ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2023, Vol. 17, No. 1, pp. 25–31. © Pleiades Publishing, Ltd., 2023.
Russian Text © The Author(s), 2023, published in Diskretnyi Analiz i Issledovanie Operatsii, 2023, Vol. 30, No. 1, pp. 28–40.

On a Countable Family of Boundary Graph Classes for the Dominating Set Problem

G. S. Dakhno[1*] and D. S. Malyshev[1,2]**

[1] *National Research University Higher School of Economics, Nizhny Novgorod Branch, Nizhny Novgorod, 603155 Russia*

[2] *Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, 603022 Russia*

*e-mail: * dahnogrigory@yandex.ru, ** dsmalyshev@rambler.ru*

Received May 29, 2022; revised October 23, 2022; accepted October 24, 2022

Abstract—A hereditary class is a set of simple graphs closed under deletion of vertices; every such class is defined by the set of its minimal forbidden induced subgraphs. If this set is finite, then the class is said to be finitely defined. The concept of a boundary class is a useful tool for the analysis of the computational complexity of graph problems in the family of finitely defined classes. The dominating set problem for a given graph is to determine whether it has a subset of vertices of a given size such that every vertex outside the subset has at least one neighbor in the subset. Previously, exactly four boundary classes were known for this problem (if $\mathbb{P} \neq \mathbb{NP}$). The present paper considers a countable set of concrete classes of graphs and proves that each its element is a boundary class for the dominating set problem (if $\mathbb{P} \neq \mathbb{NP}$). We also prove the $\mathbb{NP}$-completeness of this problem for graphs that contain neither an induced 6-path nor an induced 4-clique, which means that the set of known boundary classes for the dominating set problem is not complete (if $\mathbb{P} \neq \mathbb{NP}$).

Keywords: *hereditary graph class, computational complexity, dominating set*

DOI: 10.1134/S1990478923010039

INTRODUCTION

The present paper considers only *ordinary* graphs, i.e., undirected graphs with no loops or multiple edges. It is a continuation of the papers [1, 2], which studied the so-called boundary classes of graphs for the dominating set problem.

A graph class is said to be *hereditary* if it is closed under removal of vertices. It is well known that every hereditary and only hereditary class $\mathcal{X}$ is determined by the set $\mathcal{Y}$ of its *minimal forbidden induced subgraphs* (i.e., graphs not belonging to $\mathcal{X}$ that are minimal under removal of vertices); it is customary to write the above as follows: $\mathcal{X} = \mathrm{Free}(\mathcal{Y})$. Graphs in $\mathcal{X}$ are said to be $\mathcal{Y}$-*free*. If a hereditary class is defined by a finite set of its minimal forbidden induced subgraphs, then it is said to be *finitely defined*.

Let Π be some $\mathbb{NP}$-complete graph problem. A hereditary class in which Π is polynomially solvable is said to be Π-*easy*. A hereditary class in which the problem Π is $\mathbb{NP}$-complete is said to be Π-*hard*. Throughout the paper, it is assumed that $\mathbb{P} \neq \mathbb{NP}$, and this condition is not explicitly included in the statements of the corresponding assertions.

A hereditary class $\mathcal{X}$ is said to be Π-*limit* if there exists an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \dots$ of Π-hard graph classes such that $\mathcal{X} = \bigcap\limits_{i=1}^{\infty} \mathcal{X}_i$. An inclusion minimal Π-limit class is said to be Π-*boundary*. The concept of a boundary graph class was introduced by Alekseev [3]. The meaning of this concept is revealed by the following theorem (see [3, 4]).

Theorem 1. *A finitely defined graph class is* Π-*hard if and only if it contains some* Π-*boundary class.*

A *dominating set* of a graph $G = (V, E)$ is a subset $D \subseteq V$ such that each vertex in $V \setminus D$ has a neighbor in D. The cardinality of the least dominating set of a graph G is called its *dominance number* and is denoted by $\gamma(G)$. The *dominating set problem* (DS problem) for given graph G and number k is to verify the inequality $\gamma(G) \leq k$.

So far, exactly four boundary classes are known for the DS problem (see [1, 2]). The first of them is the class $\mathcal{T}$ consisting of all possible forests each of whose connected components is a tree with at most three leaves. The second is the class $\mathcal{D}$ of the edge graphs of the graphs in $\mathcal{T}$. To define the third and fourth boundary classes, we need operators Q and Q^* on graphs as well as the concept of hereditary closure of a graph class.

