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DYNAMICS OF 3-HOMEOMORPHISMS WITH TWO-DIMENSIONAL ATTRACTORS AND REPELLERS

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On closed orientable 3*-manifolds, we consider a class* $\mathscr{G}$ *of homeomorphisms such that the nonwandering set of each* $f \in \mathscr{G}$ *is the finite union of surfaces such that the restriction of some power* f^k *on each of these surfaces is a pseudo-Anosov homeomorphism. We prove that homeomorphisms of class* $\mathscr{G}$ *exist only on* 3*-manifolds of the form* $S_g \times \mathbb{R}/_{(J(z),r-1)}$*, where* $J : S_g \to S_g$ *is either a pseudo-Anosov homeomorphism of the surface* S_g *of genus* $g > 1$ *or a periodic homeomorphism commuting with some pseudo-Anosov homeomorphism. On such a manifold, we construct model homeomorphisms and find necessary and sufficient conditions for topological conjugacy of model mappings. Bibliography*: 13 *titles.*

1 Introduction

In this paper, we consider homeomorphisms $f : M^3 \to M^3$ given on an oriented closed 3-manifold M^3 such that the nonwandering set $NW(f)$ of each f is the disjoint union of closed surfaces. If f is an A-diffeomorphism, i.e., the set $NW(f)$ is hyperbolic and periodic points are dense in $NW(f)$, the surfaces are connected components of the basis sets, i.e., closed f-invariant subsets of $NW(f)$ possessing an everywhere dense orbit [1]. By [2], the basis sets are attractors or repellers. An f-invariant set B is called an *attractor* if there exists a closed neighborhood U of the set B such that $f(U) \subset \operatorname{int} U$ and $\bigcap\limits_{j \geqslant 0} f^j(U) = B$. In this case, the neighborhood U is

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called an *isolating neighborhood* of the attractor A. An attractor of the homeomorphism f^{-1} is called a *repeller* of the homeomorphism f. By the results of [3] and [4], each surface is a two-dimensional torus cylindrically embedded into M^3 and the restriction of some power of the diffeomorphism f on the torus is the conjugate of an Anosov diffeomorphism.

We say that $\Sigma \subset M^3$ is a *cylindrical embedding* of an oriented surface S to M^3 if there exists a homeomorphism on the image of $h : S \times [0, 1] \to M^3$ such that $\Sigma = h(S)$.

A-diffeomorphisms $f : M^3 \to M^3$ are considered in [5, 6] under the assumption that their nontrivial two-dimensional basis sets are surfaces. For such diffeomorphisms the structure of the underlying manifold was studied. In particular, it was proved that only three-dimensional manifolds of the form $\mathbb{T}^2 \times \mathbb{R}/_{(J(z),r-1)}$, where $J : \mathbb{T}^2 \to \mathbb{T}^2$ is an algebraic automorphism of a torus given by a hyperbolic unimodular matrix or the matrix $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ admit these mappings. The model mappings that are locally direct products of hyperbolic automorphisms of a torus and rough transformations of a circle were constructed on each admissible manifold. An algebraic topological conjugacy criterion for two model diffeomorphisms was found and it was proved that any structurally stable diffeomorphism with basis sets of dimension 2 is topologically conjugate to some model mapping.

This paper continues ideas of the above results to the case of homeomorphisms whose nonwandering set is the union of two-dimensional surfaces. We describe the results in more detail.

We first recall that an orientation-preserving homeomorphism $P \colon S_g \to S_g$ of a closed oriented surface of genus $g > 1$ is called a *pseudo-Anosov mapping* (a *pA*-homeomorphism) with *dilatation* $\lambda > 1$ if on S_g there exists a pair of P-invariant transversal foliations $\mathscr{F}_P^s$ and $\mathscr{F}_P^u$ with the set S of saddles and transversal measures μ^s and μ^u such that

- each saddle in S has at least three separatrices,
- $\mu^s(P(\alpha)) = \lambda \mu^s(\alpha)$ $(\mu^u(P(\alpha)) = \lambda^{-1}\mu^u(\alpha))$ for any arc α traversal to $\mathscr{F}_P^s$ $(\mathscr{F}_P^u)$.

