that $N(z) \cap N_1 = \{b'\}$; therefore, $N(z) \cap N_3 = \varnothing$.

Since $I \nsubseteq N(x)$ and $I \nsubseteq N(y)$, we can assume by symmetry considerations that $N_I(x) = \{b,c\}$, $a \in N_2$ and $c \in N_1$. The set $\{a'\}$ is completely adjacent to $N_{V'}(x) \setminus \{b'\}$ since otherwise the vertices z, a', v, some vertex in $N_{V'}(x) \setminus \{b'\}$, and also x would induce P_5. Thus, $a'b \in E$ or $a'c \in E$, and so $a' \neq a$. Let $b \in N_1$. Then $b' \in \{b,c\}$; otherwise, a', x, y, b, and c would induce $K_{2,3}$. We can assume that $b' = c$. Hence, z, c, x, y, and a induce P_5. Suppose that $b \in N_3$. If $b' = c$ then z, c, x, y, and a induce P_5. Let $b' \neq c$. Then $a'b \in E$ and $a'c \in E$. Then $b'a \in E$; otherwise, z, b', x, y, and a would induce P_5. The vertices a' and a must not be adjacent; otherwise, v, a', a, b, and c would induce $K^+_{2,3}$. The vertex b' is adjacent at least to one of the vertices a' and b since otherwise a, b', z, a', and b induce P_5. If $b'b \in E$ then $b'c \notin E$; otherwise, v, b', a, b, and c would induce $K^+_{2,3}$. Then $b'a' \in E$; otherwise, a, b', b, a', and c would induce P_5 and a', b', c, x, and y would induce $K_{2,3}$. If $b'a' \in E$ and $b'b \notin E$ then $b'c$ since otherwise x, v, b, b', and c would induce $K_{2,3}$. Then c, b', a, y, and b induce P_5.

Thus, every vertex in $V'' \setminus \{x,y\}$ is adjacent both to x and y. Hence, since $G(V'')$ is a disjoint union of complete graphs, V'' is a clique. By Lemma 11, each of a, b, and c is adjacent at least to $|V''| - 1$ vertices in V''; therefore, if $|V''| \geq 4$ then V'' has a vertex adjacent to a, b, and c simultaneously. Therefore, in this case, G contains an induced subgraph $K_{2,3}$. Thus, $|V''| \leq 3$.

Suppose that some induced subgraph $K_{1,3}$ in G is induced by vertices v', v'_1, v'_2, and v'_3, where $v' \in V'$ has degree 3 in $K_{1,3}$. Then, obviously, $\{v'_1, v'_2, v'_3\} \nsubseteq V'$ since otherwise v, v', v'_1, v'_2, and v'_3 induce a subgraph $K^+_{2,3}$. Since V'' is a clique, V'' contains exactly one of the vertices v'_1, v'_2, and v'_3. We can assume that $v'_1 = x$ and then $v'_2, v'_3 \in N_2$. In this case, v', x, y, v'_2, and v'_3 induce either $K_{2,3}$ (if $v' \in N_1$) or $K^+_{2,3}$ (if $v' \in N_3$); a contradiction.

Lemma 13 is proved. □

Theorem 1. *Problem* WVC *is solvable in time polynomial in the sum of the weights of the vertices in the class of* $\{P_5, K_{2,3}, K^+_{2,3}\}$*-free graphs.*

Proof. Problem VC is polynomially solvable for $\{P_5, K_{1,3}\}$-free graphs (see [11, Lemma 9]).

The operation of weight unitizing preserves the class $\mathrm{Free}(\{P_5, K_{1,3}\})$. Consequently, by Lemma 5, Problem WVC is solvable in time polynomial in the sum of the weights in the class $\mathrm{Free}(\{P_5, K_{1,3}\})$. Thus, we can consider only graphs in $\mathrm{Free}(\{P_5, K_{2,3}, K^+_{2,3}\})$ that contain an induced subgraph $K_{1,3}$

(note that the membership of a graph in Free($\{K_{1,3}\}$) is obviously checked in polynomial time). By Lemmas 3 and 6, we can consider only such irreducible graphs with at least 11 vertices. Lemmas 12 and 13 guarantee that each of these graphs contains at most 4 vertices that can be vertices of degree 3 in their induced subgraphs $K_{1,3}$. The antineighborhhods without the vertices themselves induce cliques on at most three vertices. To each such vertex v, we can apply the elimination of Lemma 4, and this can be done in time polynomial in the sum of the weights.

Thus, Problem WVC in the class Free($\{P_5, K_{2,3}, K_{2,3}^+\}$) is reduced to the same problem in the class Free($\{P_5, K_{1,3}\}$) in time polynomial in the sum of the weights. Thus, the theorem holds.

Theorem 1 is proved. □

4. EFFICIENT SOLVABILITY OF PROBLEM WVC FOR $\{P_5, K_{2,3}^+, W_4\}$-FREE GRAPHS

Describe the general scheme of our algorithm for $\{P_5, K_{2,3}^+, W_4\}$-free graphs. By Lemma 1, it suffices to consider only the atomic graphs from Free($\{P_5, K_{2,3}^+, W_4\}$). In this section, we prove that every such graph is either O_3-free or perfect or has at most 161 vertices. Consequently, by Lemmas 6–8, there is an algorithm polynomial in the sum of the weights solving Problem WVC for $\{P_5, K_{2,3}^+, W_4\}$-free graphs.

Let H be an atomic $\{P_5, K_{2,3}^+, W_4\}$-free graph that contains an induced subgraph $\overline{C}_7$. Since it is more convenient to work with the complement to H, consider the graph $G = \overline{H}$ which contains an induced subgraph C_7.

Lemma 14. *The graph $G = (V, E)$ is isomorphic to a 7-cycle or contains an induced 5-cycle.*

Proof. Suppose that G is C_5-free. Since $G \in$ Free($\{\overline{W}_4\}$), each vertex in G is adjacent to some two vertices in a cycle $C_7 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$. Below we take the vertices of this cycle modulo 7.

Let $u \notin V(C_7)$ and let $\{v_i, v_{i+1} \dots, v_{i+k}\}$ be the greatest set of neighbors of u on the 7-cycle consisting of consecutive vertices of the cycle. Let $1 \le k \le 5$. Thus $uv_{i-1} \notin E$ and $uv_{i+k+1} \notin E$; otherwise, we arrive to a contradiction to the choice of the set. We have $uv_{i-2} \notin E$ and $uv_{i+k+2} \notin E$; otherwise, v_{i-2}, v_{i-1}, v_i, v_{i+1}, and u or v_{i+k+2}, v_{i+k+1}, v_{i+k}, v_{i+k-1}, and u would induce $\overline{P}_5$. Therefore, $k \ne 5$. We have $uv_{i-3} \notin E$ and $uv_{i+k+3} \notin E$; otherwise, v_{i-3}, v_{i-2}, v_{i-1}, v_i, and u or v_{i+k+3}, v_{i+k+2}, v_{i+k+1}, v_{i+k}, and u would induce C_5. Therefore, $k \ne 4$. Consequently, $k \in \{1, 2, 3\}$ and $N_{V(C_7)}(u) = \{v_i, v_{i+1} \dots, v_{i+k}\}$. Thus, G is not $\overline{K}_{2,3}^+$-free. Hence, either $k = 7$ or u cannot be adjacent to two consecutive vertices of the 7-cycle. Since G is C_5-free, in the last case, $N_{V(C_7)}(u) = \{v_i, v_{i+2}\}$ for some i.

Recall that the graph H does not contain nontrivial modules. Consequently, G does not contain nontrivial modules too. Since G is $\overline{P}_5$-free, each vertex adjacent to all vertices of the 7-cycle must be adjacent to each of the vertices having exactly two neighbors on the 7-cycle.

Suppose that $u \notin V(C_7)$, $N_{V(C_7)}(u) = \{v_i, v_{i+2}\}$, and the set $\{u, v_{i+1}\}$ is not a module. Then there exists a vertex $u' \notin V(C_7)$, $N_{V(C_7)}(u') = \{v_j, v_{j+2}\}$, for which $uu' \in E$ and $v_{i+1}u' \notin E$ or $uu' \notin E$ and $v_{i+1}u' \in E$. The second case is possible only for $j = i \pm 1$; but then u, v_i, u', v_{i+3}, and v_{i+5} or u, v_{i-1}, u', v_{i+2}, and v_{i+4} induce a subgraph $\overline{W}_4$. Consider the first case. Here $j \notin \{i-1, i+1\} \cup \{i-2, i+2\}$; otherwise, the vertices u, u', v_{i+2}, v_{i+3}, and v_{i+4} or u, u', v_{i-2}, v_{i-1}, and v_i would induce a subgraph $\overline{P}_5$. But $j \notin \{i-3, i+3\}$; otherwise, v_{i-3}, u', u, v_{i+2}, and v_{i+3} or v_{i-1}, v_i, u, u', and v_{i+5} would induce a subgraph C_5. Moreover, $j \ne i$; otherwise, the vertices u, u', v_{i+1}, v_{i+3}, and v_{i+4} induce a subgraph $\overline{W}_4$; a contradiction.

Hence, there is no vertex not belonging to a 7-cycle and having exactly two neighbors on it. Thus, $V(C_7)$ is a module. Hence, G is isomorphic to a 7-cycle.

