On the topology of 3-manifolds admitting Morse-Smale diffeomorphisms with four fixed points of pairwise different Morse indices

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Abstract

In the present paper we consider class G of orientation preserving Morse-Smale diffeomorphisms f, which defined on closed 3-manifold M^3, and whose non-wandering set consist of four fixed points with pairwise different Morse indices. It follows from S. Smale and K. Meyer results that all gradient-like flows with similar properties has Morse energy function with four critical points of pairwise different Morse indices. This implies, that supporting manifold M^3 for these flows admits a Heegaard decomposition of genus 1 and hence it is homeomorphic to a lens space $L_{p,q}$. Despite the simple structure of the non-wandering set in class G there exist diffeomorphisms with wild embedded separatrices. According to V. Grines, F. Laudenbach, O. Pochinka results such diffeomorphisms do not possesses an energy function, and question about topology their supporting manifold is open. According to V. Grines, E. Zhuzhoma and V. Medvedev results M^3 is homeomorphic to a lens space $L_{p,q}$ in case of tame embedding of closures of one-dimensional separatrices of diffeomorphism $f \in G$. Moreover, the wandering set of f contains at least p non-compact heteroclinic curves. In the present paper similar result was received for arbitrary diffeomorphisms of class G. Also we construct diffeomorphisms from G with wild embedding one-dimensional separatrices on every lens space $L_{p,q}$. Such examples were known previously only on the 3-sphere.

1 Formulation of results

It's well known that the Morse-Smale systems exist on any manifolds. These systems describe regular (non-chaotic) processes in technology. They have finite hyperbolic non-wandering set, which is fully described by numbers orbits of different *Morse indices* (the dimension of their unstable manifold). A natural question arises, what we say about the supporting manifold topology of such system, if we know the structure of it non-wandering set. The classic example of an exhaustive answer to the question are the systems with two points of extremal Morse indices. In this case, it follows from Reeb's theorem [1], that the supporting manifold is homeomorphic to the n-sphere. Another example of following global properties from local ones is the equality for a gradient-like system of the alternating sum of number periodic points of different Morse indices to Euler characteristic of the ambient surface. For flows this fact follows from classical Poincare-Hopf theorem [2], [3] and for cascades it follows from the existence of a Morse energy function, proved by D. Pixton [4], and from Morse inequality.

For the dimension equals 3 this equality also is true. However, the Euler characteristic of all closed orientable 3-manifolds is equal to zero and, accordingly, it does not shed any light on the supporting space topology. A more cunning play with the numbers of periodic points of different Morse indices leads to the fact that for flows it is possible to find a connection between these numbers and the genus of the Heegaard decomposition of a 3-manifold. In the absence of a topological classification of 3-manifolds, an information about the Heegaard decomposition of a given manifold is very informative and identifying in some cases. A similar question for diffeomorphisms is open today due to the possibility of wild behaviour of the saddle point separatrices, first discovered by D. Pixton [4].

The effect of the possibly wild embedding of saddle separatrices of a Morse-Smale 3-diffeomorphism into an ambient manifold had a revolutionary impact on the understanding of the dynamics of such systems. It became clear that their description does not fit into the framework of purely combinatorial invariants, and requires the involvement of a topological apparatus. Nevertheless, a complete topological classification of Morse-Smale 3-diffeomorphisms, including the realization, was obtained in the works of C. Bonatti, V. Grines and O. Pochinka [5], [6]. However, these invariants does not answer the question, whether this 3-manifold admits a gradient-like diffeomorphism with wildly embedded saddle separatrices or not.

In papers of V. Grines, E. Zhuzhoma and V. Medvedev [7] it was established that gradient-like diffeomorphisms with tame embedded one-dimensional saddle separatrices are look like to flows, therefore the structure of the non-wandering set of such a diffeomorphism uniquely determines the Heegaard decomposition of its supporting manifold. In some partial cases this result was generalized on mildly wild embedding of separatrices by V. Grines, F. Laudenbach and O. Pochinka [8], and on diffeomorphisms with a single saddle point with wild separatrices – by C. Bonatti and V. Grines [9]. In both case the Heegaard's genus of the ambient manifold is 0, that is M^3 is the 3-sphere.

In the present paper we consider the class G of orientation-preserving Morse-Smale diffeomorphisms, defined on a closed 3-manifold M^3, whose non-wandering set consists of exactly four points of pairwise different Morse indices. It follows from the results of S. Smale [10] and K. Meyer [11] that all gradient-like flows with similar properties (see Fig. 1) has a Morse energy function with exactly four critical points of pairwise different indices. It immediately implies that the supporting manifold M^3 for such flows admits a Heegaard decomposition of genus 1 and, therefore, it is homeomorphic to a lens space $L_{p,q}$ (see, for example, [12]).

Despite the simple structure of the non-wandering set, there are diffeomorphisms in the class G with wildly embedded saddle separatrices [13] (see Fig. 2). However, all currently known examples were constructed on the 3-sphere.

One of the results of this paper is a constructive proof of the following fact.

Theorem 1. *On any lens space $L_{p,q}$ there exists a diffeomorphism $f \in G$ with wildly embedded one-dimensional saddle separatrices.*

According to [8] such diffeomorphisms do not have a Morse energy function, and the

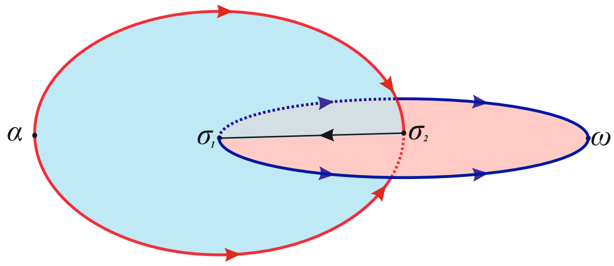

Figure 1: Gradient-like 3-flow with four points of pairwise distinct Morse indices on lens $L_{1,0} \cong \mathbb{S}^3$

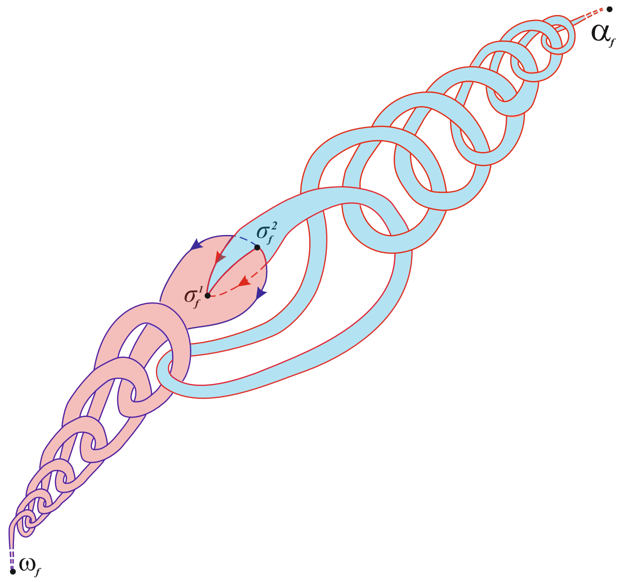

Figure 2: Diffeomorphism from the class G with wildly embedded saddle separatrices

question of the topology of their supporting manifold has remained open until today. According to [7], in the case of tame embedding of the closures one-dimensional separatrices of the diffeomorphism $f \in G$, the supporting manifold M^3 is homeomorphic to the lens space $L_{p,q}$. The wandering set of the diffeomorphism f contains at least p non-compact heteroclinic curves.

In the present paper, a similar result is obtained for arbitrary diffeomorphisms of the class G. In more detail.

Let $f \in G$. It follows from the definition of the class that the non-wandering set of f consists of exactly four points $\omega_f, \sigma_f^1, \sigma_f^2, \alpha_f$ with Morse indices $0, 1, 2, 3$, respectively. Thus f has exactly two saddle points σ_f^1, σ_f^2 of Morse indices 1 and 2, respectively, the intersection of two-dimensional manifolds of which forms a heteroclinic set

$$H_f = W^s_{\sigma_f^1} \cap W^u_{\sigma_f^2}.$$

We introduce the concept of the heteroclinic index I_f of f as follows. If the set H_f does not contain non-compact curves, then we assume $I_f = 0$. Otherwise, any non-compact curve $\gamma \subset H_f$ contains, together with any point $x \in \gamma$, a point $f(x)$. We will consider the curve

γ oriented in the direction from x to $f(x)$. We will also fix the orientation on manifolds $W^s_{\sigma_1}$ and $W^u_{\sigma_2}$. For a non-compact heteroclinic curve γ, we denote by

$$v_\gamma = (\vec{v}^1_\gamma, \vec{v}^2_\gamma, \vec{v}^3_\gamma)$$

a triple of vectors with the origin at the point $x \in \gamma$ such that $\vec{v}^1_\gamma$ – normal vector to $W^s_{\sigma_1}$, $\vec{v}^2_\gamma$ – normal vector to $W^u_{\sigma_2}$ and $\vec{v}^3_\gamma$ – tangent vector to the oriented curve γ. Let's put $I_\gamma = +1$ ($I_\gamma = -1$) in the case of right (left) orientation of v_γ. The number

$$I_f = \left| \sum_{\gamma \subset H_f} I_\gamma \right|$$

is called *the heteroclinic index of diffeomorphism* f. For an integer $p \geqslant 0$, we denote by $G_p \subset G$ a subset of diffeomorphisms $f \in G$ such that $I_f = p$ (see Fig. 3). The main result

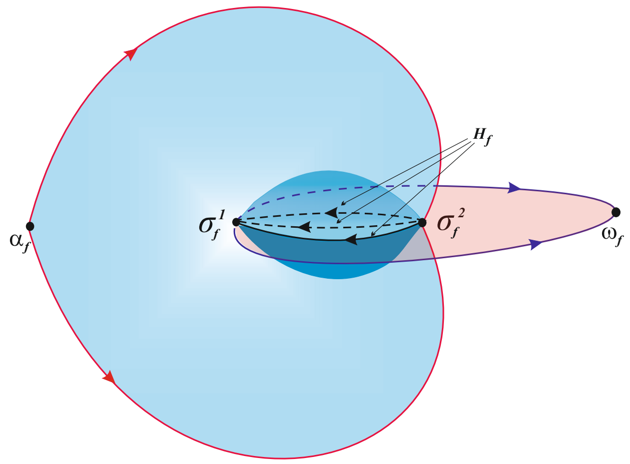

Figure 3: Diffeomorphism $f \in G$ with non-orientable set H_f consisting of three non-compact curves and heteroclinic index 1

of the work is the proof of the following fact.

Theorem 2. *If a manifold M^3 admits a diffeomorphism $f \in G_p$ then M^3 is homeomorphic to a lens space $L_{p,q}$.*

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2 Necessary definitions and facts

2.1 Topology

For any subset X of the topological space Y we will denote by $i_X : X \to Y$ the *inclusion map*. For any continuous map $\phi : X \to Y$ from the topological space X to the topological space Y will be denoted by $\phi_* : \pi_1(X) \to \pi_1(Y)$ its *induced homomorphism*.

By C^r*-embedding* ($r \geqslant 0$) of a manifold X into a manifold Y is called a map $f : X \to Y$ such that $f : X \to f(X)$ is a C^r-diffeomorphism. C^0-embedding is also called a *topological embedding.*

The topological embedding $\lambda : X \longrightarrow Y$ of an m-manifolds X into an n- manifold Y ($n \leqslant m$) is called *locally flat at a point* $\lambda(x)$, $x \in X$, if the point $\lambda(x)$ belongs to a local chart (U, ψ) of the manifold Y, that $\psi(U \cap \lambda(X)) = \mathbb{R}^m$, where $\mathbb{R}^m \subset \mathbb{R}^n$ – the set of points whose last $n - m$ coordinates are 0 or $\psi(U \cap \lambda(X)) = R^m_+$, where $\mathbb{R}^m_+ \subset \mathbb{R}^m$ – the set of points whose last coordinate is non-negative. The embedding λ is called *tame*, and a manifold X is *tamely embedded*, if λ is locally flat at every point $\lambda(x)$, $x \in X$. Otherwise, the embedding λ is called *wild*, and the manifold X is *wildly embedded.* Any point $\lambda(x)$ that is not locally flat is called a *wildness point.*

Let $\mathbb{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum\limits_{i=1}^{n} x_i^2 \leqslant 1\}$ — *is a standard n-disk (ball)*, $\mathbb{D}^0 = \{0\}$, $\mathbb{S}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum\limits_{i=1}^{n} x_i^2 = 1\}$ — it is a standard $(n-1)$-sphere, $\mathbb{S}^{-1} = \emptyset$.

