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LIE ELEMENTS AND THE MATRIX-TREE THEOREM

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Abstract. For a finite-dimensional representation V of a group G we introduce and study the notion of a Lie element in the group algebra $k[G]$. The set $\mathcal{L}(V) \subset k[G]$ of Lie elements is a Lie algebra and a G-module acting on the original representation V.

Lie elements often exhibit nice combinatorial properties. In particular, we prove a formula, similar to the classical matrix-tree theorem, for the characteristic polynomial of a Lie element in the permutation representation V of the group $G = S_n$.

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1. Introduction: Lie Elements in the Group Algebra

Let V be a finite-dimensional representation of a group G over a field k. For every $g \in G$ and every m define linear operators $\mathcal{G}_m\langle g\rangle$, $\mathcal{A}_m\langle g\rangle\colon V^{\wedge m} \to V^{\wedge m}$ as follows:

$$\mathcal{G}_m\langle g\rangle(v_1 \wedge \dots \wedge v_m) = g(v_1) \wedge \dots \wedge g(v_m),$$
$$\mathcal{A}_m\langle g\rangle(v_1 \wedge \dots \wedge v_m) = \sum_{p=1}^{m} v_1 \wedge \dots \wedge g(v_p) \wedge \dots \wedge v_m.$$

(here and below $v_1, \dots, v_m$ are arbitrary vectors in V). Also take by definition

$$\mathcal{G}_0\langle g\rangle = \mathcal{I},$$
$$\mathcal{A}_0\langle g\rangle = 0$$

for every $g \in G$. Here and below $\mathcal{I}$ means the identity operator.

Denote by $k[G]$ the group algebra of G; extend $\mathcal{G}_m$ and $\mathcal{A}_m$ by linearity to operators $k[G] \to \mathrm{End}(V^{\wedge m})$. In particular, $\mathcal{A}_0 = 0$ and $\mathcal{G}_0\langle \sum_{g\in G} a_g g\rangle = \sum_{g\in G} a_g$ (a constant regarded as an operator $k \to k$).

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Definition 1.1. An element $x \in k[G]$ satisfying

$$\mathcal{G}_m\langle x\rangle = \mathcal{A}_m\langle x\rangle \tag{1}$$

for all $m = 0, 1, \dots, \dim V$ is called a Lie element (with respect to the representation V). The set of Lie elements is denoted by $\mathcal{L}(V) \subset k[G]$.

Remark 1.2. In particular, if $x = \sum_{g\in G} a_g g$ is a Lie element then $\sum_{g\in G} a_g = \mathcal{G}_0\langle x\rangle = \mathcal{A}_0\langle x\rangle = 0$.

Example 1.3. Let $G = S_n$ (a permutation group), $k = \mathbb{C}$ and $V = \mathbb{C}^n$, the permutation representation of S_n (an element of the group permutes the coordinates of a vector $v = (x_1, \dots, x_n) \in \mathbb{C}^n$).

Lemma 1.4. $\varkappa_{ij} \overset{\text{def}}{=} 1 - (ij) \in \mathbb{C}[S_n]$ *is a Lie element.*

Here $(ij) \in S_n$ means a transposition of i and j; more generally, we will use notation like $(i_1 \dots i_k)$ for a cyclic element in S_n, that is, a permutation sending $i_1 \mapsto i_2 \mapsto \dots \mapsto i_k \mapsto i_1$ and leaving the other elements of $\{1, \dots, n\}$ fixed.

We call $\varkappa_{ij} \in \mathbb{C}[S_n]$ a *Kirchhoff difference* as a tribute to G. Kirchhoff's seminal paper [7] (1847); see Theorem 2.4 below.

Proof of Lemma 1.4. The proof is a direct computation. First, $\mathcal{G}_m\langle 1\rangle = \mathcal{I}$ and $\mathcal{A}_m\langle 1\rangle = m\mathcal{I}$. Obviously (cf. Proposition 1.7 below), one can assume $i = 1$, $j = 2$ without loss of generality. Denote by $v_1, \dots, v_n$ the standard basis in $\mathbb{C}^n$; by linearity, it is enough to check the action of $\mathcal{G}_m\langle(12)\rangle$ and $\mathcal{A}_m\langle(12)\rangle$ on $v = v_{i_1} \wedge \dots \wedge v_{i_m}$ where $1 \leqslant i_1 < \dots < i_m \leqslant n$.

Consider now three cases:

- $i_1 \geqslant 3$: here $\mathcal{G}_m\langle(12)\rangle v = v$ and $\mathcal{A}_m\langle(12)\rangle v = mv$, and therefore
$$\mathcal{G}_m\langle 1 - (12)\rangle v = 0 = \mathcal{A}_m\langle 1 - (12)\rangle v.$$
- $i_1 = 2$: here $\mathcal{G}_m\langle(12)\rangle v = v_1 \wedge v_{i_2} \wedge \dots \wedge v_{i_m}$ and
$$\mathcal{A}_m\langle(12)\rangle v = (v_1 + (m-1)v_2) \wedge v_{i_2} \wedge \dots \wedge v_{i_m},$$
so that $\mathcal{G}_m\langle 1 - (12)\rangle v = \mathcal{A}_m\langle 1 - (12)\rangle v = (v_2 - v_1) \wedge v_{i_2} \wedge \dots \wedge v_{i_m}$. The case $i_1 = 1$ and $i_2 \geqslant 3$ is similar.
- finally, $i_1 = 1$, $i_2 = 2$: here $\mathcal{G}_m\langle(12)\rangle v = -v$, so $\mathcal{A}_m\langle(12)\rangle v = (m-2)v$, and therefore $\mathcal{G}_m\langle 1 - (12)\rangle v = 2v = \mathcal{A}_m\langle 1 - (12)\rangle v$.

