Independent sets versus 4-dominating sets in outerplanar graphs

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Abstract

We show that the number of independent sets in every outerplanar graph is greater than the number of its 4-dominating sets.

Keywords— independent set, dominating set, k-dominating set, outerplanar graph

1 Introduction

Throughout this paper we consider only simple and finite graphs. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. An *independent set* is a pairwise nonadjacent subset of $V(G)$. For $k \geq 1$, a *k-dominating set* D_k is a subset of $V(G)$ such that every vertex not from D_k is adjacent to at least k vertices from D_k. We use the abbriveations 'IS' and 'k-DS' for the terms 'independent set' and 'k-dominating set' respectively. Let $i(G)$ (resp. $\partial_k(G)$) be the number of all IS (resp. k-DS) of a graph G. Denote by $\mathfrak{D}_k(G)$ the family of all k-DS of a graph G.

A planar graph is called *outerplanar* if it has an embedding in the plane such that all vertices belong to the boundary of its outer face. An outerplanar graph G is called *maximal outerplanar* (or *MOP*) if $G + uv$ is not outerplanar for any two non-adjacent vertices $u, v \in V(G)$. It is well-known that every inner face of a MOP is a triangle and every MOP with at least 3 vertices has at least 2 vertices of degree 2. Denote by $\mathcal{OP}$ and $\mathcal{MOP}$ the classes of all outerplanar and maximal outerplanar graphs, respectively.

The definition of k-dominating set implies that for every graph G, any two non-adjacent vertices $u, v \in V(G)$ and any integer $k \geq 1$ we have $\partial_k(G) \leq \partial_k(G + uv)$. Therefore, a complete graph K_n has the maximum number of k-independent sets among all n-vertex graphs. Trees with extremal numbers of k-dominating sets were described in [1] for $k = 1$ and in [2] for $k \geq 2$. In [3] and [4] for every $k \geq 2$ new upper bounds for the number of k-dominating independent sets in n-vertex graphs were presented.

The empty graph nK_1 has the maximum possible number of independent sets among all n-vertex graphs. Moon and Moser [5] described n-vertex graphs with maximal possible number of *maximal independent* (i.e. independent 1-dominating) sets. In [6] for all $n \geq 5$ the graphs $H_n \in \mathcal{MOP}$ and $H'_n \in \mathcal{MOP}$ with the maximum and minimum possible number of IS among all n-vertex MOPs were described.

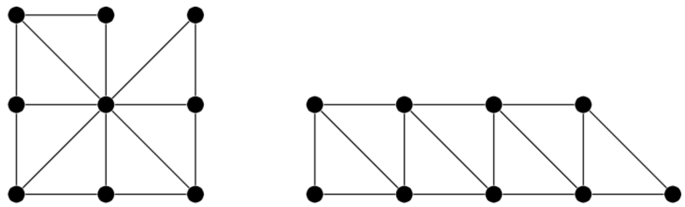

Figure 1: Graphs H'_9 and H_9

The main result of this paper is the following fact:

Theorem 1. *For every outerplanar graph G we have $i(G) > \partial_4(G)$.*

The rest of the paper is organized as follows. In Section 2 we introduce some graph terminology and present a maximal outerplanar graph partition. In Section 3 we prove Theorem 1. In Section 4 we consider a possible generalization of Theorem 1 and obtain a similar result for the class of trees.

2 Preliminaries

2.1 Basic terminology

A *tree* is a connected acyclic graph and a *leaf* is a vertex of degree one in a tree. A *support vertex* in a tree is a vertex which is adjacent to at least one leaf. A *diameter* of a connected graph is the maximum possible distance between its vertices. A simple path is called *diametral*, if its length is equal to the diameter of a graph. Clearly, the ends of any diametral path in a tree are leaves.

For a graph $G \in \mathcal{OP}$ we consider *a weak dual graph* $T(G)$, such that the vertices of $T(G)$ correspond to the inner faces of G and two vertices are adjacent if and only if the corresponding faces have a common edge. It is well-known that if $G \in \mathcal{MOP}$, then $T(G)$ is a subcubic tree.

Let G be a graph and $U \subset V(G)$ be its vertex subset. Let $G \setminus U$ be an induced subgraph of G with the set of vertices $V(G) \setminus U$.

An inner face of a MOP is called an *end face* if it has a vertex of degree 2. Note that a face f of a MOP G is an end face if and only if the corresponding vertex in $T(G)$ has degree at most one. We say that a graph G *contains* a face f, if all vertices and edges from f belong to G.

Suppose that f and f' are two adjacent inner faces of G with a common edge uv. Denote by $G[f; f']$ the maximal by inclusion subgraph of G such that it contains f, does not contain f' and uv is its outer edge. It is easy to see that if $G \in \mathcal{MOP}$, then $G[f; f'] \in \mathcal{MOP}$.

2.2 Independent and 4-dominating sets

Let G be a graph and $u, v \in V(G)$. Let $i(G, u^+)$ (resp. $i(G, u^-)$) be the number of IS of G which contain (resp. do not contain) u. We denote by $i(G, u^+, v^+)$ the number of IS of G such that $u, v \in I$

and denote the values $i(G, u^+, v^-)$, $i(G, u^-, v^+)$ and $i(G, u^-, v^-)$ in the similar way. Clearly, for any vertices $u, v \in V(G)$ we have

$$i(G) = i(G, u^+, v^+) + i(G, u^+, v^-) + i(G, u^-, v^+) + i(G, u^-, v^-).$$

Moreover, $i(G, u^+, v^+) = 0$, if and only if $uv \in E(G)$.

Let $\partial_4(G, u^+)$ (resp. $\partial_4(G, u^-)$) be the number of 4-DS in G which contain (resp. do not contain) u. We denote the values $\partial_4(G, u^+, v^+)$, $\partial_4(G, u^+, v^-)$, $\partial_4(G, u^-, v^+)$ and $\partial_4(G, u^-, v^-)$ in the similar way. Again, for any vertices $u, v \in V(G)$ we have

$$\partial_4(G) = \partial_4(G, u^+, v^+) + \partial_4(G, u^+, v^-) + \partial_4(G, u^-, v^+) + \partial_4(G, u^-, v^-).$$

Note that $\partial_4(G, u^+, v^+) > 0$ for any graph G. It is easy to see that $\partial_4(G, u^-, v^+) > 0$, if and only if $deg(u) \geq 4$. Moreover, $\partial_4(G, u^-, v^-) > 0$, if and only if either $\min(deg(u), deg(v)) \geq 5$ or $uv \notin E(G)$ and $\min(deg(u), deg(v)) \geq 4$.

2.3 MOP-partition

Consider a graph $G \in \mathcal{MOP}$ and its edge $uv \in E(G)$. Let $G_L, G_R \in \mathcal{MOP}$ be two maximal by inclusion subgraphs of G such that $E(G_L) \cap E(G_R) = uv$ and uv is an outer edge in both G_L and G_R. Clearly, if uv is not an outer edge in G than both G_L and G_R have at least 3 vertices (otherwise one of them has 2 vertices and the other coincides with G). Call a triple (G_L, G_R, uv) a *MOP-partition* of G. For the given MOP-partition (G_L, G_R, uv) we shall use the notation

$$\mathcal{I}_{00} = i(G_R, u^-, v^-),\ \mathcal{I}_{01} = i(G_R, u^-, v^+),\ \mathcal{I}_{10} = i(G_R, u^+, v^-).$$

Since $i(G_R, u^+, v^+) = 0$, we have $i(G_R) = \mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}$. Morever, if G_R has at least 3 vertices, than $\mathcal{I}_{00} > \max(\mathcal{I}_{01}, \mathcal{I}_{10})$.

We now introduce a similar concept for 4-DS. For the given MOP-partition (G_L, G_R, uv) of a graph G let $\mathcal{D}_{11} = \partial_4(G_R, u^+, v^+)$. Denote by $\mathcal{D}_{01}^k$ the number of 4-DS D' of a graph $G_R \setminus u$ such that $v \in D'$ and the vertex u has at least $\max(0, 4-k)$ neighbors from $D' \setminus v$ in G. In other words, $\mathcal{D}_{01}^k$ is the number of 'almost 4-dominating' sets D' of G_R with respect to the vertex u. Denote $\mathcal{D}_{10}^k$ in the same way as $\mathcal{D}_{01}^k$. Finally, let $\mathcal{D}_{00}^{kl}$ be the number of 4-DS D'' of a graph $G_R \setminus \{u, v\}$, such that in the graph G_R the vertices u and v have at least $\max(0, 4-k)$ and $\max(0, 4-l)$ neighbors from D'', respectively.