Proposition 2.1 ([14], Lemma 2.1)**.** *Let $\lambda \colon \mathbb{S}^2 \to M^3$ be a topological embedding that is smooth everywhere except at one point s_0, $x_0 = \lambda(s_0)$, $\Sigma = \lambda(\mathbb{S}^2)$, $y_0 \in \Sigma \setminus \{x_0\}$ be a fixed point and V be a fixed neighborhood of the sphere Σ. Then there exists a smooth 3-ball B contained in V such that $x_0 \in B$ and ∂B transversally intersects Σ along a single curve separating the points x_0 and y_0 in Σ.*

A topologically embedded into n-manifold X $(n-1)$-sphere S^{n-1} is called *cylindrical or cylindrically embedded* if there is a topological embedding $e : \mathbb{S}^{n-1} \times [-1, 1] \to X$ such that $e(\mathbb{S}^{n-1} \times \{0\}) = S^{n-1}$.

An n-manifold X is called *irreducible* if any $(n-1)$-sphere cylindrical embedded in X bounds an n-ball there.

A 3-manifold X is called *simple* if it is either irreducible or homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.

A surface F topologically embedded into a 3-manifold X is called *properly embedded* if $\partial X \cap F = \partial F$. A properly embedded into X a surface F is called *compressible* in X in one of the following two cases:

1) there is a non-contractible simple closed curve $c \subset intF$ and an embedded 2-disk $D \subset intX$ such that $D \cap F = \partial D = c$;

2) there is a 3-ball $B \subset intX$ such that $F = \partial B$.

A surface F is called *incompressible* in X if it is not compressible in X.

Proposition 2.2 ([9], Theorem 4)**.** *Let T be a two-dimensional torus smoothly embedded in the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ so that $i_{T*}(\pi_1(T)) \neq 0$. Then T is a boundary of a solid torus, smoothly embedded into $\mathbb{S}^2 \times \mathbb{S}^1$.*

Proposition 2.3 ([9], Lemma 3.1)**.** *Let S be a two-dimensional sphere cylindrical embedded in the manifold $\mathbb{S}^2 \times \mathbb{S}^1$. Then S either bounds a 3-ball there, or is ambiently isotopic to the sphere $\mathbb{S}^2 \times \{s_0\}$, $s_0 \in \mathbb{S}^1$.*

Proposition 2.4 ([15], Exercise 6). *Any two-sided compressible 2-torus T in an irreducible 3-manifold X either restricts the solid torus, or it is contained in a 3-ball there.*

Proposition 2.5 ([16], Chapter 4, sec. 5, corollary 1). *Any n-dimensional manifold cannot be separated by a subset of dimension $\leqslant n-2$.*

2.2 Morse-Smale diffeomorphisms

Let M^n be a smooth closed n-manifold with the metric d and $f : M^n \to M^n$ be an orientation-preserving diffeomorphism.

A compact f-invariant set $A \subset M^n$ is called an *attractor* of a diffeomorphism f if it has a compact neighborhood U_A such that $f(U_A) \subset int(U_A)$ and $A = \bigcap\limits_{k\geqslant 0} f^k(U_A)$. The neighborhood U_A is called *trapping. Repeller* is defined as an attractor for f^{-1}.

A point $x \in M^n$ is called a *wandering point* of the diffeomorphism f if it has a neighborhood $U_x \subset M^n$ such that $f^k(U_x) \cap U_x = \emptyset$ for any $k \neq 0$. The complement to the set of wandering points is called a *non-wandering set* of the diffeomorphism f.

If the non-wandering set of f is finite that it consists of periodic points. An isolated periodic point p of the period m_p of the diffeomorphism f is called *hyperbolic* if absolute values of all the eigenvalues of the Jacobi matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ are not unit. If all eigenvalues by modulo are less than (greater than) one, then p is called a *sink (source) point.* The sink and source points are called *nodes.* If a hyperbolic periodic point is not *nodal*, then it is called *saddle point.*

If f has a finite number of periodic points, all of them are hyperbolic, then the hyperbolic structure of the periodic point p implies that its *stable*

$$W^s_p = \{x \in M^n : \lim_{k\to+\infty} d(f^{km_p}(x), p) = 0\}$$

and *unstable*

$$W^u_p = \{x \in M^n : \lim_{k\to+\infty} d(f^{-km_p}(x), p) = 0\}$$

manifolds are smooth submanifolds, diffeomorphic to $\mathbb{R}^{q_p}$ and $\mathbb{R}^{n-q_p}$, respectively, where q_p is the number of eigenvalues of the Jacobi matrix with the absolute value greater than 1 (*Morse index of the point* p). The stable and the unstable manifolds are called *invariant manifolds.*

A number $\nu_p = +1\,(-1)$ is called an *orientation type* of the point p if the map $f^{m_p}|W^u_p$ preserves (changes) the orientation.

A connected component ℓ^u_p (ℓ^s_p) of the set $W^u_p \setminus p$ ($W^s_p \setminus p$) is called an *unstable (stable) separatrix* of the point p.

Diffeomorphism $f : M^n \to M^n$ defined on a smooth closed connected orientable n-dimensional manifold ($n \geq 1$) M^n is called *a Morse-Smale diffeomorphism* if

1. its nonwandering set Ω_f consists of a finite number of hyperbolic orbits;

2. intersection of the invariant manifolds W^s_p, W^u_q is transversal for any nonwandering points p, q.

Denote by $MS(M^n)$ the set of orientation-preserving Morse-Smale diffeomorphisms defined on an orientable n-manifold M^n.

Proposition 2.6 ([17], Theorem 2.1.1.). *Let $f \in MS(M^n)$. Then*

1. $M^n = \bigcup\limits_{p\in\Omega_f} W^u_p$;

2. *W^u_p is a smooth submanifold of the manifold M^n, diffeomorphic to $\mathbb{R}^{q_p}$ for any periodic point $p \in \Omega_f$;*

3. $cl(\ell^u_p) \setminus (\ell^u_p \cup p) = \bigcup\limits_{r\in\Omega_f : \ell^u_p \cap W^s_r \neq \emptyset} W^u_r$ *for any unstable separatrix $\ell^u_p(\ell^s_p)$ of periodic point $p \in \Omega_f$.*

If σ_1, σ_2 different saddle periodic points of the diffeomorphism $f \in MS(M^n)$, for which $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called the *heteroclinic intersection*. The path connected components of a heteroclinic intersection are called *heteroclinic points* if their dimension is 0, *heteroclinic curves* if their dimension is 1, and *heteroclinic manifolds* if their dimension is greater than 1.

The diffeomorphism $f \in MS(M^n)$ is called *gradient-like* if from the condition $W^u_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for different points $\sigma_1, \sigma_2 \in \Omega_f$ it follows that $\dim W^u_{\sigma_1} < \dim W^u_{\sigma_2}$. This is equivalent to the absence of heteroclinic points for the diffeomorphism f.

Proposition 2.7 ([17], Proposition 2.1.3.). *If a separatrix ℓ^u_σ of a saddle point σ of a diffeomorphism $f \in MS(M^n)$ does not participate in the heteroclinic intersection, then there is a unique sink point ω such that*

$$cl(\ell^u_\sigma) = \sigma \cup \ell^u_\sigma \cup \omega.$$

At the same time, $cl(\ell^u_\sigma)$ is homeomorphic to the segment if $q_p = 1$ and homeomorphic to the sphere $\mathbb{S}^{q_p}$ if $q_p > 1$.

Let's put $\hat{W}^u_p = (W^u_p \setminus p)/f^{m_p}$ and denote by $p_{\hat{W}^u_p} : W^u_p \setminus p \to \hat{W}^u_p$ natural prection.

Proposition 2.8 ([17], Theorem 2.1.3). *The projection $p_{\hat{W}^u_p}$ is a covering that induces the structure of a smooth q_p-manifolds on the space of orbits $\hat{W}^u_p$. At the same time:*

- *for $q_p = 1, \nu_p = -1$ the space $\hat{W}^u_p$ is homeomorphic to a circle;*
- *for $q_p = 1, \nu_p = +1$ the space $\hat{W}^u_p$ is homeomorphic to a pair of circles;*
- *for $q_p = 2$, $\nu_p = -1$ the space $\hat{W}^u_p$ is homeomorphic to the Klein bottle;*
- *for $q_p = 2$, $\nu_p = +1$ the space $\hat{W}^u_p$ is homeomorphic to a two-dimensional torus;*

- *for* $q_p \geqslant 3, \nu_p = -1$ *space* $\hat{W}^u_p$ *is homeomorphic to the generalized Klein bottle* $\mathbb{S}^{q_p-1}\tilde{\times}\mathbb{S}^1$*;*

- *for* $q_p \geqslant 3, \nu_p = +1$ *the space* $\hat{W}^u_p$ *is homeomorphic to* $\mathbb{S}^{q_p-1} \times \mathbb{S}^1$*.*

Let $f \in MS(M^n)$. Denote by Ω^0_f, Ω^1_f, Ω^2_f the set of sinks, saddles and sources of diffeomorphism f. We divide the set Ω^1_f into two disjoint subsets Σ_A and Σ_R such that the sets

$$A = \Omega^0_f \cup W^u_{\Sigma_A},\ R = \Omega^2_f \cup W^s_{\Sigma_R}$$

are closed and invariant. By construction, the sets A, R contain all periodic points of the diffeomorphism f. The largest dimension of the unstable (stable) manifold of periodic points from $A(R)$ is called *dimension of* $A\,(R)$.

Proposition 2.9 ([18], Theorem 1)**.** *Let* $f \in MS(M^n)$*. Then the set* A *(respectively* R*) is an attractor (repeller) of the diffeomorphism* f*. Moreover, if the dimension of the attractor* A *(repeller* R*)* $\leqslant n-2$*, then the repeller* R *(attractor* A*) is connected.*

Following [18], we will call A and R a *dual attractor and repeller* of the Morse-Smale diffeomorphism $f \in MS(M^n)$, and the set $V = M^n \setminus (A \cup R)$ – *a characteristic set*. Denote by

$$\hat{V} = V/f$$

the set of orbits of the action of the group $F = \{f^k, k \in \mathbb{Z}\}$ on the manifold V – *characteristic space*, which coincides with the set of orbits of the diffeomorphism f on V. Let

$$p_{\hat{V}} : V \to \hat{V}$$

be a natural projection that matches the point $x \in V$ with its orbit by virtue of the diffeomorphism f and endows the set $\hat{V}$ with a factortopology.

Proposition 2.10 ([18], Theorem 2)**.** *For any dual pair of attractor-repeller* A, R *of the Morse-Smale diffeomorphism* $f \in MS(M^n)$ *the following is true:*

- *the characteristic space* $\hat{V}$ *is a closed smooth orientable n-manifold, whose each connected component is either irreducible or homeomorphic to* $\mathbb{S}^{n-1} \times \mathbb{S}^1$*;*

- *projection* $p_{\hat{V}} : V \to \hat{V}$ *is a cover;*

- *a map* $\eta_{\hat{V}}$*, which assigns to each homotopy class* $[\hat{c}]$ *of loops* $\hat{c} \subset \hat{V}$ *closed at a point* $\hat{x}$ *an integer* n *such that lifting the loop* $\hat{c}$ *by* V *connects some point* $x \in p^{-1}_{\hat{V}}(\hat{x})$ *with a point* $f^n(x)$*, is a homomorphism on the fundamental group of each connected component of the space* $\hat{V}$*;*

- *if the dimension of the attractor* A *and the repeller* $R \leqslant n-2$*, then* $V, \hat{V}$ *are connected and the map* $\eta_{\hat{V}} : \pi_1(\hat{V}) \to \mathbb{Z}$ *is an epimorphism.*

A submanifold $\hat{X} \subset \hat{V}$ is called $\eta_{\hat{V}}$-*essential*, if $\eta_{\hat{V}}(i_{\hat{X}*}(\pi_1(\hat{X})) \neq \{0\}$.