Lemma is proved. □

Our first motive to write this paper was the article [5], where equation (1) for Kirchhoff differences was used to study a question in low-dimensional topology (see [5, Proposition 3.4]). Lie elements are also known to have nice combinatorial properties, which have been studied since 1847 when G. Kirchhoff [7] discovered the classical matrix-tree theorem (Theorem 2.4 below). See also its Pfaffian version by G. Masbaum and A. Vaintrob [8] (Theorem 2.6) and their numerous generalizations ([1], [6], to name just a few). The main result of this paper, Theorem 2.8, is also an analog of Theorems 2.4 and 2.6.

For any G and V the k-vector spaces $k[G]$ and $\mathrm{End}(V^{\wedge m})$ are associative algebras; consider them as Lie algebras with the commutator bracket: $[p, q] \overset{\text{def}}{=} pq - qp$.

Proposition 1.5. *Maps $\mathcal{G}_m$, $\mathcal{A}_m \colon k[G] \to \mathrm{End}(V^{\wedge m})$ are Lie algebra homomorphisms.*

Proof. Obviously, $\mathcal{G}_m \colon k[G] \to \mathrm{End}(V^{\wedge m})$ is an associative algebra homomorphism, hence a Lie algebra homomorphism. For $\mathcal{A}_m$ take $x = \sum_{g\in G} a_g g$, $y = \sum_{h\in G} b_h h$, to obtain

$$\mathcal{A}_m\langle x\rangle\,\mathcal{A}_m\langle y\rangle v_1 \wedge \dots \wedge v_m = \sum_{h\in G, 1\leqslant p\leqslant m} b_h\,\mathcal{A}_m\langle x\rangle v_1 \wedge \dots \wedge h(v_p) \wedge \dots \wedge v_m$$

$$= \sum_{g,h\in G, 1\leqslant p\leqslant m} a_g b_h v_1 \wedge \dots \wedge g(h(v_p)) \wedge \dots \wedge v_m$$

$$+ \sum_{g,h\in G, 1\leqslant p,q\leqslant m, p\neq q} a_g b_h v_1 \wedge \dots \wedge h(v_p) \wedge \dots \wedge g(v_q) \wedge \dots \wedge v_m,$$

so that $\mathcal{A}_m\langle [x, y]\rangle = [\mathcal{A}_m\langle x\rangle, \mathcal{A}_m\langle y\rangle]$. □

Corollary 1.6. *The set of Lie elements $\mathcal{L}(V) \subset k[G]$ is a Lie subalgebra.*

Proposition 1.7. *Operators $\mathcal{G}_m$ and $\mathcal{A}_m$ are conjugation-invariant: if $x \in k[G]$ and $y \in k[G]$ is invertible then for any m one has $y\,\mathcal{G}_m\langle x\rangle y^{-1} = \mathcal{G}_m\langle yxy^{-1}\rangle$ and $y\,\mathcal{A}_m\langle x\rangle y^{-1} = \mathcal{A}_m\langle yxy^{-1}\rangle$.*

The proof is straightforward.

Corollary 1.8. *The Lie algebra $\mathcal{L}(V) \subset k[G]$ is a G-module where elements of the group act by conjugation.*

2. Lie Elements in the Relection Representation of the Permutation Group

2.1. Kirchhoff differences and their commutators. Let $G = S_n$ (a permutation group), and V be its reflection (a.k.a. Coxeter or geometric) representation; $\dim V = n-1$. The permutation representation $\mathbb{C}^n$ is a sum of $V = \{(x_1, \dots, x_n) \mid x_1 + \dots + x_n = 0\}$ and a trivial representation $\mathbf{1} = \{(x, x, \dots, x) \mid x \in \mathbb{C}\}$. It follows from Remark 1.2 that any Lie element $x \in \mathcal{L}(\mathbb{C}^n)$ acts on $\mathbf{1}$ by zero. Therefore, $\mathcal{L}(V) = \mathcal{L}(\mathbb{C}^n)$; we'll denote it $\mathcal{L}_n$ for short.

For a finite-dimensional representation W of S_n denote by $\chi_x^W(t) = \det(t\mathcal{I} - x)$ the characteristic polynomial of an element $x \in \mathbb{C}[S_n]$ acting in W. It follows from the remarks above that $\chi_x^{\mathbb{C}^n}(t) = t\chi_x^V(t)$ for all $x \in \mathcal{L}_n$.

Theorem 2.1 ([2], cf. [5])**.** (1) *For all pairwise distinct $i, j, k, l \in \{1, \dots, n\}$ elements $\varkappa_{ij} \stackrel{\text{def}}{=} 1 - (ij)$, $\nu_{ijk} \stackrel{\text{def}}{=} (ijk) - (ikj)$ and $\eta_{ijkl} \stackrel{\text{def}}{=} (ijkl) + (ilkj) - (ijlk) - (iklj)$ belong to $\mathcal{L}_n$.*

(2) *Consider, for all $1 \leqslant i < j < k < l \leqslant n$, vector spaces $K_{ij}, N_{ijk}, H_{ijkl} \subset \mathbb{C}[S_n]$ spanned by all $\varkappa_{pq}$, ν_{pqr}, η_{pqrs} where the indices (p, q), (p, q, r) and (p, q, r, s) are permutations of (i, j), (i, j, k) and (i, j, k, l), respectively. Then $\dim K_{ij} = 1$, $\dim N_{ijk} = 1$ and $\dim H_{ijkl} = 2$; bases in them are $\{\varkappa_{ij}\}$, $\{\nu_{ijk}\}$ and $\{\eta_{ijkl}, \eta_{iklj}\}$.*