From the results of [7] it follows that there exists a pseudo-Anosov homeomorphism in each homotopy class of homeomorphisms of the surface S_g that contains no reducible or periodic homeomorphisms. We recall that a homeomorphism $h : S_g \to S_g$ is *reducible* by a system C of disjoint simple closed curves C_i, $i = 1, \dots, l$, that are nonhomotopic to zero and pairwise nonhomotopic to each other if C is invariant under the homomorphism h. A homeomorphism $h : S_g \to S_g$ is *periodic* if there exists $m \in \mathbb{N}$ such that $h^m = \mathrm{id}$, where id denotes the identical transformation. The least number m possessing such properties is called the *period* of the periodic homeomorphism.

We denote by $\mathscr{P}$ the set of pseudo-Anosov homeomorphisms and by $Z(P)$ the centralizer of a mapping $P \in \mathscr{P}$, i.e., $Z(P) = \{h \in \mathrm{Homeo}(S_g) : Ph = hP\}$.

Proposition 1.1 ([7, 8]). *Any homeomorphism $h \in Z(P)$ has the form $h = \iota_h p^{n_h}$, where ι_h is a periodic homeomorphism in the finite set $\mathscr{I}_P, p \in \mathscr{P}, n_h \in \mathbb{Z}$.*

We set

$$\mathscr{I} = \bigcup_{P \in \mathscr{P}} \mathscr{I}_P, \quad \mathscr{J} = \mathscr{P} \cup \mathscr{I}.$$

We consider the class $\mathscr{G}$ of homeomorphisms $f : M^3 \to M^3$ whose nonwandering sets $NW(f)$ consist of finitely many connected components $\mathscr{B}_1, \dots, \mathscr{B}_m$ possessing the following properties for $i \in \{1, \dots, m\}$:

- $\mathscr{B}_i$ is an orientable surface of a genus greater than 1 such that $\mathscr{B}_i$ is cylindrically embedded into M^3,
- there exists a natural number k_i such that $f^{k_i}(\mathscr{B}_i) = \mathscr{B}_i$ and the mapping $f^{k_i}|_{\mathscr{B}_i}$ is topologically conjugate to a pseudo-Anosov homeomorphism,
- $\mathscr{B}_i$ is either an attractor or a repeller of the homeomorphism f^{k_i}.

We denote by $\mathscr{A}$ the set of all attractors and by $\mathscr{R}$ the set of all repellers of a homeomorphism f.

Let $J\colon S_g \to S_g$ be a homeomorphism of a closed orientable surface S_g of genus $g > 1$. We set $M_J = (S_g \times \mathbb{R})/\Gamma$, where $\Gamma = \{\gamma^i, i \in \mathbb{Z}\}$ is the group of powers of the homeomorphism $\gamma\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ given by $\gamma(z,r) = (J(z), r-1)$. We denote by $\pi_{_J}\colon S_g \times \mathbb{R} \to M_J$ the natural projection.

The following assertion is proved in Section 2.

Theorem 1.1. *A manifold M^3 admits a homeomorphism f of class $\mathscr{G}$ if and only if M^3 is homeomorphic to the manifold M_J, where $J \in \mathscr{J}$.*

We construct model homeomorphisms of class $\mathscr{G}$ on each admissible manifold.

We consider tuples of numbers n, k, l such that $n, k \in \mathbb{N}$, where $l = 0$ if $k = 1$ and $l \in \{1, \ldots, k-1\}$ is mutually prime with k if $k > 1$. For every tuple n, k, l we define the diffeomorphism $\varphi_{n,k,l}\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{n,k,l}(r) = r + \frac{1}{4\pi nk}\sin(2\pi nkr) + \frac{l}{k}.$$

For $P \in \mathscr{P}$ we introduce the mapping $\overline{\varphi}_{P,n,k,l}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by

$$\overline{\varphi}_{P,n,k,l}(z,r) = (P(z), \varphi_{n,k,l}(r)).$$

The following lemma is proved in Section 3.

Lemma 1.1. *The formula*

$$\varphi_{P,J,n,k,l}(w) = \pi_{_J}(\overline{\varphi}_{P,n,k,l}(\pi_{_J}^{-1}(w))),$$

where $w \in M_J$ and $\pi_J^{-1}(w)$ is the complete preimage of a point $w \in M_J$, defines a homeomorphism $\varphi_{P,J,n,k,l}\colon M_J \to M_J$ if and only if $J \in Z(P)$.