Let $G = (V, E)$ be an arbitrary graph. We denote by $Q(G)$ the graph on the vertex set $V \cup E$ in which

$$E\bigl(Q(G)\bigr) = \bigl\{xy \mid x, y \in V,\, x \neq y\bigr\} \cup \bigl\{xe \mid x \in V,\, e \in E\colon x \text{ is incident to } e \text{ in } G\bigr\}.$$

Let $G = (V, E)$ be a *subcubic* graph, i.e., a graph with vertex degrees ≤ 3. By V' we denote the set of all its vertices of degree 3. Set $V'' = V \setminus V'$. The graph $Q^*(G)$ has the vertex set $V'' \cup E$ and the edge set

$$E\bigl(Q^*(G)\bigr) = \bigl\{xy \mid x, y \in V'',\, x \neq y\bigr\} \cup \bigl\{xe \mid x \in V'',\, e \in E\colon x \text{ is incident to } e \text{ in } G\bigr\}$$
$$\cup \bigcup_{x \in V'} \bigl\{e_1(x)e_2(x),\, e_1(x)e_3(x),\, e_2(x)e_3(x)\bigr\},$$

where $e_1(x)$, $e_2(x)$, and $e_3(x)$ are the edges incident to the vertex x.

Let $\mathcal{X}$ be an arbitrary graph class, and let F be some operator on graphs. Denote by $F(\mathcal{X})$ the set $\{F(G) \mid G \in \mathcal{X}\}$. Denote by $[\mathcal{X}]$ the *hereditary closure* of $\mathcal{X}$, i.e., the set of all graphs induced by graphs in $\mathcal{X}$. By $\mathcal{Q}$ and $\mathcal{Q}^*$ we denote the sets $[\{Q(G) \mid G \in \mathcal{T}\}]$ and $[\{Q^*(G) \mid G \in \mathcal{T}\}]$.

The classes $\mathcal{Q}$ and $\mathcal{Q}^*$ are DS-boundary. This result is generalized in the present paper. For an edge xy of an arbitrary graph, its *l-subdivision*, where l is a nonnegative integer, consists in removing this edge and adding vertices $z_1, \ldots, z_l$ and edges $xz_1, z_1z_2, \ldots, z_{l-1}z_l, z_ly$. The operation of 1-subdivision of an edge is simply called a *subdivision of an edge*. The operation inverse to the l-subdivision is called the *l-contraction*. Denote by $Q_k(G)$ the result of the $3k$-subdivision of all edges of the form xe, ye in the graph $Q(G)$, where $e = xy \in E(G)$. Denote by $\mathcal{Q}_k$ the set $[Q_k(\mathcal{T})]$. Note that $\mathcal{Q}_0 = \mathcal{Q}$. In this paper, we prove that the class $\mathcal{Q}_k$ is DS-boundary for each k.

In the paper [5], a complete classification of the complexity of the DS problem for *monogenic classes*, i.e., hereditary classes defined by exactly one forbidden induced subgraph, is obtained. It is stated in that paper in terms of an explicit description of the corresponding subgraphs. In terms of DS-boundary classes, this can be reformulated as follows: a monogenic class is DS-simple if it does not include any of the classes $\mathcal{T}$, $\mathcal{D}$, and $\mathcal{Q}$; otherwise, it is DS-hard. A similar result (with the same statement) is obtained in [6] for a family of hereditary classes defined by minimal forbidden induced subgraphs with at most 5 vertices each.

As far as the present authors are aware, so far for the DS problem there are no complete dichotomies of its complexity in families of hereditary classes defined by two minimal forbidden induced subgraphs or minimal forbidden induced subgraphs with at most 6 vertices each. In this paper, a new step is taken in both of these directions. Namely, we prove that the graph class defined by the prohibition of the induced path with 6 vertices and the complete subgraph with 4 vertices is DS-hard. Based on this and Theorem 1, it follows that there exist DS-boundary classes distinct from $\mathcal{T}$, $\mathcal{D}$, $\mathcal{Q}_k$ $(k \geq 0)$, and $\mathcal{Q}^*$.

The interested reader can refer to the survey papers [4, 7], which summarize the recent achievements in the field of boundary graph classes. In particular, the set of boundary classes may have the cardinality of continuum (see [8]), and a complete description of boundary classes for nonartificial problems on graphs may be attainable (see [9]).