(3) *Let permutation groups S_2, S_3 and S_4 act on K_{ij}, N_{ijk} and H_{ijkl} permuting indices of the elements $\varkappa_{pq}$, ν_{pqr} and η_{pqrs}. This makes K_{ij} a trivial representation of S_2, N_{ijk}, a sign representation of S_3, and H_{ijkl}, an irreducible 2-dimensional representation of S_4.*

(4) *Elements $\varkappa_{pq} \in K_{ij}$, $\nu_{pqr} \in N_{ijk}$ and $\eta_{pqrs} \in H_{ijkl}$ enjoy the following symmetries (for all p, q, r, s):*

for K_{ij}:

$$\varkappa_{qp} = \varkappa_{pq};$$

for N_{ijk}:

$$\nu_{pqr} = \nu_{qrp} = -\nu_{qpr};$$

for H_{ijkl}:

$$\begin{aligned}
&\eta_{pqrs} = -\eta_{qprs} = -\eta_{pqsr},\\
&\eta_{pqrs} = \eta_{srqp} = \eta_{rspq} = \eta_{qpsr},\\
&\eta_{pqrs} + \eta_{prsq} + \eta_{psqr} = 0.
\end{aligned}$$

Proof. Kirchhoff differences $\varkappa_{ij}$ belong to $\mathcal{L}_n$ by Lemma 1.4. ν_{ijk} and η_{ijkl} are commutators of the $\varkappa_{ij}$: $\nu_{ijk} = [\varkappa_{ij}, \varkappa_{jk}]$ and $\eta_{ijkl} = [\varkappa_{il}, \nu_{ijk}]$. So by Corollary 1.6 assertion 1 is proved.

Relations of assertion 4 can be checked immediately. A straightforward computation shows that these relations imply assertions 2 and 3. The first relation for H_{ijkl} means that the basic elements η_{ijkl}, η_{iklj} are eigenvectors of the transpositions (12) and (13), respectively, with the eigenvalue -1. □

Conjecture 2.2. *The Lie algebra $\mathcal{L}_n$ is generated by the Kirchhoff differences $\varkappa_{ij}$, $1 \leqslant i < j \leqslant n$.*

This conjecture was tested numerically for small n, but we do not know its proof at the moment.

Characterstic polynomials of Lie elements $x \in \mathcal{L}_n$ acting at V are often given by nice formulas.

Example 2.3. Let Γ be a finite graph with the vertex set $\{1, \ldots, n\}$ and the edges $e_1, \ldots, e_m$, where e_s connects vertices i_s and j_s; denote $w_\Gamma \stackrel{\text{def}}{=} w_{i_1 j_1} \ldots w_{i_m j_m}$. Also denote by $\mathcal{T}_n$ the set of trees with the vertices $1, \ldots, n$.

Consider the Lie element

$$x = \sum_{1 \leqslant i < j \leqslant n} w_{ij} \varkappa_{ij};$$

and assume $w_{ji} = w_{ij}$ for convenience.

Theorem 2.4 (matrix-tree theorem, [7])**.**

$$\det x|_V = \chi_x^V(0) = n \sum_{\Gamma \in \mathcal{T}_n} w_\Gamma.$$

There exist similar formulas for other coefficients of χ_x^V as well; for details see the review [6] and the references therein.

Example 2.5. A finite 3-graph is defined as a union of several solid triangles (called 3-edges) with some of their vertices glued. A 3-graph is called a 3-tree if it is contractible (as a topological space). The number n of vertices of a 3-tree is always odd: $n = 2m + 1$, where m is the number of 3-edges; denote by $\mathcal{T}_m^{(3)}$ the set of 3-trees with the vertices $1, \ldots, 2m + 1$.

Consider the Lie element

$$y = \sum_{1 \leqslant i < j < k \leqslant n} w_{ijk} \nu_{ijk};$$

assume for convenience $w_{jki} = w_{kij} = w_{ijk}$ and $w_{jik} = w_{ikj} = w_{kji} = -w_{ijk}$ (cf. assertion 4 of Theorem 2.1).

Let Γ be a 3-graph and $e_1, \ldots, e_m$, its 3-edges; the edge e_s is a triangle with the vertices $i_s, j_s, k_s \in \{1, \ldots, n\}$. Denote $w_\Gamma \stackrel{\text{def}}{=} w_{i_1 j_1 k_1} \ldots w_{i_m j_m k_m}$.

It is easy to observe that the operator $\nu_{ijk} \colon V \to V$ (and hence, the operator $y \colon V \to V$) is skew-symmetric with respect to the standard scalar product in V (inherited from $\mathbb{C}^n$). So if n is even and $\dim V = n - 1$ is odd, then $\det y|_V = 0$. If $n = 2m + 1$ is odd then the skew-symmetric operator $y|_V$ has a Pfaffian described below.

Folllowing [8], define a sign $\delta(\Gamma) = \pm 1$ of a 3-tree $\Gamma \in \mathcal{T}_m^{(3)}$ as follows. Denote, like above, the vertices of the s-th edge e_s of Γ as $i_s < j_s < k_s$; here $s = 1, \ldots, m$. Consider a product of the 3-cycles $\sigma \stackrel{\text{def}}{=} (i_1 j_1 k_1) \ldots (i_m j_m k_m) \in S_n$. An easy induction by m shows that σ is a cyclic permutation $(a_1 \ldots a_n)$. Now define a permutation $\tau \in S_n$ as $\tau(s) = a_s$, $s = 1, \ldots, n$; the sign $\delta(\Gamma)$ is then defined as the parity of τ. See [8] for details; in particular, it is proved there that $\delta(\Gamma)$ does not depend on the ordering of the edges of Γ.

Theorem 2.6 [8]**.** Pf $y|_V = n \sum_{\Gamma \in \mathcal{T}_n^{(3)}} \delta(\Gamma) w_\Gamma$.