From the definitions of $\mathcal{D}_{01}^k$, $\mathcal{D}_{10}^k$ and $\mathcal{D}_{00}^{kl}$ it follows immediately that for all $1 \leq k' \leq k$ and $1 \leq l' \leq l$ we have $\mathcal{D}_{01}^{k'} \leq \mathcal{D}_{01}^k, \mathcal{D}_{10}^{l'} \leq \mathcal{D}_{10}^l$ and $\mathcal{D}_{00}^{k'l'} \leq \mathcal{D}_{00}^{kl}$. We now prove a few more similar properties.

Lemma 1. *Let $G \in \mathcal{MOP}$. For the given MOP-partition (G_L, G_R, uv) the following holds:*

1. *For all $k, l \geq 0$ we have $\mathcal{D}_{11} \geq \max(\mathcal{D}_{10}^k, \mathcal{D}_{01}^k)$ and $\min(\mathcal{D}_{10}^k, \mathcal{D}_{01}^k) \geq \mathcal{D}_{00}^{kl}$.*
2. *If $deg_{G_R}(u) \geq 2$, then $2 \cdot \mathcal{D}_{10}^3 \geq \mathcal{D}_{10}^4$ and $2 \cdot \mathcal{D}_{01}^3 \geq \mathcal{D}_{01}^4$.*
3. *If $deg_{G_R}(u) = 3$, then $3 \cdot \mathcal{D}_{01}^2 \geq \mathcal{D}_{01}^3$. Moreover, if $deg_{G_R}(u) \geq 4$, then $2 \cdot \mathcal{D}_{01}^2 \geq \mathcal{D}_{01}^3$.*

Proof. **Statement 1.** We show that for all $k \geq 0$ the inequality $\mathcal{D}_{11} \geq \mathcal{D}_{10}^k$ holds. Consider the function

$$F : \mathfrak{D}_4(G_R \setminus v) \longrightarrow \mathfrak{D}_4(G_R),$$

which maps a 4-DS D of $G_R \setminus v$ into a 4-DS $D \cup \{v\}$ of G_R. Clearly, F is injective, therefore $\partial_4(G_R \setminus v) \leq \partial_4(G)$, this implies the inequality. The inequalities $\mathcal{D}_{11} \geq \mathcal{D}_{01}^k$ and $\min(\mathcal{D}_{10}^k, \mathcal{D}_{01}^k) \geq \mathcal{D}_{00}^{kl}$ are easy to prove using the same approach.

Statement 2. The inequality $deg_{G_R}(u) \geq 2$ means that the subgraph G_R has at least 3 vertices, thus $deg_{G_R}(v) \geq 2$. We denote by w the common neighbor of u and v, such that $w \in V(G_R)$ (since uv is an outer edge of $G_R \in \mathcal{MOP}$, there is exactly one such vertex). Our goal is to show that $\mathcal{D}_{10}^4 - \mathcal{D}_{10}^3 \leq \mathcal{D}_{10}^3$. The left hand side equals to the number of 4-DS of the graph $G_R \setminus v$ such that they don't have vertices from $N_{G_R}[u] \setminus v$. The right hand side equals the number of 4-DS of the same graph which contain at least 1 vertex from the set $N_{G_R}[u] \setminus v$, therefore the inequality holds. It is easy to prove that $2\mathcal{D}_{10}^3 \geq \mathcal{D}_{10}^4$, using the same approach.

Statement 3. By the definition, $\mathcal{D}_{01}^2$ is the number of 4-DS of $G_R \setminus u$ with at least two vertices from the set $N_{G_R}[u] \setminus v$. Therefore, the difference $\mathcal{D}_{01}^3 - \mathcal{D}_{01}^2$ equals to the number of 4-DS of the graph $G_R \setminus u$, with exactly one vertex from the set $N_{G_R}[u] \setminus v$. Let $N_{G_R}[u] \setminus v = \{w_1, \dots w_s\}$, where $s \geq 2$. If $s = 2$, then the number of 4-DS with both vertices w_1 and w_2 is at least half of the number of 4-DS with one fixed vertex, this yields the inequality $\mathcal{D}_{01}^3 - \mathcal{D}_{01}^2 \geq 2 \cdot \mathcal{D}_{01}^2$. If $s \geq 3$, then it is easy to see that the number of 4-DS with exactly one vertex from the set $N_{G_R}[u] \setminus v$ is less than then the number of 4-DS with at least two vertices, therefore $\mathcal{D}_{01}^3 - \mathcal{D}_{01}^2 \geq \mathcal{D}_{01}^2$, as required. □

3 Proof of Theorem 1

We call a graph $G \in \mathcal{MOP}$ *critical*, if $i(G) \leq \partial_4(G)$ and for every outerplanar graph G' such that $|V(G')| < |V(G)|$ we have $i(G') > \partial_4(G')$. In this section we show that there are no critical graphs, therefore Theorem 1 holds. It suffices to consider only maximal outerplanar graphs, since for every graph $G_0 \in \mathcal{OP}$ there exists a graph $G \in \mathcal{MOP}$ such that G_0 is a spanning subgraph of G and the inequalities $i(G_0) > i(G)$ and $\partial_4(G_0) \leq \partial_4(G)$ hold.

Therefore, we consider a graph $G \in \mathcal{MOP}$ and its weak dual graph $T(G)$ (remind that $T(G)$ is a subcubic tree). Let $x_1x_2 \dots x_k$ be some diametral path in $T(G)$. If $k \leq 3$, then there are only 3 possible MOPs up to isomorphism and it is easy to check that they are not critical. Thus we assume that $k \geq 4$.

Lemma 2. *If a graph $G \in \mathcal{MOP}$ has an edge uv such that $deg(u) = 2$ and $deg(v) = 3$, then G is not critical.*

Proof. Since $G \in \mathcal{MOP}$, the vertices u and v have the unique common neighbor a and the vertices v and a have the unique common neighbor b, other then u. Let $G_1 = G \setminus u$ and $G_3 = G \setminus \{u, a, v\}$. Then

$$i(G) = i(G, u^-) + i(G, u^+) = i(G_1) + i(G_3).$$

We now show that

$$\partial_4(G) - \partial_4(G_1) \leq \partial_4(G_3).$$

The difference $\partial_4(G) - \partial_4(G_1)$ equals to the number of 4-DS D of the graph G such that $D \setminus u$ is not a 4-DS for the graph G_1. Since v belongs to every 4-DS in both G and G_1, this is possible if and only if $a \notin D$ and exactly two vertices from the set $N(a) \setminus \{u, v\}$ belong to D. Let $\mathfrak{D}_4'(G_1)$ be the family of 4-DS D' of G such that $D' \setminus u$ is a 4-DS of G_1. Consider the function

$$F : (\mathfrak{D}_4(G) \setminus \mathfrak{D}_4'(G_1)) \longrightarrow \mathfrak{D}_4(G_3),$$

such that $F(D) = (D \cup \{b\}) \setminus \{u, v\}$. It is easy to see that F is injective, because if $D', D'' \in \mathfrak{D}_4(G) \setminus \mathfrak{D}_4'(G_1)$ are two distinct 4-DS of $\mathfrak{D}_4(G)$, then the sets $D' \setminus \{u, v, b\}$ and $D'' \setminus \{u, v, b\}$ are also distinct. Therefore,

$$\partial_4(G) \leq \partial_4(G_1) + \partial_4(G_3) < i(G_1) + i(G_3) = i(G)$$

and G is not critical. □

Lemma 2 implies that every support vertex of $T(G)$ has degree 3. In partucular, $deg(x_2) = 3$ and f_2 is adjacent to some end faces f_1 and f_1'. In the rest of the chapter we denote the faces f_1, f_1', f_2 and f_3 by $a_1a_2b_1$, $a_2a_3b_2$, $b_1b_2a_2$ and $b_1b_2c_1$ respectively.