Let U_A be a trapping neighborhood of an attractor A of a Morse-Smale diffeomorphism $f : M^n \to M^n$ and R be the dual to it repeller. Let $F_A = U_A \setminus f(U_A)$, then $cl(F_A)$ is the fundamental domain of the diffeomorphism f restriction to V. Suppose $\hat{V}_A = cl(F_A)/f$, then $\hat{V}_A$ is a smooth closed n-manifold obtained from $cl(F_A)$ by identifying boundaries due to the diffeomorphism f. Denote by $p_A : cl(F_A) \to \hat{V}_A$ the natural projection.

Consider the family $E_f \in Diff(M^n)$ of diffeomorphisms such that $\Omega_{f'} = \Omega_f$ for any diffeomorphism $f' \in E_f$ and the diffeomorphism f' coincides with the diffeomorphism f on U_A and in some neighborhood of R.

For any diffeomorphism $f' \in E_f$, we put $\hat{l}^s_{f'} = p_A(W^s_{\Sigma_A} \cap F_A)$ and $\hat{l}^u_{f'} = p_A(W^u_{\Sigma_R} \cap F_A)$.

Proposition 2.11 ([19], Lemma 1)**.** *Let $\hat{h} : \hat{V}_A \to \hat{V}_A$ be an isotopic to identity diffeomorphism. Then there exists a smooth arc $\varphi_t \subset E_f$ such that $\varphi_0 = f, \varphi_1 = f'$ and $\hat{l}^u_{f'} = \hat{h}(\hat{l}^u_f)$, $\hat{l}^s_{f'} = \hat{l}^s_f$.*

2.3 Classification of Morse-Smale 3-diffeomorphisms

Let $f \in MS(M^3)$. Let's put

$$A_f = W^u_{\Omega_0 \cup \Omega_1},\ R_f = W^s_{\Omega_2 \cup \Omega_3},\ V_f = M^3 \setminus (A_f \cup R_f).$$

By proposition 2.10 set A_f (R_f) is a connected attractor (repeller) whose topological dimension is less than or equal to 1, the set V_f is a connected 3-manifold and

$$V_f = W^s_{A_f \cap \Omega_f} \setminus A_f = W^u_{R_f \cap \Omega_f} \setminus R_f.$$

Moreover, the space $\hat{V}_f = V_f/f$ is a connected closed orientable 3-manifold and the natural projection $p_{_f} : V_f \to \hat{V}_f$ induces an epimorphism $\eta_{_f} : \pi_1(\hat{V}_f) \to \mathbb{Z}$, attributing to each homotopy class $[c] \in \pi_1(\hat{V}_f)$ of a closed curve $c \subset \hat{V}_f$ an integer n such that a lift of c connects some point $x \in V_f$ with the point $f^n(x)$. Let's put

$$\hat{L}^s_f = p_{_f}(W^s_{\Omega_1} \setminus A_f),\ \hat{L}^u_f = p_{_f}(W^u_{\Omega_2} \setminus R_f).$$

Set $S_f = (\hat{V}_f, \eta_{_f}, \hat{L}^s_f, \hat{L}^u_f)$ is called a *scheme* of the diffeomorphism $f \in MS(M^3)$.

Proposition 2.12 ([5], Theorem 1)**.** *Diffeomorphisms f, $f' \in MS(M^3)$ are topologically conjugate if and only if their schemes are equivalent, that is, there is a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that*

1) $\eta_{_f} = \eta_{_{f'}} \hat{\varphi}_*$;

2) $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$, $\hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$.

To solve the realization problem it is necessary to identify the set of all abstract schemes that can be implemented by a Morse-Smale diffeomorphism.

Let $\hat{V}$ be a simple smooth 3-manifold whose fundamental group admits an epimorphism $\eta : \pi_1(\hat{V}) \to \mathbb{Z}$, $\hat{\ell} \subset \hat{V}$ be an η-essential smooth torus and $N_{\hat{\ell}} \subset \hat{V}$ is its tubular neighborhood. Let $\hat{Y} = \mathbb{D}^2 \times \mathbb{S}^1$ and $\hat{\mu}$ be a meridian of the solid torus $\hat{Y}$ (closed curve contractible on $\hat{Y}$ and essential on $\partial\hat{Y}$) and $\zeta_{\ell} : \partial\hat{Y} \times \mathbb{S}^0 \to \partial N_{\hat{\ell}}$ be a diffeomeomorphism such that $\eta(\zeta_{\ell}(\hat{\mu} \times \mathbb{S}^0)) = 0$. It is said that the space $\hat{V}_{\hat{\ell}} = (\hat{V} \setminus int\, N_{\hat{\ell}}) \cup_{\zeta_{\ell}} (\hat{Y} \times \mathbb{S}^0)$ *is obtained from the manifold* $\hat{V}$ *by a cut-gluing operation along the torus* $\hat{\ell}$.

Structure of a smooth closed 3-manifold on the set $\hat{V}_{\hat{\ell}}$ induced by the natural projection $p_{\hat{\ell}} : (\hat{V} \setminus int\, N_{\hat{\ell}}) \sqcup (\hat{Y} \times \mathbb{S}^0) \to \hat{V}_{\hat{\ell}}$. Since any homeomorphism of the boundary of the solid torus that translates meridian to meridian can be extended to the solid torus [20], the described operation is correctly defined, that is, it does not depend (up to homeomorphism) on the choice of the tubular neighborhood $N_{\hat{\ell}}$ and the homeomorphism ζ_{ℓ}.

Similarly, a *cut-gluing operation along an η-essential smooth Klein bottle* $\hat{\ell} \subset \hat{V}$ is defined and it consists of a gluing the solid torus $\hat{Y}$ to the boundary of the manifold $\hat{V} \setminus int\, N_{\hat{\ell}}$. Also, the cut-gluing operation is generalized to the set $\hat{L} \subset \hat{V}$, which is a disjoint union of smooth η-essential tori and Klein bottles, we will denote by $V_{\hat{L}}$ the manifold obtained as a result of such an operation.

For a gradient-like diffeomorphism $f \in MS(M^3)$ each connected component $\hat{\ell}^s$ ($\hat{\ell}^u$) of the sets $\hat{L}^s_f$ ($\hat{L}^u_f$) is either a torus or a Klein bottle, η_f-essentially embedded into the manifold $\hat{V}_f$.

The scheme of any gradient-like diffeomorphism $f \in MS(M^3)$ is an abstract schema in the sense of the following definition.

A collection $S = (\hat{V}, \eta, \hat{L}^s, \hat{L}^u)$ is called an *abstract scheme* if:

1) $\hat{V}$ is a simple manifold whose fundamental group admits an epimorphism $\eta : \pi_1(\hat{V}) \to \mathbb{Z}$;

2) the sets $\hat{L}^s$, $\hat{L}^u \subset \hat{V}$ are transversally intersecting disjoint unions of smooth η-essential tori and Klein bottles;

3) each connected component of the manifolds $\hat{V}_{\hat{L}^s}$, $\hat{V}_{\hat{L}^u}$ is homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$.

Proposition 2.13 ([6], Theorem 1)**.** *For any abstract scheme $S = (\hat{V}, \eta, \hat{L}^s, \hat{L}^u)$ there is a gradient-like diffeomorphism $f \in MS(M^3)$ whose scheme S_f is equivalent to the scheme S.*

2.4 Topology of 3-manifolds admitting Morse-Smale diffeomorphisms with a given structure of a non-wandering set

Let $f \in MS(M^3)$. Let's say

$$g_f = \frac{r_f - l_f + 2}{2},$$

where r_f is the number of saddle points and l_f is the number of nodal periodic points of the diffeomorphism f. According to [17], the number g_f is a non-negative integer for any diffeomorphism $f \in MS(M^3)$.

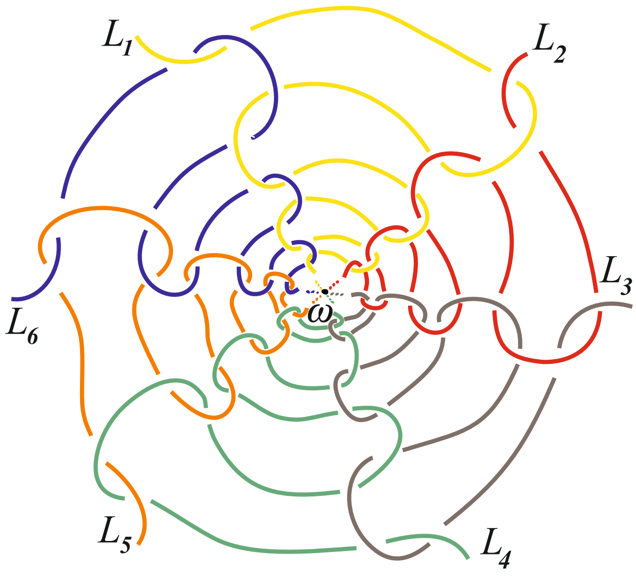

Figure 4: Wild frame of separatrix in which each separatrix is tame

If $f \in MS(M^3)$ is a gradient-like diffeomorphism. According to proposition 2.7, the closure $cl(\ell^u_\sigma)$ of any one-dimensional unstable separatrix ℓ^u_σ of the saddle point σ of the diffeomorphism f is homeomorphic to a segment that consists of this separatrix and two points: σ and some sink ω. Let L_ω be a union of unstable one-dimensional separatrices of saddle points that contain ω in their closures. According to proposition 2.8, W^s_ω is homeomorphic to $\mathbb{R}^3$ and the set $L_\omega \cup \omega$ is a union of simple arcs with a single common point ω, then by analogy with a frame of arcs in $\mathbb{R}^3$, $L_\omega \cup \omega$ is called *a frame of one-dimensional unstable separatrices.*

According to [7], a frame of separatrix $L_\omega \cup \omega$ is called *tame* if there is a homeomorphism $\psi_\omega : W^s_\omega \to \mathbb{R}^3$ such that $\psi_\omega(L_\omega \cup \omega)$ — a frame of rays in $\mathbb{R}^3$. Otherwise, the separatrix frame is called *wild* (see Fig. 4).

If α is the source of the diffeomorphism f, then the tame (wild) bundle L_α of one-dimensional stable separatrices is similarly defined.

Proposition 2.14 ([7], Theorem 4.1)**.** *If all frames of one-dimensional separatrices of a gradient-like diffeomorphism $f \in MS(M^3)$ are tame, then the ambient manifold M^3 admits a Heegaar splitting of the genus g_f.*

Proposition 2.15 ([14], Theorem 1)**.** *Let $f \in MS(M^3)$ be a Morse-Smale diffeomorphism without heteroclinic curves. Then the following statements are true:*

1) if $g_f = 0$, then M^3 – 3-sphere;

2) if $g_f > 0$, then M^3 – connected sum of g_f copies of $\mathbb{S}^2 \times \mathbb{S}^1$.

Conversely, for any non-negative integers r, l, g such that the number $g = \frac{r-l+2}{2}$ is an integer and non-negative, there is a diffeomorphism $f \in MS(M^3)$ without heteroclinic curves, with the following properties:

a) M^3 – 3-sphere if $g = 0$ and M^3 – connected sum of g copies of $\mathbb{S}^2 \times \mathbb{S}^1$ if $g > 0$;

b) the non-wandering diffeomorphism set f consists of r saddle and l node points.

Recall that *lens space* is defined as a gluing of two solid tori by means of a homeomorphism of their boundaries and is denoted by $L_{p,q}$, $p, q \in \mathbb{Z}$, where $\langle p, q\rangle$ is the homotopy type of the image of a meridian with respect to the gluing homeomorphism. Some well-known 3-manifolds are actually lens spaces, for example, the three-dimensional sphere $\mathbb{S}^3 = L_{1,0}$, the manifold $\mathbb{S}^2 \times \mathbb{S}^1 = L_{0,1}$, the projective space $\mathbb{R}P^3 = L_{1,2}$.

Proposition 2.16 ([7], Theorem 6.1). *Let $f : L_{p,q} \to L_{p,q}$ be a Morse-Smale diffeomorphism whose non-wandering set consists of exactly four points. Then*

1) f – gradient-like;

2) periodic points of the diffeomorphism f have pairwise different Morse indices;

3) if all frames of one-dimensional separatrices of f are tame, then the wandering set of diffeomorphism f contains at least p of non-compact heteroclinic curves.