1. SOME DEFINITIONS, NOTATION, AND FACTS

1.1. Graphs, Subgraphs and Operations on Them

As usual, by P_n, C_n, and K_n we denote a simple path, a simple cycle, and a complete graph on n vertices, respectively. The graphs P_n and C_n are called an *n-path* and an *n-cycle*, respectively. By $K_{p,q}$ we denote the complete bipartite graph with p vertices in one part and q vertices in the other part. The "diamond" graph is obtained by removing an edge from K_4, and the "butterfly" graph is obtained by identifying two triangles by a common vertex.

The graphs P'_k and C'_k are obtained by adding a 3-path (x, y, z) to P_k and C_k, respectively, where $x, y, z \notin V(P_k)$ and $x, y, z \notin V(C_k)$, and an edge yv, where v is the end of P_k or $v \in V(C_k)$. The graphs P''_k and C''_k are obtained by adding an edge xz to P'_k and C'_k, respectively. The graph $C^e_{k,l}$ is obtained by identifying C_k and C_l along one edge.

The graph A_1 is isomorphic to C'_3, the graph A_2 is isomorphic to C''_3, the graph A_3 is obtained from A_1 by the subdivision of yv, the graph A_4 is obtained from A_3 by removing the edge incident to its vertices of degree 2, and the graph A_5 is obtained from A_4 by subdividing its edge incident to vertices of degree 2 and 3.

Let $G = (V, E)$ be a graph, and let $V' \subseteq V$ be a subset of its vertices. By $G \setminus V'$ we denote the result of deleting all vertices belonging to the set V' from G.

By $\overline{G}$ we denote the graph complementary to the graph G. The operation of disjoint union of graphs is applied only to graphs with disjoint sets of vertices. For graphs G_1 and G_2, by $G_1 + G_2$ we denote their disjoint union. By kG we denote the disjoint union of k graphs each of which is isomorphic to the graph G.

1.2. Graph Classes

A *monotone class* of graphs is a hereditary class that is also closed with respect to removal of edges. Each such class is defined by the set of its own *minimal forbidden subgraphs* (i.e., graphs that are minimal with respect to removal of vertices and edges that do not belong to the class). If this set is finite, then it is also said to be *finitely defined*. If $\mathcal{X}$ is a monotone graph class and $\mathcal{Y}$ is the set of its forbidden subgraphs, then $\mathcal{X} = \mathrm{Free}_m(\mathcal{Y})$. By $\mathcal{G}$ we denote the set of all graphs and by $\mathcal{G}_3$, the set of subcubic graphs.

1.3. Special Vertex Subsets

An *independent set* in a graph is an arbitrary subset of $V(G)$ consisting of pairwise nonadjacent vertices. The cardinality of the largest independent set in a graph G is denoted by $\alpha(G)$. A *vertex cover* of a graph is a subset of its vertices such that each edge of the graph is incident to at least one vertex in the subset. A *clique* of a graph is any subset of its pairwise adjacent vertices. It is easy to see that for any graph $G = (V, E)$ a subset $V' \subset V$ is an independent set in G if and only if $V \setminus V'$ is a vertex cover of G or if V' is a clique in $\overline{G}$.

An *independent set problem* (IS problem) for a given graph G and a number k is to determine whether the inequality $\alpha(G) \geq k$ is satisfied. The class $\mathcal{T}$ is IS-boundary; moreover, it is the only IS-boundary class in the family of monotone classes of graphs (see Theorems 4 and 5 in [3]). The uniqueness of $\mathcal{T}$ as an IS-boundary in a family of monotone classes and the connection between independent sets/vertex covers in a graph G and dominating sets in the graph $Q(G)$ was used in [1] to prove that $\mathcal{Q}$ is DS-boundary. A similar idea will also be used in the present paper.

2. ON THE DS-BOUNDARY PROPERTY OF CLASSES $\mathcal{Q}_k$

The following assertion shows that for any k the class $[Q_k(\mathcal{G}_3)]$ is finitely defined.