Lemma 3. *If $deg(x_2) = deg(x_3) = deg(x_2') = 3$, then G is not critical.*

Proof. Suppose that b_2c_1 is the common edge of the faces f_3 and f_4. Let $G_L = G[f_3; f_4]$. Consider the (G_L, G_R, b_2c_1)-partition of G. Since G is critical, we have $i(G_R) > \partial_4(G_R)$. Our goal is to find a constant $c > 0$ such that $i(G) \geq c \cdot i(G_R)$ and $\partial_4(G) \leq c \cdot \partial_4(G_R)$. It is easy to check that the following holds:

$$i(G) = i(G_L) \cdot \mathcal{I}_{00} + i(G_L, c_1^+) \cdot \mathcal{I}_{01} + i(G_L, b_2^+) \cdot \mathcal{I}_{10} = 29 \cdot \mathcal{I}_{00} + 10 \cdot \mathcal{I}_{01} + 10 \cdot \mathcal{I}_{10}.$$

Since $\mathcal{I}_{00} \geq \max(\mathcal{I}_{01}, \mathcal{I}_{10})$, we have

$$i(G) > 16 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 16 \cdot i(G_R).$$

Moreover,

$$\begin{aligned}\partial_4(G) &= \partial_4(G, b_2^+, c_1^+) + \partial_4(G, b_2^+, c_1^-) + \partial_4(G, b_2^-, c_1^+) + \partial_4(G, b_2^-, c_1^-)\\ &= 5 \cdot \mathcal{D}_{11} + (2 \cdot \mathcal{D}_{10}^4 + \mathcal{D}_{10}^3) + (2 \cdot \mathcal{D}_{01}^4 + \mathcal{D}_{01}^3) + (\mathcal{D}_{00}^{44} + \mathcal{D}_{00}^{33}).\end{aligned}$$

By Lemma 1, we have $\partial_4(G) \leq 13 \cdot \mathcal{D}_{11} < 16 \cdot \partial_4(G_R)$. Therefore, G is not critical. □

Lemma 4. *If $deg(x_3) = 3$ and f_3 is adjacent to some end face f_2', then G is not critical.*

Proof. Denote the face f_4 by $c_1b_2c_2$ and let $G_L = G[f_3; f_4]$. Consider the (G_L, G_R, b_2c_1)-partition of graph G and the (G_L', G_R, b_2c_1)-partition of graph $G_2 = \{a_1, a_3\}$, where $G_L' = G_L \setminus \{a_1, a_3\}$. Again, our goal is to find a constant $c > 0$ such that $i(G) \geq c \cdot i(G_2)$ and $\partial_4(G) \leq c \cdot \partial_4(G_2)$. We have

$$i(G) = i(G_L) \cdot \mathcal{I}_{00} + i(G_L, c_1^+) \cdot \mathcal{I}_{01} + i(G_L, b_2^+) \cdot \mathcal{I}_{10} = 12 \cdot \mathcal{I}_{00} + 5 \cdot \mathcal{I}_{01} + 4 \cdot \mathcal{I}_{10};$$

$$i(G_2) = i(G_L') \cdot \mathcal{I}_{00} + i(G_L', c_1^+) \cdot \mathcal{I}_{01} + i(G_L', b_2^+) \cdot \mathcal{I}_{10} = 5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}.$$

Since $\mathcal{I}_{00} \geq \max(\mathcal{I}_{01}, \mathcal{I}_{10})$, we have $i(G) > \frac{16}{7} \cdot i(G_2)$. Moreover,

$$\partial_4(G) = 3 \cdot \mathcal{D}_{11} + (2 \cdot \mathcal{D}_{10}^3 + \mathcal{D}_{10}^2) + (\mathcal{D}_{01}^4 + \mathcal{D}_{01}^3) + \mathcal{D}_{00}^{32};$$

$$\partial_4(G_2) = 2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^3 + \mathcal{D}_{01}^3 + \mathcal{D}_{00}^{22}.$$

We now show that

$$3 \cdot \mathcal{D}_{11} + (2 \cdot \mathcal{D}_{10}^3 + \mathcal{D}_{10}^2) + (\mathcal{D}_{01}^4 + \mathcal{D}_{01}^3) + \mathcal{D}_{00}^{32} \leq \frac{16}{7} \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^3 + \mathcal{D}_{01}^3 + \mathcal{D}_{00}^{22}).$$

It is sufficient to prove the inequality

$$\mathcal{D}_{10}^2 + \mathcal{D}_{01}^4 + \mathcal{D}_{00}^{32} \leq \frac{11}{7} \cdot \mathcal{D}_{11} + \frac{2}{7} \cdot \mathcal{D}_{10}^3 + \frac{9}{7} \cdot \mathcal{D}_{01}^3.$$

If $|V(G_R)| = 2$, then $\mathcal{D}_{10}^2 = \mathcal{D}_{00}^{32} = 0$ and we are done. Otherwise by Lemma 1 we have $2 \cdot \mathcal{D}_{01}^3 \geq \mathcal{D}_{01}^4$ and $\mathcal{D}_{01}^3 \geq \mathcal{D}_{00}^{32}$, therefore

$$\mathcal{D}_{10}^2 + \frac{6}{7} \cdot \mathcal{D}_{01}^4 \leq \frac{11}{7} \cdot \mathcal{D}_{11} + \frac{2}{7} \cdot \mathcal{D}_{10}^3.$$

This completes the proof. □

In the rest of the chapter we assume that $deg(x_3) = 2$ and denote the face f_4 by $b_2c_1c_2$.

Lemma 5. *If $deg(x_4) = 3$, then G is not critical.*

Proof. Let $G_L = G[f_4; f_5]$. and f_3' be the face adjacent to f_4, other than f_3 and f_5. We have two cases depending on the location of f_3'.

Case 1. f_3' contains the edge b_2c_2. We consider the (G_L, G_R, c_1c_2)-partition of G.

Subcase 1. $deg(x_3') = 1$. Let $G_3 = G \setminus \{a_1, a_2, a_3\}$. Consider the (G_L', G_R, c_1c_2)-partition of G_3, where $G_L' = G_L \setminus \{a_1, a_2, a_3\}$. We have

$$i(G) = 16 \cdot \mathcal{I}_{00} + 7 \cdot \mathcal{I}_{01} + 10 \cdot \mathcal{I}_{10} > 3 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 3 \cdot i(G_3);$$

$$\partial_4(G) = 4 \cdot \mathcal{D}_{11} + (2 \cdot \mathcal{D}_{10}^{3} + \mathcal{D}_{10}^{2}) + (3 \cdot \mathcal{D}_{01}^{3} + \mathcal{D}_{01}^{2}) + (2 \cdot \mathcal{D}_{00}^{22} + \mathcal{D}_{00}^{11});$$

$$\partial_4(G_3) = 2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^{3} + \mathcal{D}_{01}^{3} + \mathcal{D}_{00}^{22}.$$

Clearly, $\partial_4(G) \leq 3 \cdot \partial_4(G_3) < 3 \cdot i(G_3) \leq i(G)$, as required.

Subcase 2. $deg(x_3') = 3$. By the previous lemma, f_3' is adjacent to some end faces f_2' and f_2''. We have

$$i(G) = 39 \cdot \mathcal{I}_{00} + 14 \cdot \mathcal{I}_{01} + 25 \cdot \mathcal{I}_{10} \geq 26 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 26 \cdot i(G_R);$$

$$\partial_4(G) = 7 \cdot \mathcal{D}_{11} + (3 \cdot \mathcal{D}_{10}^{4} + \mathcal{D}_{10}^{3}) + (4 \cdot \mathcal{D}_{01}^{4} + \mathcal{D}_{01}^{3}) + (2 \cdot \mathcal{D}_{00}^{12} + \mathcal{D}_{00}^{01}).$$

By Lemma 1, we have $\partial_4(G) \leq 19 \cdot \mathcal{D}_{11} < 26 \cdot \partial_4(G_R)$.

Subcase 3. $deg(x_3') = 2$ and $deg(b_2) = 6$. In this case f_3' is adjacent to a face f_2' which is adjacent to end faces f_1' and f_1''. Therefore,

$$i(G) = 59 \cdot \mathcal{I}_{00} + 14 \cdot \mathcal{I}_{01} + 35 \cdot \mathcal{I}_{10} \geq 36 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 36 \cdot i(G_R);$$

$$\partial_4(G) = 11 \cdot \mathcal{D}_{11} + (6 \cdot \mathcal{D}_{01}^{3} + 2 \cdot \mathcal{D}_{01}^{2}) + (3 \cdot \mathcal{D}_{10}^{4} + \mathcal{D}_{10}^{3}) + 2 \cdot (\mathcal{D}_{00}^{24} + \mathcal{D}_{00}^{13})$$
$$\leq 26 \cdot \mathcal{D}_{11} = 26 \cdot \partial_4(G_R).$$

Subcase 4. $deg(x_3') = 2$ and $deg(b_2) = 8$. Again, f_3' is adjacent to a face f_2' which is adjacent to end faces f_1' and f_1''. Therefore,

$$i(G) = 53 \cdot \mathcal{I}_{00} + 35 \cdot \mathcal{I}_{01} + 35 \cdot \mathcal{I}_{10} \geq 41 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 41 \cdot i(G_R);$$

$$\partial_4(G) \leq 4 \cdot \partial_4(G, b_2^{+}, c_2^{+}) = 4 \cdot 10 \cdot \mathcal{D}_{11} \leq 40 \cdot i(G_R).$$

Case 2. The face f_3' contains the edge c_1c_2. Consider the (G_L, G_R, b_2c_2)-partition of G.

