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CAUCHY INVARIANTS AND EXACT SOLUTIONS OF NONLINEAR EQUATIONS OF HYDRODYNAMICS

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We review exact solutions for gravity waves in deep water. All of them are obtained within the Lagrangian framework and are generalizations of Gerstner waves (to the cases of inhomogeneous pressure on the free surface and taking the rotation of the fluid into account). The Cauchy invariants are found for each type of waves.

Keywords: Lagrangian coordinates, Cauchy invariants, Gerstner wave

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In the theory of gravity waves in deep water, all exact solutions are obtained in Lagrangian coordinates. We give their survey. Different types of waves are related to the corresponding integrals of the Euler equation [1], which are called Cauchy invariants [2]–[6].

This paper is organized as follows. In Sec. 1, we discuss the properties of Cauchy invariants. Next, we analyze the waves excited by inhomogeneous pressure on the free surface of a fluid (Sec. 2) and the wave motions taking the rotation of Earth into account (Sec. 3).

1. Cauchy invariants

The equations of dynamics of an ideal incompressible fluid in Lagrangian coordinates have the form [1], [4], [5], [7]

$$\frac{D(X,Y,Z)}{D(a,b,c)} = J_0(a,b,c), \tag{1}$$

$$(\vec{R}_{tt} + \vec{g})\vec{R}_{a_i} = -\frac{1}{\rho}\nabla_{a_i} p, \qquad i = 1,2,3, \tag{2}$$

where $\vec{R} = \{X(a,b,c,t),\, Y(a,b,c,t),\, Z(a,b,c,t)\}$ is the radius vector of a fluid particle, a,b,c are the fluid particle labels (Lagrangian variables), with $a_1 = a$, $a_2 = b$, $a_3 = c$; t is time; p is the pressure; ρ is a constant density; $\vec{g}$ is the acceleration of gravity; D denotes the Jacobian, J_0 is a time-independent function, and ∇_{a_i} is the gradient with respect to the variable a_i. Equation (1) is the volume conservation equation and Eqs. (2) are momentum equations.

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We find the cross-derivatives for each of the pairs of Eqs. (2) and subtract one equation from the other. The right-hand sides cancel each other, and the left-hand sides become time derivatives. As a result of integration, we obtain three equations [1]

$$\frac{D(X_t,X)}{D(b,c)}+\frac{D(Y_t,Y)}{D(b,c)}+\frac{D(Z_t,Z)}{D(b,c)}=S_1(a,b,c), \tag{3}$$

$$\frac{D(X_t,X)}{D(c,a)}+\frac{D(Y_t,Y)}{D(c,a)}+\frac{D(Z_t,Z)}{D(c,a)}=S_2(a,b,c), \tag{4}$$

$$\frac{D(X_t,X)}{D(a,b)}+\frac{D(Y_t,Y)}{D(a,b)}+\frac{D(Z_t,Z)}{D(a,b)}=S_3(a,b,c), \tag{5}$$

where S_1, S_2, S_3 are arbitrary functions (integrals of motion). Equations (3)–(5) were first formulated by Cauchy [8], [9]. The time independence of the functions S_1, S_2, S_3 reflects the condition that circulation is preserved in a closed loop [1], [9]. Stokes called them Cauchy integrals [10], [11]. In [2], it was proposed to call them Cauchy invariants. This terminology has successfully taken root [3]–[6].

By direct differentiation of Eqs. (3)–(5), we can verify that

$$\frac{\partial S_1}{\partial a}+\frac{\partial S_2}{\partial b}+\frac{\partial S_3}{\partial c}=0.$$

This means that the Cauchy invariants cannot be set arbitrarily. We introduce a vector of Cauchy invariants $\vec{S}\{S_1, S_2, S_3\}$ such that

$$\vec{S}=S_1\vec{a}_0+S_2\vec{b}_0+S_3\vec{c}_0,$$

where $\vec{a}_0, \vec{b}_0, \vec{c}_0$ are the unit vectors along the corresponding axes. The divergence of this vector is equal to zero: $\operatorname{div}_{\vec{a}} \vec{S}=0$. System (3)–(5) can be rewritten in a more compact form

$$\vec{S}=\operatorname{rot}_{\vec{a}}\sum_{i=1}^{3}(\vec{R}_t\vec{R}_{a_i})\vec{a}_{i0}, \tag{6}$$

where the notation $\vec{a}=\{a_1,a_2,a_3\}=\{a,b,c\}$ is used and the index at the rot operation means that it is calculated in Lagrangian variables.