Homeomorphisms of the form $\varphi_{P,J,n,k,l}$ will be said to be *model*. By Proposition 1.1, Theorem 1.1, and Lemma 1.1, a model homeomorphism exists on each manifold M_J, $J \in \mathscr{J}$, and, by construction, belongs to the class $\mathscr{G}$ under consideration.

The following assertion is proved in Section 4.

Theorem 1.2. *The homeomorphisms $\varphi_{P,J,n,k,l}$ and $\varphi_{P',J',n',k',l'}$ are topologically conjugate if and only if*

(a) *$k = k'$, $n = n'$ and either $l = l'$ or $k - l = l'$,*

(b) *there exists a homeomorphism $H\colon S_g \to S_g$ such that $PH = HP'$ and either $HJ = J'H$, or $l = l'$, or $HJ = J'^{-1}H$ if $k - l = l'$.*

2 Structure of Manifolds Admitting Homeomorphisms of Class $\mathscr{G}$

Before proving Theorem 1.1, we first study the structure of the nonwandering sets of homeomorphisms of class $\mathscr{G}$.

Lemma 2.1. *For any orientation-preserving homeomorphism $f \in \mathscr{G}$ the sets $\mathscr{A}$ and $\mathscr{R}$ are not empty and consist of the same number $nk \geqslant 1$ of connected components of the same period $k \geqslant 1$. The set $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$ has $2nk$ connected components such that the boundary of each component consists exactly of one periodic attractor component and one periodic repeller component.*

Proof. We denote by $U(A)$ ($U(R)$) a cylindrical neighborhood of an attractor $A \in \mathscr{A}$ ($R \in \mathscr{R}$). We set $\dot{U}(A) = U(A) \setminus A$ ($\dot{U}(R) = U(R) \setminus R$).

We first prove that the sets $\mathscr{A}$ and $\mathscr{R}$ are not empty. Assume the contrary. Let $\mathscr{R} = \varnothing$, and let the set $\mathscr{A}$ consist of finitely many connected components. Then the manifold M^3 is represented as

$$M^3 = \bigcup_{A \in \mathscr{A}} \Big(\bigcup_{n \in \mathbb{Z}} f^n(U(A)) \Big).$$

Since M^3 is connected, the set $\mathscr{A}$ consists of only one attractor A. We consider an isolating neighborhood $\widetilde{U}(A)$ of the attractor A. We note that the α-limit set of an arbitrary point $s \notin \widetilde{U}(A)$ is contained in the nonwandering set of the homeomorphism f and, consequently, belongs to the attractor $\mathscr{A}$. Then there exists $n \in \mathbb{N}$ such that $f^{-n}(s) \in \widetilde{U}(A)$. By the definition of an isolating neighborhood of an attractor, we have $f^n(f^{-n}(s)) = s \in \widetilde{U}(A)$. We arrive at a contradiction. Consequently, the sets $\mathscr{A}$ and $\mathscr{R}$ are nonempty.

Now, we prove that the boundary of each connected component of $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$ consists exactly of one periodic attractor component and one periodic repeller component. The set $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$ is wandering and, consequently, can be represented as

$$M^3 \setminus (\mathscr{A} \cup \mathscr{R}) = \bigcup_{A \in \mathscr{A}} \Big(\bigcup_{n \in \mathbb{Z}} f^n(\dot{U}(A)) \Big) = \bigcup_{R \in \mathscr{R}} \Big(\bigcup_{n \in \mathbb{Z}} f^n(\dot{U}(R)) \Big).$$

Let V be a connected component of $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$. Since

$$V \subset \bigcup_{A \in \mathscr{A}} \Big(\bigcup_{n \in \mathbb{Z}} f^n(\dot{U}(A)) \Big), \quad V \subset \bigcup_{R \in \mathscr{R}} \Big(\bigcup_{n \in \mathbb{Z}} f^n(\dot{U}(R)) \Big)$$

and the set V is connected, there exists a unique connected component $A \in \mathscr{A}$ and a unique connected component $R \in \mathscr{R}$ such that

$$V \subset \bigcup_{n \in \mathbb{Z}} f^n(\dot{U}(A)), \quad V \subset \bigcup_{n \in \mathbb{Z}} f^n(\dot{U}(R)).$$

Consequently, $\operatorname{cl} V = A \cup V \cup R$ and $\partial V = A \cup R$.