Subcase 1. $deg(x_3') = 1$. Let $G_3 = G \setminus \{a_1, a_2, a_3\}$. Consider the (G_L', G_R, b_2c_2)-partition of G_3, where $G_L' = G_L \setminus \{a_1, a_2, a_3\}$. We have

$$i(G) = 19 \cdot \mathcal{I}_{00} + 7 \cdot \mathcal{I}_{01} + 4 \cdot \mathcal{I}_{10} > 3 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 3 \cdot i(G_3);$$

$$\partial_4(G) = 5 \cdot \mathcal{D}_{11} + 3 \cdot \mathcal{D}_{10}^{3} + \mathcal{D}_{01}^{4} + \mathcal{D}_{00}^{42} \leq 3 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^{3} + \mathcal{D}_{01}^{3} + \mathcal{D}_{00}^{22}) = 3 \cdot \partial_4(G_3).$$

By Lemma 1, we have $\mathcal{D}_{11} + 3 \cdot \mathcal{D}_{01}^{3} > \mathcal{D}_{00}^{42} + \mathcal{D}_{01}^{4}$, as required.

Subcase 2. $deg(x_3') = 3$. By the previous lemma, f_3' is adjacent to end faces f_2' and f_2''. We have

$$i(G) = 45 \cdot \mathcal{I}_{00} + 14 \cdot \mathcal{I}_{01} + 10 \cdot \mathcal{I}_{10} \geq 23 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 23 \cdot i(G_R);$$

$$\partial_4(G) = 8 \cdot \mathcal{D}_{11} + (3 \cdot \mathcal{D}_{10}^{4} + 2 \cdot \mathcal{D}_{10}^{3}) + 3 \cdot \mathcal{D}_{01}^{4} + \mathcal{D}_{00}^{43} \leq 17 \cdot \mathcal{D}_{11} < 23 \cdot \partial_4(G_R).$$

Subcase 3. $deg(x'_3) = 2$ and $deg(c_1) = 4$. f'_3 is adjacent to a face f'_2 which is adjacent to end faces f'_1 and f''_1. We have

$$i(G) = 74 \cdot \mathcal{I}_{00} + 14 \cdot \mathcal{I}_{01} + 14 \cdot \mathcal{I}_{10} \geq 34 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 34 \cdot i(G_R);$$

$$\partial_4(G) = 13 \cdot \mathcal{D}_{11} + 3 \cdot \mathcal{D}^4_{10} + 3 \cdot \mathcal{D}^4_{01} + \mathcal{D}^{44}_{00} \leq 20 \cdot \mathcal{D}_{11} < 34 \cdot \partial_4(G_R).$$

Subcase 4. $deg(x'_3) = 2$ and $deg(c_1) = 6$. f'_3 is adjacent to a face f'_2 which is adjacent to end faces f'_1 and f''_1. We have

$$i(G) = 59 \cdot \mathcal{I}_{00} + 35 \cdot \mathcal{I}_{01} + 14 \cdot \mathcal{I}_{10} > 36 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 36 \cdot i(G_R);$$

$$\partial_4(G) \leq 3 \cdot \partial_4(G, b_2^+, c_2^+) + \partial_4(G, b_2^-, c_2^-) = 3 \cdot 11 \cdot \mathcal{D}_{11} + (2 \cdot \mathcal{D}^{42}_{00} + \mathcal{D}^{31}_{00})$$

By Lemma 1, we have $\partial_4(G) \leq 36 \cdot \mathcal{D}_{11} \leq 36 \cdot \partial_4(G_R)$. □

Lemma 6. *If $deg(x_3) = deg(x_4) = 2$ and $deg(c_1) = 3$, then G is not critical.*

Proof. If $deg(x_3) = deg(x_4) = 2$ and $deg(c_1) = 3$, then f_5 contains b_2c_2. Let $G_L = G[f_3; f_4]$. For the (G_L, G_R, b_2c_1)-partition of G we have

$$i(G) = 7 \cdot \mathcal{I}_{00} + 5 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10} > 4 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 4 \cdot i(G_R);$$

$$\partial_4(G) = \partial_4(G, b_2^+, c_1^+) + \partial_4(G, b_2^-, c_1^+) = 3 \cdot \mathcal{D}_{11} + \mathcal{D}^4_{01} \leq 4 \cdot \mathcal{D}_{11} \leq \partial_4(G_R).$$

Therefore, G is not critical. □

Lemma 7. *If $deg(x_3) = deg(x_4) = 2$ and $deg(x_5) = 3$, then G is not critical.*

Proof. We denote the faces f_2, f_3, f_4 by $a_2b_1b_2, b_1b_2c_1, b_2c_1c_2$ respectively. By the previous lemma, $deg(c_1) \geq 4$ and the face f_5 corresponds to the triangle $c_1c_2d_1$. Let f'_4 be the face adjacent to f_5, other than f_4 and f_6. Let $G_L = G[f_5; f_6]$. There are two possible cases depending on the location of f'_4 in G.

Case 1. The face f'_4 contains the edge c_1d_1. Let d_0 be the third vertex of f'_4. Consider the (G_L, G_R, c_2d_1)-partition of G.

Subcase 1a. $deg(x'_4) = 1$ and $\min(deg_G(c_2), deg_G(d_1)) \geq 5$. Consider the graph $G_6 = G \setminus \{a_1, a_2, a_3, b_1, b_2, d_0\}$ and the $(G_L \cap V(G_6), G_R, c_2d_1)$-partition of G_6. We have

$$i(G) = 23 \cdot \mathcal{I}_{00} + 9 \cdot \mathcal{I}_{01} + 14 \cdot \mathcal{I}_{10} > 10 \cdot (2 \cdot \mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 10 \cdot i(G_6).$$

We now prove that

$$\partial_4(G) = 7 \cdot \mathcal{D}_{11} + (4 \cdot \mathcal{D}^3_{10} + 2 \cdot \mathcal{D}^2_{10}) + (3 \cdot \mathcal{D}^3_{01} + 3 \cdot \mathcal{D}^2_{01}) + (3 \cdot \mathcal{D}^{22}_{00} + \mathcal{D}^{12}_{00}) \leq$$

$$10 \cdot (\mathcal{D}_{11} + \mathcal{D}^2_{10} + \mathcal{D}^2_{01} + \mathcal{D}^{11}_{00}) = 10 \cdot \partial_4(G_6).$$

It remains to show that

$$3 \cdot \mathcal{D}_{11} + 8 \cdot \mathcal{D}^2_{10} + 7 \cdot \mathcal{D}^2_{01} \geq 4 \cdot \mathcal{D}^3_{10} + 3 \cdot \mathcal{D}^3_{01} + 3 \cdot \mathcal{D}^{22}_{00} + \mathcal{D}^{12}_{00}.$$

Since $\min(deg(a), deg(c)) \geq 5$, we have $3 \cdot \mathcal{D}^2_{10} \geq \mathcal{D}^3_{10}$ and $3 \cdot \mathcal{D}^2_{01} \geq \mathcal{D}^3_{01}$ by Lemma 1. Moreover, $\min(\mathcal{D}^2_{01}, \mathcal{D}^2_{10}) \geq \mathcal{D}^{22}_{00}$. If $deg(a) = 5$, then $\mathcal{D}^{12}_{00} = 0$ and we are done. If $deg(a) \geq 6$, then $2 \cdot \mathcal{D}^2_{01} \geq \mathcal{D}^3_{01}$ and we are done.

Subcase 1b. $deg(x'_4) = 1$ and $\min(deg(a), deg(c)) = 4$. Consider the graph $G_4 = G \setminus \{a_1, a_2, a_3, b_1\}$ and the $(G_L \cap V(G_4), G_R, c_2d_1)$-partition of G_4. We have

$$i(G) = 23 \cdot \mathcal{I}_{00} + 9 \cdot \mathcal{I}_{01} + 14 \cdot \mathcal{I}_{10} \geq \frac{9}{2} \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = \frac{9}{2} \cdot i(G_4).$$

We now prove that

$$\partial_4(G) = 7 \cdot \mathcal{D}_{11} + (4 \cdot \mathcal{D}^3_{10} + 2 \cdot \mathcal{D}^2_{10}) + (3 \cdot \mathcal{D}^3_{01} + 3 \cdot \mathcal{D}^2_{01}) + (3 \cdot \mathcal{D}^{22}_{00} + \mathcal{D}^{11}_{00}) \leq$$

$$\frac{9}{2} \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}) = \frac{9}{2} \cdot \partial_4(G_4).$$

It suffices to show that

$$2 \cdot \mathcal{D}_{11} + \frac{1}{2} \cdot \mathcal{D}^3_{10} + \frac{3}{2} \cdot \mathcal{D}^3_{01} \geq 2 \cdot \mathcal{D}^2_{10} + 3 \cdot \mathcal{D}^2_{01}.$$

Since $\min(deg(a), deg(c)) = 4$, we have $\min(\mathcal{D}^2_{10}, \mathcal{D}^2_{01}) = 0$, therefore the inequality holds.