3 Dynamics of diffeomorphisms of the class G

In this section, we establish some dynamic properties of the diffeomorphism $f : M^3 \to M^3$ from the class G.

Recall that the class G consists of diffeomorphisms $f \in MS(M^3)$ having exactly four non-wandering points $\omega_f, \sigma_f^1, \sigma_f^2, \alpha_f$ with Morse indices $0, 1, 2, 3$, respectively.

Due to the absence of heteroclinic points in the diffeomorphism f, one-dimensional saddle manifolds contain a unique nodal point in their closures (see, sentence 2.7). Exactly,

$$cl(W^u_{\sigma_f^1}) = W^u_{\sigma_f^1} \cup \omega_f,\ cl(W^s_{\sigma_f^2}) = W^s_{\sigma_f^2} \cup \alpha_f.$$

In this case, by proposition 2.8, the sets $A_f = cl(W^u_{\sigma_f^1})$, $R_f = cl(W^s_{\sigma_f^2})$ are pairwise disjoint topologically embedded circles (see Fig. 2, 3, 5, 6), possibly wild at the nodal points. Recall that

$$H_f = W^s_{\sigma_f^1} \cap W^u_{\sigma_f^2}.$$

If the set H_f is not empty, then, by proposition 2.6,

$$cl(W^s_{\sigma_f^1}) = W^s_{\sigma_f^1} \cup R_f,\ cl(W^u_{\sigma_f^2}) = W^u_{\sigma_f^2} \cup A_f.$$

Otherwise, according to proposition 2.7, the sets

$$cl(W^s_{\sigma_f^1}) = W^s_{\sigma_f^1} \cup \alpha_f,\ cl(W^u_{\sigma_f^2}) = W^u_{\sigma_f^2} \cup \omega_f$$

are topologically embedded disjoint two-dimensional spheres (see Fig. 6).

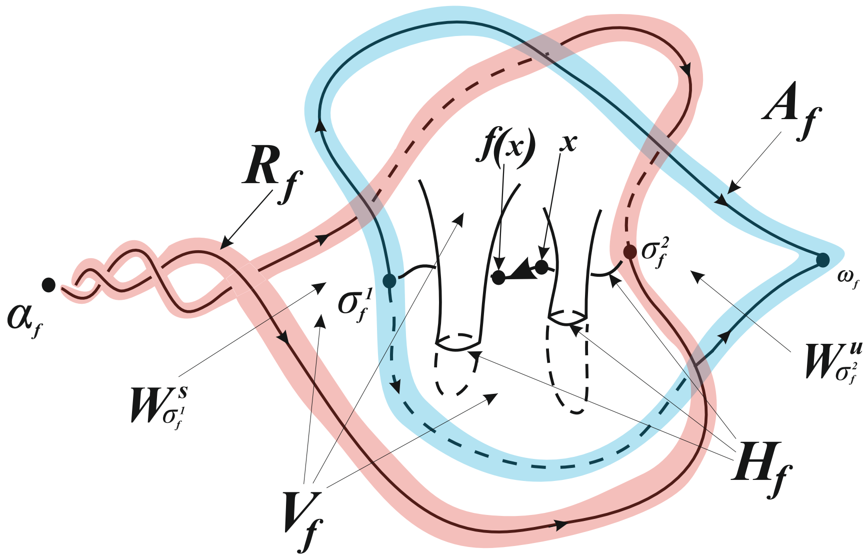

Figure 5: Phase portrait of a diffeomorphism $f \in G$ with a non-empty set H_f

3.1 Consistent neighborhoods system

Let $\mathcal{N}_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2(x_2^2 + x_3^2) \leqslant 1\}$ and $\mathcal{N}_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1^2 + x_2^2)x_3^2 \leqslant 1\}$. Define in the neighborhood of $\mathcal{N}_1$ a pair of transversal foliations $\mathcal{F}_1^u, \mathcal{F}_1^s$ as follows:

$$\mathcal{F}_1^u = \bigcup_{(c_2,c_3)\in Ox_2x_3} \{(x_1, x_2, x_3) \in \mathcal{N}_1 : (x_2, x_3) = (c_2, c_3)\},$$

$$\mathcal{F}_1^s = \bigcup_{c_1\in Ox_1} \{(x_1, x_2, x_3) \in \mathcal{N}_1 : x_1 = c_1\}.$$

Define in the neighborhood of $\mathcal{N}_2$ a pair of transversal foliations $\mathcal{F}_2^u, \mathcal{F}_2^s$ as follows:

$$\mathcal{F}_2^u = \bigcup_{c_3\in Ox_3} \{(x_1, x_2, x_3) \in \mathcal{N}_3 : x_3 = c_3\},$$

$$\mathcal{F}_2^s = \bigcup_{(c_1,c_2)\in Ox_1x_2} \{(x_1, x_2, x_3) \in \mathcal{N}_3 : (x_1, x_2) = (c_1, c_2)\}.$$

We define diffeomorphisms $\nu_i : \mathbb{R}^3 \to \mathbb{R}^3$ by formulas:

$$\nu_1(x_1, x_2, x_3) = \left(2x_1, \frac{x_2}{2}, \frac{x_3}{2}\right), \ \nu_2 = a_1^{-1}.$$

Note that for $i \in \{1, 2\}$, the set $\mathcal{N}_i$ is invariant with respect to diffeomorphism ν_i, which translates leaves of the foliation $\mathcal{F}_i^u$ ($\mathcal{F}_i^s$) into leaves of the same foliation.

By [21], the saddle point σ_f^i of the diffeomorphism $f \in G$ has *a linearizing neighborhood* N_f^i equipped with the homeomorphism $\mu_i : N_f^i \to \mathcal{N}_i$, conjugating the diffeomorphism $f|_{N_f^i}$ with the diffeomorphism $\nu_i|_{\mathcal{N}_i}$ and being a diffeomorphism on $N_f^i \setminus (W^s_{\sigma_f^i} \cup W^u_{\sigma_f^i})$. Foliations $\mathcal{F}_i^u, \mathcal{F}_i^s$ are induced by the homeomorphism μ_i^{-1}, f-invariant foliations of F_i^u, F_i^s on the

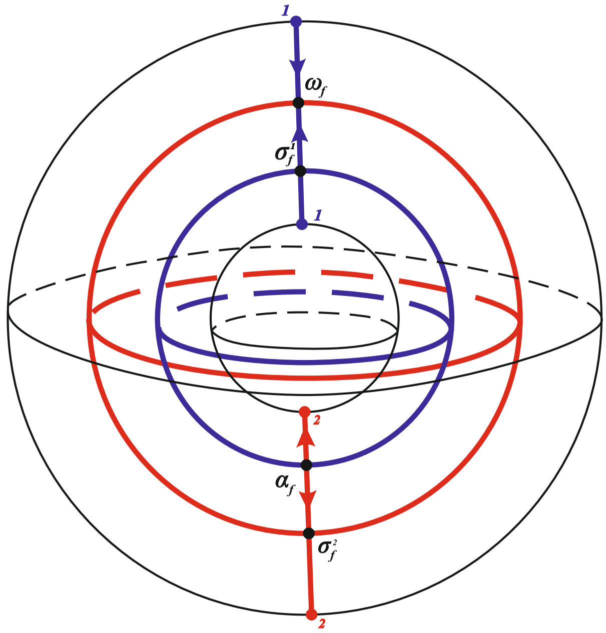

Figure 6: Phase portrait of a diffeomorphism $f \in G$ with an empty set H_f

linearizing neighborhood N_f^i. For any point $x \in N_f^i$ we will denote by $F_{i,x}^u$ ($F_{i,x}^s$) a unique leaf of the foliation F_i^u (F_i^s) passing through the point x.

If the set H_f is empty, then the set N_f of disjoint linearizing neighborhoods N_f^1, N_f^2 of saddle points of the diffeomorphism f is called a *consistent neighborhoods system*, and the foliations F_i^s, F_i^u $(i = 1, 2)$ – *consistent*.

If $H_f \neq \emptyset$, then we choose f-invariant tubular neighborhood $N_{H_f} \subset M^3$ of curves of the set H_f, equipped with f-invariant $C^{1,1}$-foliation F, consisting of two-dimensional disks, transversal to H_f. For any point $x \in N_{H_f}$, we will denote by F_x a unique leaf of the foliation F passing through the point x.

The union N_f of linearizing neighborhoods of N_f^1, N_f^2 saddle points of the diffeomorphism f is called a *consistent neighborhoods system*, and the foliations F_i^s, F_i^u $(i = 1, 2)$, *are consistent* if for any point $x \in (N_f^1 \cap N_f^2 \cap N_{H_f})$ and the leaf F_x of the foliation F passing through the point x, the conditions are met (see Fig. 7):

$$F_{1,x}^s \cap F_x = F_{2,x}^s \cap (N_f^1 \cap N_{H_f}), \;\; F_{2,x}^u \cap F_x = F_{1,x}^u \cap (N_f^2 \cap N_{H_f}).$$

Proposition 3.1 ([5], Theorem 1)**.** *For any diffeomorphism $f \in G$ there is a consistent neighborhoods system.*

3.2 Quotients

Consider the characteristic spaces $V_{\omega_f} = W_{\omega_f}^s \setminus \omega_f$ and $\hat{V}_{\omega_f} = V_{\omega_f}/f$. By proposition 2.8, $\hat{V}_{\omega_f}$ is diffeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$. By proposition 2.10, the projection $p_{\omega_f} : V_{\omega_f} \to \hat{V}_{\omega_f}$

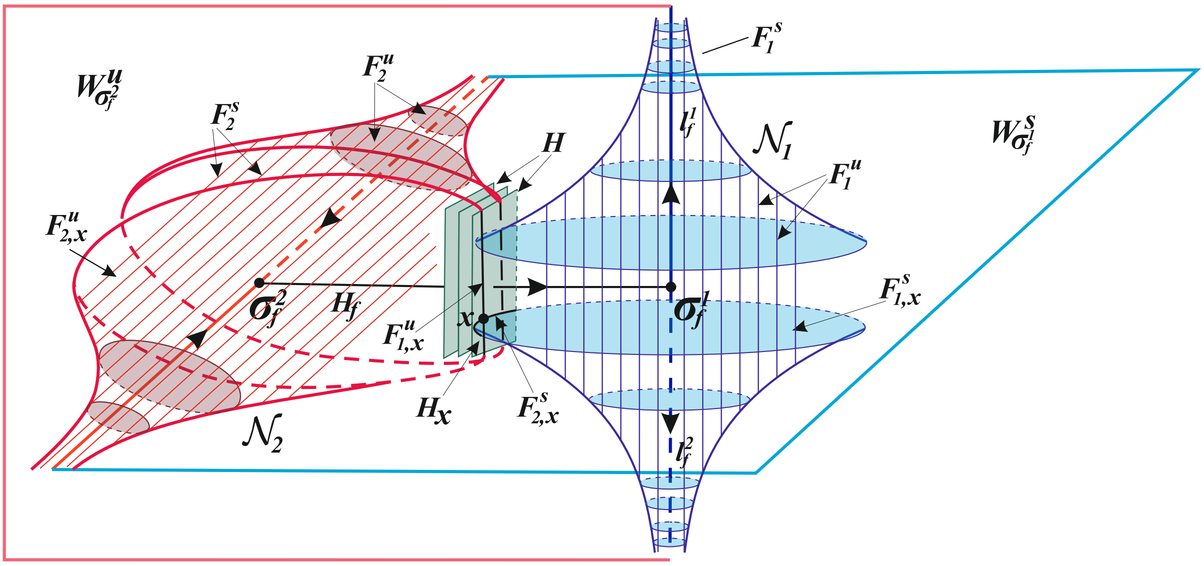

Figure 7: Consistent neighborhoods system

is covering which generate an epimorphism $\eta_{\omega_f} : \pi_1(V_{\omega_f}) \to \mathbb{Z}$. Let's put (see Fig. 8)

$$\hat{A}_f = p_{\omega_f}(A_f).$$

By proposition 2.8, $\hat{A}_f$ consists of a pair of disjoint knots $L^1_f \sqcup L^2_f$ such that the map

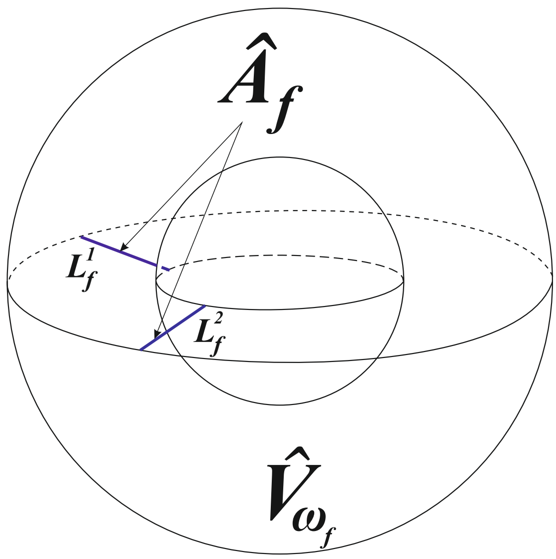

Figure 8: Space $\hat{V}_{\omega_f}$

$i_{L^i_f *} : \pi_1(L^i_f) \to \pi_1(\hat{V}_{\omega_f})$, $i \in \{1, 2\}$ is an isomorphism. Moreover,

$$N_{\hat{A}_f} = p_{\omega_f}(N^1_f \cap V_{\omega_f})$$

is a disjoint union of tubular neighborhoods $N_{L^1_f}$, $N_{L^1_f}$ of knots L^1_f, L^2_f, accordingly.