The Cauchy invariants are related to the vorticity $\vec{\omega}(\omega_x,\omega_y,\omega_z)$ as

$$\vec{\omega}=J_0^{-1}(S_1\vec{R}_a+S_2\vec{R}_b+S_3\vec{R}_c). \tag{7}$$

If we choose the initial coordinates of fluid particles X_0, Y_0, Z_0 as Lagrangian variables, then the following equalities hold:

$$S_1=\omega_{x0}, \qquad S_2=\omega_{y0}, \qquad S_3=\omega_{z0}. \tag{8}$$

Here, $\omega_{x0}, \omega_{y0}, \omega_{z0}$ are the initial vorticity components. This result was obtained by Cauchy [1], [10]. In the general case, the Cauchy invariants are related to vorticity as

$$S_1=J_0(\vec{\omega}\nabla a), \qquad S_2=J_0(\vec{\omega}\nabla b), \qquad S_3=J_0(\vec{\omega}\nabla c). \tag{9}$$

For 2D flows, X and Y depend only on a, b, t, and $Z=c$. As can be seen from (3), (4), the Cauchy invariants S_1 and S_2 are equal to zero. Similarly, the vorticity components ω_x and ω_y are also zero (see (7)). The component ω_z has the form

$$\omega_z=\frac{1}{J_0}\left|\frac{D(X_t,X)}{D(a,b)}+\frac{D(Y_t,Y)}{D(a,b)}\right|=\frac{S_3(a,b)}{J_0(a,b)}. \tag{10}$$

It is a function of the Lagrangian coordinates and is independent of time, which means that the vorticity of fluid particles is preserved. The invariant S_3 is generally proportional to ω_z.

2. Generalized Gerstner waves

In the theory of water waves, pressure is traditionally required to be constant on a free surface. However, this condition can be violated in the presence of wind. Its effect can be modeled as the action of inhomogeneous and nonstationary pressure on the free surface.

We consider fluid motion in the XY plane. We introduce complex coordinates of a fluid particle trajectory

$$W = X + iY, \qquad \overline{W} = X - iY, \qquad X = X(a, b, t), \qquad Y = Y(a, b, t)$$

and complex Lagrangian coordinates

$$\chi = a + ib, \qquad \bar{\chi} = a - ib.$$

In this case, system of equations (1), (5) can be written as the condition that two Jacobians are time-independent [4], [12], [13],

$$\frac{D(W, \overline{W})}{D(\chi, \bar{\chi})} = J_0(\chi, \bar{\chi}), \qquad \frac{D(W_t, \overline{W})}{D(\chi, \bar{\chi})} = \frac{i}{2} S_3. \tag{11}$$

By direct substitution, we can see that the expression

$$W = G(\chi) e^{i\lambda t} + F(\bar{\chi}) e^{i\mu t}, \tag{12}$$

where G and F are analytic functions and λ and μ are real numbers, is an exact solution of system (11). The functions G and F are to a substantial degree arbitrary because their choice is only limited by the condition that the Jacobian J_0 be nonzero in the flow domain.