Subcase 2. $deg(x'_4) = 3$. The face f'_4 is adjacent to some faces f'_3 and f''_3. By the previous lemmas, both f'_3 and f''_3 are end faces. Therefore,

$$i(G) = 55 \cdot \mathcal{I}_{00} + 18 \cdot \mathcal{I}_{01} + 35 \cdot \mathcal{I}_{10} \geq 36 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 36 \cdot i(G_R);$$

$$\partial_4(G) \leq 2 \cdot \partial_4(G, c_2^+, d_2^+) + 2 \cdot \partial_4(G, c_2^+, d_2^-) = 2 \cdot (11 \cdot \mathcal{D}_{11} + 4 \cdot \mathcal{D}^4_{10} + 2 \cdot \mathcal{D}^3_{10})$$
$$\leq 34 \cdot \mathcal{D}_{11} < 36 \cdot \partial_4(G_R).$$

In the remaining subcases we consider the induced subgraph G' of G with the vertex set $(V(G) \setminus V(G_L)) \cup \{b_2, c_2, d_2\}$ and the (G'_L, G_R, c_1d_1)-partition of G, where G'_L is an induced subgraph of G_L with the vertex set $\{b_2, c_1, c_2, d_0, d_1\}$.

Subcase 3. $deg(x'_4) = deg(x'_3) = 2$. The face f'_3 is adjacent to some face f'_2. By Lemma 3 f'_2 is adjacent to end faces f'_1 and f''_1. Moreover, by the previous lemmas, the faces f'_3 and f'_4 does not contain any vertices of degree 3.

Subcase 3a. $deg(c_1) \geq 6$. If $deg(c_1) \geq 7$, then $deg(d_0) = 3$, a contradiction. Suppose that $deg(c_1) = 6$.

$$i(G) = 106 \cdot \mathcal{I}_{00} + 63 \cdot \mathcal{I}_{01} + 63 \cdot \mathcal{I}_{10} > 20 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 20 \cdot i(G');$$

$$\partial_4(G) \leq 25 \cdot \mathcal{D}_{11} + (12 \cdot \mathcal{D}^3_{10} + 10 \cdot \mathcal{D}^2_{10}) + (12 \cdot \mathcal{D}^3_{01} + 10 \cdot \mathcal{D}^2_{01})$$
$$+9 \cdot \mathcal{D}^{22}_{00} + 3 \cdot \mathcal{D}^{21}_{00} + 3 \cdot \mathcal{D}^{12}_{00} + 4 \cdot \mathcal{D}^{11}_{00}$$
$$\leq 25 \cdot \mathcal{D}_{11} + 22 \cdot \mathcal{D}^3_{10} + 22 \cdot \mathcal{D}^3_{01} + 19 \cdot \mathcal{D}^{22}_{00}$$
$$< 20 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}) = 20 \cdot \partial_4(G').$$

Subcase 3b. $deg(c_1) = 5$. If $deg_{G_L}(d_1) \geq 5$, then $deg(d_0) = 3$ and we use Lemma 6. Suppose that $deg_{G_L}(d_1) = 4$ and $deg_{G_L}(d_0) = 5$. We have

$$i(G) = 116 \cdot \mathcal{I}_{00} + 45 \cdot \mathcal{I}_{01} + 63 \cdot \mathcal{I}_{10} > 23 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 23 \cdot i(G');$$

$$\partial_4(G) = 28 \cdot \mathcal{D}_{11} + (8 \cdot \mathcal{D}^4_{10} + 14 \cdot \mathcal{D}^3_{10} + 2 \cdot \mathcal{D}^2_{10}) + (12 \cdot \mathcal{D}^3_{01} + 10 \cdot \mathcal{D}^2_{01})$$
$$+(6 \cdot D^{23}_{00} + 3 \cdot D^{22}_{00} + 2 \cdot D^{13}_{00} + 1 \cdot D^{12}_{00}) \leq$$
$$\leq 28 \cdot \mathcal{D}_{11} + 8 \cdot \mathcal{D}_{11} + 16 \cdot \mathcal{D}^3_{10} + 22 \cdot \mathcal{D}^3_{01} + 8 \cdot \mathcal{D}_{11} + 4 \cdot \mathcal{D}^{22}_{00}$$

$$\leq 23 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^{3} + \mathcal{D}_{01}^{3} + \mathcal{D}_{00}^{22}) \leq 23 \cdot \partial_4(G').$$

Subcase 4. $deg(x_4') = 2$, $deg(x_3') = 3$. This is possible only if f_3' is adjacent to end faces, otherwise we use Lemma 5.

Subcase 4a. $deg(c_1) = 7$. The face f_3' is adjacent to two end faces by Lemmas 3 and 6. We have

$$i(G) = 73 \cdot \mathcal{I}_{00} + 45 \cdot \mathcal{I}_{01} + 49 \cdot \mathcal{I}_{10} > 14 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 14 \cdot i(G');$$

$$\partial_4(G) \leq 15 \cdot \mathcal{D}_{11} + 15 \cdot \mathcal{D}_{10}^{3} + 15 \cdot \mathcal{D}_{01}^{3} + 15 \cdot \mathcal{D}_{00}^{22}$$
$$< 14 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^{3} + \mathcal{D}_{01}^{3} + \mathcal{D}_{00}^{22}) \leq 14 \cdot \partial_4(G').$$

Subcase 4b. $deg(c_1) = 5$, $deg(x_4') = 2$, $deg(x_3') = 2$. The face f_3' is adjacent to two end faces by Lemmas 3 and 6. We have

$$i(G) = 88 \cdot \mathcal{I}_{00} + 18 \cdot \mathcal{I}_{01} + 49 \cdot \mathcal{I}_{10} > 15 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 15 \cdot i(G');$$

$$\partial_4(G) \leq 18 \cdot \mathcal{D}_{11} + 6 \cdot \mathcal{D}_{10}^{4} + (9 \cdot \mathcal{D}_{01}^{3} + 9 \cdot \mathcal{D}_{01}^{2}) + (3 \cdot \mathcal{D}_{00}^{23} + \mathcal{D}_{00}^{13})$$
$$< 15 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^{3} + \mathcal{D}_{01}^{3} + \mathcal{D}_{00}^{22}) \leq 14 \cdot \partial_4(G').$$

Case 2. The face f_4' corresponds to the triangle $c_2d_1d_2$. We consider the (G_L, G_R, c_1d_1)-partition of G.

Subcase 1. $deg(x_4') = 1$. Consider the graph $G_4 = G \setminus \{a_1, a_2, a_3, b_1\}$ and the $(G_L \cap V(G_4), G_R, c_2d_1)$-partition of G_4. We have

$$i(G) = 25 \cdot \mathcal{I}_{00} + 9 \cdot \mathcal{I}_{01} + 10 \cdot \mathcal{I}_{10} \geq \frac{34}{7} \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = \frac{34}{7} \cdot i(G_4).$$

Moreover,

$$\partial_4(G) = 7 \cdot \mathcal{D}_{11} + 4 \cdot \mathcal{D}_{10}^{3} + (2 \cdot \mathcal{D}_{01}^{4} + \mathcal{D}_{01}^{3}) + (2 \cdot \mathcal{D}_{00}^{32} + \mathcal{D}_{00}^{22}) \leq$$
$$\frac{34}{7} \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^{3} + \mathcal{D}_{01}^{3} + \mathcal{D}_{00}^{22}) = \frac{34}{7} \cdot \partial_4(G_4).$$

It suffices to show that

$$\frac{19}{7} \cdot \mathcal{D}_{11} + \frac{6}{7} \cdot \mathcal{D}_{10}^{3} + \frac{27}{7} \cdot \mathcal{D}_{01}^{3} \geq 2 \cdot \mathcal{D}_{01}^{4} + 2 \cdot \mathcal{D}_{00}^{32}.$$

By Lemma 1, we have $\mathcal{D}_{11} > \mathcal{D}_{01}^{4}$ and $\mathcal{D}_{01}^{3} > \mathcal{D}_{00}^{32}$, thus the inequality holds.

Subcase 2. $deg(x_4') = 3$. We assume that f_4' is adjacent to end faces f_3' and f_3'' (it was shown in the previous case that the other configurations are not possible). Therefore,

$$i(G) = 59 \cdot \mathcal{I}_{00} + 18 \cdot \mathcal{I}_{01} + 25 \cdot \mathcal{I}_{10} \geq 34 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 34 \cdot i(G_R);$$

$$\partial_4(G) = 12 \cdot \mathcal{D}_{11} + (4 \cdot \mathcal{D}_{10}^{4} + 3 \cdot \mathcal{D}_{10}^{3}) + (4 \cdot \mathcal{D}_{01}^{4} + 4 \cdot \mathcal{D}_{01}^{4}) + 2 \cdot \mathcal{D}_{00}^{33} + \mathcal{D}_{00}^{23}$$
$$\leq 30 \cdot \mathcal{D}_{11} < 34 \cdot \partial_4(G_R).$$

In the remaining subcases we consider the induced subgraph G'' of G with the vertex set $(V(G) \setminus V(G_L)) \cup \{b_2, c_2, d_2\}$ and the (G_L', G_R, c_1d_1)-partition of G, where G_L' is an induced subgraph of G_L with the vertex set $\{b_2, c_2, d_2, c_1, d_1\}$.