Lemma 14 is proved. □

Clearly, every $\{P_5, W_4, C_5, \overline{C}_7\}$-free graph is perfect. Therefore, every $\{P_5, K^+_{2,3}, W_4\}$-free graph not containing induced subgraphs C_5 and $\overline{C}_7$ is perfect. Thus, by Lemmas 1 and 8, Problem WVC is solvable for these graphs in polynomial time. Recalling Lemmas 6 and 14, it remains to consider only $\{P_5, K^+_{2,3}, W_4\}$-free graphs containing an induced C_5. This is in essence the main contents of this section, where practically all the lemmas are aimed to finding the exact structure of the arising graphs. There are rather many such lemmas, and they all are of technical nature.

Let $G = (V, E)$ be an atomic $\{P_5, K^+_{2,3}, W_4\}$-free graph that contains an induced cycle $C = (v_1, v_2, v_3, v_4, v_5)$. Henceforth, the indices of the vertices in the cycle are understood modulo 5. Introduce the following notations for the graph G:

- $V_i = \{x \notin V(C) \mid N_{V(C)}(x) = \{v_i, v_{i+2}\}\}$,
- $V'_i = \{y \notin V(C) \mid N_{V(C)}(y) = \{v_i, v_{i+1}, v_{i+2}\}\}$,,
- $V''_i = \{z \notin V(C) \mid N_{V(C)}(z) = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}\}$,
- $V'''_i = \{t \notin V(C) \mid N_{V(C)}(t) = \{v_i, v_{i+2}, v_{i+3}\}\}$,
- V'''' is the set of the vertices each of which is adjacent to all vertices in $V(C)$,
- $S = V \setminus \Big(V(C) \cup \bigcup\limits_{i=1}^{5}(V_i \cup V'_i \cup V''_i \cup V'''_i) \cup V''''\Big)$,
- $\widehat{V}$ is the set of all vertices not belonging to C and, simultaneously, having neighbors both on C and in S.

Clearly, every vertex not belonging to C and having a neighbor on C belongs to the set

$$\bigcup_{i=1}^{5}\big(V_i \cup V'_i \cup V''_i \cup V'''_i\big) \cup V''''.$$

Recalling that G is a $\{P_5, K^+_{2,3}, W_4\}$-free graph, it is not hard to verify the next

Lemma 15. *The following hold:*

(i) *For each i, the sets V'_i, V''_i, and V'''' are cliques.*

(ii) *The set $\bigcup\limits_{i=1}^{5}(V_i \cup V'_i)$ is completely nonadjacent to S.*

(iii) *For each i, we have*

(1) *V_i is completely adjacent to $V_{i-1} \cup V_{i+1} \cup V'''_{i+1}$ and completely nonadjacent to*

$$V'_i \cup V'_{i+2} \cup V'_{i+3} \cup V''_{i-1} \cup V''_i \cup V'''';$$

moreover, $V'_{i-1} \cup V'_{i+1} \cup V''_{i+1} \cup V''_{i+3} = \varnothing$ if $V_i \neq \varnothing$;

(2) *V'_i is completely adjacent to*

$$V'_{i-1} \cup V'_{i+1} \cup V''_{i-1} \cup V''_i \cup V'''_{i+1} \cup V'''_{i+3} \cup V'''_{i+4} \cup V''''$$

and completely nonadjacent to $V''_{i+2} \cup V'''_i$ cupV'''_{i+2}; moreover, $V'''_{i+1} = \varnothing$ if $V'_i \neq \varnothing$;

(3) *V''_i is completely adjacent to $V''_{i-1} \cup V''_{i+1} \cup V''''$ and completely nonadjacent to $V''_{i-2} \cup V''_{i+2} \cup V'''_i \cup V'''_{i-2}$; moreover, $V'''_{i+1} \cup V'''_{i+2} \cup V'''' = \varnothing$ if $V''_i \neq \varnothing$;*

(4) *V'''_i is completely adjacent to $V'''_{i-1} \cup V'''_{i+1}$ and completely nonadjacent to V''''.*

Lemma 16. *V_i is independent for each i.*

Proof. Suppose that V_i is not independent. Denote by $H^* = (V^*, E^*)$ an arbitrary connected component of $G(V_i)$ with at least two vertices. Lemma 15 (items (ii) and (iii), (1)) implies that the elements of V^* can have neighbors only in the set

$$V^* \cup \{v_i, v_{i+2}\} \cup V''_{i+2} \cup \bigcup_{j=1,\, j\neq i+1}^{5} V'''_i.$$

Since V^* is not a nontrivial module in G, there are adjacent vertices $a, b \in V^*$ and $c \notin V_i$ such that $ac \notin E$ and $bc \in E$. Clearly,

$$c \in V''_{i+2} \cup \bigcup_{j=1,\, j\neq i+1}^{5} V'''_i.$$

If $c \in V''_{i+2}$ then v_i, v_{i+2}, a, b, and c induce W_4.

If $c \in V'''_{i+3} \cup V'''_{i+4}$ then a, b, c, v_{i+3}, and v_{i+4} induce P_5.

If $c \in V'''_i \cup V'''_{i+2}$ then v_i, v_{i+2}, a, b, and c induce W_4; a contradiction.

Hence, V_i is an independent set. Lemma 16 is proved. $\square$

In the following four lemmas, we consider the situation when $S = \varnothing$ while proving that either $G \in \mathrm{Free}(\{O_3\})$ or G has few vertices:

Lemma 17. *If* $\bigcup_{i=1}^{5} \left(V_i \cup V'''_i\right) \cup S = \varnothing$ *then* G *is* O_3*-free.*

Proof. Suppose the contrary. Let $\{a, b, c\}$ be an independent set of G. Clearly, $|C \cap \{a, b, c\}| \leq 1$.

If $v_i \in \{a, b, c\}$ (say, $v_i = a$) then, by Lemma 15 (items (i), (iii), (2), and (iii), (3)), b and c are adjacent. This also implies that $V'''' \cap \{a, b, c\} = \varnothing$; therefore, $C \cap \{a, b, c\} = \varnothing$ and two elements of the set $\{a, b, c\}$ belong to one of the sets $\bigcup_{i=1}^{5} V'_i$ and $\bigcup_{i=1}^{5} V''_i$. Let these be the vertices a and b.

If $a, b \in \bigcup_{i=1}^{5} V'_i$ then, by Lemma 15 (items (i) and (iii), (2)), we assume that $a \in V'_i$ and $b \in V'_{i+2}$. This and items (i), (iii), (2), and (iii), (3) of Lemma 15 imply that $c \in V''_{i+3}$; therefore, a, v_{i+1}, c, v_{i+3}, and b induce P_5.

If $a, b \in \bigcup_{i=1}^{5} V''_i$ then, by Lemma 15 (items (i) and (iii), (3)), we may assume that $a \in V''_i$ and $b \in V''_{i+2}$. By items (i), (iii), (2) and (iii), (3) of Lemma 15, $c \in V'_{i+4}$. Then a, v_{i+1}, c, v_{i+4}, and b induce P_5. Hence, the initial assumption fails. Lemma 17 is proved. $\square$

Lemma 18. *If* $S = \varnothing$ *and for some* i *vertices* $a \in V'_i$ *and* $b \in V'_{i+2}$ *are adjacent then either* $G \in \mathit{Free}(\{O_3\})$ *or* $|V| \leq 17$.

Proof. Owing to Lemma 15 (item iii, (2)) and the W_4-freeness of G, we have $V'''' = \varnothing$. Since $G \in \mathrm{Free}(\{K^+_{2,3}\})$, by Lemma 15 (item iii, (2)), V'''_i is a clique.

By Lemma 15 (item iii, (1)),

$$V_{i+1} = V_{i+3} = V_{i+4} = \varnothing.$$

At the same time, if there exists a vertex $c \in V_i \cup V_{i+2}$ then c is adjacent neither to a nor to b by Lemma 15 (item iii, (1)); but then either v_{i+3}, b, a, v_i, and c or v_{i+1}, a, b, v_{i+4}, and c induce P_5. Therefore, $V_i = V_{i+2} = \varnothing$.

By Lemma 15 (item iii, (2)), $V'''_{i+1} = V'''_{i+3} = \varnothing$. Clearly, there is no vertex $c \in V'''_{i+2}$; otherwise, $ac \notin E$ and $bc \notin E$ by Lemma 15 (iii, (2)), and the vertices c, v_i, a, b, and v_{i+3} induce P_5. By Lemma 15 (items (iii), (2) and (iii), (4)) and the W_4-freeness of G, either $V'''_i = \varnothing$ or $V'''_{i+4} = \varnothing$.

Symmetry considerations and Lemma 17 enable us to assume that $V'''_i \neq \varnothing$. Consequently, $V'''_{i+4} = \varnothing$ and also $V''_{i+3} = V''_{i+4} = \varnothing$ by Lemma 15 (item iii, (3)). Since $G \in \mathrm{Free}(\{K^+_{2,3}\})$ by Lemma 15 (item iii, (2)), V'''_i is a clique. If there exists a vertex $c \in V''_i$ then $ac \in E$ and $\{c\}$ is completely nonadjacent to V'''_i by Lemma 15 (item iii, (2)). The vertices b and c must be adjacent; otherwise, a, b, c, v_{i+2}, and v_{i+3}

would induce W_4. Therefore, v_{i+3}, v_{i+4}, b, c, and an arbitrary vertex in V_i''' would induce $K_{2,3}^+$. Hence, $V_i''=\varnothing$. If there exists a vertex $c\in V_{i+2}''$ then $ac\notin E$, $bc\in E$, and $\{c\}$ is completely nonadjacent to V_i''' by items (iii), (2) and (iii), (3) of Lemma 15. Then v_{i+2}, a, b, c, and an arbitrary vertex in V_i''' induce some $K_{2,3}^+$. Consequently, $V_{i+2}''=\varnothing$.