A particle in flows (12) moves along a radius-$|F|$ circle whose center in turn rotates along a circle with the radius $|G|$. If the ratio of frequencies μ and λ is positive, the particle trajectory is an epicycloid, and if it is negative, the trajectory is a hypocycloid; the number of petals in the curves depends on the frequency ratio. Such orbits were followed by planets in the Ptolemaic picture of the world, which is why this type of flows is called Ptolemaic [12], [13]. The Cauchy invariant has the form

$$S_3 = \lambda |G'|^2 - \mu |F'|^2. \tag{13}$$

Gerstner waves belong to the set of Ptolemaic motions and are written as

$$W = \chi + iAe^{i(k\bar{\chi} - \omega t)}, \qquad \operatorname{Im} \chi \leqslant 0, \tag{14}$$

where A is the amplitude, k is the wave number, and $\omega = \sqrt{gk}$ is the wave frequency [14]. Fluid particles move in circles. The wave has a trochoidal profile and propagates in the horizontal direction. The properties of Gerstner waves and their generalizations in hydrodynamics and geophysics are discussed in detail in [15], [16].

The pressure is constant on the profile of a Gerstner wave. However, this condition can be violated in the presence of wind. The effect of this violation can be modeled by inhomogeneous and nonstationary pressure defined on the free surface. The problem thus reduces to studying the influence of boundary conditions of that type on the wave evolution.

We consider the generalizations of Gerstner waves of this kind. We assume that the flow domain in the Lagrangian variables occupies the lower half-plane, and the fluid motion is described by the expression

$$W = G(\chi) + F(\bar{\chi}) e^{-\omega t}. \tag{15}$$

This motion belongs to the family of Ptolemaic flows, but G can differ from a linear function and F can differ from the exponential (see (14)). The function G defines the level with respect to which the particles on the free surface rotate, and the module of F defines the radius of their circular rotation (the wave amplitude). The particles are at rest in deep regions, and hence the following condition must be satisfied:

$$|F| \to 0 \quad \text{as} \quad b \to -\infty.$$

Because F is an analytic function, it reaches its maximum on the free surface. Hence, it follows that particles on the surface oscillate with the largest amplitude.

Wave solution (15) corresponds to the pressure distribution

$$\frac{p-p_0}{\rho} = -glm(G + Fe^{-i\omega t}) + \frac{1}{2}\omega^2|F|^2 + \mathrm{Re}\Big(e^{i\omega t}\int \omega^2 G'\vec{F}\,d\chi\Big), \tag{16}$$

where p_0 is a constant pressure on the free surface. In the general case, the pressure varies periodically with time and is nonuniform along the free surface: $\mathrm{Im}\,\chi = 0$. For Gerstner wave (14), pressure (16) has the form

$$p = p_0 - \rho g b - \frac{\rho}{2}\omega^2 A^2(1 - e^{2kb}).$$

In essence, we have a whole class of exact solutions that describe the complex free surface dynamics for inhomogeneous and harmonically varying pressure on it. The vorticity of waves (15) is given by

$$\omega_Z = \frac{2\omega|F'|^2}{|G'|^2 - |F'|^2},$$

and the Cauchy invariant is

$$S_3 = 2\omega|F'|^2. \tag{17}$$

Different examples of generalized Gerstner waves (15) are studied in a series of papers [17]–[21]. The details are summarized in Table 1. The Ptolemaic solutions allow a broad class of nonstationary phenomena to be analyzed on a model level. We discuss the dynamics of a rogue wave on the Gerstner wave background.

Table 1. Examples of generalized Gerstner waves (α and β are constants that are different in each example)