Proposition 3.2 ([17], Lemma 4.4). *If at least one of the sets $\hat{V}_{\omega_f} \setminus int\, N_{L_f^1}$, $\hat{V}_{\omega_f} \setminus int\, N_{L_f^2}$ is not homeomorphic to a solid torus, then the manifold $W^u_{\sigma_f^1}$ is wildly embedded into the supporting manifold M^3.*

By proposition 2.9, the sets A_f and R_f are dual attractor and repeller, respectively, for the diffeomorphism f. Let's put

$$V_f = M^3 \setminus (A_f \cup R_f).$$

By proposition 2.10, the characteristic space $\hat{V}_f = V_f/f$ is a smooth simple orientable 3-manifold, and the natural projection $p_f : V_f \to \hat{V}_f$ is a cover inducing an epimorphism

$$\eta_f : \pi_1(\hat{V}_f) \to \mathbb{Z}.$$

Let's put (see Fig. 9)

$$T^s_f = p_f(W^s_{\sigma_1}),\ T^u_f = p_f(W^u_{\sigma_2}), C_f = p_f(H_f).$$

By propositions 2.8 and 2.10, the sets T^s_f, T^u_f are smoothly embedded 2-tori in $\hat{V}_f$ such

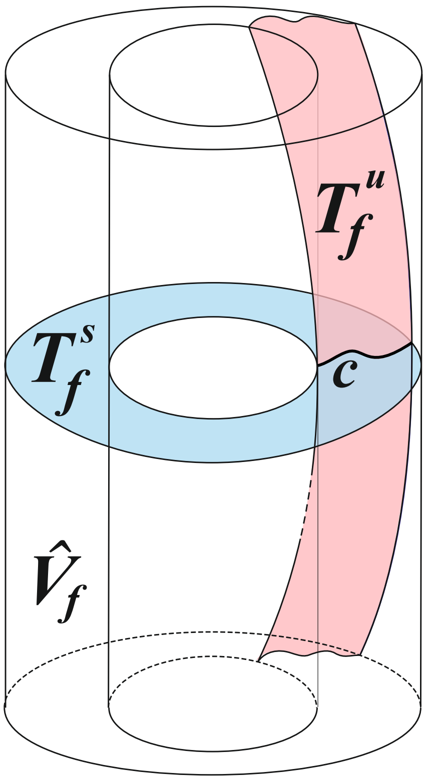

Figure 9: Space $\hat{V}_f$

that $\eta_f(i_{T^s_f *}(\pi_1(T^s_f))) = \eta_f(i_{T^u_f *}(\pi_1(T^u_f))) \cong \mathbb{Z}$. Moreover

$$N_{T^s_f} = p_f(N^1_f \cap V_f),\ N_{T^u_f} = p_f(N^2_f \cap V_f)$$

are tubular neighborhoods of the tori T^s_f, T^u_f, accordingly.

3.3 The scheme of $f \in G$

As was mentioned above the collection

$$S_f = (\hat{V}_f, \eta_f, T^s_f, T^u_f)$$

is the scheme of $f \in G$ and, by proposition 2.12, is a complete invariant of the topological classification.

The main result of the section is the following lemma.

Lemma 3.1 ([19], Lemma 2)**.** *For any diffeomorphism $f \in G$, the following is true:*

1. *the manifold $\hat{V}_f$ is irreducible and the tori T^s_f, T^u_f are incompressible in it;*
2. *the set C_f consists of a finite number of smoothly embedded closed curves, while $\eta_f([c]) = 0$ if and only if the curve $c \subset C_f$ is a projection of a compact heteroclinic curve;*
3. *any curve $c \subset C_f$ such that $\eta_f([c]) = 0$ is contractible (or non-contractible) simultaneously on both tori T^s_f, T^u_f.*

Proof. Let us prove successively all the statements of the lemma.

1. By virtue of the sentence 2.9, the manifold $\hat{V}_f$ is simple. Since the torus T^s_f is η_f-essential in $\hat{V}_f$, then it does not lie in a 3-ball. Let's show from the opposite that the torus T^s_f does not bound the solid torus in $\hat{V}_f$. It follows from proposition 2.2, that the manifold $\hat{V}_f$ is not homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and, therefore, is irreducible. Then, by proposition 2.4, the torus T^s_f is incompressible in $\hat{V}_f$.

If we assume that $\hat{V}_f \cong \mathbb{S}^2 \times \mathbb{S}^1$, then the torus T^s_f bounds a solid torus there by proposition 2.2, and therefore $\hat{V}_f \setminus T^s_f$ consists of two connected components. On the other hand, by proposition 2.6,

$$M^3 = W^s_{\omega_f} \cup W^s_{\sigma^1_f} \cup W^s_{\sigma^2_f} \cup W^s_{\alpha_f}.$$

Then $V_f \setminus W^s_{\sigma^1_f} = W^s_{\omega_f} \setminus A_f$ and, hence, the manifolds $\hat{V}_f \setminus T^s_f$ and $\hat{V}_{\omega_f} \setminus \hat{A}_f$ are homeomorphic. Since a one-dimensional submanifold does not divide a manifold of dimension three (see proposition 2.5), the set $\hat{V}_{\omega_f} \setminus \hat{A}_f$ is connected (see Fig. 8). We got a contradiction with the fact that a connected manifold is homeomorphic to an non-connected one.

2. It follows directly from the definition of the epimorphism η_f that $\eta_f([c]) = 0$ if and only if c is a projection of a compact curve.

3. Suppose that some curve $c \subset C_f$ is contractible on the torus T^u_f and essential on the torus T^s_f. Then, by definition, the torus T^s_f is compressible in $\hat{V}_f$, which contradicts the proven point 1. $\square$

Denote by C^0_f a subset of C_f consisting of contractible on T^u_f curves. Let's call the heteroclinic curves from the set $H^0_f = p_f^{-1}(C^0_f)$ *inessential*, the remaining heteroclinic curves will be called *essential*.

4 Trivialization of the dynamics of diffeomorphisms from G

Recall that for any diffeomorphism $f \in G$, we introduced the concept of a heteroclinic index I_f of f and for an integer $p \geqslant 0$ we denote by $G_p \subset G$ a subset of diffeomorphisms $f \in G$ such that $I_f = p$. Note that any essential compact heteroclinic curve $\gamma \subset H_f$ bounds a disk $d_\gamma \subset W^s_{\sigma^1_f}$ containing the saddle σ^1_f. We will consider any such curve oriented so that when moving along it, the disk d_γ remains on the left. Then for the curve γ, similarly to a non-compact curve, an orientation v_γ is determined.

The set H_f is called *orientable* if it consists only of essential heteroclinic curves with the same orientation. Otherwise, we will call the set H_f *not orientable* (see Fig. 3). Denote by $G^+_p \subset G_p$, $p \geqslant 0$ a subset of diffeomorphisms $f \in G_p$ with an orientable set H_f. Thus, for any diffeomorphism $f \in G^+_p$, the set H_f is either empty, or consists either only of non-compact or only of compact heteroclinic curves (see Fig. 10).

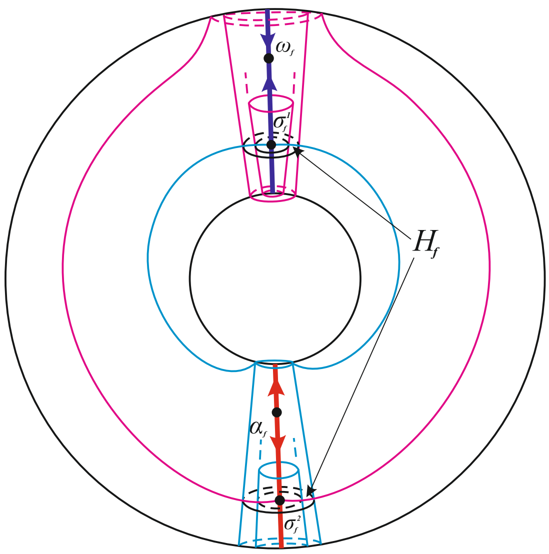

Figure 10: Diffeomorphism $f \in G^+_0$ with an orientable set H_f consisting of an infinite set of compact curves

The main result of this chapter is the proof of the following theorem.

Theorem 3. *For any diffeomorphism $f : M^3 \to M^3$ from the class G_p, $p \geqslant 0$ in the set $Diff(M^3)$ there is an arc connecting the diffeomorphism f with some diffeomorphism $f_+ \in G^+_p$.*

The proof of the theorem will directly follow from the lemmas 4.1, 4.2, proved below.

4.1 Disappearance of inessential heteroclinic curves

Denote by $\tilde{G}_p \subset G_p$ a subclass of diffeomorphisms f for which the set H^0_f is empty.

The main result of this section is the proof of the following fact.

Lemma 4.1. *For any diffeomorphism $f : M^3 \to M^3$ from the class G_p there exists an arc in the set $Diff(M^3)$ connecting the diffeomorphism f with some diffeomorphism $\tilde{f} \in \tilde{G}_p$.*

Proof. Let $f \in G_p$. If $H^0_f = \emptyset$, then the lemma is proved. Otherwise, by lemma 3.1, for any

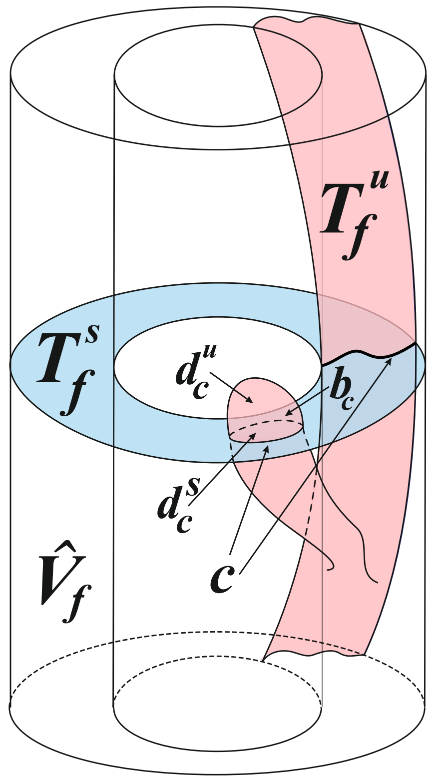

Figure 11: Construction of the 3-ball b_c

curve $c \subset C^0_f$ there exists a unique disk d^s_c such that $d^s_c \subset T^s_f$, $c = \partial d^s_c$ and a similar disk $d^u_c \subset T^u_f$ curve $c = \partial d^u_c$ (see Fig. 11).