The article [4] describes a technique (called discrete path integration) giving a uniform proof of Theorems 2.4 and 2.6 and of some more similar statements as well. We are going to use this technique to prove the main result of the paper, Theorem 2.8.

2.2. The main theorem. Theorem 2.8 is a formula for the characteristic polynomial of the Lie element

$$z = \sum_{1 \leqslant i < j < k < l \leqslant n} \xi_{ijkl} \colon V \to V, \tag{2}$$

where $\xi_{ijkl} \in H_{ijkl}$ are arbitrary, that is, $\xi_{ijkl} = w_{ijkl} \eta_{ijkl} + w_{iklj} \eta_{iklj}$ for some $w_{ijkl}, w_{iklj} \in \mathbb{C}$ (see Theorem 2.1 above).

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$-matrices, and $I \subseteq \{1, \ldots, n\}$. Their *$I$-shuffle* is defined as a $n \times n$-matrix $(A, B)_I = (u_{ij})$, where

$$u_{ij} = \begin{cases} a_{ij}, & i \in I, \\ b_{ij}, & i \notin I. \end{cases}$$

Definition 2.7. The *shuffle determinant* of the matrices A and B is

$$\operatorname{sdet}(A,\, B) \stackrel{\text{def}}{=} \sum_{I \subseteq \{1,\ldots,n\}} \det(A,\, B)_I \det(A,\, B)_{\bar{I}},$$

where bar means the complement: $\bar{I} \stackrel{\text{def}}{=} \{1,\, \ldots,\, n\} \setminus I$.

Let $1 \leqslant r \leqslant n$ and let $\mathcal{E}$ be a r-element set of 4-tuples $(i_1,\, j_1,\, k_1,\, l_1),\, \ldots,\, (i_r,\, j_r,\, k_r,\, l_r)$. Consider the vector space $H_{\mathcal{E}} = \bigotimes_{s=1}^{r} H_{i_s j_s k_s l_s}$ of dimension 2^r and define a linear functional $\Phi_r \colon H_{\mathcal{E}} \to \mathbb{C}$ as follows. Let $A_{\mathcal{E}}$, $B_{\mathcal{E}}$ be $r \times n$-matrices with the elements

$$\begin{aligned} &(A_{\mathcal{E}})_{si_s} = 1, \quad (A_{\mathcal{E}})_{sj_s} = -1,\\ &(B_{\mathcal{E}})_{sk_s} = 1, \quad (B_{\mathcal{E}})_{sl_s} = -1,\\ &(A_{\mathcal{E}})_{ij} = (B_{\mathcal{E}})_{ij} = 0 \quad \text{for all other } i,\, j; \end{aligned} \tag{3}$$

here $s = 1,\, \ldots,\, r$. For a r-element set $J \subseteq \{1,\, \ldots,\, n\}$ denote by $A_{\mathcal{E}}^J$ and $B_{\mathcal{E}}^J$ the $r \times r$-submatrices of $A_{\mathcal{E}}$ and $B_{\mathcal{E}}$, respectively, containing all the r rows and the columns listed in J. Then take by definition

$$\Phi_r(\eta_{i_1 j_1 k_1 l_1} \otimes \cdots \otimes \eta_{i_r j_r k_r l_r}) = \sum_{J \subseteq \{1,\ldots,n\}, \#J = r} \operatorname{sdet}(A_{\mathcal{E}}^J,\, B_{\mathcal{E}}^J). \tag{4}$$

and extend Φ_r to the whole $H_{\mathcal{E}}$ by linearity.

Theorem 2.8. *Let z be defined by (2) and Φ_r, by (4). Then*

$$\chi_z^{\mathbb{C}^n}(t) = t^n + \mu_1 t^{n-1} + \cdots + \mu_{n-1} t,$$

where

$$\mu_r = \Phi_r(z^{\otimes r})$$

for every $r = 1,\, \ldots,\, n-1$.

See Section 4 for the proof.

If $r = n - 1$ then the definition of Φ_r can be simplified:

Proposition 2.9. *If $r = n - 1$ then all the summands in (4) are equal, so one may take*

$$\Phi_{n-1}(\eta_{i_1 j_1 k_1 l_1} \otimes \cdots \otimes \eta_{i_{n-1} j_{n-1} k_{n-1} l_{n-1}}) = n \operatorname{sdet}(A_{\mathcal{E}}^J,\, B_{\mathcal{E}}^J).$$

for any subset $J \subset \{1,\, \ldots,\, n\}$ of cardinality $(n-1)$, e.g., $J = \{1,\, \ldots,\, n-1\}$.

The proof of the proposition is also in Section 4.

3. Shuffle Determinant

Here are basic properties of the shuffle determinant of Definition 2.7:

Theorem 3.1. (1) $\operatorname{sdet}(A,\, B)$ *is a polynomial of variables a_{ij} and b_{ij}, $1 \leqslant i,\, j \leqslant n$, with integer coefficients, bihomogeneous of degree n (thus, its total degree is $2n$).*

(2) $\operatorname{sdet}(B,\, A) = \operatorname{sdet}(A,\, B)$.

