ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2023, Vol. 17, No. 1, pp. 213–224. © Pleiades Publishing, Ltd., 2023.
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On the Number of Minimum Total Dominating Sets in Trees

D. S. Taletskii[1,2*]

[1] *National Research University Higher School of Economics, Nizhny Novgorod Branch, Nizhny Novgorod, 603155 Russia*

[2] *Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, 603022 Russia*

*e-mail: *dmitalmail@gmail.com*

Received June 16, 2022; revised October 4, 2022; accepted October 6, 2022

Abstract—The minimum total dominating set (MTDS) of a graph is a vertex subset D of minimum cardinality such that every vertex of the graph is adjacent to at least one vertex of D. In this paper we obtain a sharp upper bound for the number of MTDSs in the class of n-vertex 2-caterpillars. We also show that for all $n \geq 1$ every n-vertex tree has less than $(\sqrt{2})^n$ MTDSs.

Keywords: *extremal combinatorics, tree, 2-caterpillar, minimum total dominating set*

DOI: 10.1134/S1990478923010234

INTRODUCTION

The *dominating set* of a graph is a vertex subset D such that any vertex not in D is adjacent to at least one vertex in D. The *total dominating set* of a graph is a vertex subset D' such that any vertex of the graph is adjacent to at least one vertex in D'. The dominating set is called *minimum* if it is of the least cardinality. We use the abbreviations DS, MDS, TDS, and MTDS for the terms "dominating set," "minimum dominating set," "total dominating set," and "minimum total dominating set," respectively. The *total dominance number* $\gamma_t(G)$ of a graph G is the cardinality of each of its MTDSs. Let $\vartheta(G)$ denote the number of all MTDSs in the graph G.

In 2006, Bród and Skupień [1] described trees containing the maximum and minimum number of DMs among all n-vertex trees. Later, Krzywkowski and Wagner[2] described trees and connected graphs containing the minimum number of TDSs. The question of whether a tree with dominance number γ can contain more than 2^γ MDSs remained open until 2017, when an example of such a tree was given in [3]. On the other hand, Alvarado et al. [4] showed that every tree with dominance number γ contains at most 2.4606^γ MDSs. For all $k \geq 2$, the paper [5] describes trees that contain the maximum and minimum number of k-DSs (i.e., sets D_k such that each vertex of a tree not in D_k is adjacent to at least k vertices in D_k).

To date, the question of the structure of trees containing the maximum possible number of MDSs and MTDSs remains open. It was shown in [6] in 2019 that each n-vertex tree contains less than $95^{n/13}$ *minimal* (i.e., inclusion-minimal) DSs. In addition, an example of an n-vertex tree containing more than $0.649 \times 95^{n/13}$ minimal DSs is given for any $n \geq 1$. The methods proposed in [6] can also be applied to other classes of graphs, but using them to enumerate sets of fixed cardinality (including MDSs and MTDSs) is not possible in the opinion of the present author.

In 2019, Henning et al. [7] obtained three upper bounds for the number of MTDSs in trees and forests. Namely, for an n-vertex forest F with total dominance number γ_t, they proved the inequality

$$\vartheta(F) \leq \min\left(\left(8\sqrt{e}\right)^{\gamma_t}\left(\frac{n-\gamma_t/2}{\gamma_t/2}\right)^{\gamma_t/2},\ \left(1+\sqrt{2}\right)^{n-\gamma_t},\ 1.4865^n\right).$$

In the present paper, we prove the strict inequality $\vartheta(T) < (\sqrt{2})^n$ for all n-vertex trees. In addition, a sharp upper bound for the number of MTDSs for the class of n-vertex 2-caterpillars is obtained.

1. SOME DEFINITIONS AND NOTATION

As usual, the vertex and edge sets of a simple undirected graph G are denoted by $V(G)$ and $E(G)$, respectively. The *open neighborhood* $N(v)$ of a vertex v is the set consisting of all adjacent vertices, and the *the closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$.

Let $G \setminus V_0$ denote the subgraph of G induced by the vertices of the set $V(G) \setminus V_0$. In the case of $V_0 = \{v\}$, we will use the notation $G \setminus v$ instead of $G \setminus \{v\}$. Let $G - e$ denote the graph obtained by removing the edge $e \in E(G)$ from the graph G.

A tree vertex is called a *preleaf* if it is adjacent to at least one leaf. Let us say that a tree vertex is *preterminal* if all but one of its neighbors are leaves. The *diameter* $\operatorname{diam}(T)$ of a tree T is the maximum possible distance between its vertices. A simple path $P = v_1v_2\dots v_m$ of a tree T is said to be *diametrical* if it consists of $\operatorname{diam}(T) + 1$ pairwise distinct vertices. A tree is called a *k-caterpillar* if the distance from each of its vertices to some simple path, called the *backbone*, is at most k. We assume that the backbone of a k-caterpillar is a diametrical path. The *star graph* S_m is the $(m + 1)$-vertex tree containing a vertex of degree m (here $m \geq 0$).

By $\overline{a, b}$ we denote the set of all integers in the interval $[a; b]$. Let $P = v_1v_2\dots v_m$ be chosen in the tree T. For each $i \in 2, \dots, m$, denote by T_i the inclusion-maximal subtree T that contains v_i and does not contain v_{i-1}. We assume that the subtree T_1 coincides with T. We set $\widehat{T}_i = T_i \setminus T_{i+1}$. By $\mathcal{D}_{T,P}(v_i)$ we denote the distance in the tree $\widehat{T}_i$ from the vertex v_i to the nearest leaf other than v_i. If $\widehat{T}_i$ consists of one vertex, then we set $\mathcal{D}_{T,P}(v_i) = 0$. Note that if the vertex v lies on the backbone of a k-caterpillar, then $\mathcal{D}_{T,P}(v) \leq k$. In the case where the choice of the tree T and the path P is clear from the context, we use the notation $\mathcal{D}(v)$ instead of $\mathcal{D}_{T,P}(v)$.

Recall that $\vartheta(G)$ denotes the number of MTDSs in a graph G. We assume that $\vartheta(K_1) = 0$. The numbers of MTDSs in G containing and not containing a vertex v will be denoted by $\vartheta_+(G, v)$ and $\vartheta_-(G, v)$, respectively. A vertex v of the graph G is said to be *universal* if $\vartheta_+(G, v) = \vartheta(G)$ and *idle* if $\vartheta_-(G, v) = \vartheta(G)$. As usual, $G_1 \cup G_2$ denotes the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. It is easy to see that $\vartheta(G_1 \cup G_2) = \vartheta(G_1)\vartheta(G_2)$ for disjoint graphs G_1 and G_2.

We say that a tree T *separable* if it is possible to remove an edge from it in such a way that the number of MTDSs in the resulting forest remains the same and *inseparable* otherwise. We say that an n-vertex tree (2-caterpillar) is *maximal* if it contains the maximum possible number of MTDSs among all n-vertex trees (n-vertex 2-caterpillars, respectively).