Subcase 3. $deg(x_4') = deg(x_3') = 2$. As in the previous case, f_3' is adjacent to some face f_2' which is adjacent to end faces f_1' and f_1''. Again, we assume that the faces f_3' and f_4' does not contain vertices of degree 3.

Subcase 3a. $deg(c_2) \geq 5$. If $deg(c_2) = 7$, then $deg(d_2) = 3$, a contradiction. Thus we assume that $deg(c_2) = 5$ and d_2 belongs to f_2' and f_3'. Therefore,

$$i(G) = 116 \cdot \mathcal{I}_{00} + 63 \cdot \mathcal{I}_{01} + 45 \cdot \mathcal{I}_{10} \geq 23 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 20 \cdot i(G'');$$

$$\partial_4(G) \leq 24 \cdot \mathcal{D}_{11} + 12 \cdot (\mathcal{D}_{10}^4 + \mathcal{D}_{10}^3) + 8 \cdot (\mathcal{D}_{01}^4 + \mathcal{D}_{01}^3) + 12 \cdot \mathcal{D}_{00}^{32}$$
$$< 20 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^3 + \mathcal{D}_{01}^3 + \mathcal{D}_{00}^{22}) = 20 \cdot \partial_4(G'').$$

Subcase 3b. $deg(c_2) = 4$. If $deg(d_2) = 3$, the we apply Lemma. We assume that $deg(d_2) = 5$. Therefore,

$$i(G) = 130 \cdot \mathcal{I}_{00} + 45 \cdot \mathcal{I}_{01} + 45 \cdot \mathcal{I}_{10} > 24 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 24 \cdot i(G'');$$

$$\partial_4(G) = 25 \cdot \mathcal{D}_{11} + (8 \cdot \mathcal{D}_{10}^4 + 4 \cdot \mathcal{D}_{10}^3) + (8 \cdot \mathcal{D}_{01}^4 + 4 \cdot \mathcal{D}_{01}^3) + (4 \cdot \mathcal{D}_{00}^{22} + 2 \cdot \mathcal{D}_{00}^{32} + 2 \cdot \mathcal{D}_{00}^{23} + \mathcal{D}_{00}^{22});$$
$$\partial_4(G) \leq 24 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^3 + \mathcal{D}_{01}^3 + \mathcal{D}_{00}^{22}) = 24 \cdot \partial_4(G'').$$

Subcase 4. $deg(x_4') = 2$, $deg(x_3') = 3$. As in the previous case, it is possible only if f_3' is adjacent to two end faces.

Subcase 4a. $deg(c_2) = 6$. We have

$$i(G) = 77 \cdot \mathcal{I}_{00} + 45 \cdot \mathcal{I}_{01} + 35 \cdot \mathcal{I}_{10} > 15 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 15 \cdot i(G'');$$

$$\partial_4(G) \leq 16 \cdot \mathcal{D}_{11} + (8 \cdot \mathcal{D}_{10}^3 + 3 \cdot \mathcal{D}_{10}^2) + (6 \cdot \mathcal{D}_{01}^4 + 6 \cdot \mathcal{D}_{01}^3 + \mathcal{D}_{01}^2) + (4 \cdot \mathcal{D}_{00}^{32} + 2 \cdot \mathcal{D}_{00}^{22} + 2 \cdot \mathcal{D}_{00}^{21} + \mathcal{D}_{00}^{11})$$
$$\leq 16 \cdot \mathcal{D}_{11} + 11 \cdot \mathcal{D}_{10}^3 + 13 \cdot \mathcal{D}_{01}^4 + 9 \cdot \mathcal{D}_{00}^{32} \leq 15 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^3 + \mathcal{D}_{01}^3 + \mathcal{D}_{00}^{22}) \leq 15 \cdot \partial_4(G'').$$

Subcase 4b. $deg(c_2) = 4$. We have

$$i(G) = 98 \cdot \mathcal{I}_{00} + 18 \cdot \mathcal{I}_{01} + 35 \cdot \mathcal{I}_{10} > 16 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 16 \cdot i(G'');$$

$$\partial_4(G) \leq 18 \cdot \mathcal{D}_{11} + 4 \cdot \mathcal{D}_{10}^4 + (6 \cdot \mathcal{D}_{01}^4 + 3 \cdot \mathcal{D}_{01}^3) + 2 \cdot (\mathcal{D}_{00}^{34} + \mathcal{D}_{00}^{24})$$
$$\leq 18 \cdot \mathcal{D}_{11} + 4 \cdot \mathcal{D}_{10}^4 + 9 \cdot \mathcal{D}_{01}^4 + 3 \cdot \mathcal{D}_{00}^{34}$$
$$< 16 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^3 + \mathcal{D}_{01}^3 + \mathcal{D}_{00}^{22}) \leq 16 \cdot \partial_4(G'').$$

□

Lemma 8. *If $deg(x_3) = deg(x_4) = deg(x_5) = deg(x_6) = 2$, then G is not critical.*

Proof. We denote the faces f_2, f_3, f_4 by $a_2b_1b_2, b_1b_2c_1, b_2c_1c_2$ respectively. By Lemma 6, we have $deg(c_1) \geq 4$. There are three possible cases.

Case 1. $deg(c_1) \geq 5$. In this case $deg(c_2) = 3$, the face f_5 contains c_1c_2 and c_1 belongs to f_6. Let $G_L = G[f_4; f_5]$. Consider the (G_L, G_R, c_1c_2)-partition of G.

$$i(G) = 9 \cdot \mathcal{I}_{00} + 7 \cdot \mathcal{I}_{01} + 5 \cdot \mathcal{I}_{10} \geq 7 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 7 \cdot i(G_R);$$

$$\partial_4(G) = 4 \cdot \mathcal{D}_{11} + (2 \cdot \mathcal{D}_{01}^3 + \mathcal{D}_{01}^2) \leq 7 \cdot \mathcal{D}_{11} \leq 7 \cdot \partial_4(G_R).$$

In the remaining cases we denote the faces f_5 and f_6 by $c_1c_2d_1$ and $c_2d_1d_2$ respectively.

Case 2. $deg(c_2) \geq 5$. In this case $deg(d_1) = 3$. Let $G_L = G[f_5; f_6]$. Consider the (G_L, G_R, c_2d_1)-partition of G.

$$i(G) = 14 \cdot \mathcal{I}_{00} + 9 \cdot \mathcal{I}_{01} + 7 \cdot \mathcal{I}_{10} > 10 \cdot (\mathcal{I}_{00} + \mathcal{I}_{01} + \mathcal{I}_{10}) = 10 \cdot i(G_R);$$

$$\partial_4(G) = \partial_4(G, c_2^+, d_1^+) + \partial_4(G, c_2^-, d_1^+) = 6 \cdot \mathcal{D}_{11} + 3 \cdot \mathcal{D}_{01}^3 + \mathcal{D}_{01}^2 \le \partial_4(G_R).$$

Case 3. $deg(c_1) = deg(c_2) = 4$. Let $G_L = G[f_4; f_5]$. Consider the (G_L, G_R, c_1c_2)-partition of G and the (G'_L, G_R, c_1c_2)-partition of the graph $G_3 = G \setminus \{a_1, a_2, a_3\}$, where $G'_L = G_L \setminus \{a_1, a_2, a_3\}$.