(3) *Let* $X \stackrel{\text{def}}{=} \operatorname{diag}(x_1, \dots, x_n)$ *be a diagonal matrix with* $x_1, \dots, x_n$ *as diagonal entries. Then* $\operatorname{sdet}(A, B) = [x_1 \dots x_n : \det(A + BX)^2]$ *(that is,* $\operatorname{sdet}(A, B)$ *is equal to the coefficient at the monomial* $x_1 \dots x_n$ *in the polynomial* $\det(A + BX)^2$*).*
(4) $\operatorname{sdet}(CA, CB) = \operatorname{sdet}(A, B) \det C^2$ *for any* $n \times n$*-matrix* C*. In particular, if* B *is invertible then* $\operatorname{sdet}(A, B) = \operatorname{sdet}(B^{-1}A, \mathcal{I}) \det B^2$*.*
(5) $\operatorname{sdet}(A, \mathcal{I}) = (-1)^n \sum_{\sigma \in S_n} (-2)^{\nu(\sigma)} a_{1\sigma(1)} \dots a_{n\sigma(n)}$*, where* $\nu(\sigma)$ *is the number of independent cycles in* σ*.*

Remark . Let $W = (\mathbb{C}^2)^{\otimes n}$ be the natural representation of S_n (by permuting factors in any decomposable tensor). It is easy to see then that the character of W is given by $\chi_W(\sigma) = 2^{\nu(\sigma)}$. Now it follows from Property 5 that $\operatorname{sdet}(A, \mathcal{I})$ is equal, up to a sign, to the immanant of the matrix A associated with the representation $W \otimes \varepsilon$ (ε is the sign representation). The authors wish to thank the reviewer for this remark.

Proof. Assertions 1 and 2 are obvious from Definition 2.7.

Assertion 3: denote by $a_1, , \dots, a_n$ and $b_1, , \dots, b_n$ columns of the matrices A and B, respectively; we will be writing $\det(a_1, \dots, a_n)$ instead of $\det A$, and similarly for other matrices. The determinant of a matrix is a multilinear function of its columns, so one has

$$[x_1 \dots x_n : \det(A + BX)^2] = [x_1 \dots x_n : \det(a_1 + x_1 b_1, \dots, a_n + x_n b_n)^2]$$
$$= \sum_{I \subseteq \{1,\dots,n\}} [x_I : \det(a_1 + x_1 b_1, \dots, a_n + x_n b_n)][x_{\bar{I}} : \det(a_1 + x_1 b_1, \dots, a_n + x_n b_n)]$$

(where $x_I \stackrel{\text{def}}{=} \prod_{i \in I} x_i$)

$$= \sum_{I \subseteq \{1,\dots,n\}} \det(w_1, \dots, w_n) \det(w'_1, \dots, w'_n)$$

(where $w_i = b_i$, $w'_i = a_i$ if $i \in I$ and vice versa if $i \notin I$)

$$= \sum_{I \subseteq \{1,\dots,n\}} \det(A, B)_I \det(A, B)_{\bar{I}} = \operatorname{sdet}(A, B).$$

Assertion 4 follows from 3: $\det(CA + CBX)^2 = \det(A + BX)^2 \det C^2$. The matrix C does not depend on $x_1, \dots, x_n$, so the same equality takes place for coefficients at $x_1 \dots x_n$.

To prove assertion 5 note that $\det(A, \mathcal{I})_I$ is the diagonal minor of the matrix A formed by the rows and the columns listed in I. Hence,

$$\det(A, \mathcal{I})_I = \sum_{\substack{\sigma \in S_n \\ \sigma(j) = j \, \forall j \in \bar{I}}} \operatorname{sgn}(\sigma) \prod_{i \in I} a_{i\sigma(i)},$$

where $\operatorname{sgn} \sigma = 1$ or -1 depending on the parity of σ. Therefore

$$\det(A, \mathcal{I})_I \det(A, \mathcal{I})_{\bar{I}} = \sum_{\substack{\sigma \in S_n \\ \sigma(i) \in I \, \forall i \in I \\ \sigma(i) \in \bar{I} \, \forall i \in \bar{I}}} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}.$$

Summation over $I \subseteq \{1, \dots, n\}$ gives

$$\begin{aligned}\operatorname{sdet}(A, \mathcal{I}) &= \sum_I \det(A, \mathcal{I})_I \det(A, \mathcal{I})_{\bar{I}} \\ &= \sum_{\sigma \in S_n} \#\{I \subseteq \{1, \dots, n\} \mid \sigma(i) \in I \,\forall i \in I\} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}.\end{aligned}$$

The subset I invariant with respect to σ (that is, such that $\sigma(i) \in I$ for all $i \in I$) is a union of several independent cycles of σ; so the number of such subsets is $2^{\nu(\sigma)}$. On the other hand, $\operatorname{sgn}(\sigma) = (-1)^{n+\nu(\sigma)}$, which finishes the proof. □

Give now a more detailed description of $\operatorname{sdet}(A, B)$ as a polynomial of a_{ij} and b_{ij}. By assertion 1 of Theorem 3.1 any term of the polynomial looks like $c \cdot a_{i_1 j_1} \dots a_{i_n j_n} b_{k_1 l_1} \dots b_{k_n l_n}$, where $c \in \mathbb{Z}$. Denote by $\Gamma \stackrel{\text{def}}{=} \Gamma(i_1, j_1, \dots, k_n, l_n)$ a directed graph with the vertices $1, \dots, n$ and the $2n$ edges $(i_1 j_1), \dots, (i_n j_n)$, $(k_1 l_1), \dots, (k_n l_n)$.

Theorem 3.2. (1) *Every vertex of the graph Γ is incident to exactly four edges, two of them entering the vertex and two, leaving it.*

(2) *The coefficient c at the monomial depends on the graph Γ only and is equal to $\pm 2^{m(\Gamma)}$, where $m(\Gamma) \in \mathbb{Z}_{>0}$ is the number of connected components of an auxiliary graph Γ' determined by Γ.*

The proof below contains the exact contruction of the graph Γ'.