Since $G\in \mathrm{Free}(\{K_{2,3}^+\})$, by Lemma 15 (i), no vertex in V_i''' is adjacent to two vertices in V_{i+1}''. Consequently, by Lemma 15 (item iii, (2)), for each vertex $x\in V_{i+1}''$ we have $|N_{V_i'''}(x)|\le 1$ since otherwise $N_{V_i'''}(x)$ is a nontrivial module. If there exist vertices $x_1,x_2\in V_{i+1}''$ adjacent respectively to the vertices $y_1,y_2\in V_i'''$ then $x_1y_2, x_2y_1\notin E$ and v_{i+2}, x_1, x_2, y_1, and y_2 induce W_4. Hence, between V_i''' and V_{i+1}'', there is at most one edge. Therefore, by Lemma 15 (item iii, (3)), $|V_{i+1}''|\le 2$; otherwise, G would contain a nontrivial module.

Recall that $V_{i+1}''\cup V_i'''$ is completely adjacent to V_{i+2}' by Lemma 15 (item iii, (2)). Thus, $\{a\}$ is completely adjacent to V_{i+1}'' since otherwise some of its vertices and a, b, v_{i+1}, and v_{i+2} induce W_4. Therefore, V_{i+1}'' is completely nonadjacent to V_i''' since otherwise a, b, and v_{i+4} together with a pair of adjacent vertices, one of which belongs to V_{i+1}'' and the other, to V_i''', induce $K_{2,3}^+$. There are no such nonadjacent vertices $c\in V_{i+1}''$ and $a'\in V_i'$; otherwise, $a'\ne a$ and $\{c\}$ is completely adjacent to V_i''' for avoiding the induction of P_5 by a', v_{i+1}, c, v_{i-2}, and a vertex in V_i'''; a contradiction. No vertex $a'\in V_i'$ can be adjacent to vertices $b',b''\in V_{i+2}'$ since otherwise, by items (i) and (iii), (2) of Lemma 15, the vertices v_{i+4}, a', b', and b'' together with an arbitrary vertex V_i''' induce $K_{2,3}^+$. If there exists an edge $a^*b^*\ne ab$, where $a^*\in V_i'$ and $b^*\in V_{i+2}'$, then $b=b^*$. Indeed, if $b\ne b^*$ then $a\ne a^*$ and $ab^*\notin E$, $ba^*\notin E$, and v_{i+2}, a, b, a^*, and b^* induce W_4.

Owing to Lemma 15 (item iii, (2)), the set V_{i+4}' must be empty since otherwise $V_i'''=\varnothing$. The set V_{i+3}' is completely nonadjacent to V_i' since otherwise G would contain an induced subgraph P_5. By items (i) and (iii), (2) of Lemma 15, for every vertex $x\in V_{i+2}'$, $N_{V_i'}(x)$ is a modulus of G; therefore, it contains at most one vertex. Hence, $V_i'\setminus\{a\}$ and $V_{i+2}'\setminus\{b\}$ are modules in G and $\max(|V_i'|,|V_{i+2}'|)\le 2$. By analogy, $\max(|V_{i+1}'|,|V_{i+3}'|)\le 2$. Lemma 15 (item ii, (2)) and the facts that there is at most one edge between V_i''' and V_{i+1}'' and G contains no nontrivial modules imply that $|V_i'''|\le 2$. Thus, $|V|\le 17$.

Lemma 18 is proved. □

Lemma 19. *If $S=\varnothing$ and for some i vertices $a\in V_i'$ and $b\in V_{i+1}''$ are nonadjacent then either $G\in$ Free$(\{O_3\})$ or $|V|\le 14$.*

Proof. Items (iii), (1) and (iii), (3) of Lemma 15 imply that

$$\bigcup_{j=1,\ j\ne i+2}^{5} V_j=\varnothing,\qquad V_{i-2}'''\cup V_{i+1}'''\cup V_{i+2}'''\cup V''''=\varnothing.$$

Note that if there exists a vertex $c\in V_{i+2}$ then, by Lemma 15 (item iii, (1)), $ca\notin E$ and $cb\notin E$. Then c, v_{i-1}, b, v_{i+1}, and a induce P_5. Consequently, $V_{i+2}=\varnothing$. If there exist vertices $c_1\in V_{i-1}'''$ and $c_2\in V_i'''$ then, by items (iii), (2), (iii), (3), and (iii), (4) of Lemma 15,

$$ac_1\in E,\qquad ac_2\notin E,\qquad bc_1\notin E,\qquad c_1c_2\in E.$$

At the same time, $c_2b\in E$ since otherwise c_2, v_{i-2}, b, v_{i+1}, and a induce P_5. Then v_{i+1}, c_1, v_{i+2}, b, and c_2 induce W_4. Thus, at least one of the sets V_{i-1}''' and V_i''' is empty. But, by Lemma 17, we can assume that at least one of these sets is nonempty.

Further we will separately consider the two cases: $V_{i-1}'''\ne\varnothing$ and $V_i'''\ne\varnothing$. Note that, by Lemma 18, we can assume that, for every j, the set V_j' is completely nonadjacent to V_{j+2}'.

• Suppose that $c\in V_{i-1}'''$. Then $V_i'''=\varnothing$. Note that $V_{i-2}'=\varnothing$ by Lemma 15 (item iii, (2)). Next, from Lemma 15 (item iii, (3)) we have $V_{i-2}''\cup V_{i+2}''=\varnothing$ and $V_{i-1}''=\varnothing$ since otherwise b, c, v_{i+1}, and v_{i+2} together with an arbitrary element in V_{i-1}'' would induce $K_{2,3}^+$.

By Lemma 15 (item iii, (2)), the set V_i' is completely adjacent to $V_i'' \cup V_{i-1}'''$ and $V_{i-1}' \cup V_{i+2}'$ is completely nonadjacent to V_{i-1}'''. By Lemma 15 (item iii, (3)), V_{i+1}'' is completely adjacent to V_i'' and completely nonadjacent to V_{i-1}'''. The set V_i' is completely nonadjacent to V_{i+1}'' since otherwise an element in V_i', an adjacent element of V_{i+1}'', the vertices c, v_i, and v_{i+1} would induce $K_{2,3}^+$. The set V_i'' is completely nonadjacent to V_{i-1}'''; otherwise, an element of V_i'', an adjacent element of V_{i-1}''', and also the vertices v_i, v_{i+1}, and b would induce $K_{2,3}^+$. The set V_{i-1}' is completely adjacent to V_i''; otherwise, some its element and a nonadjacent element in V_i'' and also v_i, v_{i+2}, and c would induce P_5. The set V_i'' is completely adjacent to V_{i+2}'; otherwise, an element of V_{i+2}', a nonadjacent element of V_i'', and also v_{i-2}, v_{i+1}, and c would induce P_5. Lemma 15 (item iii, (2)) implies that each of the sets V_{i-1}', V_i', V_{i+1}', V_{i+2}', V_i'', V_{i+1}'', and V_{i-1}''' is a module in G; therefore, it contains at most one vertex. Consequently, $|V| \leq 12$.

• Let $c \in V_i'''$. Then $V_{i-1}''' = \varnothing$. By items (iii), (2) and (iii), (3) of Lemma 15, we have $V_{i-1}' = V_{i-2}'' = V_{i-1}'' = \varnothing$. By items (iii), (2) and (iii), (3) of Lemma 15, V_i''' is completely adjacent to V_{i+1}' and completely nonadjacent to V_{i+2}''; therefore, V_{i+1}' is completely nonadjacent to V_{i+2}'' since an element of V_{i+2}'' and an element of V_{i+1}' adjacent to it, v_{i+1}, v_{i+2}, and c induce $K_{2,3}^+$.

By Lemma 15 (item iii, (2)), V_i' is completely nonadjacent to V_i'''. If a vertex $b' \in V_{i+1}''$ is nonadjacent to all vertices in V_i' then $\{b'\}$ is completely adjacent to V_i''' since otherwise b', a vertex from V_i''' nonadjacent to b', a vertex from V_i' nonadjacent to b', v_{i+1}, and v_{i-2} induce P_5. Consequently, $\{b\}$ is completely adjacent to V_i'''. By Lemma 15 (item iii, (3)), V_i'' is completely adjacent to V_{i+1}'' and completely adjacent to V_i''', which, in the case of $V_i'' \neq \varnothing$, means that an arbitrary element in V_i'' together with v_{i-1}, v_{i-2}, b, and c induce $K_{2,3}^+$. Therefore, $V_i'' = \varnothing$.

By Lemma 15 (i), V_{i+1}'' is a clique; therefore, no vertex of V_i''' is adjacent to two vertices in V_{i+1}'' since otherwise G would contain an induced subgraph $K_{2,3}^+$. Consequently, V_i''' is completely nonadjacent to $V_{i+1}'' \setminus \{b\}$. The set $V_{i+1}'' \setminus \{b\}$ is completely adjacent to V_i' since an element of $V_{i+1}'' \setminus \{b\}$, a nonadjacent element of V_i' together with c, v_{i+1}, and v_{i-2} induce P_5. The set $\{b\}$ is completely adjacent to V_{i-2}' because a, v_{i+1}, b, and v_{i-1}, and an element of V_{i-2}' nonadjacent to b induce P_5. The set $V_{i+1}'' \setminus \{b\}$ is completely adjacent to V_{i-2}' since an element of $V_{i+1}'' \setminus \{b\}$, a nonadjacent element of V_{i-2}' together with c, v_{i-1}, and v_{i+2} induce P_5.