Let a vertex $v \in V(T)$ be chosen in a tree T. Denote by $\widehat{\gamma}_t(T, v)$ the cardinality of the smallest vertex subset $D \subseteq V(T)$ such that every vertex in $V(T)$, except possibly v, is adjacent to at least one vertex in D. It is easily seen that the inequality $\gamma_t(T) - 1 \leq \widehat{\gamma}_t(T, v) \leq \gamma_t(T)$ holds. Denote by $\widehat{\vartheta}(T, v)$ the number of subsets $D \subseteq V(T)$ of cardinality $\widehat{\gamma}_t(T, v)$ such that each vertex $V(T)$, possibly except for the vertex v, is adjacent to at least one vertex in D. Note that if $\widehat{\gamma}_t(T, v) = \gamma_t(T)$, then $\widehat{\vartheta}(T, v) \geq \vartheta(T)$, because in this case each MTDS T has cardinality $\widehat{\gamma}_t(T, v)$. We define the quantities $\widehat{\vartheta}_+(T, v)$ and $\widehat{\vartheta}_-(T, v)$ by analogy with $\vartheta_+(T, v)$ and $\vartheta_-(T, v)$.

Let a set D be a TDS of a tree T, and let $\operatorname{diam}(T) \geq 3$. Denote by $L(T)$ the set of leaves of T whose neighbors are preterminal vertices. Consider the mapping $\varphi\colon L(T) \to V(T)$ taking each leaf $l \in L(T)$ to the only nonleaf vertex at distance 2 from it. Denote by $\varphi(D)$ the set obtained by replacing each leaf $l \in L(T)$ in D by the vertex $\varphi(l)$. Since D contains all preleaves of T, it follows that $\varphi(D)$ is a TDS, while $|\varphi(D)| \leq |D|$. Thus, for any MTDS D the set $\varphi(D)$ is an MTDS as well.

We say that a vertex v of a tree T is *φ-universal* if $v \in \varphi(D)$ for any MTDS $D \subseteq V(T)$. Note that every universal vertex is φ-universal, and every nonleaf vertex adjacent to at least one preterminal vertex is φ-universal.

Figure 1 shows a tree that is a 2-caterpillar and its diametrical path $v_1v_2\dots v_9$ is the backbone of the caterpillar. Note that $\mathcal{D}(v_4) = 0$, $\mathcal{D}(v_2) = 1$, and $\mathcal{D}(v_5) = 2$. In addition, the nonleaf vertices v_3, v_5, v_6, and v_7 (and only they) are adjacent to preterminal vertices, and so they are φ-universal.

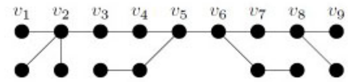

Fig. 1. Example of 2-caterpillar with backbone $v_1v_2\dots v_9$.

2. PRELIMINARIES

2.1. Class of Elementary Forests

We say that a forest F is *elementary* if each of its connected components is a star graph. We say that an elementary n-vertex *forest* is maximal if it contains the maximum possible number of MTDSs among all such forests. It is clear that $\vartheta(S_k) = k$ for all $k \geq 0$.

Lemma 1. *For $n = 4k + r \geq 12$, $r \in \{0, 1, 2, 3\}$, the n-vertex maximal elementary forest is unique, is isomorphic to the forest $F_n = (k - r)S_3 \cup rS_4$, and contains $f(n) = 4^r \cdot 3^{k-r}$ MTDSs.*

Proof. Let a star S_k be the least connected component of a forest F. If F contains a star S_m such that $k + 1 < m$, then the subgraph $S_k \cup S_m$ can be replaced by the subgraph $S_{k+1} \cup S_{m-1}$ and the number of MTDSs of the forest F will increase, which contradicts its maximality. Then for some integers $a \geq 1$ and $b \geq 0$ the equality $F = aS_k \cup bS_{k+1}$ holds. Let us show that for any n-vertex forest F not isomorphic to F_n, there exists a replacement of some of its subgraphs by a forest with the same number of vertices and a larger number of MTDSs.

Case of $k = 0$. Replace the entire forest F with the tree S_{n-1}.

Case $k = 1$. Replace the entire forest F with the forest $(b - 1)S_{k+1} \cup S_{(a+1)(k+1)}$.

Case of $k = 2$. If F contains the forest $2S_2$, then replace it with the tree S_5. Otherwise, F contains the forest $S_2 \cup 2S_3$; replace it with the forest $S_4 \cup S_5$.

Case of $k = 3$. If $b \leq 4$, then the condition of the lemma is satisfied; otherwise, replace the forest $4S_4$ with the forest $5S_3$.

Case of $k = 4$. If $a \leq 3$ and $b = 0$, then the condition of the lemma is satisfied. Otherwise, F contains one of the forests $4S_4$, $2S_4 \cup S_5$, or $2S_5$; replace them with the forests $5S_3$, $4S_3$, or $3S_3$, respectively.

Case of $k = 5$. If F contains the forest $2S_5$, then replace it with the forest $3S_3$. Otherwise, F contains the forest $S_5 \cup S_6$; replace it with the forest $2S_3 \cup S_4$.

Case of $k = 6$. If F contains the forest $2S_6$, then replace it with the forest $S_3 \cup 2S_4$. Otherwise, F contains the tree S_7; replace it with the forest $S_2 \cup S_4$.

Case of $k \geq 7$. Replace the tree S_k with the forest $S_2 \cup S_{k-3}$.

Thus, for any $n \geq 12$ and any n-vertex forest F other than F_n, there exists at least one replacement that increases the number of MTDSs in it. The proof of Lemma 1 is complete. □

Note that for $n \in \{4, 5, 8, 9, 10\}$ the maximal forest is unique and isomorphic to F_n, for $n = 11$ the only maximal forest is $S_4 \cup S_5$, and for $n \in \{1, 2, 3, 6, 7\}$ the tree S_{n-1} is a maximal forest (possibly not the only one). Thus, for all positive integers n, the inequality $\vartheta(F_n) \leq f(n)$ holds, which is strict for $n \in \{1, 2, 3, 6, 7, 11\}$.

2.2. Universal and Idle Vertices

Lemma 2. *If a tree T contains two adjacent nonleaf vertices u and v such that u is idle and v is either idle or universal, then T is separable.*

Proof. If the vertex v is idle, then remove the edge uv and denote the resulting forest by F_1. Obviously, $\gamma_t(F_1) \geq \gamma_t(T)$. On the other hand, each MTDS of the tree T does not contain the vertices u and v, and so it is a TDS in the forest F_1, whence $\gamma_t(F_1) \leq \gamma_t(T)$. Then $\gamma_t(F_1) = \gamma_t(T)$, with each MTDS of the tree T being an MTDS of the forest F and vice versa, whence $\vartheta(F_1) = \vartheta(T)$.