Lemma 1. *For each $k \geq 1$, the equality*

$$\left[Q_k(\mathcal{G}_3)\right] = \mathrm{Free}(\mathcal{X}_k)$$

holds, where

$$\mathcal{X}_k = \{K_{1,5}, C_4, \ldots, C_{6k+2}, C_{6k+4}, \ldots, C_{12k+5}, \text{diamond}, \text{butterfly},$$

$$K_{1,3} + C_3, K_{1,3} + P_{6k+2}, C_3 + P_{6k+2}, 2K_{1,3}, 2C_3, 2P_{6k+2},$$
$$P'_{6k+3}, C'_{6k+3}, P''_{6k+2}, C^e_{6k+3,6k+3}, A_1, A_2, A_3, A_4, A_5\}.$$

Proof. It can be proved that each element of the set $\mathcal{X}_k$ is a minimal forbidden induced subgraph of the class $[Q_k(\mathcal{G}_3)]$, so that $[Q_k(\mathcal{G}_3)] \subseteq \mathrm{Free}(\mathcal{X}_k)$. Let us show that this inclusion is an equality. Consider an arbitrary graph $G \in \mathrm{Free}(\mathcal{X}_k)$. Since

$$2P_{6k+2}, C_4, \ldots, C_{6k+2}, C_{6k+4}, \ldots, C_{12k+5} \in \mathcal{X}_k,$$

it follows that all induced cycles of G are 3- or $(6k + 3)$-cycles, and each of its induced paths has at most $12k + 4$ vertices. Since

$$C_4, C_5, C_6, K_{1,3} + C_3, 2K_{1,3}, 2C_3, \text{diamond}, \text{butterfly}, A_2, A_3, A_4, A_5 \in \mathcal{X}_k,$$

we see that any two vertices of the graph G of degree at least three are adjacent.

Let G not contain triangles. If $G \in \mathrm{Free}(\{C_{6k+3}\})$, then, since $2K_{1,3} \in \mathcal{X}_k$, the graph G is a forest in which all but possibly one of the components are simple paths. Each such $\{K_{1,5}, 2P_{6k+2}, K_{1,3} + P_{6k+2}, P'_{6k+3}\}$-free graph belongs to $[Q_k(\mathcal{G}_3)]$. If G contains an induced $(6k + 3)$-cycle , then it is unique, because

$$2P_{6k+2}, C'_{6k+3}, C^e_{6k+3,6k+3} \in \mathcal{X}_k.$$

Each such $\{K_{1,5}, 2P_{6k+2}, K_{1,3} + P_{6k+2}, P'_{6k+3}\}$-free graph belongs to $[Q_k(\mathcal{G}_3)]$.

Suppose that G contains triangles. Consider the largest clique V' in G; it contains at least three vertices. It is easily seen that each connected component of $G \setminus V'$ is a path with at most $6k + 1$ vertices, and each of their inner vertices is not adjacent to any vertex of V', while each of their terminal vertices has at most one neighbor in V'. Each such $\{K_{1,5}, \text{butterfly}, C^e_{6k+3,6k+3}\}$-free graph belongs to $[Q_k(\mathcal{G}_3)]$. The proof of the lemma is complete. $\square$

The next lemma is a generalization of Lemma 11 in [3] (note that the latter actually deals with monotone classes rather than just hereditary ones).

Lemma 2. *For any number k and monotone class $\mathcal{X}$, the* IS *problem in the class $\mathcal{X}$ is polynomially equivalent to the* DS *problem in the class $[Q_k(\mathcal{X})]$.*

Proof. A triple subdivision (i.e., 3-subdivision) of any edge of an arbitrary graph increases its dominance number exactly by one (see, e.g., Lemma 3 in [5]), and so $\gamma(Q_k(G)) = \gamma(Q_0(G)) + 2k|E(G)|$ for any graph G. When proving Lemma 11 in [3], it was shown (see items (a) and (b)) that for any nonempty graph G the dominance number of the graph $Q_0(G)$ is equal to the cardinality of the smallest vertex cover of G, which is the same as $|V(G)| - \alpha(G)$. Thus, the relation

$$\gamma\big(Q_k(G)\big) = \big|V(G)\big| + 2k\big|E(G)\big| - \alpha(G)$$

holds for any nonempty graph G.

It is easy to see that for any graph H isomorphic to $Q_k(G)$ one can compute G in a time polynomial in $|V(H)|$. To this end, it suffices to find an inclusion-maximal clique in the graph H (it will be the largest and corresponds to $V(G)$), remove it from H, and find the connected components of the result, which will be paths with $6k + 1$ vertices. The neighbors of the ends of these paths in the clique correspond to the edges of G. It is clear that for any graph G the graph $Q_k(G)$ is computed in a time polynomial in $|V(G)|$. Thus, the IS problem in any class $\mathcal{Y}$ (not necessarily even hereditary) is polynomially equivalent to the DS problem in the class $Q_k(\mathcal{Y})$.