Among the curves of the set C^0_f, we choose an *innermost curve* c, that is, such that $int\, d^s_c \cap C^0_f = \emptyset$. Since $d^s_c \cap d^u_c = c$, the set $d^s_c \cup d^u_c$ is a two-dimensional sphere cylindrical embedded into the manifold $\hat{V}_f$. By lemma 3.1, the manifold $\hat{V}_f$ is irreducible and, therefore, this sphere bounds a three-dimensional ball b_c there. Denote by $T^u_{f,c}$ a two-dimensional torus obtained by smoothing the torus $(T^u_f \setminus d^u_c) \cup d^s_c$. Then there is an isotopic to identity diffeomorphism $\hat{h} : \hat{V}_f \to \hat{V}_f$ such that $\hat{h}(T^u_f) = T^u_{f,C^0_f}$. Then by proposition 2.11 there is an arc $\zeta_t \subset E_f$ such that $\zeta_0 = f$ and $T^u_{\zeta_1} = T^u_{f,c}$, $T^s_{\zeta_1} = T^s_f$.

Repeating this process for each innermost curve, we get a required diffeomorphism $\tilde{f} \in \tilde{G}_p$. $\square$

4.2 Disappearance of non-orientable heteroclinic curves

The main result of this section is the proof of the following fact.

Lemma 4.2. *For any diffeomorphism $f : M^3 \to M^3$ from the class $\tilde{G}_p$ there exists an arc in the set $Diff(M^3)$ connecting the diffeomorphism f with some diffeomorphism $f_+ \in G_p^+$.*

Proof. Let $f \in \tilde{G}_p$. If the set H_f is either empty or orientable, then the lemma is proved.

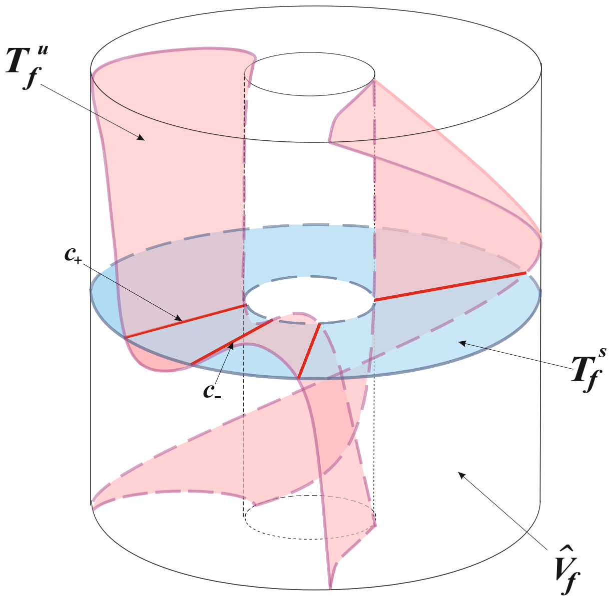

Figure 12: Projections of non-compact heteroclinic curves with different orientations into the space $\hat{V}_f$

Otherwise, C_f consists of essential pairwise homotopy on each of the tori T_f^s, T_f^u (see, for example, [20]) curves and among them there are curves c_+, c_- with the positive, negative orientation, accordingly (see Fig. 12, 13).

We show that the number of curves in H_f can be reduced by at least two.

To do this, put $Y_f = p_{\omega_f}(W^u_{\sigma_f^2} \cap V_{\omega_f})$ and $\tilde{Y}_f = Y_f \setminus int\,(N_{L_f^1} \sqcup N_{L_f^2})$. Then $\tilde{Y}_f$ consists of a finite number of annuli whose boundaries lie on the tori $T_f^1 = \partial N_{L_f^1}$, $T_f^2 = \partial N_{L_f^2}$. Due to the non-orientability of the set H_f, there is a connected component K^u of the set $\tilde{Y}_f$ having boundary circles on the same connected component of the set $T_f^1 \sqcup T_f^2$, for certainty we assume that on T_f^1. Then the circles ∂K^u divide the torus T_f^1 into two annuli, each of which K^s forms a two-dimensional torus T_{K^u} when combined with the annulus K^u. We show that K^s can be chosen such that the torus T_{K^u} is a boundary of a solid torus Q_{K^u}, whose the interior avoids $N_{L_f^1} \sqcup N_{L_f^2}$ in $\hat{V}_{\omega_f}$ (see Fig. 14, 15).

Since the torus T_f^1 is η_{ω_f}-essential in $\hat{V}_{\omega_f}$, then the annulus K^u can be chosen so that the torus T_{K^u} is also η_{ω_f}-essential in $\hat{V}_{\omega_f}$. By proposition 2.2, the torus T_{K^u} bounds a solid torus Q_{K^u} in $\hat{V}_{\omega_f}$. If $N_{L_f^1} \subset Q_{K^u}$, then, by construction, $cl(Q_{K^u} \setminus N_{L_f^1})$ is also a solid torus bounded by a torus constructed by the second annulus K^s. Therefore, everywhere else we assume that $Q_{K^u} \cap N_{L_f^1} = K^s$.

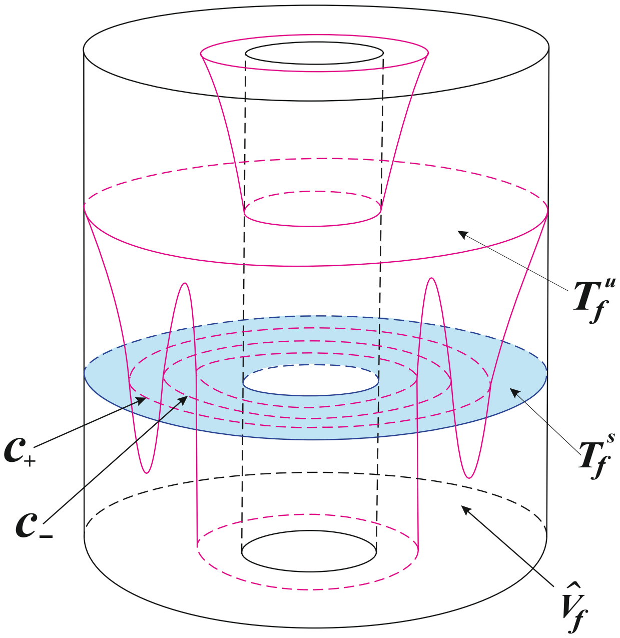

Figure 13: Projections of compact heteroclinic curves with different orientations into the space $\hat{V}_f$

Thus, every annulus K^u is associated with a torus T_{K^u}, bounding the solid torus Q_{K^u} in $\hat{V}_{\omega_f}$. Since, by proposition 2.4, any torus homotopically nontrivially embedded in a solid torus bounds a unique solid torus there, then among all such solid tori Q_{K^u} there exists $Q_{K_0^u}$ whose interior does not intersect with annuli K^u. Then the interior of the torus $Q_{K_0^u}$ does not intersect with the tori $N_{L_f^1} \sqcup N_{L_f^2}$ and $Q_{K_0^u} \cap Y_f = K_0^u$. Denote by K_0^s the second half of the torus $T_{K_0^u}$.

Denote by $\mathcal{K}_0^u$ the connected component of the set $T_f^u \setminus C_f$ such that $p_f(p_{\omega_f}^{-1}(K_0^u)) \subset \mathcal{K}_0^u$ and through $\mathcal{K}_0^s$ the connected component of the set $T_f^s \setminus C_f$ such that the annulus $p_f(p_{\omega_f}^{-1}(K_0^s))$, $\mathcal{K}_0^s$ lie in the same connected component N_0^s of the set $N_{T_f^s} \setminus T_f^u$. Denote by $\mathcal{K}_0'^s$ the connected component of the set $\partial N_{T_f^s} \cap N_0^s$, different from $p_f(p_{\omega_f}^{-1}(K_0^s))$. Let's put $\mathcal{K}_0'^u = \mathcal{K}_0^u \cup (cl(N_0^s) \cap T_f^u)$.

By construction $\partial \mathcal{K}_0^s = \partial \mathcal{K}_0^u = c_+ \sqcup c_-$, where c_+, $c_- \subset C_f$ are non-contractible curves with the positive, negative orientation, respectively. In addition, the torus $\mathcal{T}_0 = \mathcal{K}_0^u \cup \mathcal{K}_0^s$ bounds in $\hat{V}_f$a solid torus $\mathcal{Q}_0$, whose interior does not intersect with the set $T_f^u \cup T_f^s$. Denote by $T_f'^u$ a two-dimensional torus obtained by smoothing the torus $(T_f^u \setminus \mathcal{K}_0'^u) \cup \mathcal{K}_0'^s$. Since the torus $\mathcal{T}_0' = \mathcal{K}_0'^u \cup \mathcal{K}_0'^s$ bounds in $\hat{V}_f$ a solid torus $\mathcal{Q}_0'$, whose interior does not intersect with the torus T_f^u, then there is an isotopic to identity diffeomorphism $\hat{h} : \hat{V}_f \to \hat{V}_f$ such that $\hat{h}(T_f^u) = T_f'^u$. Then by proposition 2.11 there is an arc $\zeta_t \subset E_f$ such that $\zeta_0 = f, \zeta_1 = f'$ and $T_{f'}^u = T_f'^u$, $T_{f'}^s = T_f^s$.

Thus, the diffeomorphism $f' \in G$ is given on the same manifold M^3 as the diffeomorphism f, but has two less heteroclinic curves. Continuing this process, we will construct an arc connecting the diffeomorphism f with some diffeomorphism $f_+ \in G_p^+$. $\square$

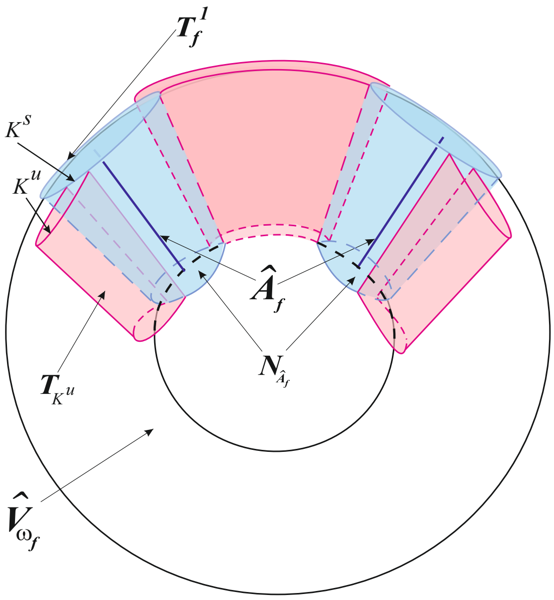

Figure 14: Projections of invariant saddle manifolds into the space $\hat{V}_{\omega_f}$ corresponding to the figure 12

5 Topology of a 3-manifold admitting diffeomorphisms of the class G

In this section we prove Theorems 2 and 1.

5.1 Lens space as an ambient manifold for diffeomorphisms of class G

Let us prove that if a manifold M^3 admits a diffeomorphism $f \in G_p$ then M^3 is homeomorphic to a lens space $L_{p,q}$.

Proof. By theorem 3, without lost of generality, we will assume that $f \in G_p^+$, that is, the set of H_f of heteroclinic curves of the diffeomorphism f is orientable and the heteroclinic index is $p \geqslant 0$. Let 's consider the following cases separately: 1) $p = 0$, 2) $p > 0$.

1) In the case $p = 0$, the set H_f is either empty or consists only of compact curves bounding disks on $W^s_{\sigma_f^1}$ containing the saddle σ_f^1, and all curves in H_f have the same orientation (see Fig. 10). If the set H_f is empty, then, by proposition 2.15, the ambient manifold M^3 is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ (see Fig. 6).