If, however, the vertex v is universal, then we remove all the edges incident to the vertex u except for the edge uv, and denote the resulting forest by F_2. As in the previous case, it is easy to check that the equalities $\gamma_t(F_2) = \gamma_t(T)$ and $\vartheta(F_2) = \vartheta(T)$ hold. The proof of Lemma 2 is complete. □

Lemma 3. *If a tree T contains universal vertices u and v such that $dist(u, v) = 3$, then T is separable.*

Proof. By assumption, there exist vertices u' and v' in T such that there exists a path $uu'v'v$. Let us denote by F the forest obtained by removing the edge $u'v'$ from T and show that $\vartheta(T) = \vartheta(F)$. Obviously, $\gamma_t(F) \geq \gamma_t(T)$. Let us prove that if D' is an MTDS T, then it is an MTDS of F. Since the vertices u and v are in D', it follows that each of the vertices u' and v' in the forest F has a neighbor in D'. Thus, D' is a TDS in the forest F, whence $\gamma_t(F) = \gamma_t(T)$ and $\vartheta(F) \geq \vartheta(T)$. On the other hand, since $\gamma_t(F) = \gamma_t(T)$, it follows that each MTDS of the forest F is an MTDS for the tree T, whence $\vartheta(F) = \vartheta(T)$, as desired. The proof of Lemma 3 is complete. □

Lemma 4. *The following statements are true for any tree T.*

1. *If T contains a vertex v adjacent to a leaf v' and also to a preterminal vertex u, then $\vartheta(T_1) \geq \vartheta(T)$, where T_1 is the tree obtained by removing the leaf v' from T.*
2. *If T contains a vertex v adjacent to at least two preterminal vertices u_1 and u_2, then $\vartheta(T_2) \geq \vartheta(T)$, where T_2 is the tree obtained by removing the vertex u_2 and all leaves adjacent to it from T.*

Proof. Let us prove the first assertion of the lemma. Since the vertex v' is a leaf in T, it follows that $\gamma_t(T_1) \leq \gamma_t(T)$. On the other hand, v is φ-universal in T_1. Then there exists an MTDS $D \ni v$ of the tree T_1. It is obvious that D is also a TDS for T, whence $\gamma_t(T_1) = \gamma_t(T)$, and each MTDS of T is also an MTDS for T_1, whence $\vartheta(T_1) \geq \vartheta(T)$, as desired.

The second assertion can be proved in a similar way. The proof of Lemma 4 is complete. □

Lemma 5. *Let a tree T contain a vertex u that is not a preleaf. If all neighbors of u are adjacent to at least one φ-universal vertex, then u is idle.*

Proof. Assume, on the contrary, that there exists an MTDS D containing u. Consider the set $\varphi(D)$. By assumption, all vertices of the open neighborhood $N(u)$ have at least one neighbor in $\varphi(D)$ distinct from u, and at least one of these vertices is itself included in $\varphi(D)$. Then the set $\varphi(D) \setminus \{u\}$ is also a TDS, which contradicts the minimality of D. The proof of Lemma 5 is complete. □

Lemma 6. *If a tree T contains an idle nonleaf vertex u not adjacent to universal vertices and $\gamma_t(T \setminus u) = \gamma_t(T)$, then $\vartheta(T \setminus u) > \vartheta(T)$.*

Proof. Let $u_1, u_2, \ldots, u_k$ be the neighbors of the vertex u in T. Denote by T_i^* the inclusion-maximal subtree of T containing vertex u_i and not containing u. Since the vertex u is idle and $\gamma_t(T \setminus u) = \gamma_t(T)$, for any MTDS D of the tree T the set $D\ capV(T_i^*)$ is an MTDS of the tree T_i^*. Since for any $i \in 1, \ldots, k$ there exists an MTDS D_i of the tree T that does not contain v_i, it follows that the set $D_i \cap V(T_i)$ is an MTDS of the tree T_i and does not contain v_i, whence $\vartheta_-(T_i, v_i) > 0$. Thus, we have the inequality

$$\vartheta(T) = \prod_{i=1}^{k} \vartheta(T_i^*) - \prod_{i=1}^{k} \vartheta_-(T_i^*, v_i) < \prod_{i=1}^{k} \vartheta(T_i^*) = \vartheta(T \setminus u).$$

The proof of Lemma 6 is complete. □

Lemmas 2 and 6 imply the following assertion.

Corollary 1. *For any $n, k \geq 1$, if an n-vertex tree T is a k-caterpillar and contains an idle nonleaf vertex u such that $\gamma_t(T \setminus u) = \gamma_t(T)$, then either T is separable or it is neither a maximal tree nor a maximal k-caterpillar.*

This assertion will apply both to 2-caterpillars and to arbitrary trees.

3. CLASS OF 2-CATERPILLARS

Lemma 7. *For $n \geq 3$, for any n-vertex 2-caterpillar T there exists an $(n+1)$-vertex 2-caterpillar T' such that $\vartheta(T) < \vartheta(T')$.*

Proof. The proof is carried out by induction on the number n of vertices. One can readily verify that the assertion of the lemma is true for $n \in 3, \dots, 6$. Assume that for $n \geq 7$ there exists some maximal n-vertex 2-caterpillar T for which the assertion is false but for any 2-caterpillar T'' with fewer vertices the strict inequality $\vartheta(T'') < \vartheta(T)$ holds.

Assume that $\operatorname{diam}(T) \leq 4$. If $\operatorname{diam}(T) = 2$, then T is isomorphic to S_{n-1}. If $\operatorname{diam}(T) = 3$, then T contains exactly two nonleaf vertices, with $\gamma_t(T) = 2$ and $\vartheta(T) = 1$. However, if $\operatorname{diam}(T) = 4$, then the central vertex T is universal, whence $\vartheta(T) = 1$. Thus, the inequality $\vartheta(T) \leq \vartheta(S_{n-1}) < \vartheta(S_n)$ holds; this is impossible by assumption.

Assume that $\operatorname{diam}(T) \geq 5$. Denote by $v_1 v_2 \dots v_k$ the backbone of T. The following cases are possible.

Case 1. In T there exists at least one nonidle leaf l adjacent to some vertex u. We attach a new leaf l' to u and denote the resulting tree by T'. Then $\vartheta(T') = \vartheta_-(T, l) + 2\vartheta_+(T, l) > \vartheta(T)$; this is a contradiction.

When considering cases 2–4, we will assume that all leaves T are idle and the vertex v_3 is universal (since all neighbors of v_2 except for v_3 are idle leaves).

Case 2. The inequality $\deg(v_3) \geq 3$ holds. By Lemma 4, in the tree T there exists a subtree T' such that $\vartheta(T') \geq \vartheta(T)$; a contradiction.

When considering cases 3 and 4, we assume that $\deg(v_3) = 2$.

Case 3. The vertex v_4 is universal or idle. If v_4 is universal, then for any MTDS D of the tree T the set $D \setminus \{v_2\}$ is an MTDS of the tree T_2, whence $\vartheta(T) \leq \vartheta(T_2)$; a contradiction. If, however, v_4 is idle, then $\deg(v_4) = 2$, because otherwise v_4 would be φ-universal. Then for any MTDS D of the tree T the set $D \setminus \{v_2, v_3\}$ is an MTDS in T_5, whence $\vartheta(T) \leq \vartheta(T_5)$; a contradiction.