First, we show that $\mathcal{I}_{00} \le \mathcal{I}_{01} + \mathcal{I}_{10}$. Denote by G'_R the induced subgraph of G_R with the vertex set $V(G_R) \setminus \{c_1, c_2\}$. Clearly,

$$\mathcal{I}_{00} = i(G'_R),\ \mathcal{I}_{10} = i(G'_R, d_1^-),\ \mathcal{I}_{01} = i(G'_R, d_1^-, d_2^-).$$

Therefore, $\mathcal{I}_{10} > \mathcal{I}_{01}$ and $\mathcal{I}_{00} - \mathcal{I}_{10} = i(G'_R, d_1^+) \le \mathcal{I}_{01}$. We have

$$i(G) = 9 \cdot \mathcal{I}_{00} + 7 \cdot \mathcal{I}_{01} + 5 \cdot \mathcal{I}_{10} \ge \frac{10}{3} \cdot (3 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 1 \cdot \mathcal{I}_{10}) = \frac{10}{3} \cdot i(G_1).$$

Moreover,

$$\partial_4(G) = 4 \cdot \mathcal{D}_{11} + 3 \cdot \mathcal{D}_{10}^2 + 2 \cdot D_{01}^3,\ \partial_4(G') = \mathcal{D}_{11} + \mathcal{D}_{10}^2 + D_{01}^3.$$

It remains to show that $\partial_4(G) \le \frac{10}{3} \cdot \partial_4(G')$ or $\mathcal{D}_{11} \le 2 \cdot \mathcal{D}_{01}^3$. Indeed, $\mathcal{D}_{11} - \mathcal{D}_{01}^3$ equals to the number of 4-DS of G_R which contain c_1 and c_2 and does not contain d_1 and $\mathcal{D}_{01}^3$ equals to the number of 4-DS which contain c_1, c_2 and d_1. For every 4-DS D of G_R such that $d_1 \notin D$, the set $D \cup \{d_1\}$ is also a 4-DS of G_R, thus the inequality holds. □

Lemma 9. *If $deg(x_3) = deg(x_4) = deg(x_5) = 2$ and $deg(x_6) = 3$, then G is not critical.*

Proof. Denote the faces f_5 and f_6 by $c_1c_2d_1$ and $c_2d_1d_2$ respectively. Let f'_5 be the face adjacent to f_6 other than f_5 and f_7. By Lemma 8, $deg(c_1) = 4$ and $deg(c_2) \ge 5$. Let $G_L = G[f_4; f_5]$. We assume that G_L contains at most one face of degree 3 except f_2, which is adjacent to two end faces. By previous lemmas, G_L contains no vertices of degree 3. There are two possible cases depending on the location of f'_5 in G.

Case 1. f'_5 contains the edge c_2d_2. Denote by c_3 the third vertex of f'_5. In each of the following subcases we consider the partition (G_L, G_R, d_1d_2) of the graph G and the partition (G'_L, G_R, d_1d_2) of the graph G', where G' is a spanning subgraph of G with the vertex set $(V(G) \setminus V(G_L)) \cup \{c_1, c_2, c_3, d_1, d_2\}$. Clearly,

$$\partial_4(G') = 2 \cdot \mathcal{D}_{11} + \mathcal{D}_{10}^3 + \mathcal{D}_{01}^3 + \mathcal{D}_{00}^{22}.$$

It is not hard to check using Lemma 1, that in all subcases below we have $i(G) > \partial_4(G)$, therefore G is not critical.

Subcase 1. $deg(x'_5) = 1$.

$$i(G) = 35 \cdot \mathcal{I}_{00} + 14 \cdot \mathcal{I}_{01} + 18 \cdot \mathcal{I}_{10} > 7 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 7 \cdot i(G');$$

$$\partial_4(G) = 10 \cdot \mathcal{D}_{11} + (6 \cdot \mathcal{D}_{10}^3 + 3 \cdot \mathcal{D}_{10}^2) + (4 \cdot \mathcal{D}_{01}^3 + 4 \cdot \mathcal{D}_{01}^2) + 4 \cdot \mathcal{D}_{00}^{22}.$$

Subcase 2. $deg(x'_5) = 3$ and f'_5 is adjacent to some faces f'_4 and f''_4. By the previous lemmas, both f'_4 and f''_4 are end faces. Therefore,

$$i(G) = 70 \cdot \mathcal{I}_{00} + 28 \cdot \mathcal{I}_{01} + 45 \cdot \mathcal{I}_{10} > 14 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 7 \cdot i(G');$$

$$\partial_4(G) = 22 \cdot \mathcal{D}_{11} + (6 \cdot \mathcal{D}_{10}^4 + 4 \cdot \mathcal{D}_{10}^3) + (8 \cdot \mathcal{D}_{01}^3 + 4 \cdot \mathcal{D}_{01}^2) + (4 \cdot \mathcal{D}_{00}^{23} + 4 \cdot \mathcal{D}_{00}^{12}).$$

Subcase 3. $deg(x'_5) = 2$ and f'_4 is adjacent to end faces f'_3 and f''_3. Two configurations are possible:

Subcase 3a. $deg(c_2) = 7$ and c_2 belong to f'_4.

$$i(G) = 98 \cdot \mathcal{I}_{00} + 70 \cdot \mathcal{I}_{01} + 63 \cdot \mathcal{I}_{10} > 19 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 19 \cdot i(G');$$

$$\partial_4(G) \leq 22 \cdot (\mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}) \leq 19 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}).$$

Subcase 3b. $deg(c_2) = 5$ and c_2 does not belong to f'_4.

$$i(G) = 98 \cdot \mathcal{I}_{00} + 28 \cdot \mathcal{I}_{01} + 63 \cdot \mathcal{I}_{10} > 18 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 7 \cdot i(G');$$

$$\partial_4(G) = 26 \cdot \mathcal{D}_{11} + 10 \cdot \mathcal{D}^4_{10} + (12 \cdot \mathcal{D}^3_{01} + 8 \cdot \mathcal{D}^2_{01}) + 4 \cdot \mathcal{D}^{24}_{00};$$

Subcase 4. $deg(x'_5) = deg(x'_4) = 2$ and f'_3 is adjacent to end faces f'_2 and f''_2. Since G_L has no vertices of degree 3 in G, only two configurations are possible:

Subcase 4a. $deg(c_2) = 6$.

$$i(G) = 161 \cdot \mathcal{I}_{00} + 98 \cdot \mathcal{I}_{01} + 81 \cdot \mathcal{I}_{10} > 32 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 32 \cdot i(G');$$

$$\partial_4(G) \leq 36 \cdot (\mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}) \leq 36 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}).$$

Subcase 4b. $deg(c_2) = 5$.

$$i(G) = 175 \cdot \mathcal{I}_{00} + 70 \cdot \mathcal{I}_{01} + 81 \cdot \mathcal{I}_{10} > 35 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 35 \cdot i(G');$$

$$\partial_4(G) = 39 \cdot \mathcal{D}_{11} + (12 \cdot \mathcal{D}^4_{10} + 14 \cdot \mathcal{D}^3_{10}) + (16 \cdot \mathcal{D}^3_{01} + 16 \cdot \mathcal{D}^2_{01}) + 8 \cdot \mathcal{D}^{33}_{00} + 4 \cdot \mathcal{D}^{23}_{00};$$

$$\partial_4(G') = 2 \cdot \mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}.$$

Subcase 5. $deg(x'_5) = deg(x'_4) = deg(x'_3) = 2$ and f'_2 is adjacent to end faces f'_1 and f''_1. Again, since G_L has no vertices of degree 3 in G, only two configurations are possible:

Subcase 5a. $deg(c_2) = 6$.

$$i(G) = 245 \cdot \mathcal{I}_{00} + 126 \cdot \mathcal{I}_{01} + 126 \cdot \mathcal{I}_{10} > 49 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 49 \cdot i(G');$$

$$\partial_4(G) \leq 52 \cdot (\mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}) \leq 52 \cdot (2 \cdot \mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}).$$

Subcase 5b. $deg(c_2) = 5$.

$$i(G) = 259 \cdot \mathcal{I}_{00} + 98 \cdot \mathcal{I}_{01} + 126 \cdot \mathcal{I}_{10} > 51 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 52 \cdot i(G');$$

$$\partial_4(G) = 52 \cdot \mathcal{D}_{11} + (18 \cdot \mathcal{D}^4_{10} + 18 \cdot \mathcal{D}^3_{10} + 3 \cdot \mathcal{D}^2_{10}) + (24 \cdot \mathcal{D}^3_{01} + 16 \cdot \mathcal{D}^2_{01}) + 12 \cdot \mathcal{D}^{23}_{00} + 4 \cdot \mathcal{D}^{22}_{00}.$$

Case 2. f'_5 contains the edge d_1d_2. Denote by d_3 the third vertex of f'_5. In each of the following subcases we consider the partition (G_L, G_R, d_1d_2) of the graph G and the partition (G'_L, G_R, d_1d_2) of the graph G', where G' is a spanning subgraph of G with the vertex set $(V(G) \setminus V(G_L)) \cup \{c_1, c_2, d_1, d_2, d_3\}$. Clearly, we have

$$\partial_4(G') = 2 \cdot \mathcal{D}_{11} + \mathcal{D}^3_{10} + \mathcal{D}^3_{01} + \mathcal{D}^{22}_{00}.$$

It is not hard to check using Lemma 1, that in all subcases below we have $i(G) > \partial_4(G)$.