Proof. Assertion 1: take some $I \subseteq \{1, \dots, n\}$. If $i \in I$ then the elements a_{ij} (for all j) are in the i-th column of $(A, B)_I$; if $i \notin I$, then they are in the i-th column of $(A, B)_{\bar{I}}$. Thus, exactly one of $a_{i_1 j_1}, \dots, a_{i_n j_n}$ is a_{ij} for some j, which implies $\{i_1, \dots, i_n\} = \{1, \dots, n\}$; similarly, $\{k_1, \dots, k_n\} = \{1, \dots, n\}$. So, every vertex of Γ is an initial vertex of two edges. At the same time, for every $j \in \{1, \dots, n\}$ every monomial of $\det(A, B)_I$ contains exactly one letter x_{ij}, where $x = a$ or b, for some $i \in \{1, \dots, n\}$; the same is true for $\det(A, B)_{\bar{I}}$ — hence, every vertex of Γ is a terminal vertex for two edges.

Assertion 2: note first that the monomial is *not* determined uniquely by the graph Γ since one cannot tell which edges correspond to a_{ij} and which to b_{kl}. Prove that this ambiguity does not influence the coefficient.

By assertion 1, every mononial in $\operatorname{sdet}(A, B)$ is equal to

$$x = a_{1j_1} a_{2j_2} \dots a_{nj_n} b_{1l_1} b_{2l_2} \dots b_{nl_n}$$

for some $j_1, \dots, j_n, l_1, \dots, l_n$. It is enough to show that the coefficient at x is the same as the coefficient at the monomial $x' = b_{1j_1} a_{2j_2} \dots a_{nj_n} a_{1l_1} b_{2l_2} \dots b_{nl_n}$. Note that for every $I \subseteq \{1, \dots, n\}$ the contribution of the term $\det(A, B)_I \det(A, B)_{\bar{I}}$ to the coefficient at x is equal to the contribution of $\det(A, B)_{I'} \det(A, B)_{\bar{I}'}$ to the coefficient at x', where $I' \stackrel{\text{def}}{=} I \triangle \{1\}$. But $I \mapsto I \triangle \{1\}$ is an invertible operation (indeed, an involution) on the set of subsets of $\{1, \dots, n\}$, so the coefficients at x and at x' are equal.

To obtain a formula for the coefficient take a monomial x as above and paint every edge (ij) of the graph Γ blue if the corresponding letter comes from I (that is, $i \in I$ and the letter is b_{ij} or $i \in \bar{I}$ and the letter is a_{ij}) and red if it comes from $\bar{I}$. A blue-red painting of the edges of Γ corresponds to a subset $I \subseteq \{1, \dots, n\}$ if each vertex is initial and terminal for exactly one red and one blue edge; if I exists, then it is obviously unique. The subgraphs of Γ formed by red and blue edges are graphs of some permutations; call them σ_r and σ_b, respectively. The contribution of the term $\det(A, B)_I \det(A, B)_{\bar{I}}$ into the coefficient is equal to the product of parities of σ_r and σ_b.

Consider a graph Γ' whose vertices are edges of Γ; two vertices are connected by an edge if the corresponding edges of Γ share the same initial vertex or the same terminal vertex. By assertion 1, every vertex of Γ' is incident to exactly two edges — hence, Γ' is a union of nonintersecting cycles. Red and blue vertices alternate in the cycle; therefore, each cycle in Γ' has even length.

The graph $\Gamma = \Gamma(i_1, j_1, \dots, k_n, l_n)$ determines Γ'. To fix a subset I one should paint vertices of Γ' so that the colors alternate in every cycle. For each cycle there are obviously two such paintings possible; thus, the number of subsets I for the graph G is 2^m, where m is the number of cycles (connected components) in Γ'.

Let now $I_1, I_2 \subseteq \{1, \dots, n\}$ be two sets making nonzero contributions to the coefficient at the monomial x and such that the corresponding colorings differ on one cycle of the graph Γ' only; let this cycle be $e_1 \dots e_{2s}$. Then permutations $(\sigma_1)_r$ and $(\sigma_2)_r$ differ by a product of transpositions $(e_1 e_2)(e_3 e_4) \dots (e_{2s-1} e_{2s})$, and their parities differ by $(-1)^s$. The same is true for permutations $(\sigma_1)_b$ and $(\sigma_2)_b$, so the terms $\det(A, B)_{I_1} \det(A, B)_{\bar{I}_1}$ and $\det(A, B)_{I_2} \det(A, B)_{\bar{I}_2}$ make equal contributions of ± 1 into the coefficient. This finishes the proof. □

4. Proof of Theorem 2.8 and Final Remarks

4.1. Proofs

Proof of Proposition 2.9. The set $J \subset \{1, \dots, n\}$ of cardinality $(n-1)$ is $\{1, \dots, n\} \setminus \{k\}$ for some k; denote $A^J \stackrel{\text{def}}{=} A_k$ and $B^J \stackrel{\text{def}}{=} B_k$ for short. By definition, $\operatorname{sdet}(A_k, B_k) = \sum_{I \subseteq \{1,\dots,n\}} \det(A_k, B_k)_I \det(A_k, B_k)_{\bar{I}}$. Denote by $\xi_1, \dots, \xi_n$ the columns of the $(n-1) \times n$-matrix $(A, B)_I$; one has $\xi_1 + \dots + \xi_n = 0$. The matrix $(A_{k+1}, B_{k+1})_I$ is obtained from $(A_k, B_k)_I$ by replacement of the column ξ_{k+1} with $-\xi_1 - \dots - \xi_{k-1} - \xi_{k+1} - \dots - \xi_n$. Since all the rows ξ_i except ξ_{k+1} are present in $(A_{k+1}, B_{k+1})_I$, the determinant of $(A_{k+1}, B_{k+1})_I$ is the same as if the replacement row were still $-\xi_{k+1}$. Thus, $\det(A_{k+1}, B_{k+1})_I = -\det(A_k, B_k)_I$. The subset I is arbitrary, so $\det(A_{k+1}, B_{k+1})_{\bar{I}} = -\det(A_k, B_k)_{\bar{I}}$, too, and therefore $\operatorname{sdet}(A_{k+1}, B_{k+1}) = \operatorname{sdet}(A_k, B_k)$, proving the proposition. □

Proof of Theorem 2.8. Let $v, \alpha \in \mathbb{C}^n$ be nonzero vectors. Denote by $M[\alpha, v] \colon \mathbb{C}^n \to \mathbb{C}^n$ a rank 1 linear operator defined as $M[\alpha, v](u) = (\alpha, u)v$, $u \in \mathbb{C}^n$, where $(\cdot, \cdot)$ is the standard ($\mathbb{C}$-valued) scalar product in $\mathbb{C}^n$.