Wave model	$G(\chi)$	$F(\overline{\chi})$	Reference
Oscillating standing soliton	χ	$\frac{\beta}{(\overline{\chi}+i)^n}$; $\beta > 0, n \geqslant 2$	[17]
Oscillating soliton on the background of a Gerstner wave	χ	$iAe^{ik\overline{\chi}} + \frac{\beta}{(\chi+i)^n}$	[17]
Breather overturning on calm water	$\chi - \frac{i\beta}{(\chi-i)^2}$	$\frac{i\beta}{(\overline{\chi}+i)^2}$	[18]
Nonstationary Gerstner waves	$\chi + \frac{\beta}{\chi - i\alpha}$	$iAe^{ik\overline{\chi}}$	[19]
Rogue wave inside a packet of a Gerstner wave	$\chi + \frac{i}{k}\ln\big(1+P\big(\frac{\chi}{\alpha}\big)\big)$; $P\big(\frac{\chi}{\alpha}\big) = \frac{i\beta}{i\alpha-\chi}$	$iA\big(1+P\big(\frac{\chi}{\alpha}\big)\big)e^{ik\overline{\chi}}$	[20]
Rogue wave on the background of a Gerstner wave	$\chi - \frac{i\beta}{(\chi-i\alpha)^2}$	$-iAe^{ik\overline{\chi}} + \frac{i\beta}{(\chi+i\alpha)^2}$	[21]

Figure 1 shows the dynamics of a wave surface for expression (15) with functions G and F that correspond to the last row in Table 1. Numerical computations were carried out in the case $A = 0.5\,\mathrm{m}$, $k = 0.074\,\mathrm{m}$, $\alpha = 12\mathrm{m}$, $\beta = 328\,\mathrm{m}^3$,$\omega = \sqrt{gk} = 0.85\,\mathrm{s}^{-1}$, and $\lambda = 84.9\,\mathrm{m}$. At the initial moment $t = 0$, the shape of the free surface (the upper curve) exactly coincides with the Gerstner wave profile. Later on, a peak starts to grow on the profile, reaching a maximum at the moment $t = \pi/\omega$, and then decreasing and disappearing near the end of the period. The largest peak height, $h = 2\beta/\alpha^2 + A \approx 5.1\,\mathrm{m}$, is eight times the Gerstner wave amplitude A. This is why the peak formation can be considered the birth of a rogue wave (see [22] for details of rogue wave formation). The reason is the pressure applied at the surface. The lowest curve in Fig. 1 shows the deviation of the free surface pressure from atmospheric pressure p_0. At each free surface point, pressure varies with time, but its negative jump in the region of the wave peak is about 100 mm Hg.

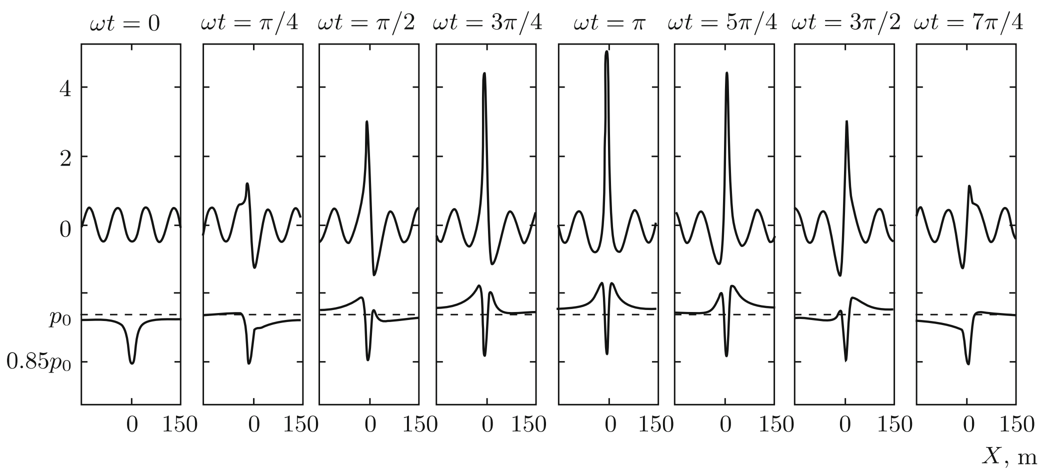

Fig. 1. Formation of a rogue wave on the background of a Gerstner wave.

We note that the form of the Cauchy invariant for the considered wave (see Table 1 and (17)) is a complex function of a and b.