We show that the number of components of all attractors in $\mathscr{A}$ coincides with the number of components of all repellers in $\mathscr{R}$. We fix a component A_1 of an attractor in $\mathscr{A}$. Then A_1 belongs to the boundary of both domains $V_1, V_2 \subset M^3 \setminus (\mathscr{A} \cup \mathscr{R})$. Assume that $\partial V_1 = A_1 \cup R_1$ and $\partial V_2 = A_2 \cup R_2$. Then either R_1 and R_2 coincide and the required assertion is true or there exist domains V_3 and V_4 such that $R_1 \subset \partial V_3$ and $R_2 \subset \partial V_4$. We denote by A_2 the boundary component of V_4 different from R_2 and by A_3 the boundary component of V_3 different from R_1. There are two cases: either $A_2 = A_3$ and the required assertion is valid or there exist domains V_5 and V_6 whose boundaries contain A_3 and A_2 respectively. Arguing as above, and taking into account that the number of connected components of the nonwandering set is finite, we find that the number of periodic components of all attractors coincides with the number of periodic components of all repellers.

We prove that all components of the set $\mathscr{A} \cup \mathscr{R}$ have the same period. For this purpose we first show that if there exists a component of the set $\mathscr{A} \cup \mathscr{R}$ with period 1, then all components of the set $\mathscr{A} \cup \mathscr{R}$ have period 1.

For the sake of definiteness we assume that some connected component A of the set $\mathscr{A}$ has period 1. Let V be a domain in $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$ such that $\partial V = A \cup R$, where R is a connected component of the set $\mathscr{R}$. We show that R also has period 1. Assume the contrary, i.e., $f(R) \neq R$. We set $\widetilde{V} = f(V)$ and note that $\partial \widetilde{V} = f(A) \cup f(R) = A \cup f(R)$ which implies $V \cap \widetilde{V} \neq \varnothing$. We consider a cylindrical neighborhood $U(A)$ of the attractor A such that $U(A) \subset V \cup A \cup \widetilde{V}$. We denote by Q and $\widetilde{Q}$ the connected components of the set $U(A) \setminus A$ such that $Q \subset V$ and $\widetilde{Q} \subset \widetilde{V}$ respectively. Then $f(Q) \subset \widetilde{V}$ and $f(\widetilde{Q}) \subset V$. Since the diffeomorphism f preserves the orientation of M^3, we arrive at a contradiction with the fact that the restriction of the diffeomorphism f on A preserves the orientation of A.

We assume that $\mathscr{A} \cup \mathscr{R}$ has components of different periods. We denote by k the least period of connected components of $\mathscr{A} \cup \mathscr{R}$, i.e., at least one connected component of the nonwandering set $NW(f^k)$ of the homeomorphism f^k has period 1. Then, as above, all connected components of the set $NW(f^k)$ have period 1, which means that all connected components of the set $\mathscr{A} \cup \mathscr{R}$ for the homeomorphism f have period k. □

Lemma 2.2. *For any homeomorphism $f \in \mathscr{G}$ the closure of each connected component of the set $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$ is homeomorphic to $S_g \times [0,1]$.*

Proof. Without loss of generality we assume that the period of connected components of the nonwandering set is $k = 1$ (otherwise, we can consider the homeomorphism f^k). Let A (R) be an attractor (a repeller) lying in the nonwandering set of f, and let A (R) be homeomorphic to the surface S_a (S_r). Since A (R) is a cylindrically embedded surface, there exists a closed neighborhood $U(A)$ ($U(R)$) and a homeomorphism h_A (h_R) such that $h_A \colon U(A) \to S_a \times [-1,1]$ ($h_R \colon U(R) \to S_r \times [-1,1]$); moreover, $h_A(A) = S_a \times \{0\}$ ($h_R(R) = S_r \times \{0\}$). We set

$$U_A^1 = h_A^{-1}(S_a \times [-1,0]), \quad U_A^2 = h_A^{-1}(S_a \times [0,1]),$$
$$N_A^1 = h_A^{-1}(S_a \times \{-1\}), \quad N_A^2 = h_A^{-1}(S_a \times \{1\})$$

$$\Big(U_R^1 = h_R^{-1}(S_r \times [-1,0]), \quad U_R^2 = h_R^{-1}(S_r \times [0,1]),$$
$$N_R^1 = h_R^{-1}(S_r \times \{-1\}), \quad N_R^2 = h_R^{-1}(S_r \times \{1\})\Big).$$