The arguments of the last two paragraphs and items (iii), (2) and (iii), (3) of Lemma 15 imply that each of the sets

$$V_{i-2}',\ N_{V_i'}(b),\ N_{V_i'}^-(b),\ V_{i+1}',\ V_{i+2}',\ V_{i+1}'' \setminus \{b\},\ V_{i+2}'',\ V_i'''$$

is a module in G; therefore, it contains at most one vertex. Thus, $|V| \leq 14$.

Lemma 19 is proved. □

Lemma 20. *If $S = \varnothing$ then either G is O_3-free or $|V| \leq 161$.*

Proof. Items (iii), (1) and (iii), (2) of Lemma 15 imply that the set $V_i' \cup \{v_{i+1}\}$ for $V_i' \neq \varnothing$ is not a nontrivial module if and only if either there is an edge between V_i' and $V_{i+2}' \cup V_{i+3}'$ or V_i' is not completely adjacent to $V_{i+1}'' \cup V_{i+3}''$. If one of these situations is realized then we arrive to the case of Lemmas 18 and 19, which means the validity of this assertion. Therefore, we may assume that

$$V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5' = \varnothing.$$

Since $G \in \mathrm{Free}(\{W_4\})$, the graph $G(V_i''')$ is P_3-free for each i; i.e., $G(V_i''')$ is a disjoint sum of complete graphs. Since $G \in \mathrm{Free}(\{K_{2,3}^+\})$, $G(V_i''')$ has one or two connected components. By Lemma 15 (i), V_{i+1}'' is a clique. Since $G \in \mathrm{Free}(\{K_{2,3}^+\})$, no vertex in V_i''' is adjacent to two vertices in V_{i+1}'' and also no two nonadjacent vertices in V_i''' can have a common neighbor in V_{i+1}'' or have a vertex in V_{i+1}'' nonadjacent to them simultaneously.

Let Q be the set of vertices of some connected component of $G(V_i''')$, where $|Q| \geq 2$. Then, by Lemma 15 (item iii, (1)), V_{i-1} is empty. Items (iii), (1), (iii), (3), and (iii), (4) of Lemma 15 imply that each vertex in the set

$$V_{i-1} \cup \bigcup_{j=1,\, j\neq i+1}^{5} V_j'' \cup V_{i-1}''' \cup V_{i+1}''' \cup V''''$$

is either adjacent to each vertex in Q or adjacent to none of them. If a vertex in $V_{i-2} \cup V_i \cup V_{i-2}''' \cup V_{i+2}'''$ has neighbors in Q but is adjacent not to each vertex in Q then G contains an induced subgraph W_4. Consequently, each vertex from this set is either adjacent to each vertex in Q or adjacent to none of them.

Since $G \in \mathrm{Free}(\{K_{2,3}^+\})$, no vertex in V_{i+1} is adjacent to two vertices in Q and no vertex in Q is adjacent to two vertices in V_{i+1} by Lemma 16. This, Lemma 16 and the P_5-freeness of G imply that there are at most two edges between Q and V_{i+1}. Similarly, there are at most two edges between Q and V_{i+2}. The W_4-freeness of G implies that only one vertex $x \in V_{i+1}''$ can have neighbors in Q; therefore, either $|V_{i+1}''| \leq 2$ or $|V_{i+1}''| \geq 3$ and $V_i''' = Q$, which stems from the arguments of the end of the second paragraph. In the second case, by items (i), (iii), (1) and (iii), (3) of Lemma 15, the set $V_{i+1}'' \setminus \{x\}$ is a nontrivial module of G.

If $|Q| \geq 7$ then Q must have two vertices adjacent to each vertex in $V_{i+1} \cup V_{i+2}$ and either simultaneously adjacent to x or simultaneously nonadjacent to x. Consequently, in this case, G has a nontrivial module; therefore, $|Q| \leq 6$. Hence, $|V_j'''| \leq 12$ for each j since every such set consist of at most two cliques with at most 6 vertices each.

Show that $|V_i''| \leq 2$ for each i. By items (iii), (1) and (iii), (3) of Lemma 15, each vertex of the set

$$V'''' \cup \bigcup_{j=1,\, j\neq i-2}^{5} V_j \cup \bigcup_{j=1,\, j\neq i}^{5} V_j'' \cup \bigcup_{j=1,\, j\neq i-1}^{5} V_j'''$$

is either adjacent to each vertex in V_i'' or adjacent to none of them. By Lemma 15 (i), V_i'' is a clique. In view of the W_4-freeness of G, it follows that each vertex in V_{i-2} is either adjacent to all vertices in V_i'' or adjacent to none of them. Therefore, V_i'' is a module in G if $V_{i-1}''' = \varnothing$ and has at most one vertex. The arguments in the previous paragraph imply that $|V_i''| \leq 2$ if $V_{i-1}''' \neq \varnothing$.

Consider the set V_i. By Lemma 16, it is independent. By Lemma 15 (i), V_i is completely adjacent to $V_{i-1} \cup V_{i+1} \cup V_{i+1}'''$ and completely nonadjacent to

$$\bigcup_{j=1,\, j\neq i+2}^{5} V_j''.$$

Each vertex in V_i is either adjacent to all vertices in V_{i+2}'' or adjacent to none of them. Since $G \in \mathrm{Free}(\{W_4\})$; for every two adjacent elements in V_i''', each element of V_i is either simultaneously adjacent to them or simultaneously nonadjacent to them. Since $G \in \mathrm{Free}(\{K_{2,3}^+\})$, no vertex in V_i''' is adjacent to two vertices in V_i. Similar assertions also hold for V_{i+2}'''. Therefore, V_i contains at most four elements each of which has a neighbor in $V_i''' \cup V_{i+2}'''$. We proved above that there are at most four edges between V_i and V_{i-1}''' and also between V_i and V_{i-2}'''.

Denote by $\widetilde{V}_i$ the set of all vertices in V_i not having neighbors in $V_{i-2}''' \cup V_{i-1}''' \cup V_i''' \cup V_{i+2}'''$. If $|V_i| \geq 13$ then $|\widetilde{V}_i| \geq |V_i| - 12$. Recall that, by Lemma 16, the sets V_{i-2}, V_i, and V_{i+2} are independent. Hence, since $G \in \mathrm{Free}(\{P_5\})$, for every two vertices $a, b \in V_i$, one of the sets $N_{V_{i-2}}(a)$ and $N_{V_{i-2}}(b)$ is a subset of the other. The same holds for $N_{V_{i+2}}(a)$ and $N_{V_{i+2}}(b)$. Thus, all vertices in V_i having neighbors in V_{i-2} (respectively, in V_{i+2}) have a common neighbor therein. Hence, since $G \in \mathrm{Free}(\{K_{2,3}^+\})$, V_i has at most two vertices such that the vertex has a neighbor in V_{i-2} (respectively, in V_{i+2}). Therefore, $|\widetilde{V}_i| \leq 5$ since otherwise G would contain a nontrivial module. Hence, $|V_j| \leq 17$ for each j.

Items (iii), (1)–(4) of Lemma 15 imply that $|V'''''| \leq 1$. Thus,

$$|V| \leq 5 + 5 \cdot 17 + 5 \cdot 2 + 5 \cdot 12 + 1 = 161.$$

Lemma 20 is proved. □

In the following six lemmas, we consider the situation when $S \neq \varnothing$ while proving that G has few vertices. Since $G \in \mathrm{Free}(\{P_5\})$, we have

Lemma 21. *Each vertex $a \in S$ adjacent to $b \in V_i'''$ has no neighbor in S nonadjacent to b.*

Refer to the vertices of the set $N(v_1) \cup N(v_2) \cup N(v_3) \cup N(v_4) \cup N(v_5)$ as the *dominating* cycle of C. Henceforth, we assume that the cycle C dominates the maximal number of vertices among all induced 5-cycles of G. By Lemma 15 (ii), we have

$$\widehat{V} \subseteq \bigcup_{i=1}^{5} (V_i'' \cup V_i''') \cup V''''.$$

Lemma 22. *Suppose that there is a vertex $x \in S$ adjacent to $y \in V_i'''$. Then the following properties are fulfilled simultaneously:*

(i) *The set $\{y\}$ is completely adjacent to $V_{i+1} \cup V_{i+2} \cup V_{i+1}''$; moreover, $|V_{i+1}| \leq 1$ and $|V_{i+2}| \leq 1$.*

(ii) *There are no adjacent vertices a and b such that $a \in V_j'$ and $b \in V_{j+2}'$ for some j.*

(iii) *Every vertex in $V_{i-2}''' \cup V_{i+2}'''$ is adjacent to exactly one of the vertices x and y.*

Proof. (i) If there exists $x' \in V_{i+1} \cup V_{i+2}$ then it is nonadjacent to x by Lemma 15 (ii). Consequently, $x'y \in E$ since otherwise x, y, x', v_i, and also v_{i-1} or v_{i+1} induce P_5; therefore, $\{y\}$ is completely adjacent to $V_{i+1} \cup V_{i+2}$. By Lemma 16, each of the sets V_{i+1} and V_{i+2} is independent. Therefore, if at least one of them has two elements then, together with v_{i-2}, v_{i+2}, and y, these two elements induce a subgraph $K_{2,3}^+$.