Consider an arbitrary graph $G \in [Q_k(\mathcal{X})] \setminus Q_k(\mathcal{X})$. We can assume that G contains a clique V^* with at least three vertices; otherwise G is a disjoint union of only simple paths or a $(6k + 3)$-cycle and simple paths, and for such graphs the DS problem is solved in linear time. For the same reasons, we can assume that G is different from a complete graph, and so G consists of V^*, several induced $(6k + 3)$-cycles, each of which has exactly one common edge with V^*, as well as several simple paths, each of which has at most $6k + 2$ vertices and also a vertex common with V^*.

In the graph G, we perform 3-contractions as long as possible. We obtain some graph G', which is uniquely determined. It is clear that $\gamma(G) - \gamma(G') = k'$, where k' is the number of 3-contractions performed in G. The graph G' consists of V^*, several triangles, each of which has exactly one edge in common with V^*, and the induced 2-, 3-, and 4-paths, each of which has exactly one common vertex with V^*. If there are no such paths in G', then $G' = Q_0(H')$ for some graph $H' \in \mathcal{X}$ (recall that $\mathcal{X}$ is monotone).

Assume that the set of above-mentioned paths in G' is nonempty. It is clear that there exists a least dominating set G' containing, for each i-path, where $1 \le i \le 4$, the ith vertex from the end of the i-path belonging to V^*. Consider the set $\widetilde{V}$ of those vertices in G' that are not dominated by these vertices of the i-paths, and also the subgraph G'' of the graph G' induced by $\widetilde{V}$ and all their neighbors in V^*. If G'' is complete, then $\gamma(G) = 1$. Otherwise, there exists a graph $H'' \in \mathcal{X}$ such that $G'' = Q_0(H'')$ and $\gamma(G'') = |V(H'')| - \alpha(H'')$. Thus, the DS problem in the class $[Q_k(\mathcal{X})] \setminus Q_k(\mathcal{X})$ is polynomially reduced to the IS problem in the class $\mathcal{X}$. The proof of the lemma is complete. $\square$

Theorem 2. *The class $\mathcal{Q}_k$ is* DS-*boundary for each k.*

Proof. The set of subcubic graphs that do not contain cycles of length $\le i$ will be denoted by $\mathcal{X}_i$. For any i, the class $\mathcal{X}_i$ is monotone and IS-hard (see [10]), $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \cdots$, and $\bigcap_{i=1}^{\infty} \mathcal{X}_i = \mathcal{T}$, so by Lemma 2 for any k and i the class $[Q_k(\mathcal{X}_i)]$ is DS-hard. It is easily seen that

$$\big[Q_k(\mathcal{X}_1)\big] \supseteq \big[Q_k(\mathcal{X}_2)\big] \supseteq \cdots \quad \text{and} \quad \bigcap_{i=1}^{\infty} \big[Q_k(\mathcal{X}_i)\big] = \mathcal{Q}_k;$$

therefore, the class $\mathcal{Q}_k$ is DS-limit for each k.

Let us prove that the class $\mathcal{Q}_k$ is DM-boundary for any k. Assume, on the contrary, that there exists a sequence $\mathcal{Y}_{1,k} \supseteq \mathcal{Y}_{2,k} \supseteq \ldots$ of DS-hard classes such that $\bigcap_{i=1}^{\infty} \mathcal{Y}_{i,k} = \mathcal{Q}'_k \subset \mathcal{Q}_k = [Q_k(\mathcal{T})]$. Then for some $G \in \mathcal{T}$ we have

$$\mathcal{Q}'{}_k \subseteq \big[Q_k(\mathcal{G}_3)\big] \cap \mathrm{Free}\big(\{Q_k(G)\}\big) \subseteq \Big[Q_k\Big(\mathrm{Free}_m\big(\{G\}\big)\Big)\Big].$$

By Lemma 1, the class $[Q_k(\mathcal{G}_3)] \cap \mathrm{Free}(\{Q_k(G)\})$ is finitely defined. Therefore, there exists an i' such that the following inclusion holds:

$$\mathcal{Y}_{i',k} \subseteq \Big[Q_k\Big(\mathrm{Free}_m\big(\{G\}\big)\Big)\Big].$$