We fix an attractor A. Since the nonwandering set of f consists only of attractors and repellers, there exists a natural number m such that $f^{-m}(N_A^1)$ belongs to a neighborhood of some repeller $R_1 \subset NW(f)$ and $f^{-m}(N_A^2)$ belongs to a neighborhood of some repeller $R_2 \subset NW(f)$, where R_1

and R_2 are homeomorphic to S_{r_1} and S_{r_2} respectively (we note that $R_1 = R_2$ for $n = 1$). Without loss of generality we can assume that $f^{-m}(N_A^1) \subset \text{int}\, U_{R_1}^1$ and $f^{-m}(N_A^2) \subset \text{int}\, U_{R_2}^2$. We show that R_1 and $N_{R_1}^1$ belong to different connected components of the set $U_{R_1}^1 \setminus f^{-m}(N_A^1)$. Assume the contrary. By [9, Lemma 3.1], $f^{-m}(N_A^1)$ is the boundary of some domain $D_A^1 \subset \text{int}\, U_{R_1}^1$. By [10, Lemma 1], $R_1 \subset \text{int}\, f^{-m}(U_A^1)$. Since the surface R_1 is invariant, we arrive at a contradiction. Thus, the set $U_{R_1}^1 \setminus f^{-m}(N_A^1)$ consists of two connected components. By [9, Theorem 3.1], we have $a \geqslant r_1$ (the genus of $f^{-m}(N_A^1)$ is not less than that of R_1). Similarly, acting by the mapping f on the surface $N_{R_1}^1$, we find $r_1 \geqslant a$. Consequently, $a = r_1$ and the surfaces A and R_1 are homeomorphic.

Further, we set $a = r_1 = g$. By [9, Theorem 3.2], the closure of each connected component of the set $U_{R_1}^1 \setminus f^{-m}(N_A^1)$ is homeomorphic to $S_g \times [0,1]$. (We note that the assumptions of Theorem 3.2 in [9] include the smoothness condition on $f^{-m}(N_A^1)$. However, the result remains valid if the surface is tame embedded.) Then the surfaces R_1 and $f^{-m}(N_A^1)$ bound a closed domain in M^3 that is homeomorphic to $S_g \times [0,1]$. Since the set $f^{-m}(U_A^1)$ is also homeomorphic to $S_g \times [0,1]$, from [10, Lemma 2] it follows that the connected component of the set $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$, bounded by A and R_1, is homeomorphic to the direct product $S_g \times [0,1]$. Similarly, we can show that the connected component of the set $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$, bounded by A and R_2, is homeomorphic to the direct product $S_g \times [0,1]$. Arguing in the same way for all attractors in $NW(f)$, we conclude that each connected component of the set $M^3 \setminus (\mathscr{A} \cup \mathscr{R})$ is homeomorphic to $S_g \times [0,1]$. □

Proof of Theorem 1.1. *Sufficiency.* The required assertion follows from the existence of model homeomorphisms on each manifold M_J (cf. Lemma 1.1).

Necessity. Let the manifold M^3 admit a homeomorphism f in the class $\mathscr{G}$. By Lemma 2.1, all connected components of the wandering set $NW(f)$ have the same period $k \in \mathbb{N}$. Without loss of generality we assume that $k = 1$ (otherwise, we can consider the homeomorphism f^k). We fix an attractor A. Then $f(A) = A$ and, in view of Lemma 2.2 and [10, Lemma 2], $\text{cl}\,(M^3 \setminus A)$ is homeomorphic to $S_g \times [0,1]$. Then the manifold M^3 is covered by the space $S_g \times \mathbb{R}$ with a covering mapping $q_f : S_g \times \mathbb{R} \to M^3$ such that $q_f^{-1}(A) = S_g \times \mathbb{Z}$, whereas the homeomorphism f lifts to a homeomorphism $\overline{f} : S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ ($q_f \overline{f} = f q_f$) such that $\overline{f}(S_g \times \{0\}) = S_g \times \{0\}$ (see, for example, [11, Statement 10.35]).