If there exists a vertex $z \in V_{i+1}''$ nonadjacent to y then $zx \in E$ since otherwise z, v_i, v_{i+1}, y, and x would induces P_5. Then the induced 5-cycle (v_i, v_{i+1}, z, x, y) dominates more vertices than C. Indeed, $\{y\}$ is completely adjacent to V_{i+2} and completely adjacent to V_{i+2}' (the last holds by Lemma 15 (item iii, (2)). We have a contradiction to the choice of C. Consequently, $\{y\}$ is completely adjacent to V_{i+1}''.

(ii) Suppose the contrary. Symmetry considerations and the fact that $V_{i+4}' = \varnothing$ imply that, by Lemma 15 (item iii, (2)), it suffices to consider the case of $a \in V_i'$ and $b \in V_{i+2}'$. By items (ii) and (iii), (2) of Lemma 15, $ax \notin E$, $bx \notin E$, $ay \notin E$, and $by \in E$. But then v_{i+1}, a, b, y, and x induce P_5.

(iii) Let $z \in V_{i-2}''' \cup V_{i+2}'''$. We can assume without loss of generality that $z \in V_{i-2}'''$. The vertex z must be adjacent to at least one of the vertices x and y since otherwise x, y, v_{i-2}, z, and v_{i+1} would induce P_5. At the same time, z cannot be adjacent to x and y simultaneously; otherwise, x, y, z, v_{i-2}, and v_i would induce $K_{2,3}^+$.

Lemma 22 is proved. □

The proof of the following auxiliary assertion is the longest and the most technically difficult proof of the article. Unfortunately, due to the continuity of the proof, the authors do not know how to reduce it or partition into some independent smaller parts.

Lemma 23. *Suppose that there exists a vertex $x \in S$ adjacent to vertices $y \in V_i'''$ and $y' \in V_{i-2}'''$. Then $|V| \leq 16$.*

Proof. Clearly, $N_{V'''_{i-2}}(x)$ is completely nonadjacent to $N_{V'''_i}(x)$ since otherwise G would contain an induced subgraph $K^+_{2,3}$. By items (iii)(2) and (iii)(3) of Lemma 15,

$$V'_{i-1} = V'_{i+2} = \bigcup_{j=1,\, j\neq i}^{5} V''_j = \varnothing.$$

Hence, owing to items (ii), (iii),(1), and (iii),(2) of Lemma 15 and Lemma 22(ii), we conclude that the set $V'_j \cup \{v_{j+1}\}$ is a module in G for each j. Thus, V'_j is empty for all j because G is atomic.

By Lemma 15(item iii,(3)), either $V''_i = \varnothing$ or $V'''' = \varnothing$. By items (iii),(3) and (iii),(4) of Lemma 15, the set $V''_i \cup V''''$ is completely nonadjacent to $V'''_{i-2} \cup V'''_i$. Consequently, $\{x\}$ is completely adjacent to $V''_i \cup V''''$; otherwise, G would contain an induced subgraph P_5. Thus, $|V''_i \cup V''''| \le 1$ since otherwise, by Lemma 15(i), G contains an induced subgraph $K^+_{2,3}$. The set V'''_{i-1} is completely adjacent to $V'''_{i-2} \cup V'''_i$ by Lemma 15(item iii,(4)). Hence, owing to the $\{W_4, K^+_{2,3}\}$-freeness of G, we have $|V'''_{i-1}| \le 1$.

Check that each vertex $y'' \in V'''_{i+1} \cup V'''_{i+2}$ is adjacent to x. Suppose the contrary. Without loss of generality, we can assume that $y'' \in V'''_{i+2}$ is nonadjacent to x. Then $y''y' \in E$ by Lemma 15(item iii,(4)). By Lemma 22(item iii), $y''y \in E$. Then y, y', y'', v_i, and v_{i-1} induce $K^+_{2,3}$. Hence, $\{x\}$ is completely adjacent to $V'''_{i+1} \cup V'''_{i+2}$, which and the $K^+_{2,3}$-freeness of G imply that V'''_{i+1} and V'''_{i+2} are independent sets. Consequently, by Lemma 15(item iii,(4)), owing to the $K^+_{2,3}$-freeness of G, we have $|V'''_{i+1}| \le 1$ and $|V'''_{i+2}| \le 1$.

Check that if $N^-_{V'''_i}(x)$ is nonempty then it is completely nonadjacent to $N_{V'''_i}(x)$ and completely adjacent to $N_{V'''_{i-2}}(x)$; moreover,

$$N^-_{V'''_{i-2}}(x) = \varnothing, \qquad N^-_{V'''_i \cap \widehat{V}}(x) = \varnothing.$$

Suppose that $y_1 \in V'''_i$ is nonadjacent to x. Consider arbitrary vertices $y_2 \in N_{V'''_i}(x)$ and $y_3 \in N_{V'''_{i-2}}(x)$ that necessarily exist, and $y_4 \in N^-_{V'''_{i-2}}(x)$. Then $y_1y_2 \notin E$ since otherwise $y_1y_3 \notin E$; so that the vertices v_i, v_{i-2}, y_1, y_2, and y_3 should not induce W_4, and y_1, y_2, x, y_3, and v_{i+1} induce P_5. The vertices y_1 and y_3 must be adjacent; otherwise, y_1, v_{i+2}, y_2, x, and y_3 would induce P_5. Similarly, $y_3y_4 \notin E$ and $y_2y_4 \in E$. Then $y_1y_4 \in E$ since otherwise v_{i+1}, y_1, y_2, y_3, and y_4 would induce P_5; but then v_i, v_{i+1}, y_1, y_2, and y_4 induce W_4. Suppose that y_1 is adjacent to a vertex $x^* \in S$. Then $x^*y_3 \notin E$ since otherwise v_{i-2}, v_i, y_1, y_3, and x^* would induce $K^+_{2,3}$; and $x^*y_2 \notin E$ since otherwise y_2, x^*, y_1, y_3, and v_{i+1} would induce P_5. The vertices x and x^* are nonadjacent since otherwise x^*, x, y_3, v_{i+1}, and v_{i+2} would induce P_5. Consequently, x^*, y_1, v_{i-2}, y_2, and x induce P_5.

Since G is $\{P_5, K^+_{2,3}\}$-free; therefore, the above implies that every vertex $x^* \in S \setminus \{x\}$ either has no neighbors in

$$\bigcup_{j=1,\, j\neq i-1}^{5} V'''_j$$

or

$$N(x^*) \cap \bigcup_{j=1,\, j\neq i-1}^{5} V'''_j = N(x) \cap \bigcup_{j=1,\, j\neq i-1}^{5} V'''_j,$$

and $\{x^*\}$ is completely adjacent to $V''_i \cup V''''$.

Since G is a connected P_5-free graph and V'''' is completely nonadjacent to $V'''_1 \cup V'''_2 \cup V'''_3 \cup V'''_4 \cup V'''_5$ by Lemma 15(item iii,(4)); therefore, each vertex in S is adjacent to a vertex in $\widehat{V}$. Show that every two adjacent vertices $x_1, x_2 \in S$ satisfy $N_{\widehat{V}}(x_1) = N_{\widehat{V}}(x_2)$. Suppose the contrary. Then the symmetric difference of $N_{\widehat{V}}(x_1)$ and $N_{\widehat{V}}(x_2)$ must consist of a unique element $y^* \in V''''$; otherwise, G would contain an induced P_5. Thus, we can assume that $N_{\widehat{V}}(x_2) \supset N_{\widehat{V}}(x_1)$ and $N_{\widehat{V}}(x_2) \setminus N_{\widehat{V}}(x_1) = \{y^*\}$; therefore,

$$N(x_1) \cap \bigcup_{j=1,\, j\neq i-1}^{5} V_j''' = \varnothing$$

since otherwise $x_1y^* \in E$. Consequently, $x_2y \in E$; otherwise, x_1, x_2, y^*, v_{i-2}, and y would induce P_5. Then x_1, x_2, y, v_i, and v_{i-1} induce P_5. Thus, every two adjacent vertices in S have identical sets of neighbors in $\widehat{V}$. Due to the atomicity of G, it follows that S is an independent set; otherwise, the vertex set of some connected component of $G(S)$ constitutes a nontrivial module.

Check that either $|V_i'' \cup V_{i-1}''' \cup V''''| \le 1$ or $|V| \le 10$. The inequality $|V_i'' \cup V_{i-1}''' \cup V''''| \le 1$ fails only if $V_{i-1}''' = \{z_1\}$ and $z_2 \in V_i'' \cup V''''$. If $z_2 \in V_i''$ then $z_1z_2 \notin E$; otherwise, v_i, v_{i+1}, y', z_1, and z_2 would induce W_4; and $xz_1 \in E$; otherwise, x, z_2, v_i, v_{i-1}, and z_1 would induce P_5. Therefore, by Lemma 22 (i), the induced 5-cycle $(x, z_2, v_i, v_{i-1}, z_1)$ dominates more vertices than C. Suppose that $z_2 \in V''''$. Then $yz_2 \notin E$, $y'z_2 \notin E$, and $z_1z_2 \notin E$ by Lemma 15 (iii, (4)). The set V_i is empty since otherwise each its element is adjacent to y' and nonadjacent to x and z_2 (by items (ii) and (iii), (1) of Lemma 15 and Lemma 22 (i)); and hence this element and v_{i-1}, z_2, v_{i+1}, and y' induce P_5. By analogy, $V_{i+1} = \varnothing$. If $V_{i+2}''' = \varnothing$ then the induced 5-cycle $(z_2, v_{i+1}, z_1, y, v_{i-2})$ dominates more vertices than C, which follows from Lemma 22 (i); therefore, we can assume that $V_{i+2}''' \neq \varnothing$. Likewise, we can assume that $V_{i+1}''' \neq \varnothing$; otherwise, the induced 5-cycle $(z_2, v_{i+2}, z_1, y, v_i)$ dominates more vertices than C. Lemma 15 (item iii, (4)) and the $K_{2,3}^+$-freeness of G imply that $V_j''' = \{y_j\}$ for each j. The results of the third paragraph imply that $(y_1, y_2, y_3, y_4, y_5)$ is an induced 5-cycle to whose every vertex x is adjacent. Obviously, if some vertex x' is adjacent to at least one of the vertices y_1–y_5 then x' is adjacent to each of these vertices. Consequently, G contains an induced subgraph W_4. The vertex z_2 cannot be adjacent to a vertex in $S \setminus \{x\}$ since otherwise $\{z_2\}$ would be a separating clique due to the independence of S.