The class $\mathrm{Free}_m(\{G\})$ is monotone and does not include the class $\mathcal{T}$. By Theorem 2 in [3], the class $\mathrm{Free}_m(\{G\})$ is IS-simple. Therefore, the class $\mathcal{Y}_{i',k}$ is DS-simple by Lemma 2. We obtain a contradiction with the assumption that $\mathbb{P} \neq \mathbb{NP}$. The proof of the theorem is complete. $\square$

3. $\mathbb{NP}$-COMPLETENESS OF THE DS PROBLEM IN THE CLASS $\mathrm{Free}(\{P_6, K_4\})$

Let $G = (V, E)$ be an arbitrary graph, where $V = \{v_1, v_2, \ldots, v_n\}$. Denote by $R(G)$ the graph obtained from G with the help of the transformations described below.The proposed construction is similar to the construction in [11], which proves the $\mathbb{NP}$-completeness of the DS problem in the class of *chordal bipartite graphs* , i.e., graphs in the class $\mathrm{Free}(\{C_3, C_5, C_6, \ldots\})$.

Each vertex $v_i \in V$ is transformed into a graph (V_i, E_i), where

$$V_i = \{x_i, y_i, z_i, a_i, b_i\}, \quad E_i = \{a_i x_i, x_i y_i, y_i b_i, a_i z_i, x_i z_i, y_i z_i, b_i z_i\}.$$

Each edge $v_i v_j \in E$ is transformed into a pair of vertices p_{ij}, q_{ij}. We have

$$V(R(G)) = \bigcup_{i=1}^{n} V_i \;\cup \bigcup_{v_i v_j \in E} \{p_{ij}, q_{ij}\},$$

$$E(R(G)) = \bigcup_{i=1}^{n} E_i \cup \bigcup_{v_i v_j \in E} \{p_{ij}x_i,\, p_{ij}y_j,\, q_{ij}y_i,\, q_{ij}x_j\} \cup \bigcup_{\substack{i,j\in 1,\dots,n,\\ i\neq j}} \{x_iy_j, x_iz_j, y_iz_j\}.$$

Lemma 3. *For any graph G, one has the relation*

$$\gamma\big(R(G)\big) = 2\big|V(G)\big| - \alpha(G).$$

Proof. First, we prove that $\gamma(R(G)) \leq n + k$, where $k = n - \alpha(G)$ is the cardinality of the least vertex cover of the graph G. Indeed, if $\{v_{i_1}, \dots, v_{i_k}\}$ is the least vertex cover of the graph G, then

$$\{x_{i_1}, y_{i_1}, \dots, x_{i_k}, y_{i_k}\} \cup \big\{z_i \mid i \notin \{i_1, \dots, i_k\}\big\}$$

is a dominating set of $R(G)$.

Now let us prove that $\gamma(R(G)) \geq n + k$. Let us show that there exists a least dominating set of the graph $R(G)$ such that for any $i \in 1, \dots, n$ the vertices x_i and y_i belong or do not belong to it simultaneously. Indeed, if for some smallest dominating set D of the graph $R(G)$ one has $x_i \in D$ and $y_i \notin D$, then either $b_i \in D$ or $z_i \in D$, because b_i is dominated by the set D. If $v \in D$, where $v \in \{b_i, z_i\}$, then $(D \setminus \{v\}) \cup \{y_i\}$ is a dominating set of the graph $R(G)$.

Let D be the least dominating set of the graph $R(G)$ such that for any $i \in 1, \dots, n$ either $x_i, y_i \notin D$ or $x_i, y_i \in D$. We set $I = \{i \mid x_i, y_i \in D\}$. It follows from the minimality of D that if $p_{ij} \in D$, then $x_i, x_j, y_i, y_j \notin D$ and $q_{ij} \in D$. Then $(D \setminus \{p_{ij}, q_{ij}\}) \cup \{x_i, y_i\}$ is also a dominating set of $R(G)$, and so we can assume that none of the vertices p_{ij} and q_{ij} belongs to D. Therefore, the set $\{v_i \mid i \in I\}$ is a vertex cover of the graph G, and hence $|I| \geq k$. For any $i \in \{1, 2, \dots, n\} \setminus I$, the set D simultaneously dominates the vertices a_i and b_i, and so $a_i, b_i \in D$, or $z_i \in D$, for any such i. Since D is minimal, it follows that $z_i \in D$ for any such i. Therefore,

$$\gamma\big(R(G)\big) = |D| = 2|I| + n - |I| \geq n + k.$$

The proof of the lemma is complete. □

Lemma 4. *One has the inclusion* $R(\mathcal{G}) \subseteq \mathrm{Free}(\{P_6, K_4\})$.