Case 4. The vertex v_4 is not universal and not idle. Assume that $\deg(v_4) \geq 3$. Then v_4 is not a preleaf and is adjacent to at least one preterminal vertex all whose other neighbors are idle leaves. However, v_4 is universal, which is impossible; therefore, $\deg(v_4) = 2$. Assume that the vertex v_5 is not idle. Then for any MTDS $D \ni v_5$ the set $(D \setminus \{v_3\}) \cup \{v_1\}$ is also an MTDS; this contradicts the universality of v_3. Thus, v_5 is idle and $\deg(v_5) = 2$. Let us show that $\deg(v_6) = 2$. Let $\deg(v_6) > 2$. Then the vertex v_6 is φ-universal and by Lemma 5 the vertex v_4 is idle, which is impossible. Since v_5 is idle and $\deg(v_6) = 2$, it follows that v_7 is universal.

Note that the inequality $\vartheta_+(T, v_4) \leq \vartheta_+(T, v_6)$ holds, because each MTDS of T contains exactly one of the vertices v_4 or v_6; moreover, if the MTDS D contains v_4, then the set $(D \setminus \{v_4\}) \cup \{v_6\}$ is an MTDS as well. Consider an n-vertex forest $S_3 \cup T_5$ and denote by T' the tree obtained by adding a leaf of the tree S_3 to the vertex v_7 of the tree T_5. Obviously, T' is a separable 2-caterpillar. Then

$$\vartheta(T') = 3\vartheta(T_5) \geq 3\vartheta_+(T, v_6) > \vartheta_+(T, v_4) + \vartheta_+(T, v_6) = \vartheta(T).$$

Thus, T is not a maximal 2-caterpillar; this contradicts the assumption.

The proof of Lemma 7 is complete. $\square$

Theorem 1. *For $n \geq 12$, every maximal n-vertex 2-caterpillar T contains a maximal elementary forest F_n as a spanning subgraph. The equality $\vartheta(T) = f(n)$ holds.*

Proof. It follows from the reasoning in the previous lemma that if $\operatorname{diam}(T) \leq 4$, then $\vartheta(T) \leq \vartheta(S_{n-1}) \leq f(n)$. The equality $\vartheta(T) = f(n)$ is possible only if T is isomorphic to S_3 or S_4. Thus, we will assume that $n \geq 6$ and $\operatorname{diam}(T) \geq 5$.

By $v_1, \ldots, v_k$ we denote the backbone of T and by p, the index of the leftmost vertex of the backbone that is different from v_2 and has degree greater than 2 (if $T = P_n$, then we set $p = k+1$). Denote by q the number of the second vertex on the left of the backbone with this property (if there is no such vertex, then we set $q = k+1$). We assume that if T is different from P_n, then $p \le \lfloor \frac{k+1}{2} \rfloor$ (otherwise, we will rename the vertices of the backbone in reverse order).

Assume that there exists a maximal 2-caterpillar T that either contains more than $f(n)$ MTDSs or contains exactly $f(n)$ MTDSs and does not contain the forest F_n as a spanning subgraph. We can assume that T is inseparable, because there are no 2-caterpillars with fewer vertices that have this property. Denote by a the number of leaves adjacent to v_2 (that is, $a = \deg(v_2) + 1$). Let us consider several cases depending on the value of p.

Case $p = 3$. By Lemmas 4 and 7, the tree T is not a maximal 2-caterpillar; a contradiction.

Case $p = 4$. If $\mathcal{D}(v_4) = 2$, then the vertex v_4 is adjacent to some preleaf u_4 that does not lie on the backbone T. Then $\operatorname{dist}(v_2, u_4) = 3$ and T is separable by Lemma 3; a contradiction. Now let $\mathcal{D}(v_4) = 1$. Let us show that in this case all leaves adjacent to v_4 are idle. Assume that this is not true and there exists some MTDS D containing a leaf v_4' adjacent to v_4. Then the set $\varphi(D) \setminus \{v_4'\}$ is also a TDS and is less than D in cardinality; a contradiction. Since T is maximal, by Lemma 7 the vertex v_4 is adjacent to the only idle leaf v_4'. Consider several cases depending on the value of the quantity q.

Case of $p = 4$ and $q = 5$. If $\mathcal{D}(v_5) = 1$, then v_5 is universal and T is separable, because $\operatorname{dist}(v_2, v_5) = 3$. If $\mathcal{D}(v_5) = 2$, then we remove the vertices v_4 and v_4' from the tree and denote the resulting forest by F. The equality $\gamma_t(F) + 1 = \gamma_t(T)$ holds, and for any MTDS D of the tree T, the set $D \setminus \{v_4\}$ is an MTDS of the forest F. Moreover, each connected component of F is a 2-caterpillar. Then

$$f(n) > f(a+2)f(n-a-4) \ge \vartheta(F) \ge \vartheta(T);$$

this contradicts the assumption about the maximality of T.

Case of $p = 4$ and $q = 6$. If $\mathcal{D}(v_6) = 2$ and v_6 is adjacent to some preleaf u_6, then $\operatorname{dist}(v_4, u_6) = 3$ and T is separable. If $\mathcal{D}(v_6) = 1$, then we act by analogy with the previous case. Let us remove the vertices v_4 and v_4' from the tree and denote the resulting forest by F. Then for any MTDS D of the tree T the set $D \setminus \{v_4\}$ is an MTDS of the forest F, whence $f(n) > \vartheta(F) \ge \vartheta(T)$; a contradiction.

Case of $p = 4$ and $q = 7$. If $\mathcal{D}(v_7) = 1$, then v_7 is universal and T is separable, because $\operatorname{dist}(v_4, v_7) = 3$. Let $\mathcal{D}(v_7) = 2$ and v_7 be adjacent to some preterminal vertex u_7 not lying on the backbone T. The vertices v_4 and u_7 are universal, while vertices v_3 and v_7 are φ-universal. Then by Lemma 5 the vertices v_5 and v_6 are idle and T is separable by Lemma 2; a contradiction.

Case of $p = 4$ and $q = 8$. Since the vertex v_8 is either universal or φ-universal, it follows by Lemma 5 that the vertex v_6 is idle. One can readily verify that $\gamma_t(T) = \gamma_t(T_6) + 3$. Then each MTDS of T containing the vertex v_7 contains v_3 and does not contain v_5, whence $\vartheta(T) = \vartheta_+(T, v_5) + \vartheta_+(T, v_7)$ and $\vartheta_+(T, v_7) \le \vartheta(T_6)$. If the vertex v_5 is idle, then T is separable by Lemma 2; a contradiction. However, if v_5 is not idle, then $\vartheta_+(T, v_5) = (a+1)\vartheta_-(T_7, v_7)$, because if some MTDS of T contains the vertex v_5, then it also contains exactly one vertex from the open neighborhood $N(v_2)$. Thus,

$$\vartheta(T) = \vartheta_+(T, v_7) + \vartheta_+(T, v_5) \le \vartheta(T_6) + (a+1)\vartheta_-(T_7, v_7).$$

Since $f(n-a-5) \ge \vartheta(T_6)$ and $f(n-a-6) \ge \vartheta_-(T_7, v_7) \ge \vartheta(T_7)$, we have

$$f(n) > f(n-a-5) + (a+1)f(n-a-6) \ge \vartheta(T);$$

this contradicts the assumption about the maximality of T.