Henceforth, we assume that $|V| \ge 11$. Recall that each vertex in S is adjacent to a vertex in $\widehat{V}$ and S is independent. Since S is independent and $|V_i'' \cup V_{i-1}''' \cup V''''| \le 1$, we have

$$\left\{x^* \in S \mid N(x^*) \cap \bigcup_{j=1,\, j\neq i-1}^{5} V_j''' = \varnothing\right\} = \varnothing;$$

otherwise, $V_i'' \cup V_{i-1}''' \cup V''''$ would be a separating clique.

Show that we can assume that $V_{i-2} \neq \varnothing$. Suppose that $V_{i-2} = \varnothing$. By Lemma 15 (item iii, (1)) and Lemma 22 (item i), the set V_{i-1} is completely adjacent to $V_{i-2}''' \cup V_i'''$; therefore, we can assume that $V_{i-1} = \varnothing$; otherwise, by Lemma 15 (item ii), for the cycle $(x, y, y', v_{i+2}, v_{i+1})$, which dominates the same number of vertices as C, this case is equivalent to the case of $V_{i-2} \neq \varnothing$ for C. Likewise, $V_{i+2} = \varnothing$. By Lemma 16, each of the sets V_i and V_{i+1} is independent. Thus, by Lemma 22 (i), in view of the $K_{2,3}^+$-freeness of G, we have $\max(|V_i|, |V_{i+1}|) \le 1$.

Suppose that $S \setminus \{x\}$ is nonempty and $x^* \in S \setminus \{x\}$. Then either $N_{\widehat{V}}(x) = N_{\widehat{V}}(x^*)$ or one of the vertices x and x^* is adjacent to an element of V_{i-1}''' and the other is not. In the second case, $xx^* \notin E$ since $G \in \mathrm{Free}(\{P_5\})$; and, for the cycle $(x, y, v_{i+2}, v_{i+1}, y')$ that dominates the same number of vertices as C, this case is equivalent to the case of $V_{i-2} \neq \varnothing$ for C. Consequently, we can assume that $S = \{x\}$ since G is atomic.

Verify that $\max(|V_{i-2}'''|, |V_i'''|) \le 2$. Consider only the case of V_i'''. In view of the $K_{2,3}^+$-freeness of G and the result of the fourth paragraph, the set $N_{V_i'''}(x)$ either consists of a single vertex and $N^-_{V_i'''}(x)$ is a clique or $N_{V_i'''}(x)$ consists of two nonadjacent vertices, but then $N^-_{V_i'''}(x) = \varnothing$. In view of the W_4-freeness of G and Lemma 22 (i), the set V_{i+1} is completely adjacent to $N_{V_i'''}(x)$ and completely nonadjacent to $N^-_{V_i'''}(x)$. The result of the third paragraph and the P_5-freeness of G imply that $N^-_{V_i'''}(x)$ is completely adjacent to V_{i+2}'''. By Lemma 22 (i), V_i is completely adjacent to V_{i-2}''', whence, in view of the result of the fourth paragraph and the W_4-freeness of G, it follows that $V_i \cup S$ is completely nonadjacent to $N^-_{V_i'''}(x)$. These observations and items (iii),(3) and (iii),(4) of Lemma 15 imply that $N^-_{V_i'''}(x)$ is a module in G; therefore, it contains more than one element.

We obtain the inequality

$$|V| \le |V(C)| + |V_i| + |V_{i+1}| + |V'''_{i-2}| + |V'''_i| + |V'''_{i+1}| + |V'''_{i+2}| + |V''_i \cup V'''_{i-1} \cup V'''| + |S| \le 15.$$

Henceforth, we assume that $V'_{i-2} \neq \varnothing$.

Check that every two vertices $a \in V_{i-2}$ and $b \in V'''_i \cup V'''_{i-2}$ are nonadjacent. Without loss of generality, we can assume that $b \in V'''_i$. If $ay' \in E$ then a, y', b, v_{i-2}, and v_i induce a subgraph W_4 (if $by' \notin E$) or $by' \in E$. In the second case, $b \neq y$ and $ay \notin E$ (otherwise, a, v_i, v_{i-2}, y', and y would induce W_4), $yb \notin E$ (otherwise, v_{i-2}, v_i, a, b, and y would induce W_4), $ax \notin E$ (by Lemma 15 (ii)), $bx \notin E$ (otherwise, x, b, y', v_{i-2}, and v_i would induce a subgraph $K^+_{2,3}$) and x, y, v_{i+2}, b, and a induce a subgraph P_5. Hence, $ay' \notin E$. Then $y'b \in E$ since otherwise a, b, v_{i+2}, v_{i+1}, and y' would induce P_5. Consequently, $b \neq y$ and $bx \notin E$; otherwise, x, y', b, v_{i-2}, and v_i would induce $K^+_{2,3}$. The vertices a and y are nonadjacent since otherwise a, y, x, y, and v_{i+1} would induce P_5. The vertices b and y are nonadjacent since otherwise a, b, y, y', and v_{i+2} would induce $K^+_{2,3}$. By Lemma 15 (ii), $ax \notin E$; therefore, a, b, v_{i+2}, y, and x induce a subgraph P_5.

Check that there are no adjacent vertices a and b the first of which belongs to V_{i-2} and the second, to $V_i \cup V_{i+1}$. From symmetry considerations, we can consider only the case when $b \in V_i$. By Lemma 15 (ii), we obtain $ax \notin E$ and $bx \notin E$. The vertex a is nonadjacent to y and y' simultaneously. By Lemma 22 (1), $y'b \in E$. At the same time, $by \in E$ since otherwise a, b, y', x, and y induce a subgraph P_5. Consequently, a, b, y, y', and v_i induce $K^+_{2,3}$.

Recall that V_{i-2} is completely nonadjacent to $V_i \cup V_{i+1} \cup V''_i \cup V''_{i-2}$. Moreover, since, by the results of the third paragraph, the set $\{x\}$ is completely adjacent to $V'''_{i+1} \cup V'''_{i+2}$; by Lemma 22 (i), we infer that V_{i-2} is completely adjacent to $V'''_{i+1} \cup V'''_{i+2}$. By Lemma 16, V_{i-2} is independent. In view of these circumstances and items (ii) and (iii),(1) of Lemma 15, we can assume that $V_{i-2} = \{a'\}$ since otherwise G contains a nontrivial module. The same considerations imply that a' is adjacent to some element in $V''_i \cup V''''$; otherwise, $\{a', v_{i-1}\}$ is a nontrivial module in G. We can assume that $V''_i \varnothing$ and $V'''' \neq \varnothing$. Indeed, if $V''_i \neq \varnothing$ then $V'''' = \varnothing$, and $(a', v_i, v_{i+1}, v_{i+2}, v_{i-2})$ is an induced 5-cycle whose all vertices are adjacent to some element in V''_i. This situation is equivalent to the fact that $V'''' \neq \varnothing$.

By the result of the seventh paragraph, we conclude that $|V''''| = 1$ and $V''_i = V'''_{i-1} = \varnothing$. Lemma 15 (item iii (4)), Lemma 16, Lemma 22 (i), and the fact that $V_{i-2} = \{a'\}$ and $G \in \mathrm{Free}(\{K^+_{2,3}\})$ imply that $|V_j| \le 1$ for each j, $|V'''_{i-2}| \le 1$, and $|V'''_i| \le 1$. The results of the fifth and eighth paragraphs and the atomicity of G imply that $S = \{x\}$. Thus,

$$|V| \le |V(C)| + \left| \bigcup_{j=1}^{5} V_j \right| + \left| \bigcup_{j=1,\, j \neq i-1}^{5} V'''_j \right| + |V''''| + |S| \le 16.$$

Lemma 23 is proved. ∎

Lemma 24. *If a vertex $x \in S$ is adjacent to vertices $y_1, y_2 \in V'''_i$ then $|V| \le 21$.*

Proof. The $K^+_{2,3}$-freeness of G, items (iii),(3) and (iii),(4) of Lemma 15, and Lemma 22 (i) imply that $y_1y_2 \notin E$ and the sets V_{i+1}, V_{i+2}, V''_i, V''_{i+1}, V''_{i+2}, V'''_{i-1}, V'''_{i+1}, and V'''' are empty. By Lemma 15 (item iii, (1)), $V''_{i-1} = V''_{i-2} = \varnothing$. By Lemma 22 (iii) and Lemma 23, we can assume that each element of the set $V'''_{i-2} \cup V'''_{i+2}$ is adjacent to y_1 and y_2 simultaneously. Therefore, $V'''_{i-2} \cup V'''_{i+2} = \varnothing$ since otherwise G would contain an induced subgraph W_4. Items (ii), (iii),(1), and (iii),(2) of Lemma 15 and Lemma 22 (i) imply that $V'_j = \varnothing$; otherwise, $V'_j \cup \{v_{j+1}\}$ is a nontrivial module in G. Items (ii) and (iii),(1) of Lemma 15 imply that $V_{i-1} = \varnothing$ since otherwise $V_{i-1} \cup \{v_i\}$ would be a nontrivial module in G.