Lemma 4.1. *Let $v_1, \dots, v_n$ be the standard basis in $\mathbb{C}^n$ (orthonormal with respect to $(\cdot, \cdot)$). Then the Lie element $\eta_{ijkl} = (ijkl) + (ilkj) - (ijlk) - (iklj)$ acts in the permutation representation $\mathbb{C}^n$ as $M[v_i - v_j, v_l - v_k] + M[v_l - v_k, v_i - v_j]$.*

The proof is an immediate check.

Lemma 4.1 allows to derive Theorem 2.8 from [4, Corollary 2.4]. To keep up with the notation of [4], let's take by definition

$$\begin{aligned} e_{s,0} &= v_{i_s} - v_{j_s}, \\ e_{s,1} &= v_{k_s} - v_{l_s}, \\ \alpha_{s,0} &= v_{k_s} - v_{l_s}, \\ \alpha_{s,1} &= v_{i_s} - v_{j_s}, \end{aligned} \tag{5}$$

so that $z = \sum_{s=1}^{m} \sum_{u \in \{0,1\}} M[e_{s,u}, \alpha_{s,u}]$. Now Corollary 2.4 of [4] implies that

$$\begin{aligned} \mu_r &= \sum_{s_1,\dots,s_r=1}^{m} w_{i_{s_1} j_{s_1} k_{s_1} l_{s_1}} \dots w_{i_{s_r} j_{s_r} k_{s_r} l_{s_r}} \sum_{u_1,\dots,u_r=0}^{1} \det\bigl((\alpha_{s_p,u_p}, e_{s_q,u_q})\bigr)_{p,q=1}^{r} \\ &= \sum_{s_1,\dots,s_r=1}^{m} w_{i_{s_1} j_{s_1} k_{s_1} l_{s_1}} \dots w_{i_{s_r} j_{s_r} k_{s_r} l_{s_r}} \\ &\qquad \times \sum_{u_1,\dots,u_r=0}^{1} \sum_{\substack{J \subseteq \{1,\dots,n\} \\ \#J = r}} \det(\alpha_{s_p,u_p})_{p \in J} \det(e_{s_p,u_p})_{p \in J}; \end{aligned}$$

in the last equation by $\det(c_1, \dots, c_r)$ we mean a determinant of a $r \times r$ matrix having vectors $c_1, \dots, c_r \in \mathbb{C}^r$ as columns. Instead of indices $u_1, \dots, u_r \in \{0, 1\}$ consider a set $I \stackrel{\text{def}}{=} \{i \mid u_i = 1\} \subseteq \{1, \dots, r\}$. Taking (5) and (3) into account one can write

$$\begin{aligned} \mu_r &= \sum_{s_1,\dots,s_r=1}^{m} w_{i_{s_1} j_{s_1} k_{s_1} l_{s_1}} \dots w_{i_{s_r} j_{s_r} k_{s_r} l_{s_r}} \sum_{\substack{J \subseteq \{1,\dots,n\} \\ \#J = r}} \sum_{I \subseteq \{1,\dots,r\}} \det(A_{\mathcal{E}}^{J})_I \det(B_{\mathcal{E}}^{J})_{\bar{I}} \\ &= \sum_{s_1,\dots,s_r=1}^{m} w_{i_{s_1} j_{s_1} k_{s_1} l_{s_1}} \dots w_{i_{s_r} j_{s_r} k_{s_r} l_{s_r}} \sum_{\substack{J \subseteq \{1,\dots,n\} \\ \#J = r}} \operatorname{sdet}(A_{\mathcal{E}}^{J}, B_{\mathcal{E}}^{J}). \end{aligned}$$

Theorem 2.8 is proved. □

4.2. Final remarks and further research

4.2.1. *Geometry of* 4*-graphs.* For every $1 \leqslant i < j < k < l \leqslant n$ consider two different tetrahedra with the vertices i, j, k, l, and call them 4-edges T_1 and T_2. A 4-graph is defined a union of several 4-edges glued by vertices.

The main result of the paper, Theorem 2.8, expresses a coefficient μ_r of the characteristic polynomial as a homogeneous (degree r) polynomial of the coefficients w_{ijkl}. If one puts a coefficient w_{ijkl} on the 4-edge T_1 and w_{iklj}, on the 4-edge T_2, then this polynomial becomes a sum over the set of all 4-graphs with r edges. The summand corresponding to a graph $\mathcal{E}$ is the product of weights of all the edges of $\mathcal{E}$ times an integer coefficient $c_{\mathcal{E}}$ described in the theorem (sum of shuffle determinants of minors of the matrices $A_{\mathcal{E}}$ and $B_{\mathcal{E}}$).