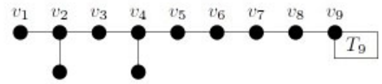

Fig. 2. Structure of the 2-caterpillar T for $p = 4$, $q \geq 9$, and $a = 2$.

CASE OF $p = 4$ AND $q \geq 9$. In this case, T has the structure shown in Fig. 2. Suppose that $\gamma_t(T_9) > \widehat{\gamma}_t(T_9, v_9)$. Then v_8 is universal and v_6 is idle by Lemma 5. If the MTDS D contains the vertex v_5, then it does not contain v_7; otherwise the set $\varphi(D) \setminus \{v_5\}$ would be a TDS as well, which is impossible. If D contains v_7, then it does not contain v_5 and hence contains v_3. Then

$$\begin{aligned}\vartheta(T) &= \vartheta_+(T, v_5) + \vartheta_+(T, v_7) = (a+1)\vartheta_-(T_7, v_7) + \vartheta_+(T_6, v_8)\\ &\leq (a+1)\vartheta(T_7) + \vartheta(T_6) \leq (a+1)f(n-a-6) + f(n-a-5) < f(n).\end{aligned}$$

Assume that $\gamma_t(T_9) = \widehat{\gamma}_t(T_9, v_9)$. We introduce the notation

$$A_1 = \vartheta_+(T_9, v_9), \quad A_2 = \vartheta(T_9), \quad B_1 = \widehat{\vartheta}_+(T_9, v_9), \quad B_2 = \widehat{\vartheta}(T_9, v_9).$$

Note that $A_1 \leq A_2$ and $B_1 \leq B_2$. Moreover, since $\gamma_t(T_9) = \widehat{\gamma}_t(T_9, v_9)$, it follows that $A_2 \leq B_2$. Assume that there exists an MTDS D containing at least three vertices in the set $\{v_5, v_6, v_7, v_8\}$. Then the set

$$\big(\varphi(D) \setminus \{v_5, v_6, v_7, v_8\}\big) \cup \{v_7, v_8\}$$

is an MTDS as well and is less than D in terms of cardinality; a contradiction. Thus, for any MTDS D the intersection $D \cap \{v_5, v_6, v_7, v_8\}$ coincides with one of the sets $\{v_5, v_6\}$, $\{v_6, v_7\}$, $\{v_5, v_8\}$, or $\{v_7, v_8\}$. Then

$$\vartheta(T) = (a+1)A_1 + A_2 + (a+1)B_1 + B_2.$$

Denote by T_6' the tree obtained from T_6 by adding the leaf v_7' to the vertex v_7. Then the forest $F = S_{a+3} \cup T_6'$ satisfies the relation

$$\begin{aligned}f(n) \geq \vartheta(F) &= (a+3)\big(\vartheta_+(T_6', v_6) + \vartheta_+(T_6', v_7') + \vartheta_+(T_6', v_8)\big)\\ &= (a+3)\big(2\vartheta_+(T_6', v_6) + \vartheta_+(T_6', v_8)\big) = (a+3)(2A_2 + B_2) > \vartheta(T);\end{aligned}$$

this contradicts the assumption about the maximality of T.

CASE OF $p = 5$. If $\mathcal{D}(v_5) = 1$, then the vertex v_5 is universal and T is separable, because $\mathrm{dist}(v_2, v_5) = 3$. However, if $\mathcal{D}(v_5) = 2$, then v_5 is adjacent to some preterminal vertex u_5 that does not lie on the backbone of T. By Lemma 5, the vertex v_4 is idle in T, and by Corollary 1, T is either separable or nonmaximal; a contradiction.

CASE OF $p = 6$. The vertex v_6 is φ-universal. Then the vertex v_4 is idle by Lemma 5, and by Corollary 1, T is either separable or nonmaximal; a contradiction.

CASE OF $p = 7$. It is clear that the vertices v_3 and v_7 are φ-universal. Then by Lemma 5 the vertex v_5 is idle. Therefore, v_3 and v_7 are universal. Each MTDS T contains exactly one vertex from the set $\{v_4, v_6\}$ (because if the TDS D contains both vertices, then the set $D \setminus \{v_4\}$ is also a TDS). Moreover, $\vartheta_+(T, v_4) \leq \vartheta_+(T, v_6)$, because for any MTDS D' the set $(D' \setminus \{v_4\}) \cup \{v_6\}$ is an MTDS as well. Then

$$\vartheta(T) = \vartheta_+(T, v_4) + \vartheta_+(T, v_6) \leq 2\vartheta(T_5) \leq 2f(n-a-3) < f(n).$$

CASE $p \geq 8$. We argue by analogy with the case of $p = 4$ and $q \geq 9$. If $\gamma_t(T_8) > \widehat{\gamma}_t(T_8, v_8)$, then v_5 is idle, while v_3 and v_7 are universal. If the MTDS D contains the vertex v_4, then it

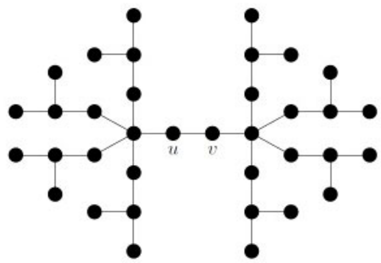

Fig. 3. Caterpillar T^*_{36}.

does not contain v_6; otherwise, the set $\varphi(D)\setminus\{v_4\}$ would also be a TDS, which is impossible. If v_4 is idle, then T is separable by Lemma 2; otherwise,

$$\vartheta(T)=\vartheta_+(T,v_4)+\vartheta_+(T,v_6)=\vartheta_-(T_6,v_6)+\vartheta_+(T_5,v_7)$$
$$\le\vartheta(T_6)+\vartheta(T_5)\le f(n-a-4)+f(n-a-3)<f(n).$$

Assume that $\gamma_t(T_8)=\widehat{\gamma}_t(T_8,v_8)$. We introduce the notation

$$A_1=\vartheta_+(T_8,v_8),\quad A_2=\vartheta(T_8),\quad B_1=\widehat{\vartheta}_+(T_8,v_8),\quad B_2=\widehat{\vartheta}(T_8).$$

By analogy with the previous case, for any MTDS D the intersection $D\cap\{v_5,v_6,v_7,v_8\}$ coincides with one of the sets $\{v_5,v_6\}$, $\{v_6,v_7\}$, $\{v_5,v_8\}$, or $\{v_7,v_8\}$. Recall that $\deg(v_2)=a+1$. Then

$$\vartheta(T)=(a+1)(A_1+A_2)+B_1+B_2.$$

Let us remove all leaves adjacent to v_2 and add $a-1$ new leaves to the vertex v_3; then we attach the leaf v_6' to the vertex v_6. The resulting n-vertex 2-caterpillar T' satisfies $\vartheta(T')=(a+1)(2A_2+B_2)$. Since $\gamma_t(T_8)=\widehat{\gamma}_t(T_8,v_8)$, it follows that $A_2>0$, whence $\vartheta(T')>\vartheta(T)$; this contradicts the maximality of T.