3. Exact solutions for waves taking Earth's rotation into account

We choose the reference frame on rotating Earth as shown in Fig. 2. Its origin is at a latitude Φ, the X axis is directed eastward, the Y axis is directed northward, and the Z axis is directed vertically upward. In this reference frame, the vector of Earth's rotation $\vec{\Omega}$ lies in the plane YZ. In the rotating reference frame, each particle is affected by the Coriolis and centrifugal forces in addition to the gravity force, and the equation of motion takes the form [23]

$$\vec{R}_{tt} + 2\vec{\Omega} \times \vec{R}_t = -\frac{1}{\rho}\nabla p + \nabla\Phi - \vec{\Omega} \times (\vec{\Omega} \times \vec{R}), \tag{18}$$

where $\Phi = -gZ$ is the geopotential. The centrifugal force has a gradient character, and Eq. (18) can be rewritten as

$$\begin{aligned} &\vec{R}_{tt} + 2\vec{\Omega} \times \vec{R}_t = -\nabla H, \\ &H = \frac{p}{\rho} - \Phi + \Phi_{\mathrm{c}}, \qquad \Phi_{\mathrm{c}} = -\frac{1}{2}(\vec{\Omega} \times \vec{R})^2, \end{aligned} \tag{19}$$

where Φ_{c} is the potential of centrifugal forces.

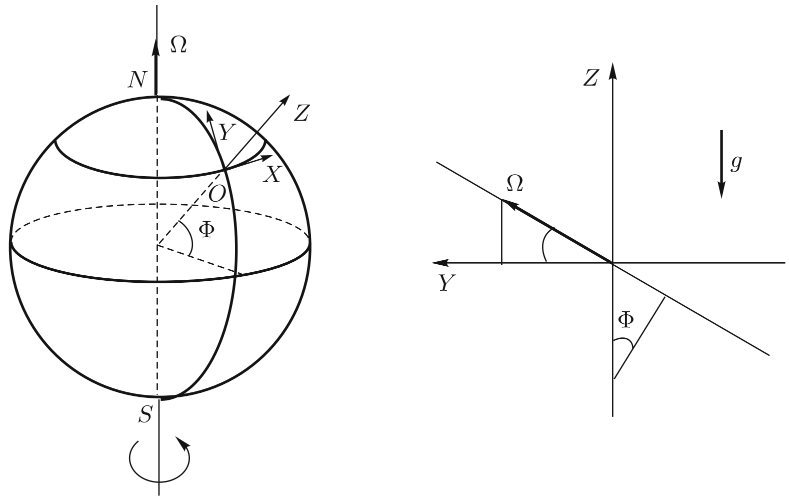

Fig. 2. Coordinate system on Earth's surface.

Taking a dot product of Eq. (19) and $\vec{R}_{a_i}$, we obtain momentum equations in the Lagrangian coordinates:

$$\vec{R}_{tt}\vec{R}_{a_i} + 2(\vec{\Omega}, \vec{R}_t, \vec{R}_{a_i}) = -H_{a_i}, \qquad i = 1, 2, 3, \qquad \{a_i\} = \{a, b, c\}. \tag{20}$$

Together with continuity equation (1), three equations (20) make a system of equations of an ideal incompressible fluid in a rotating reference frame. The second term in the left-hand side is the scalar triple product.

We study two types of wave motion such that

1. the projections of Earth's angular velocity can be considered constant in the entire flow domain: the Coriolis parameters $f = 2\Omega_Z = 2\Omega \sin\Phi$ and $\tilde{f} = 2\Omega_Y - 2\Omega\cos\Phi$ are assumed to be constant (the f-plane approximation);

2. near-equatorial flows are in the band of low latitudes $\Phi Y/R$, where R is Earth's radius and the Coriolis parameters are $f = \beta Y$, $\beta = 2\Omega/R$, and $\tilde{f} = 2\Omega$ (the β-plane approximation).