Matrix-tree theorems 2.4 and 2.6 have similar structure with ordinary graphs and 3-graphs in place of the 4-graphs. In Theorem 2.4 the coefficient $c_{\mathcal{E}}$ is equal to

n if $\mathcal{E}$ is a tree and is zero otherwise. In Theorem 2.6 one has $\mu_{n-1} = \left(\operatorname{Pf} y|_V\right)^2$, so $c_{\mathcal{E}} = n^2 \sum \delta(G_1)\delta(G_2)$, where the sum is taken over all representations $\mathcal{E} = G_1 \sqcup G_2$ of $\mathcal{E}$ as a union of two 3-trees. The formula for $c_{\mathcal{E}}$ in Theorem 2.8 is explicit, but unlike theorems 2.4 and 2.6 it is not related to the geometry of the underlying 4-graph. Finding such a relation would be an interesting combinatorial problem to solve.

4.2.2. *Structure of $\mathcal{L}_n$ as a Lie algebra and as a S_n-module.* For any group G and its representation V the elements $x \in \mathcal{L}(V) \subset k[G]$ act in the representation V. This action may have a kernel; denote it $K(V) \subset \mathcal{L}(V)$ (and $K_n \subset \mathcal{L}_n$ if V is the permutation representation of S_n).

Conjecture 4.2. $\dim \mathcal{L}_n/K_n = (n-1)!$. *The repeated commutators of Kirchhoff differences*

$$[[\dots[[\varkappa_{1i_1}, \varkappa_{2i_2}], \varkappa_{3i_3}], \dots], \varkappa_{n-1,i_{n-1}}]$$

for all $i_1, \dots, i_{n-1}$ such that $s+1 \leqslant i_s \leqslant n$ for all $s = 1, \dots, n-1$ form a basis in $\mathcal{L}_n/K_n$.

We tested the conjecture numerically for small n; yet it is not proved at the moment of writing.

For any n consider the embedding $\iota_n \colon S_n \to S_{n+1}$ of S_n to S_{n+1} as a stabilizer of $(n+1)$; extend it by linearity to the algebra homomorphism $\iota_n \colon \mathbb{C}[S_n] \to \mathbb{C}[S_{n+1}]$.

Proposition 4.3. $\iota_n(\mathcal{L}_n) \subset \mathcal{L}_{n+1}$.

Proof. Let $u = \sum_{\sigma \in S_n} a_\sigma \sigma \in \mathcal{L}_n$; consider the action of $\mathcal{G}_m\langle \iota_n(u)\rangle$ and $\mathcal{A}_m\langle \iota_n(u)\rangle$ on $x \stackrel{\text{def}}{=} x_{i_1} \wedge \dots \wedge x_{i_m}$, where $1 \leqslant i_1 < \dots < i_m \leqslant n+1$. If $i_m \leqslant n$ then $\mathcal{G}_m\langle \iota_n(u)\rangle(x) = \mathcal{G}_m\langle u\rangle(x) = \mathcal{A}_m\langle u\rangle(x) = \mathcal{A}_m\langle \iota_n(u)\rangle(x)$, hence $\iota_n(u) \in \mathcal{L}_{n+1}$.

Let now $i_m = n+1$. Then $\mathcal{G}_m\langle \iota_n(u)\rangle(x) = \mathcal{G}_{m-1}\langle u\rangle(x_{i_1} \wedge \dots \wedge x_{i_{m-1}}) \wedge x_{n+1}$. On the other hand,

$$\mathcal{A}_m\langle \iota_n(u)\rangle(x) = \left(\sum_{\sigma \in S_n} a_\sigma \sum_{p=1}^{n} x_{i_1} \wedge \dots \wedge x_{\sigma(i_p)} \wedge \dots \wedge x_{i_{m-1}}\right) \wedge x_{n+1} + \sum_{\sigma \in S_n} a_\sigma \cdot x.$$

By Remark 1.2 the last term in the equation above is zero. Thus,

$$\mathcal{A}_m\langle \iota_n(u)\rangle(x) = \mathcal{A}_{m-1}\langle u\rangle(x_{i_1} \wedge \dots \wedge x_{i_{m-1}}) \wedge x_{n+1},$$

and therefore $\mathcal{G}_m\langle \iota_n(u)\rangle(x) = \mathcal{A}_m\langle \iota_n(u)\rangle(x)$, which means $\iota_n(u) \in \mathcal{L}_{n+1}$. $\square$

Proposition 4.3 allows to consider the inductive limit $\mathcal{L}_\infty$ of $\mathcal{L}_2 \stackrel{\iota_2}{\hookrightarrow} \mathcal{L}_3 \stackrel{\iota_3}{\hookrightarrow} \dots$. It is a representation of the group S_∞ of finitely supported permutations of $\{1, 2, \dots\}$ and a Lie subalgebra of $\mathbb{C}[S_\infty]$ (conjecturally, generated by the Kirchhoff differences $\varkappa_{ij} = 1 - (ij)$, $1 \leqslant i < j$). Very few is known yet about both structures on $\mathcal{L}_\infty$.

4.2.3. *Lie elements and embedded graphs.* Let $(i_1 j_1), \dots, (i_m j_m)$ be a sequence of transpositions in S_n or, which is the same, the numbered edges of a graph Γ with the vertices $1, \dots, n$. There exists a uniquely defined embedding of Γ into a sphere M with handles and holes sending the vertices of Γ to the boundary of M; distribution of the vertices among components of the boundary coincides with the cyclic structure of the permutation $\sigma = (i_1 j_1) \dots (i_m j_m)$. The embedding is described in [5]; the paper [3] (in preparation) containes a more detailed analysis of its properties, as well as a generalization to nonorientable surfaces. This construction and the Lie element property of $1 - (ij)$ allow, in particular, to obtain a formula for the number of "minimal" (one-faced) embeddings of any graph; see [5, Theorem 2]. Probably, there exists a version of this theory for embedded 3-graphs, 4-graphs etc.; Lie element properties of ν_{ijk} and η_{ijkl} will give some important information about the embeddings.

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