The proof of Theorem 1 is complete. $\square$

Note that the assertion of the theorem does not hold for the class of 3-caterpillars. Figure 3 shows a 36-vertex 3-caterpillar T^*_{36} with central vertices u and v that is obtained by attaching four copies of the star S_3 to each end of the path P_4. Denote by M_k the number of MTDSs of T^*_{36} containing k central vertices. Then

$$\vartheta(T^*_{36})=M_2+M_1+M_0=3^8+2\cdot 3^4\cdot(3^4-2^4)+(3^4-2^4)^2>3^9=f(36).$$

4. CLASS OF ARBITRARY TREES

It is well known that there exist n-vertex forests containing at least $(\sqrt{2})^n$ minimum dominating sets (for example, the forest $\frac{n}{2}P_2$ is suitable for even n). However, as will be shown in this section, each n-vertex tree T contains fewer than $(\sqrt{2})^n$ minimum *total* dominating sets.

Lemma 8. *For $n,k\ge 1$, if an n-vertex elementary forest F contains at least k connected components, then $(\sqrt{2})^n\ge(4/3)^k\vartheta(F)$.*

Proof. Consider the function $g(m)=(\sqrt{2})^{m+1}/m$ defined on the set of positive integers. Since $g(m)$ is monotone increasing as $m\ge 3$, the inequality $g(m)\ge 4/3$ holds.

Let $F=S_{m_1}\cup S_{m_2}\cdots\cup S_{m_k}$ (we assume that $m_1,m_2,\dots,m_k>0$; otherwise $\vartheta(F)=0$ and there is nothing to prove). Then

$$\frac{(\sqrt{2})^n}{\vartheta(F)}=\prod_{i=1}^{k}\frac{(\sqrt{2})^{m_i+1}}{m_i}=\prod_{i=1}^{k}g(m_i)\ge\left(\frac{4}{3}\right)^k.$$

The proof of Lemma 8 is complete. $\square$

We say that a maximal n-vertex tree is *critical* if it contains at least $(\sqrt{2})^n$ MTDSs and for any $n' < n$ every n'-vertex tree contains fewer than $(\sqrt{2})^{n'}$ MTDSs. It is clear from the definition that every critical tree is inseparable. Since all trees of diameter at most 4 are 2-caterpillars, it follows by Theorem 1 and Lemma 8 that they are noncritical. Thus, we assume that each critical tree has a diameter of at least 5.

Lemma 9. *For any critical tree T and any diametrical path $v_1v_2\dots v_k$, one has $\mathcal{D}(v_4) = 3$.*

Proof. By Lemma 4, we have $\mathcal{D}(v_3) = 0$. Let $\mathcal{D}(v_4) \le 2$. Then three cases are possible.

CASE OF $\mathcal{D}(v_4) = 0$. Recall that $\deg(v_2) = a + 1$. Then

$$\vartheta(T) = \vartheta_-(T, v_3) + \vartheta_+(T, v_3) \le a\vartheta(T_4) + \widehat{\vartheta}(T_4, v_4).$$

Note that equality is achieved if $\vartheta_-(T, v_3) > 0$; otherwise $\vartheta(T) = \widehat{\vartheta}(T_4, v_4)$. Let us show that $\gamma_t(T) = \gamma_t(T_5) + 2$. On the one hand, each MTDS T contains at least two vertices in the set $N[v_2]$, whence $\gamma_t(T) \ge \gamma_t(T_5) + 2$. On the other hand, for any MTDS D_5 of tree T_5 the set $D_5 \cup \{v_2, v_3\}$ is an MTDS of the tree T, whence $\gamma_t(T) \le \gamma_t(T_5) + 2$. Further, two cases are possible.

CASE OF $\gamma_t(T_4) > \gamma_t(T_5)$. We have $\gamma_t(T) = \gamma_t(T_4) + 1$. Then the vertex v_3 is universal in T (otherwise there would be an MTDS of T of cardinality $\gamma_t(T_4) + 1$ containing the vertices v_1 and v_2, which is impossible). If there exists an MTDS D of the tree T containing v_5, then the set $D \setminus \{v_2, v_3\}$ is an MTDS for the tree T_4, and $\gamma_t(T) = \gamma_t(T_4) + 2$; this is a contradiction. Hence v_5 is idle in T. Note that if the MTDS D of the tree T contains the vertex v_4, then it does not contain other vertices of $N(v_5)$; otherwise the set $D \setminus \{v_4\}$ also would be a TDS. The vertex v_6 is not idle in T (otherwise it is separable by Lemma 2), and for each MTDS $D \ni v_6$ the set $D \setminus \{v_2, v_3\}$ is an MTDS of T_5. Then v_6 is not idle in T_5 and $\partial_+(T, v_6) \le \partial_+(T_5, v_6)$. Moreover, for any MTDS $D \ni v_4$ the set $(D \setminus \{v_4\}) \cup \{v_6\}$ is also an MTDS, whence $\vartheta_+(T, v_4) \le \vartheta_+(T, v_6)$. Consequently,

$$\vartheta(T) \le \vartheta_+(T, v_4) + \vartheta_+(T, v_6) \le 2\vartheta_+(T_5, v_6) \le 2\vartheta(T_5) < 2(\sqrt{2})^{n-a-3} < (\sqrt{2})^n.$$

CASE OF $\gamma_t(T_4) = \gamma_t(T_5)$. It can readily be seen that

$$\vartheta(T_4) \le \vartheta(T_5), \quad \widehat{\vartheta}_-(T_4, v_4) = \vartheta(T_5).$$

Let us show that $\widehat{\vartheta}_+(T_4, v_4) \le \vartheta(T_5)$. Since the vertex v_4 is a leaf in the tree T_4, we have $\gamma_t(T_5) = \widehat{\gamma}_t(T_4, v_4)$. Consider a set $D \subseteq V(T_4)$ of cardinality $\gamma(T_5)$ such that $v_4 \in D$ and each vertex of $V(T_4) \setminus \{v_4\}$ has a neighbor in D. By definition, there exist $\widehat{\vartheta}_+(T_4, v_4)$ such sets. In this case, $v_6 \notin D$ (otherwise $D \setminus \{v_4\}$ would be an MTDS of T_5, which is impossible). Then the set $(D \setminus \{v_4\}) \cup \{v_6\}$ is an MTDS of T_5, whence $\widehat{\vartheta}_+(T_4, v_4) \le \vartheta(T_5)$. By assumption, $\vartheta(T_5) < (\sqrt{2})^{n-a-3}$. Then

$$\vartheta(T) \le a\vartheta(T_4) + \widehat{\vartheta}(T_4, v_4) \le (a + 2)\vartheta(T_5) < (a + 2)(\sqrt{2})^{n-a-3} < (\sqrt{2})^n.$$

CASE OF $\mathcal{D}(v_4) = 2$. The vertex v_4 is adjacent to some preleaf u_4. Then $\operatorname{dist}(v_2, u_4) = 3$, and the tree T is separable by Lemma 3.