The representation of the vector $\vec{\Omega}$ is different in each case, but a general result can be formulated for these cases [24], [25]. We eliminate the gradient term from Eqs. (20) by taking cross derivatives and, after intermediate computations, obtain the equations

$$\vec{R}_{ta_i}\vec{R}_{a_j} - \vec{R}_{ta_j}\vec{R}_{a_i} + 2(\vec{\Omega}, \vec{R}_{a_i}, \vec{R}_{a_j}) = S_k(a, b, c), \qquad i, j = 1, 2, 3, \quad i \neq j \neq k, \tag{21}$$

where pairs a_i, a_j are selected from the triple of coordinates a, b, c by cyclic permutations. Equation (21) is equivalent to the conservation condition for three invariants S_1, S_2, S_3. If $\vec{\Omega} = 0$, this equation coincides with system (3)–(5).

3.1. Gerstner waves in a rotating fluid. In the f-plane approximation, Pollard found the exact solution [26]

$$\begin{cases} X = a - \dfrac{Am}{k} e^{mc} \sin[k(a - Ut)], \\ Y = b + f\dfrac{Am}{k^2 U} e^{mc} \cos[k(a - Ut)], \\ Z = c + Ae^{mc} \cos[k(a - Ut)], \end{cases} \tag{22}$$

where A and m are positive constants, and k and U are the respective wave number and phase velocity of the wave. Inserting (22) into continuity equation (1), we obtain

$$J_0 = 1 - m^2 A^2 e^{2mc}.$$

We assume that $c = c_0$ defines a free surface. The flow domain is given by the condition $c \le c_0 < 0$. To preserve the one-to-one character of map (22) ($J_0 \ne 0$), the inequality $A \le 1/[me^{mc}]$ must hold. It ensures that there are no self-crossings in the wave profile (in a Gerstner wave, the role of the parameter m is played by the wave number).

Inserting (22) into expressions (21), we compute the values of generalized Cauchy invariants

$$S_1 = 0, \qquad S_2 = m(k^2 - m^2)UA^2 e^{2mc} + \tilde{f}(1 - m^2 A^2 e^{2mc}), \qquad S_3 = f.$$

Equation (21) also defines the parameter m as

$$m^2 = \frac{k^4 U^2}{k^2 U^2 - f^2}. \tag{23}$$

Thus, a single free parameter A, which defines the wave amplitude, remains in solution (22).

Wave oscillations of fluid particles decay exponentially with depth, ensuring the bottom impermeability condition ($c \to -\infty$). To find the pressure, we insert expressions (22) into Eqs. (20) and neglect the centrifugal force. The expression for the pressure takes the form

$$p - p_0 = \rho \frac{mgA^2}{2}[e^{2mc} - e^{2mc_0}] - \rho g(c - c_0). \tag{24}$$

Just as for a Gerstner wave, the pressure depends only on the vertical Lagrangian coordinate. In deriving expression (24), we find the wave dispersion relation by requiring that the pressure be time-independent at the free surface:

$$U^2(k^2 U^2 - f^2) = (g - \tilde{f}U)^2.$$

If rotation is absent (the Coriolis parameters are zero), the last expression coincides with that for Gerstner waves. The wave travels from west to east, and its crests are parallel to the Y axis.

It follows from relations (22), (23) that

$$(X - a)^2 + (Y - b)^2 + (Z - c)^2 = \frac{m^2 A^2}{k^2} e^{2mc},$$
$$Y - f\frac{m}{k^2 U} Z - b + f\frac{m}{k^2 U} c = 0.$$

On the one hand, particles move over the surface of the sphere, and on the other, remain in the plane that makes an angle $\gamma = \arctan(fm/k^2 U)$ with the Z axis. Therefore, their trajectories are circles lying in this plane. The center of each such circle is located at a point (a, b, c), which does not coincide with the initial particle position, and the rotation radius is mAe^{2mc}/k. By setting $c = c_0$ in (22), we obtain a parametric representation of the surface wave profile: for every fixed parameter b, this is a smooth trochoid in the plane tilted at the angle γ to the Z axis At the equator, $f = 0$, $m = k$, and the Pollard solution transforms into the Gerstner solution (the role of b is now played by the coordinate c). At the equator, the particles oscillate in the plane XZ; for $f \ne 0$, the plane of their oscillations is inclined in each hemisphere toward the respective pole. However, as Pollard himself concluded [26], the magnitude of this angle is extremely small.