If the set H_f is not empty, then each connected component K^u of the set $\tilde{Y}_f$ is a smooth two-dimensional annulus having one boundary component on the torus T_f^1, and the other – on the torus T_f^2 and each of the circles is the meridian of the solid torus $N_{L_f^1}$, $N_{L_f^2}$, respectively. Denote by $\delta_1 \subset N_{L_f^1}$, $\delta_2 \subset N_{L_f^2}$ two-dimensional disks bounded by these meridians and having exactly one intersection point with the nodes L_f^1, L_f^2, respectively. Then the set $S = K^u \cup d_1 \cup d_2$ is a two-dimensional sphere cylindrical embedded in the manifold $\hat{V}_{\omega_f}$. Since the sphere S has a single intersection point with each of the knots

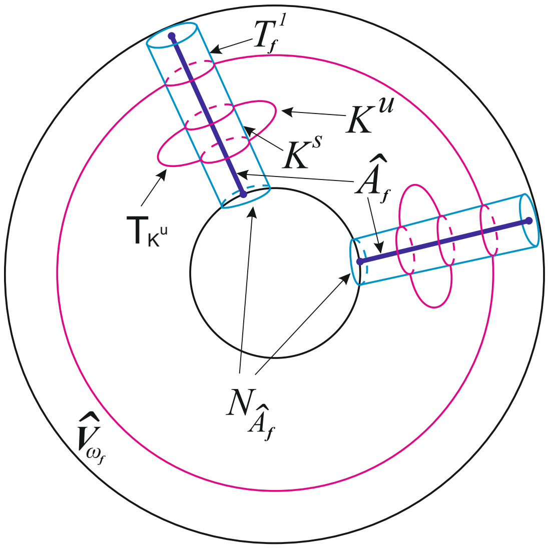

Figure 15: Projections of invariant saddle manifolds into the space $\hat{V}_{\omega_f}$ corresponding to the figure 13

L_f^1, L_f^2, then, by proposition 2.3, it is ambiently isotopic to the sphere $\mathbb{S}^2 \times \{s_0\}$, $s_0 \in \mathbb{S}^1$. Let's choose a sphere $\tilde{S}$, close to the sphere S, so that the intersection of $\tilde{S} \cap N_{L_f^i}$, $i = 1, 2$ is a two-dimensional disk $\tilde{d}_i$ having a single intersection point with the knot L_f^i and $(\tilde{S} \cap Y_f) \subset int\,(\tilde{d}_1 \sqcup \tilde{d}_2)$.

Then the sphere $\bar{S}$, which is a connected component of the set $p_{\omega_f}^{-1}(\tilde{S})$, bounds a 3-ball $B \subset W_{\omega_f}^s$ containing ω_f in its interior. In this case, the intersection of $\bar{S} \cap W_{\sigma_f^2}^u$ belongs to the disjoint union of two disks $\Delta_1 \subset p_{\omega_f}^{-1}(\tilde{d}_1)$, $\Delta_2 \subset p_{\omega_f}^{-1}(\tilde{d}_2)$ (see Fig. 16). Let's put $I = W_{\sigma_1}^u \setminus int\,B$. From the properties of a consistent neighborhoods system and orientability of heteroclinic curves, it follows that there exists a tubular neighborhood N_I of an arc I such that the intersection of $\partial N_I \cap W_{\sigma_f^2}^u$ consists of a single closed curve μ_2. Then the set $Q_1 = B \cup N_I$ is homeomorphic to a solid torus and the curve μ_2 is its meridian.

Since the curve μ_2 is homotopic on $W_{\sigma_f^2}^u \setminus \sigma_f^2$ by the heteroclinic diffeomorphism curve f, then it bounds a disk δ_2 containing the saddle σ_f^2. Let's choose a tubular neighborhood $N_{\delta_2} \subset M^3 \setminus int\,Q_1$ of the disk δ_2 so that $N_{\delta_2} \cap W_{\sigma_f^2}^u = \delta_2$ and $N_{\delta_2} \cap \partial Q_1$ is an annulus on the torus ∂Q_1, which is a tubular neighborhood of the curve μ_2. Since the curve μ_2 is essential on the torus ∂Q_1, the set $S_\alpha = \partial(Q_1 \cup N_{\delta_2})$ is homeomorphic to the 2-sphere. By construction, the sphere S_α does not intersect with unstable manifolds of saddle points and, therefore, by proposition 2.6, lies in W_α^u, where it bounds a 3-ball B_α.

Thus, the set $Q_2 = M^3 \setminus int\,Q_1$ is so that by cutting it across the disk δ_2, a 3-ball is obtained. This means that Q_2 is a solid torus, the curve μ_2 is its meridian, and $M^3 = Q_1 \cup Q_2$ is the lens space $L_{0,1} \cong \mathbb{S}^2 \times \mathbb{S}^1$ (see Fig. 10).

2) In the case $p > 0$, due to the orientability of the set H_f, each connected component K^u of the set $\tilde{Y}_f$ is an η_{ω_f}-essential annulus having boundary circles on different tori. Then each connected component of the boundary of the set $\hat{N}_f = p_{\omega_f}(N_f^1 \cup N_f^2)$ is an η_{ω_f}-essential two-dimensional torus in $\mathbb{S}^2 \times \mathbb{S}^1$ (see Fig. 17). According to proposition 2.2, each such

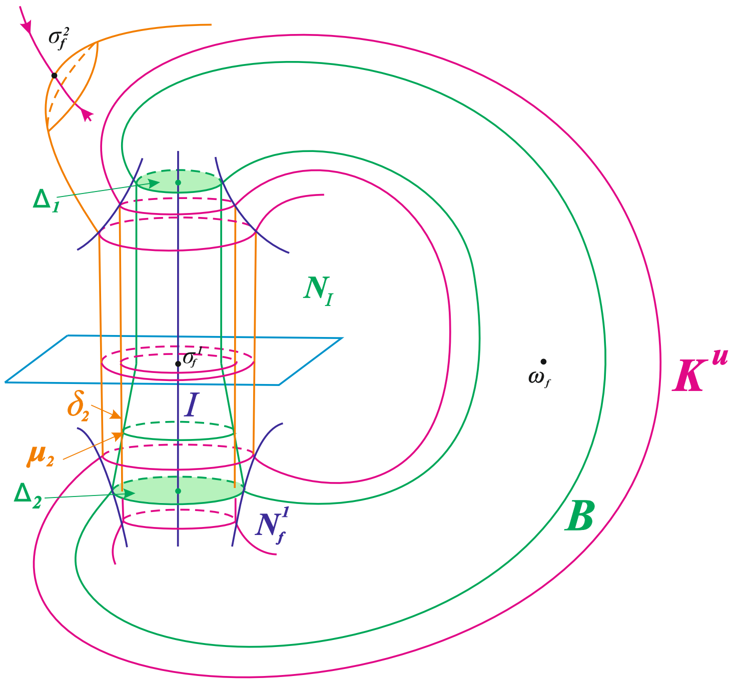

Figure 16: Construction of the ball B

torus bounds in $\hat{V}_{\omega_f}$ a solid torus, which implies that $\hat{N}_f$ belongs to the interior of the solid torus $\hat{J} \subset \hat{V}_{\omega_f}$.

Let $J = p_{\omega_f^{-1}}(\hat{J})$. Since J is an f-invariant solid cylinder whose boundary does not intersect the invariant manifolds of saddle points, then, by proposition 2.6, $\partial J \subset W^u_{\alpha_f}$. Let's choose a 2-disk $d \subset (J \cap W^u_{\alpha_f})$ so that $\partial d \subset \partial J$ and d divides J into two connected components. Select a point $y_0 \in int\, d$. Denote by J_{ω_f} the closure of the connected component containing ω_f. Then J_{ω_f} is a 3-ball on the manifold M^3, which is tame everywhere, except, perhaps, the point ω_f. Let's put $S_{\omega_f} = \partial J_{\omega_f}$. According to proposition 2.1 there exists a smooth 3-ball $B \subset M^3$ such that $\omega_f \in int\, B$ and ∂B transversally intersects S_{ω_f} along a single curve separating in S_{ω_f} points ω_f and y_0. Without lost of generality, we assume that ∂B intersects the cylinder J along the disk Δ transversally intersecting N^1_f along two disks and N^2_f – along p disks (see Fig. 18).

Let's put $I = W^u_{\sigma_1} \setminus int\, B$. From the properties of the consistent neighborhoods system and orientability of heteroclinic curves, it follows that there exists a tubular neighborhood N_I of an arc I such that $N_I \cap \Delta = N^1_f \cap \Delta$, $W^s_{\sigma_1}$ intersects with Q_1 by one 2-disk whose boundary μ_1 intersects with $W^u_{\sigma_2}$ exactly at p points and the intersection of $\partial N_I \cap W^u_{\sigma^2_f}$ consists exactly of p curves. Then the set $Q_1 = B \cup N_I$ is homeomorphic to a solid torus and $W^u_{\sigma_2} \cap \partial Q_1 = W^u_{\sigma_2} \cap (\Delta \cup \partial N_I)$. Since $cl(W^u_{\sigma_2}) \setminus W^u_{\sigma_2} = cl(W^u_{\sigma_1}) \subset int\, Q_1$, then $W^u_{\sigma_2} \cap \partial Q_1$ consists of closed curves. Since the intersection of the disk $W^u_{\sigma_2}$ with the torus ∂Q_1 is oriented, it consists of a single curve μ_2 (see Fig. 19).

Since the curve μ_2 intersects all heteroclinic curves of the diffeomorphism f in an orientable way, it bounds a disk δ_2 containing the saddle σ^2_f. Reasoning similarly to the case

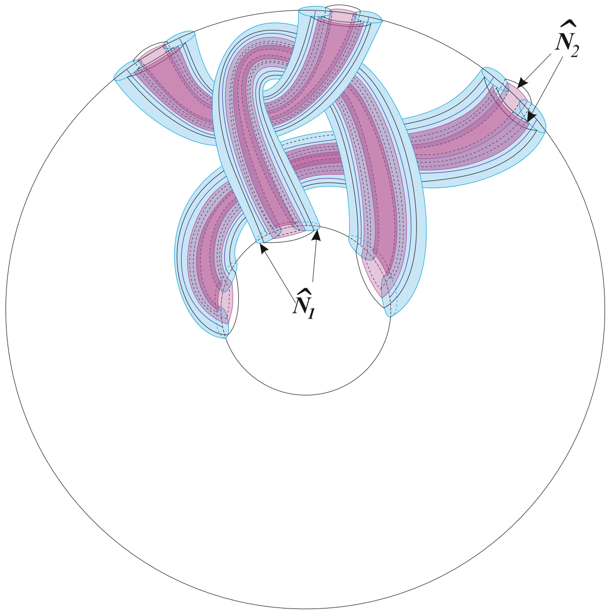

Figure 17: Set $\hat{N}_f$

of $p = 0$, we get that $M^3 = Q_1 \cup Q_2$ is the lens space $L_{p,q}$, where $\langle p, q \rangle$ is the homotopy type of the curve μ_2 on the torus ∂Q_1. □

5.2 Construction of diffeomorphisms with wildly nested separatrices on each lens space

In this section, we constructively prove theorem 1: on any lens space $L_{p,q}$ there exists a diffeomorphism $f \in G$ with wildly embedded one-dimensional saddle separatrices.

5.2.1 Construction on the lens $L_{0,1} \cong \mathbb{S}^2 \times \mathbb{S}^1$

Let L_1, $L_2 \subset \mathbb{S}^2 \times \mathbb{S}^1$ be two disjoint knots from generator class (Hopf knots), trivial and non-trivial, respectively. Let N_{L_1}, N_{L_2} be their pairwise disjoint tubular neighborhoods. Let's choose on the torus $T_i = \partial N_{L_i}$, $i = 1, 2$ generators λ_i, μ_i so that the parallel λ_i is a Hopf knot, and μ_i is the meridian of the solid torus N_{L_i}. Let $\tilde{N}_{L_1} \supset N_{L_1}$ be also a tubular neighborhood of the knot L_1 that does not intersect with N_{L_2} and $\tilde{T} = \partial \tilde{N}_{L_1}$ (see Fig. 20).