CASE OF $\mathcal{D}(v_4) = 1$. In this case, v_4 is adjacent to some leaf v_4' and is not adjacent to preleaves. We assume that the vertex v_5 is not idle and not universal in T (otherwise T would be separable by Lemmas 2 and 3). There are two cases.

CASE OF $\deg(v_4) = 3$. Note that if the MTDS of the tree T contains the vertex v_5, then it can contain any vertex of the neighborhood $N(v_2)$. If, however, the MTDS does not contain v_5, then it contains the vertex v_3. Then

$$\vartheta(T) = \vartheta_+(T, v_5) + \vartheta_-(T, v_5) = (a + 1)\vartheta_+(T_4, v_5) + \vartheta(F_5) \le (a + 1)\vartheta(T_4) + \vartheta(F_5).$$

If $\vartheta(T_4) \geq 2\vartheta(F_5)$, then $\vartheta(T) \leq (a+3/2)\vartheta(T_4) < (a+3/2)(\sqrt{2})^{n-a-2}$. If, however, $\vartheta(T_4) < 2\vartheta(F_5)$, then $\vartheta(T) < (2a+3)\vartheta(F_5) < (2a+3)(\sqrt{2})^{n-a-5}$. One can readily verify that in both cases for all integer $a \geq 1$ one has the inequality $\vartheta(T) < (\sqrt{2})^n$.

Case of $\deg(v_4) \geq 4$. By F_0 we denote the forest $T \setminus (V(T_5) \cup \{v_4, v_4'\})$. Set $Q = \vartheta(F_0)$ and $q = |V(F_0)|$. Note that F_0 is an elementary forest that consists of at least $\deg(v_4) - 2 \geq 2$ connected components. Then by Lemma 8 we have $(\sqrt{2})^q/Q \geq 16/9$. Since $\vartheta(T_4) \leq (\sqrt{2})^{n-q}$ and $\vartheta(F_5) \leq (\sqrt{2})^{n-q-3}$, we have

$$\begin{aligned}\vartheta(T) &= \vartheta_+(T, v_5) + \vartheta_-(T, v_5)\\ &\leq Q\big(\vartheta(T_4) + \vartheta(F_5)\big) \leq \frac{9}{16}(\sqrt{2})^q\big((\sqrt{2})^{n-q} + (\sqrt{2})^{n-q-3}\big) < (\sqrt{2})^n.\end{aligned}$$

The proof of Lemma 9 is complete. □

Lemma 10. *For any critical tree T and any of its diametrical paths $v_1v_2\ldots v_k$, one has $\mathcal{D}(v_5) \in \{0, 4\}$.*

Proof. By Lemmas 4 and 9, $\mathcal{D}(v_3) = 0$ and $\mathcal{D}(v_4) = 3$. Suppose that $\mathcal{D}(v_5) \notin \{0, 4\}$. Three cases are possible.

Case of $\mathcal{D}(v_5) = 1$. The vertex v_5 is preleaf. Then T is separable, because $\mathrm{dist}(v_2, v_5) = 3$; a contradiction.

Case of $\mathcal{D}(v_5) = 2$. The vertex v_5 is adjacent to some preleaf vertex u_5 different from v_4 and v_6. Since $\mathcal{D}(v_4) = 3$, all neighbors of the vertex v_4 are adjacent to preleaves. Then, by Lemma 5, the vertex v_4 is idle. By Corollary 1, the tree T is not critical; a contradiction.

Case of $\mathcal{D}(v_5) = 3$. The vertex v_5 is adjacent to some vertex u_4 whose neighbor u_3 is a preleaf. Note that if u_3 is not a preterminal vertex, then it is adjacent to some preterminal vertex u_2 as well as to the leaf u_3'. Then the tree T is not critical by Lemma 4; a contradiction. Hence the vertex u_3 is preterminal, and the vertex u_4 is φ-universal in T. Then the vertex v_4 is idle by Lemma 5, and, by Corollary 1, the tree T is not critical, a contradiction.

The proof of Lemma 10 is complete. □

Theorem 2. *For $n \geq 1$, the inequality $\vartheta(T) < (\sqrt{2})^n$ holds for any n-vertex tree T.*

Proof. For $n \leq 9$, all n-vertex trees are 2-caterpillars, and so they satisfy the condition of the theorem. Suppose that for $n \geq 10$ there exists an n-vertex critical tree T. Denote by $P = v_1v_2\ldots v_k$ some diametrical path T. Then by Lemmas 4, 9, and 10 we have $\mathcal{D}(v_3) = 0$, $\mathcal{D}(v_4) = 3$, and $\mathcal{D}(v_5) \in \{0.4\}$. We assume that the vertex v_5 is adjacent to the vertex v_6 and also to some vertices $v_4^1, \ldots, v_4^k$ (here $v_4 = v_4^1$). There are 6 possible cases depending on the value of $\mathcal{D}(v_6)$.

Case of $\mathcal{D}(v_6) = 1$. The vertex v_6 is a preleaf. Then, by Lemma 5, the vertex v_4 is idle and, by Corollary 1, the tree T is not critical; a contradiction.

Case of $\mathcal{D}(v_6) = 2$. The vertex v_6 is adjacent to some preleaf u_5. Then, by Lemma 5, the vertex v_5 is idle. Denote by F_0 an elementary forest $T \setminus (V(T_6) \cup N[v_5])$. If the vertex v_6 is not idle in T_6, then $\gamma_t(T) = \gamma_t(T_6) + \gamma_t(F_0)$. Then the vertex v_4 is idle in T and T is separable by Lemma 2; a contradiction. If, however, the vertex v_6 is idle in T_6, then $\gamma_t(T) = \gamma_t(T_6) + \gamma_t(F_0) + 1$. For each $i \in 1, \ldots, k$, from T we delete all edges incident with the vertex v_4^i except for the edge $v_4^iv_5$. Then the resulting forest F satisfies the equality $\gamma_t(F) = \gamma_t(T_6) + \gamma_t(F_0) + 2$, and for any MTDS D of the tree T the set $D \cup \{v_5\}$ is an MTDS of the forest F. Thus, $(\sqrt{2})^n > \vartheta(F) \geq \vartheta(T)$; a contradiction.

Case of $\mathcal{D}(v_6) = 3$. The vertex v_6 is adjacent to some vertex u_5 whose neighbor u_4 is a preleaf. Let us assume that u_4 is not a preterminal vertex. If there exists a diametrical path $P' = u_1u_2u_3u_4u_5v_6\ldots v_k$ in T, then $\mathcal{D}_{T,P'}(u_4) = 1$; this contradicts Lemma 9. Otherwise, the vertex u_4 is adjacent to at least one leaf u_4' and at least one preterminal vertex. Then, by Lemma 4, the tree T is not critical; a contradiction.