Proof. In each graph in $R(\mathcal{G})$, the degrees of the vertices a_i, b_i, p_{ij}, q_{ij} are equal to 2, and so they cannot be contained in the subgraph K_4. In each graph in $R(\mathcal{G})$, the subsets

$$X = \{x_i\}_{i=1}^n,$$
$$Y = \{y_i\}_{i=1}^n,$$
$$Z = \{z_i\}_{i=1}^n$$

are independent; hence $R(\mathcal{G}) \subseteq \mathrm{Free}(\{K_4\})$.

Note that in each graph in $R(\mathcal{G})$, each of the vertices a_i, b_i, p_{ij}, q_{ij} has exactly two neighbors, and these are adjacent. It is easily seen that the subgraph of each graph in $R(\mathcal{G})$ induced by the subset $X \cup Y \cup Z$ is $\{P_4\}$-free, which means that $R(\mathcal{G}) \subseteq \mathrm{Free}(\{P_6\})$. The proof of the lemma is complete. □

Theorem 3. *The class* $\mathrm{Free}(\{P_6, K_4\})$ *is* DS-*hard.*

Proof. The graph $R(G)$ is computed from the graph G in a time polynomial in $|V(G)|$, and so by Lemma 3 the IS problem $\mathbb{NP}$-complete in the class $\mathcal{G}$ is polynomially reduced to the DS problem in the class $R(\mathcal{G})$. Thus, the DS problem is $\mathbb{NP}$-complete in the class $R(\mathcal{G})$, which, by Lemma 4, is a subset of the class $\mathrm{Free}(\{P_6, K_4\})$. The proof of the theorem is complete. □

Note that $P_6 \in \mathcal{T}$, $P_6 \in \mathcal{D}$, $K_4 \in \mathcal{Q}_k$ $(k \geq 0)$, and $K_4 \in \mathcal{Q}^*$. Based on this, the following assertion is a corollary of Theorems 1 and 3.

Corollary 1. *There exist* DS-*boundary classes other than* $\mathcal{T}$, $\mathcal{D}$, $\mathcal{Q}_k$ $(k \geq 0)$, *and* $\mathcal{Q}^*$.

FUNDING

This research was carried out within the framework of the Basic Research Program at the National Research University "Higher School of Economics."

REFERENCES

1. V. E. Alekseev, D. V. Korobitsyn, and V. V. Lozin, "Boundary classes of graphs for the dominating set problem," Discrete Math. **285** (1–3), 1–6 (2004).
2. D. S. Malyshev, "A complexity dichotomy and a new boundary class for the dominating set problem," J. Combin. Optim. **32**, 226–243 (2016).
3. V. E. Alekseev, "On easy and hard hereditary classes of graphs with respect to the independent set problem," Discrete Appl. Math. **132**, 17–26 (2003).
4. V. E. Alekseev, R. Boliac, D. V. Korobitsyn, and V. V. Lozin, "NP-hard graph problems and boundary classes of graphs," Theor. Comput. Sci. **389**, 219–236 (2007).
5. D. V. Korobitsyn, "On the complexity of determining the dominance number in monogenic classes of graphs," Diskretn. Mat. **2**, 90–96 (1990).
6. D. S. Malyshev, "A dichotomy for the dominating set problem for classes defined by small forbidden induced subgraphs," Discrete Appl. Math. **203**, 117–126 (2016).
7. D. S. Malyshev and P. M. Pardalos, "Critical hereditary graph classes: A survey," Optim. Lett. **10**, 1593–1612 (2016).
8. D. S. Malyshev, "Continuum sets of boundary classes of graphs for coloring problems," Diskretn. Anal. Issled. Oper. **16**, 41–51 (2009).
9. D. S. Malyshev, "Classes of graphs critical for the edge list-ranking problem," Diskretn. Anal. Issled. Oper. **20**, 59–76 (2013) [J. Appl. Ind. Math. **8** (2), 245–255 (2014)].
10. O. J. Murphy, "Computing independent sets in graphs with large girth," Discrete Appl. Math. **35**, 167–170 (1992).
11. H. Müller and A. Brandstädt, "The NP-completeness of Steiner tree and dominating set for chordal bipartite graphs," Theor. Comput. Sci. **53**, 257–265 (1987).