Denote by $\hat{V}$ a manifold obtained from $\mathbb{S}^2 \times \mathbb{S}^1 \setminus int\,(N_{L_1} \cup N_{L_2})$ by identifying the boundary tori by means of a diffeomorphism that translates the meridian μ_1 into the meridian μ_2. Denote by $q : \mathbb{S}^2 \times \mathbb{S}^1 \setminus int\,(N_{L_1} \cup N_{L_2}) \to \hat{V}$ the natural projection. Let's put $\hat{L}^s = q(\tilde{T})$ and $\hat{L}^u = q(T_2)$. Note that the fundamental group $\pi_1(\hat{V})$ admits an epimorphism $\eta : \pi_1(\hat{V}) \to \mathbb{Z}$, which assigns to the homotopy class of a closed curve in $\hat{V}$ the number of its revolutions around $q(\lambda_1)$. At the same time, the tori $\hat{L}^s$, $\hat{L}^u$ are η-essential. Let 's put

$$S = (\hat{V}, \eta, \hat{L}^s, \hat{L}^u).$$

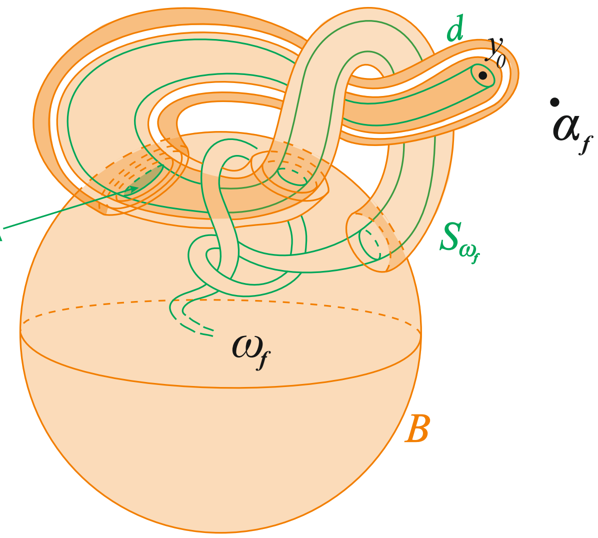

Figure 18: Construction of a ball B

By construction, the manifold $\hat{V}_{\hat{L}^s}$ is homeomorphic to the initial manifold $\mathbb{S}^2\times\mathbb{S}^1$. Since the torus $\tilde{T}$ bounds two solid tori in $\mathbb{S}^2\times\mathbb{S}^1$, then the manifold $\hat{V}_{\hat{L}^u}$ is also homeomorphic to the manifold $\mathbb{S}^2\times\mathbb{S}^1$.

Thus, the scheme S is an abstract scheme. By proposition 2.13, the scheme S is realizable by some gradient-like diffeomorphism $f\in MS(M^3)$ such that the schemes S_f and S are equivalent. Since the sets $\hat{L}^s$, $\hat{L}^u$, $\hat{V}_{\hat{L}^s}$, $\hat{V}_{\hat{L}^u}$ are connected, the diffeomorphism f has exactly four non-wandering points of pairwise different Morse indices, that is, $f\in G$. Since the tori $\hat{L}^s$, $\hat{L}^u$ do not intersect, the set H_f is empty. According to the theorem 2, the ambient manifold of the diffeomorphism f is homeomorphic to the lens space $L_{0,1}\cong\mathbb{S}^2\times\mathbb{S}^1$. According to proposition 3.2, the manifold $W^u_{\sigma^1_f}$ is wildly embedded in the supporting manifold.

5.2.2 Construction on the lens $L_{p,q}$, $p\neq 0$

Let $p\neq 0$ and $q\neq 0$ be mutually simple with p. On the three-dimensional torus

$$\mathbb{T}^3=\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1=\left\{\left(e^{i2\pi x},e^{i2\pi y},e^{i2\pi z}\right)\,:\,x,y,z\in\mathbb{R}\right\}$$

let's set the generators

$$a=\mathbb{S}^1\times\{e^{i2\pi 0}\}\times\{e^{i2\pi 0}\},\ b=\{e^{i2\pi 0}\}\times\mathbb{S}^1\times\{e^{i2\pi 0}\},\ c=\{e^{i2\pi 0}\}\times\{e^{i2\pi 0}\}\times\mathbb{S}^1.$$

Let's define two-dimensional tori $\tilde{T}^s,\tilde{T}^u\subset\mathbb{T}^3$ as follows:

$$\tilde{T}^s=\left\{\left(e^{i2\pi x},e^{i2\pi y},e^{i2\pi z}\right)\,:\,z=0\right\},\ \tilde{T}^u=\left\{\left(e^{i2\pi x},e^{i2\pi y},e^{i2\pi z}\right)\,:\,z=\frac{p}{q}y\right\}.$$

Let's choose tubular neighborhoods of these tori $N_{\tilde{T}^s}$, $N_{\tilde{T}^u}$. By construction, the closure of each connected component of the set $\mathbb{T}^3\setminus(N_{\tilde{T}^s}\cup N_{\tilde{T}^u})$ is a solid torus with a generator

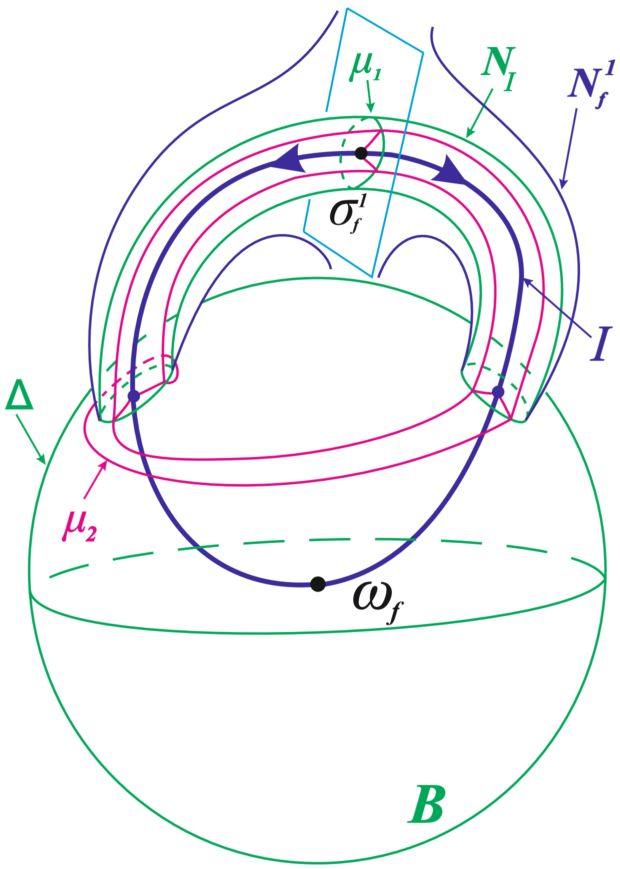

Figure 19: Curves μ_1, μ_2

homotopic to the knot a. Let's choose one such component W and denote by μ_W the meridian of the solid torus W (see Fig. 21).

Let $L \subset \mathbb{S}^2 \times \mathbb{S}^1$ be a non-trivial Hopf knot, N_L be its tubular neighborhood with meridian μ_{N_L} and $\zeta : \partial N_L \to \partial W$ is a diffeomorphism that translates the meridian μ_{N_L} into the meridian μ_W. Let's put

$$\hat{V} = (\mathbb{T}^3 \setminus int\, W) \cup_\zeta (\mathbb{S}^2 \times \mathbb{S}^1 \setminus int\, N_L).$$

Denote by $q : (\mathbb{T}^3 \setminus int\, W) \sqcup (\mathbb{S}^2 \times \mathbb{S}^1 \setminus int\, N_L) \to \hat{V}$ the natural projection. Let's put $\hat{L}^s = q(\tilde{T}^s)$, $\hat{L}^u = q(\tilde{T}^u)$. Note that the fundamental group $\pi_1(\hat{V})$ admits an epimorphism $\eta : \pi_1(\hat{V}) \to \mathbb{Z}$, which assigns to the homotopy class of a closed curve in $\hat{V}$ the number of its revolutions around $q(a)$. At the same time, the tori $\hat{L}^s$, $\hat{L}^u$ are η-essential. Let 's put

$$S = (\hat{V}, \eta, \hat{L}^s, \hat{L}^u).$$

Let's check the validity of the abstract scheme by showing that the manifold $\hat{V}_{\hat{L}^s}$ is homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ (for the manifold $\hat{V}_{\hat{L}^u}$, the proof is similar).

By construction, the manifold $\mathbb{T}^3 \setminus int\, N_{\tilde{T}^s}$ is homeomorphic to $\mathbb{T}^2 \times [0, 1]$. Gluing a solid torus to each component of the connectivity of this manifold so that the meridian of the solid torus is glued to the curve which is homotopic to b, we get the manifold $\mathbb{S}^2 \times \mathbb{S}^1$.

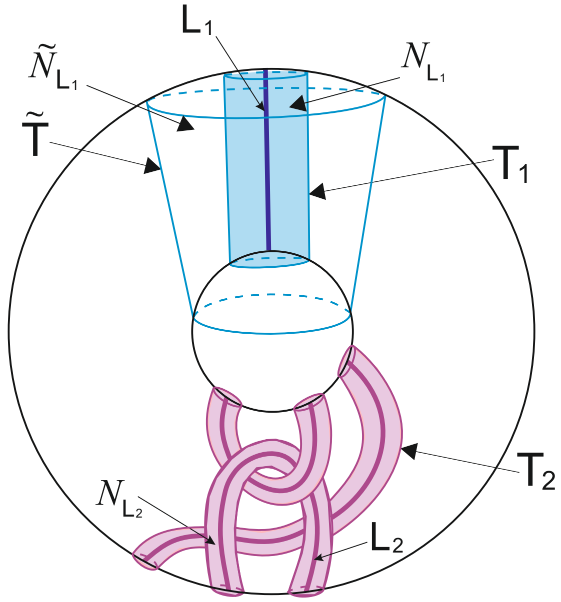

Figure 20: Construction of a diffeomorphism $f \in G_0$ with wildly embedded separatrices

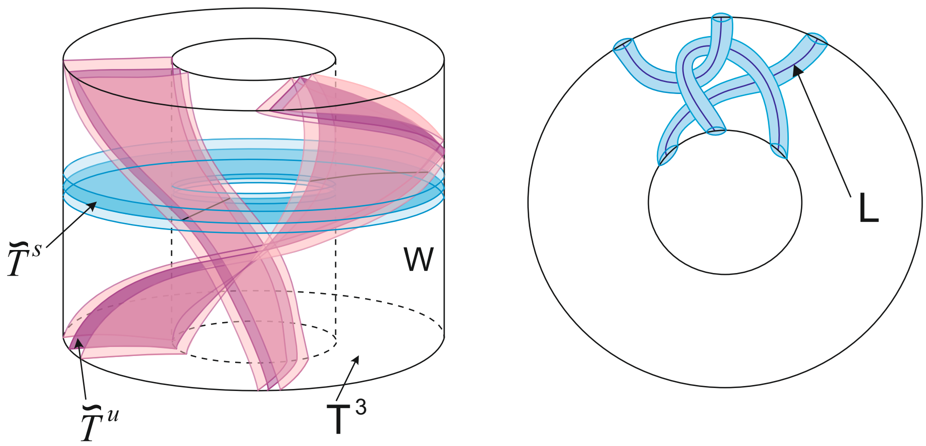

Figure 21: Construction of a diffeomorphism $f \in G_2$ with wildly embedded separatrices

In this case, the resulting manifold is a gluing along the boundary of two solid tori W and $\mathbb{S}^2 \times \mathbb{S}^1 \setminus int\, W$. Then the manifold $\hat{V}_{\hat{L}^s}$ is obtained by gluing the manifolds $\mathbb{S}^2 \times \mathbb{S}^1 \setminus int\, W$, $\mathbb{S}^2 \times \mathbb{S}^1 \setminus int\, N_L$ along the boundary by means of a diffeomorphism that translates the meridian μ_{N_L} into meridian μ_W. Since $\mathbb{S}^2 \times \mathbb{S}^1 \setminus int\, W$ is a solid torus, $\hat{V}_{\hat{L}^s}$ is homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$.

Thus, the scheme S is an abstract scheme. By proposition 2.13, the scheme S is realizable by some gradient-like diffeomorphism $f \in MS(M^3)$ such that the schemes S_f and S are equivalent. Since the sets $\hat{L}^s$, $\hat{L}^u$, $\hat{V}_{\hat{L}^s}$, $\hat{V}_{\hat{L}^u}$ are connected, the diffeomorphism f has exactly four non-wandering points of pairwise different Morse indices, that is, $f \in G$. Since the tori $\hat{L}^s$, $\hat{L}^u$ intersect orientably along p η-essential curves, the set H_f is orientable and consists of p non-compact heteroclinic curves. According to theorem 2, the ambient manifold of the diffeomorphism f is homeomorphic to the lens space $L_{p,q}$. According to proposition 3.2, the manifold $W^u_{\sigma^1_f}$ is wildly embedded into the supporting manifold.

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