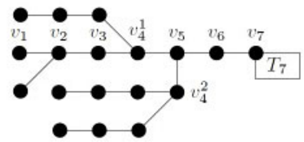

Fig. 4. Structure of tree T in the case of $\mathcal{D}(v_6) = 0$.

Thus, the vertex u_4 is preterminal in T. Then the vertex u_5 is φ-universal and by Lemma 5, the vertex v_5 is idle in T. We act by analogy with the previous case. If the vertex v_6 is not idle in T_6, then the vertex v_4 is idle in T and, by Lemma 2, the tree T is separable; a contradiction. Otherwise, for each $i \in 1, \ldots, k$ we remove all edges incident to the vertex v_4^i except for the edge $v_4^i v_5$. Then for any MTDS D of the tree T the set $D \cup \{v_5\}$ is an MTDS of the resulting forest F, whence $(\sqrt{2})^n > \vartheta(F) \geq \vartheta(T)$; a contradiction.

Case of $\mathcal{D}(v_6) = 5$. In this case, there exists some diametrical path $P' = u_1u_2u_3u_4u_5v_6\ldots v_k$, where u_5 is different from v_5 and v_7. By Lemmas 9 and 10, $\mathcal{D}_{T,P'}(u_4) = 3$ and $\mathcal{D}_{T,P'}(u_5) \in \{0, 4\}$. Assume that there exists an MTDS D that does not contain v_6. Then the set

$$\Big(\varphi(D) \setminus \big(N(v_5) \cup N(u_5)\big)\Big) \cup \{v_6\}$$

is also a TDS and is less than D in cardinality; a contradiction. Thus, the vertex v_6 is universal in T. Then the vertex v_4 is idle by Lemma 5 and, by Corollary 1, the tree T is not critical; a contradiction.

Case of $\mathcal{D}(v_6) = 0$. In this case, T has the structure depicted in Fig. 4. We introduce the notation

$$A_1 = \vartheta_-(T_7, v_7), \quad A_2 = \vartheta(F_7), \quad B_1 = \vartheta_+(T_7, v_7), \quad B_2 = \widehat{\vartheta}_+(T_7, v_7).$$

Denote by T_5' the inclusion-maximal subtree T that contains the vertex v_5 but does not contain the vertices $v_4^1, \ldots, v_4^k$. Denote by F_0 the elementary forest obtained by deleting all vertices of the subtree T_6 and the neighborhood $N[v_5]$ from the tree T. Let $Q = \vartheta(F_0)$ and $q = |V(F_0)|$. Obviously, $\gamma_t(T) = \gamma_t(T_5') + \gamma_t(F_0)$. Then each MTDS of the tree T contains exactly two vertices of the set $N[v_5] \cup N[v_6]$, which are either adjacent or at a distance of 3 from each other. Note that if $A_1 \geq A_2$, then the pair $\{v_4^i, v_5\}$ cannot be included in the MTDS. Similarly, if $B_1 \geq B_2$, then the pair $\{v_5, v_6\}$ cannot be included in the MTDS. Then

$$\vartheta(T) \leq Q\big(k \cdot \min(A_1, A_2) + A_2 + k \cdot \min(B_1, B_2) + B_2\big).$$

Since $\vartheta(T_5') = (A_2 + B_2) < (\sqrt{2})^{n-q-k}$, we have $(A_2 + B_2)(\sqrt{2})^{q+k} < (\sqrt{2})^n$. Then it suffices to prove that

$$(A_2 + B_2)(\sqrt{2})^{q+k} > Q(k+1)(A_2 + B_2).$$

Since $\mathcal{D}(v_4) = 3$, it follows that the forest F contains at least two connected components and $(\sqrt{2})^q/Q \geq (4/3)^2$. On the other hand, $(\sqrt{2})^k/(k+1) \geq 2/3$, whence we obtain the desired inequality.

Case of $\mathcal{D}(v_6) = 4$. In this case, T has the structure shown in Fig. 5. If there exists some diametrical path $P' = u_1u_2u_3u_4u_5v_6v_7\ldots v_s$, where the vertex u_5 differs from v_5, then we apply the arguments from the case of $\mathcal{D}(v_6) = 5$. Otherwise, the vertex v_6 is adjacent to some vertices $u_5^1, \ldots, u_5^m$ (where $m \geq 1$) all of whose neighbors except for the vertex v_6 have degree 2 and are adjacent to some terminal vertices. Let us introduce the notation A_1, A_2, B_1, B_2, Q, q by analogy with the previous case. Denote by F_0' the elementary forest obtained by removing all vertices of the subtree T_7 and the neighborhood $N[v_6]$ from the tree T_6. Let $W = \vartheta(F_0')$ and $w = |V(F_0')|$. Each MTDS of the tree T contains exactly two vertices of the set $N[v_5] \cup N[v_6]$ that are either adjacent or at a distance of 3 from each other. Then

$$\vartheta(T) \leq QW\big(k(m+1)\min(A_1, A_2) + mA_2 + A_2 + k \cdot \min(B_1, B_2) + B_2\big).$$

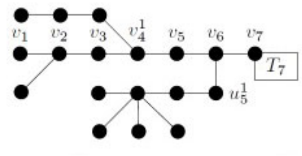

Fig. 5. Structure of tree T in the case of $\mathcal{D}(v_6) = 4$.

Denote by T_5'' the inclusion-maximal subtree of the tree T_5 that contains the vertices v_5 and v_6 and does not contain the vertices $v_4^1, \ldots, v_4^k$ and $u_5^1, \ldots, u_5^m$. Then

$$\vartheta(T_5'') = \vartheta_+(T_5'', v_5) + \vartheta_+(T_5'', v_7) = A_2 + B_2 < (\sqrt{2})^{n-q-w-k-m}.$$

Let us show that

$$(A_2 + B_2)(\sqrt{2})^{q+w+k+m} > QW\big((k+1)mA_2 + (m+1)A_2 + kB_2 + B_2\big).$$

We assume that $A_2 \leq B_2$. Then it suffices to prove the inequality

$$(\sqrt{2})^{q+w+k+m} > QW(k+1)(m+1).$$

By Lemma 9, the forest $F \cup F'$ contains at least three connected components. Then

$$\frac{(\sqrt{2})^{q+w+k+m}}{QW(k+1)(l+1)} = \frac{(\sqrt{2})^{q+w}}{QW} \frac{(\sqrt{2})^{k+m}}{(k+1)(m+1)} \geq \left(\frac{4}{3}\right)^3 \left(\frac{2}{3}\right)^2 > 1.$$

Thus, $\vartheta(T) < (\sqrt{2})^n$.

The proof of Theorem 2 is complete. $\square$

FUNDING

This research was supported by the Russian Science Foundation, project no. 21–11–00194.

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