3.2. Trapped waves in a near-equatorial domain. Close to the equator, in the β-plane approximation, Eqs. (21) can be rewritten as [25]

$$\begin{aligned}
&\frac{D(X_t,X)}{D(b,c)}+\frac{D(Y_t,Y)}{D(b,c)}+\frac{D(Z_t,Z)}{D(b,c)}+2\Omega\frac{D(Z,X)}{D(b,c)}+\beta Y\frac{D(X,Y)}{D(b,c)}&&=S_1(a,b,c),\\
&\frac{D(X_t,X)}{D(c,a)}+\frac{D(Y_t,Y)}{D(c,a)}+\frac{D(Z_t,Z)}{D(c,a)}+2\Omega\frac{D(Z,X)}{D(c,a)}+\beta Y\frac{D(X,Y)}{D(c,a)}&&=S_2(a,b,c),\\
&\frac{D(X_t,X)}{D(a,b)}+\frac{D(Y_t,Y)}{D(a,b)}+\frac{D(Z_t,Z)}{D(a,b)}+2\Omega\frac{D(Z,X)}{D(a,b)}+\beta Y\frac{D(X,Y)}{D(a,b)}&&=S_3(a,b,c),
\end{aligned}\tag{25}$$

Constantin [27] found an exact solution of this system in the form

$$\begin{cases}
X=a-\dfrac{1}{k}e^{k[c-h(b)]}\sin[k(a-Ut)],\\[2ex]
Y=b,\\[2ex]
Z=c+\dfrac{1}{k}e^{k[c-h(b)]}\cos[k(a-Ut)].
\end{cases}\tag{26}$$

where $h(b)=\beta b^2/(2(kU+2\Omega))$, and the phase velocity is

$$U=\frac{\sqrt{\Omega^2+kg}-\Omega}{k}.$$

Relations (26) describe equatorial surface waves propagating eastwards at a speed U. These are periodic spatial waves whose amplitude decreases exponentially in the meridional direction. Hence, they are called trapped. For $h=0$, expressions (26) become the Gerstner solution. An additional exponentially decaying factor in the amplitude is a signature of this solution.

The expressions for the generalized Cauchy invariants of waves (26) are as follows [23]:

$$\begin{aligned}
&S_1=0, \qquad S_2=2\Omega-2(kU+\Omega)e^{2\xi},\\
&S_3=\beta b\left[1-\frac{2(kU+\Omega)}{kU+2\Omega}e^{2\xi}\right], \qquad \xi=k[c-h(b)].
\end{aligned}\tag{27}$$

The zonal component of the vector $\vec{S}\{S_1,S_2,S_3\}$ is equal to zero. The vorticity $\vec{\omega}$ for waves (26) is determined by the equalities

$$\begin{aligned}
&\vec{\omega}=S_0^{-1}\{-bkU^2g^{-1}\beta e^{\xi}\sin\theta,-2kUe^{2\xi},bkU^2g^{-1}\beta(e^{\xi}\cos\theta-e^{2\xi})\},\\
&J_0^{-1}=1-e^{2\xi}, \qquad \theta=k(a-Ut).
\end{aligned}\tag{28}$$

All three of its components are nonzero, while the zonal and vertical components depend on time. The comparison of formulas (27) and (28) shows a visual difference between the vorticity vector and the vector of Lagrangian invariants.

4. Conclusions

Cauchy invariants are an important concept in the Lagrangian approach. They were discovered at the beginning of the 19th century, but subsequently were thoroughly forgotten. This paper is an attempt to draw renewed attention to them. We discussed the form and properties of the Cauchy invariants using exact solutions for gravity waves in deep water as an example.

Conflicts of interest. The authors declare no conflicts of interest.

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