Since $G \in \text{Free}(\{W_4\})$, we have $V_i''' = Q_1 \sqcup Q_2$ for the cliques Q_1 and Q_2; moreover, $y_1 \in Q_1$ and $y_2 \in Q_2$. Since $G \in \text{Free}(\{K_{2,3}^+\})$, we have $N_{Q_1}(x) = \{y_1\}$ and $N_{Q_2}(x) = \{y_2\}$. Hence, since G is P_5-free, at least one of the cliques Q_1 and Q_2 contains exactly one vertex. Assume that $Q_2 = \{y_2\}$. Since $G \in \text{Free}(\{W_4\})$, each of the vertices $V_{i-2} \cup V_i$ is either adjacent to all vertices in Q_1 or adjacent to none of them. Thus, we can speak of the adjacency of the vertices in $V_{i-2} \cup V_i$ with Q_1 and Q_2. Consequently, $|Q_1 \setminus \widehat{V}| \le 1$ because $Q_1 \setminus \widehat{V}$ is a module in G. Since $G \in \text{Free}(\{W_4\})$, no vertex in $V_{i-2} \cup V_i$ is adjacent to Q_1 and Q_2 simultaneously. Since $G \in \text{Free}(\{K_{2,3}^+\})$, no vertex in V_i''' is adjacent to two and more vertices in V_{i-2} or in V_i.

By Lemma 21, each element of S is adjacent to some vertex in V_i'''. The set S is independent. Indeed, otherwise, by Lemma 21, the vertices of every connected component of $G(S)$ together with at least two vertices would constitute a nontrivial module in G. Since S is independent and G is $K_{2,3}^+$-free and contains no separating cliques; therefore, each vertex in S is adjacent to exactly two vertices in V_i''', one of which is y_2. Hence, $|Q_1 \cap \widehat{V}| \le 2$ because if $Q_1 \cap \widehat{V}$ contains two vertices a_1 and a_2 different from y_1 then there exist vertices $b_1, b_2 \in S$ such that

$$x \notin \{b_1, b_2\}, \qquad a_1b_1, a_2b_2 \in E, \qquad a_1b_2, a_2b_1 \notin E, \qquad b_1y_1, b_2y_1 \notin E, \qquad b_1y_2, b_2y_2 \in E.$$

Then b_1, y_2, b_2, a_2, and y_1 induce P_5. Since $|Q_1 \cap \widehat{V}| \le 2$, S is an independent set, and each vertex in S is adjacent to y_2 and to exactly one more vertex in $Q_1 \cap \widehat{V}$; therefore, we have $|S| \le 2$ since otherwise G would contain a nontrivial module.

Thus, $|V_i'''| \le 4$ and $|S| \le 2$. Recall that no vertex in V_i''' is adjacent to two or more vertices in V_{i-2} or V_i. Consequently, if $|V_{i-2}| \ge 5$ then there exist at least $|V_{i-2}| - 4$ vertices each of which is adjacent to no vertex in V_i'''. Analogously, if $|V_i| \ge 5$ then there exist at least $|V_i| - 4$ vertices each of which is adjacent to no vertex in V_i'''. If a and b are vertices each of which has no neighbor in V_i''' then $ab \notin E$. Indeed, otherwise, by Lemma 15 (ii), a, b, v_{i+2}, y_1, and x would induce P_5. Thus, $|V_{i-2}| \le 5$ and $|V_i| \le 5$ since otherwise G would contain a nontrivial module consisting of two vertices from V_{i-2} or two vertices from V_i; therefore, $|V| \le 21$.

Lemma 24 is proved. $\square$

We will assume that at least one of the sets V'''' and $\bigcup_{i=1}^{5} V_i$ is empty. Indeed, if $v' \in V_i$ then $C' = (v_{i-2}, v_{i-1}, v_i, v', v_{i+2})$ is an induced 5-cycle of G; moreover, it dominates the same number of vertices as C. By items (iii), (1) and (iii), (3) of Lemma 15, each element in V'''' is adjacent to four vertices in C' and there is no vertex adjacent to all its vertices simultaneously.

Lemma 25. *Suppose that there is $x \in S$ adjacent to $y \in V_i'''$; moreover, $V_i'' \cup V_{i+2}'' \cup V'''' \ne \varnothing$. Then $|V| \le 16$.*

Proof. If there exists a vertex $x' \in S$ adjacent to a vertex $y' \in V_j''$ then for each j' there are no adjacent vertices $a \in V_{j'}$ and $b \in V_{j'+2}$. Indeed, we can consider only the cases of $j \in \{j', j'+3\}$. In both cases, $ay' \notin E$ and $by' \notin E$ by items (iii), (1) of Lemma 15. By Lemma 15 (ii), we have $ax' \notin E$ and $bx' \notin E$. In the first case, y', $v_{j'+3}$, $v_{j'+4}$, b, and a, and, in the second case, x', y', $v_{j'+4}$, b, and a induce P_5.

Suppose that there exists a vertex $y' \in V_i'' \cup V_{i+2}''$. Symmetry considerations imply that it suffices to consider only the case when $y' \in V_{i+2}''$. Items (iii), (3) and (iii), (4) of Lemma 15 and the $K_{2,3}^+$-freeness of G imply that $yy' \notin E$ and $V_i'' = V'''' = \varnothing$. By Lemma 15 (iii, (1)), $V_{i-2}'' = V_{i-1}'' = \varnothing$. Each vertex in S adjacent to an element in V_i''' must be adjacent to all elements in V_{i+2}''. Otherwise, v_{i-1}, a vertex in V_{i+2}'', and a vertex from V_i''' nonadjacent to it, v_{i+2}, and some vertex in S induce P_5. Consequently, $y'x \in E$ and $V_{i+2}'' = \{y'\}$ since otherwise x, two arbitrary vertices in V_{i+2}'', v_i, and v_{i+2} induce $K_{2,3}^+$. If there exists a vertex $y'' \in V_{i+1}''$ then y'' must be adjacent to y and y' by Lemma 15 (iii, (3)) and Lemma 22 (i), but then y, y', y'', v_{i+1}, and v_{i+2} induce $K_{2,3}^+$. Therefore, $V_{i+1}'' = \varnothing$. If there exists a vertex $y''' \in V_i'''$ and $y''' \ne y$ then, by Lemma 15 (iii, (3)) and the $K_{2,3}^+$-freeness of G, we have $y'y''' \notin E$ and $yy''' \in E$. Since $G \in \text{Free}(\{K_{2,3}^+\})$, we have $xy''' \notin E$. Therefore, v_{i-1}, y', x, y, and y''' induce P_5. Hence, $V_i''' = \{y\}$.

Owing to items (ii), (iii), (1), (iii), (2) of Lemma 15 and Lemma 22 (ii), we have $V'_j = \varnothing$ for all j. This is obvious for $j \in \{i-2, i, i+2\}$ since otherwise the set $V'_j \cup \{v_{j+1}\}$ would be a nontrivial module in G. Check that $V'_j = \varnothing$ for $j \in \{i-1, i+1\}$. From symmetry considerations, examine only $j = i+1$. If $\widetilde{y} \in V'_{i+1}$ then $\widetilde{y}y' \in E$ and $\widetilde{y}x \notin E$ by items (ii) and (iii), (2) of Lemma 15 and $\widetilde{y}y' \notin E$ since otherwise v_{i+1}, v_{i+2}, y, y', and $\widetilde{y}$ would induce $K^+_{2,3}$. Then v_{i-1}, y', x, y, and $\widetilde{y}$ induce P_5. By Lemma 15 (iii, (1)), $V_{i-1} = V_{i+1} = \varnothing$.

If $y^* \in V'''_{i-2}$ then, by Lemma 22 (iii) and Lemma 23, we can assume that $yy^* \in E$ and $xy^* \notin E$. By Lemma 15 (iii, (3)), $y'y^* \notin E$. Then v_{i-1}, y', v_{i+2}, y, and y^* induce P_5. If $y^* \in V'''_{i-1} \cup V'''_{i+1}$ then $yy^* \in E$ by Lemma 15 (iii,(4)). If $y^* \in V'''_{i-1}$ then $y^*y' \notin E$ by Lemma 15 (iii, (3)) and either v_{i+1}, y^*, y, x, and y' (if $xy^* \notin E$) or v_{i+1}, y^*, x, y', and v_{i-2} (if $xy^* \in E$) induce P_5. If $y^* \in V'''_{i+1}$ then $y^*y' \notin E$ since otherwise y^*, y', y, v_{i-2}, and v_{i+2} would induce W_4. If $xy^* \notin E$ then v_{i+1}, y^*, v_{i-2}, y', and x induce P_5. If $xy^* \in E$ then $(y^*, v_{i+2}, v_{i+1}, y', x)$ is an induced 5-cycle in G that, by Lemma 22 (i), dominates more vertices than C. Hence, $V'''_{i-2} = V'''_{i-1} = V'''_{i+1} = \varnothing$.