Subcase 1. $deg(x'_5) = 1$.

$$i(G) = 37 \cdot \mathcal{I}_{00} + 14 \cdot \mathcal{I}_{01} + 14 \cdot \mathcal{I}_{10} > 7 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 7 \cdot i(G');$$

$$\partial_4(G) = 10 \cdot \mathcal{D}_{11} + 6 \cdot \mathcal{D}^3_{10} + (3 \cdot \mathcal{D}^4_{01} + \mathcal{D}^3_{01}) + (3 \cdot \mathcal{D}^{32}_{00} + \mathcal{D}^{22}_{00}).$$

Subcase 2. $deg(x'_5) = 3$, f'_5 is adjacent to end faces f'_4 and f''_4.

$$i(G) = 70 \cdot \mathcal{I}_{00} + 28 \cdot \mathcal{I}_{01} + 45 \cdot \mathcal{I}_{10} > 14 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 7 \cdot i(G');$$

$$\partial_4(G) = 16 \cdot \mathcal{D}_{11} + (6 \cdot \mathcal{D}_{10}^4 + 4 \cdot \mathcal{D}_{10}^3) + (6 \cdot \mathcal{D}_{01}^4 + 5 \cdot \mathcal{D}_{01}^3 + \mathcal{D}_{01}^2) + (3 \cdot \mathcal{D}_{00}^{23} + \mathcal{D}_{00}^{12}).$$

Subcase 3. $deg(x'_5) = 2$ and f'_4 is adjacent to end faces f'_3 and f''_3. Two configurations are possible:

Subcase 3a. $deg(d_1) = 6$, d_1 belongs to f'_4.

$$i(G) = 116 \cdot \mathcal{I}_{00} + 70 \cdot \mathcal{I}_{01} + 49 \cdot \mathcal{I}_{10} > 23 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 23 \cdot i(G');$$

$$\partial_4(G) = 22 \cdot \mathcal{D}_{11} + 12 \cdot \mathcal{D}_{10}^3 + 4 \cdot \mathcal{D}_{10}^3 + (9 \cdot \mathcal{D}_{01}^4 + 6 \cdot \mathcal{D}_{01}^3 + \mathcal{D}_{01}^2) + 4 \cdot (6 \cdot \mathcal{D}_{00}^{32} + 2 \cdot \mathcal{D}_{00}^{22}).$$

Subcase 3b. $deg(d_1) = 4$, d_1 does not belong to f'_4.

$$i(G) = 98 \cdot \mathcal{I}_{00} + 28 \cdot \mathcal{I}_{01} + 49 \cdot \mathcal{I}_{10} > 18 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 7 \cdot i(G');$$

$$\partial_4(G) = 26 \cdot \mathcal{D}_{11} + 6 \cdot \mathcal{D}_{10}^4 + (3 \cdot \mathcal{D}_{01}^3 + 2 \cdot \mathcal{D}_{01}^2) + 3 \cdot \mathcal{D}_{00}^{34} + 2 \cdot \mathcal{D}_{00}^{24}.$$

Subcase 4. $deg(x'_5) = deg(x'_4) = 2$ and f'_3 is adjacent to two end faces f'_2 and f''_2.

Subcase 4a. $deg(d_1) = 5$.

$$i(G) = 171 \cdot \mathcal{I}_{00} + 98 \cdot \mathcal{I}_{01} + 63 \cdot \mathcal{I}_{10} > 33 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 25 \cdot i(G');$$

$$\partial_4(G) = 36 \cdot \mathcal{D}_{11} + (18 \cdot \mathcal{D}_{10}^3 + 14 \cdot \mathcal{D}_{10}^2) + (12 \cdot \mathcal{D}_{01}^4 + 10 \cdot \mathcal{D}_{01}^3 + 2 \cdot \mathcal{D}_{01}^2) + 9 \cdot \mathcal{D}_{00}^{32} + 3 \cdot \mathcal{D}_{00}^{22}.$$

Subcase 4b. $deg(d_1) = 4$.

$$i(G) = 189 \cdot \mathcal{I}_{00} + 70 \cdot \mathcal{I}_{01} + 63 \cdot \mathcal{I}_{10} > 35 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 35 \cdot i(G');$$

$$\partial_4(G) = 36 \cdot \mathcal{D}_{11} + (12 \cdot \mathcal{D}_{10}^4 + 6 \cdot \mathcal{D}_{10}^3) + (12 \cdot \mathcal{D}_{01}^4 + 4 \cdot \mathcal{D}_{01}^3) + 12 \cdot \mathcal{D}_{00}^{33}.$$

Subcase 5. $deg(x'_5) = deg(x'_4) = deg(x'_3) = 2$ and x'_2 is adjacent to two end faces x'_1 and x''_1.

Subcase 5a. $deg(d_1) = 5$.

$$i(G) = 259 \cdot \mathcal{I}_{00} + 126 \cdot \mathcal{I}_{01} + 98 \cdot \mathcal{I}_{10} > 51 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 51 \cdot i(G');$$

$$\partial_4(G) = 52 \cdot \mathcal{D}_{11} + (24 \cdot \mathcal{D}_{10}^3 + 16 \cdot \mathcal{D}_{10}^2) + (18 \cdot \mathcal{D}_{01}^4 + 5 \cdot \mathcal{D}_{01}^3 + 4 \cdot \mathcal{D}_{01}^2) + 12 \cdot \mathcal{D}_{00}^{32} + 4 \cdot \mathcal{D}_{00}^{22}.$$

Subcase 5b. $deg(d_1) = 4$.

$$i(G) = 277 \cdot \mathcal{I}_{00} + 98 \cdot \mathcal{I}_{01} + 98 \cdot \mathcal{I}_{10} > 52 \cdot (5 \cdot \mathcal{I}_{00} + 2 \cdot \mathcal{I}_{01} + 2 \cdot \mathcal{I}_{10}) = 52 \cdot i(G');$$

$$\partial_4(G) = 52 \cdot \mathcal{D}_{11} + (18 \cdot \mathcal{D}_{10}^4 + 6 \cdot \mathcal{D}_{10}^3) + (18 \cdot \mathcal{D}_{01}^4 + 6 \cdot \mathcal{D}_{01}^3) + 16 \cdot \mathcal{D}_{00}^{33}.$$

□

Lemmas 2–9 imply the main result of this paper.

Theorem 1. *For every outerplanar graph G we have $i(G) > \partial_4(G)$.*

4 Concluding remarks

It seems that the following generalization of Theorem 1 is true.

Conjecture 1. *For every graph G with the average vertex degree at most $k \geq 1$ the inequality $i(G) \geq \partial_k(G)$ holds. Moreover, equality occurs if and only if G is k-regular.*

Although we are unable to prove this statement even for $k = 4$, it is easy to obtain a similar result for the class of trees.

Theorem 2. *For every tree T we have $i(T) > \partial_2(T)$.*

Proof. Clearly, for every tree with at most 3 vertices the inequality holds. Let T be a n-vertex tree such that $i(T) \leq \partial_2(T)$ and for every tree T' such that $|V(T')| < |V(T)|$ we have $i(T') > \partial_2(T')$. Consider a diametral path $X = x_1x_2x_3 \dots x_k$ in T. If $k \leq 3$ then $\partial_2(T) \leq 2$ and $i(G) \geq 5$, thus we assume that $k \geq 4$. Let T_2 (resp. T_3) be the maximal by inclusion subtree of T such that $x_2, x_3 \in V(T_2)$ and $deg(x_2) = 1$ (resp. $x_3, x_4 \in V(T_3)$ and $deg(x_3) = 1$). Since all neighbors of x_2, except possibly x_3, belong to every 2-DS of T, we have

$$\partial_2(T) = \partial_2(T, x_2^+) + \partial_2(T, x_2^-) \leq \partial_2(T_2) + \partial_2(T_3).$$

On the other hand,

$$i(T) = i(T, x_1^-) + i(T, x_1^+) \geq i(T_2) + i(T_3).$$

Since $i(T_2) > \partial_2(T_2)$ and $i(T_3) > \partial_2(T_3)$, we have $i(T) > \partial_2(T)$.

□

5 Acknowledgements

This work was performed at the Saint Petersburg Leonhard Euler International Mathematical Institute and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075–15–2022–287).

References

[1] D. Bród, Z. Skupien, Trees with extremal numbers of dominating sets. *Australas. J Comb.*, 35, 273–290, 2006.

[2] D. S. Taletskii, Trees with extremal numbers of k-dominating sets. *Discrete Mathematics*, 345, 112656, 2022.

[3] Z. L. Nagy, On the Number of k-Dominating Independent Sets. *Journal of Graph Theory*, 84(4), 566–580, 2015.

[4] D. Gerbner, B. Keszegh, A. Methuku, B. Patk'os, M. Vizer. An improvement on the maximum number of k-dominating independent sets. *Journal of Graph Theory*, 91(4), 88–97, 2017.

[5] J. W. Moon, L. Moser, On cliques in graphs. *Israel Journal of Mathematics*, 3, 23–28, 1965.

[6] A. F. Alameddine, Bounds on the Fibonacci number of a maximal outerplanar graph, *The Fibonacci Quarterly,* 36.3, 206–210, 1998.