By Lemma 15 (iii, (1)), $\{y'\}$ is completely nonadjacent to $V_{i-2} \cup V_{i+2}$. By Lemma 16, V_i is independent. Since $G \in \mathrm{Free}(\{W_4\})$, we have $|N_{V_i}(y')| \le 1$. Recall that, by Lemma 15 (ii), the set $V_{i-2} \cup V_i \cup V_{i+2}$ is completely nonadjacent to S. By Lemma 22 (i), the set $\{y\}$ is completely adjacent to V_{i+2}. At the same time, $\{y\}$ is completely nonadjacent to $V_{i-2} \cup N_{V_i}(y')$ since otherwise some its element and y, x, y', and v_{i-1} induce P_5. Consequently, if $V'''_{i+2} = \varnothing$ then each of the sets V_{i-2}, $N^-_{V_i}(y')$, and V_{i+2} is a module in G; therefore, it has at most one element.

Suppose that $V'''_{i+2} \neq \varnothing$. Then V'''_{i+2} is completely adjacent to $\{y'\}$ by Lemma 15 (iii, (3)). By item (iii) of Lemma 22 and Lemma 23, we can assume that each element of V'''_{i+2} is adjacent to y and nonadjacent to x. Since the graph G is W_4-free, V'''_{i+2} is a clique. Since G is $K^+_{2,3}$-free, V_{i+2} is completely adjacent to V'''_{i+2} and $V_{i-2} \cup N_{V_i}(y')$ is completely adjacent to V'''_{i+2}. The set V'''_{i+2} is either completely adjacent to $N_{V_i}(y')$ or completely nonadjacent to it since otherwise v_{i+2} and y', an element of $N_{V_i}(y')$, an arbitrary element of $N_{V_i \cap V'''_{i+2}}(y')$, and an arbitrary element of $V'''_{i+2} \setminus N_{V_i}(y')$ would induce P_5.

Suppose that a vertex $x'' \in S$ is adjacent to a vertex $y'' \in V'''_{i+2}$. Clearly, x'' is not adjacent to y since $G \in \mathrm{Free}(\{K^+_{2,3}\})$. Then $x''y' \in E$; otherwise, the vertices y', x, y, y'', and x'' induce P_5. At the same time, $xx'' \notin E$; otherwise, x'', x, y, v_i, and v_{i+1} induce P_5. Note that (y', x'', y'', y, x) is an induced 5-cycle dominating more vertices than the cycle C. Consequently, each of the sets V_{i-2}, $N^-_{V_i}(x)$, V_{i+2}, and V'''_{i+2} is a module in G and contains at most one vertex.

Due to the connectedness and P_5-freeness of G, each vertex in S is adjacent to a vertex in $\widehat{V}$. By Lemma 21, each element in S adjacent to y has no neighbor in S that is adjacent to y. By the P_5-freeness of G, the same assertion also holds for y'. Since neither $\{y\}$ nor $\{y'\}$ is a separating clique in G, we have $N_S(y) = N_S(y')$. Hence, $S = \{x\}$; otherwise, S is a nontrivial module in G. Thus, we have the inequality

$$|V| \le |V(C)| + |V_{i-2}| + |V_{i+2}| + |N_{V_i}(x)| + |N^-_{V_i}(x)| + |V''_{i+2}| + |V'''_i| + |V'''_{i+2}| + |S| \le 13.$$

Let $y' \in V''''$. Then

$$\bigcup_{j=1}^{5} V''_j = \varnothing, \qquad \bigcup_{j=1}^{5} V'_j = \varnothing,$$

where the first holds by Lemma 15 (iii, (3)), and the second, by items (ii), (iii), (1), (iii), (2) of Lemma 15 and by Lemma 22 (ii).

By our assumptions before the statement of the lemma, we have $V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 = \varnothing$. By analogy with the arguments in the second paragraph, it is not hard to show that $yy' \notin E$, $y'x \in E$, $V'''_i = \{y\}$, and $V'''' = \{y'\}$. Each element of the set $V'''_{i-2} \cup V'''_{i+2}$ is nonadjacent to y' by Lemma 15 (iii, (4)). Owing to Lemma 22 (iii) and Lemma 23, we can assume that each element of the set

$V'''_{i-2} \cup V'''_{i+2}$ is adjacent to y and nonadjacent to x. Therefore, $V'''_{i-2} \cup V'''_{i+2} = \varnothing$ since otherwise every its element together with an element of $\{v_{i-1}, v_{i+1}\}$, y, x, and y' would induce P_5.

Suppose that $y^* \in V'''_{i-1} \cup V'''_{i+1}$. Without loss of generality, we can assume that $y^* \in V'''_{i-1}$.

By Lemma 15 (iii, (4)), $y^*y \in E$. If $y^*x \notin E$ then, by Lemma 15 (iii, (4)), the cycle (y', v_{i+1}, y^*, y, x) is induced and dominates more vertices than C. Therefore, for each vertex $x' \in S$ adjacent to y, the set $\{x'\}$ is completely adjacent to $V'''_{i-1} \cup V'''_{i+1} \cup V''''$. This and the $K_{2,3}$-freeness of G imply that $|V'''_{i-1}| \leq 1$ and $|V'''_{i+1}| \leq 1$.

Since $G \in \text{Free}(\{P_5\})$ and x is completely adjacent to $V'''_{i-1} \cup V'''_{i} \cup V'''_{i+1} \cup V''''$, each vertex in S that has a neighbor in $\widehat{V}$ is either adjacent to all vertices in $V'''_{i-1} \cup V'''_{i} \cup V'''_{i+1} \cup V''''$ or adjacent to y' and not adjacent to any of the vertices in $V'''_{i-1} \cup V'''_{i} \cup V'''_{i+1}$. Since $\{y'\}$ is not a separating clique and $G \in \text{Free}(\{P_5\})$, each vertex in S is adjacent to each element of the set $V'''_{i-1} \cup V'''_{i} \cup V'''_{i+1} \cup V''''$. Since G does not contain nontrivial modules, $S = \{x\}$. Therefore, we have

$$|V| \leq |V(C)| + |V'''_{i-1}| + |V'''_{i+1}| + |V''''| + |S| \leq 9.$$

Lemma 25 is proved. □

Lemma 26. *If $S \neq \varnothing$ then $|V| \leq 21$.*

Proof. Suppose to the contrary that $|V| \geq 22$. Show that, in this case, $\widehat{V}$ is a separating clique in G. Recall that

$$\widehat{V} \subseteq \bigcup_{i=1}^{5} (V''_i \cup V'''_i) \cup V''''.$$

If $\widehat{V}$ is not a separating clique then $\widehat{V}$ contains two nonadjacent vertices a and b. The vertex a has a neighbor $a' \in S$, and the vertex b has a neighbor $b' \in S$.

Let $a \in V'''_i$. Then $V''_{i-2} = V''_{i-1} = \varnothing$ by Lemma 15 (iii, (3)). If $V''_i \cup V''_{i+2} \cup V'''' \neq \varnothing$ then $|V| \leq 16$ by Lemma 25; therefore, we can assume that $V''_i \cup V''_{i+2} \cup V'''' = \varnothing$. By Lemma 15 (iii, (4)), we have $b \notin V'''_{i-1} \cup V'''_{i+1}$. By Lemma 22 (iii) and Lemma 23, we can assume that each element in $\widehat{V} \cap V'''_i$ is adjacent to each element in $V'''_{i-2} \cup V'''_{i+2}$. Therefore, $b \in V'''_i$. If $a'b \in E$ or $b'a \in E$ then $|V| \leq 21$ by Lemma 24. If $a'b \notin E$ or $b'a \notin E$ then either a', a, v_{i+2}, b, and b' (if $a'b' \notin E$) or b', a', a, v_i, and v_{i+1} induce P_5.

Thus, we assume that

$$\{a, b\} \cap \bigcup_{i=1}^{5} V'''_i = \varnothing.$$

By Lemma 15 (iii, (3)), $a \notin V''''$ and $b \notin V''''$. Let $a \in V''_i$. Then $b \notin V''_{i-1} \cup V''_{i+1}$ by Lemma 15 (iii, (3)); i.e., $b \in V''_{i-2} \cup V''_{i+2}$. Without loss of generality, we can assume that $b \in V''_{i+2}$. Clearly, $a'b \notin E$ and $b'a \notin E$ since otherwise a, b, v_{i-1}, and v_{i+1} together with a' or b' induce P_5. Consequently, $a' \neq b'$. If $a'b' \notin E$ then b', b, v_i, a, and a' induce P_5, and if $a'b' \in E$ then b, b', a, a', and v_{i+1} induce P_5.

Thus, $\widehat{V}$ constitutes a separating clique in G; a contradiction to the fact that $|V| \geq 22$.

Lemma 26 is proved. □

The main result of this section is

Theorem 2. *Problem* WVC *is solvable in time polynomial in the sum of the vertices for $\{P_5, K^+_{2,3}, W_4\}$-free graphs.*

Proof. Owing to Lemma 1, we can consider Problem WVC only for atomic graphs of the class Free($\{P_5, K^+_{2,3}, W_4\}$). By Lemma 14, each graph of this class either contains a induced C_5 or has 7 vertices or is perfect. Reckoning with this and Lemmas 6 and 8, we can assume that we consider the atomic graphs of class Free($\{P_5, K^+_{2,3}, W_4\}$) containing an induced C_5.

Consider the graph $G = (V, E)$ of this type. If $S = \varnothing$ then, by Lemma 20, either G is O_3-free or $|V| \leq 161$. Consequently, to (G, w) we can apply an algorithm polynomial in the sum of the weights, which exists by Lemmas 16 and 17. If $S \neq \varnothing$ then, by Lemma 26, $|V| \leq 21$. Therefore, to (G, w) we can apply an algorithm polynomial in the sum of the weights, which exists by Lemma 16. Thus, the assertion holds.

Theorem 2 is proved. □

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