For $i \in \mathbb{Z}$ we set $q_{f,i} = q_f|_{S_g \times \{i\}} : S_g \times \{i\} \to A$. Then the mapping $q_{f,i}$ is a homeomorphism and determines the homeomorphism $P_i : S_g \to S_g$ by

$$(P_i(z), i) = q_{f,i}^{-1} f q_{f,i}(z, i). \tag{2.1}$$

Since $f|_A$ is a pseudo-Anosov homeomorphism, P_i is also a pseudo-Anosov homeomorphism. Since $\overline{f}(S_g \times [0,1]) = S_g \times [0,1]$, the homeomorphisms P_0 and P_1 are homotopic (cf., for example, [12, Theorem 5.15.3]). Consequently (see, for example, [7]), there exists a homeomorphism $\psi : S_g \to S_g$ that is homotopic to the identity mapping and such that

$$\psi P_0 = P_1 \psi. \tag{2.2}$$

We define the homeomorphism $\widetilde{J} : S_g \to S_g$ by

$$(\widetilde{J}(z), 0) = q_{f,0}^{-1} q_{f,1}(z, 1). \tag{2.3}$$

Then the manifolds $M_{\widetilde{J}}$ and M^3 are homeomorphic by a homeomorphism sending the equivalence class of $S_g \times \mathbb{R}$ by the action of $(\widetilde{J}(z), r-1)$ to a point $q_f(z, r)$. Furthermore, (2.1) and (2.3) imply

$$\widetilde{J}P_1 = P_0\widetilde{J}. \tag{2.4}$$

We set $J = \widetilde{J}\psi$. Expressing P_1 from (2.2) as $P_1 = \psi P_0 \psi^{-1}$ and substituting into (2.4), we get $\widetilde{J}\psi P_0 \psi^{-1} = P_0 \widetilde{J}$ or $JP_0 = P_0 J$. Then $J \in Z(P_0)$ and, consequently, $J \in \mathscr{J}$. Since the homeomorphisms J and $\widetilde{J}$ are isotopic, M_J is homeomorphic to $M_{\widetilde{J}}$ (cf., for example, [13, Proposition 1]). Hence M^3 is homeomorphic to M_J, where $J \in \mathscr{J}$. □

3 Properties of Model Homeomorphisms

We show that the formula $\varphi_{P,J,n,k,l}(w) = \pi_J(\overline{\varphi}_{P,n,k,l}(\pi_J^{-1}(w)))$, where $w \in M_J$ and $\pi_J^{-1}(w)$ is the complete preimage of the point $w \in M_J$, defines a homeomorphism $\varphi_{P,J,n,k,l} : M_J \to M_J$ if and only if $J \in Z(P)$.

Proof of Lemma 1.1. We recall that for a homeomorphism $J : S_g \to S_g$ we denote by M_J the space of orbits $(S_g \times \mathbb{R})/\Gamma$, where Γ is the infinite cyclic group of powers of the homeomorphism $\gamma\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ given by $\gamma(z, r) = (J(z), r - 1)$. We denote by $\alpha : \mathbb{R} \to \mathbb{R}$ the diffeomorphism given by

$$\alpha(r) = r - 1. \tag{3.1}$$

Then

$$\gamma(z, r) = (J(z), \alpha(r)). \tag{3.2}$$

We recall that the homeomorphism $\overline{\varphi}_{P,n,k,l} : S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ has the form

$$\overline{\varphi}_{P,n,k,l}(z, r) = (P(z), \varphi_{n,k,l}(r)), \tag{3.3}$$

where $P : S_g \to S_g$ is a pseudo-Anosov homeomorphism and $\varphi_{n,k,l} : \mathbb{R} \to \mathbb{R}$ is the diffeomorphism given by

$$\varphi_{n,k,l}(r) = r + \frac{1}{4\pi nk}\sin(2\pi nkr) + \frac{l}{k}. \tag{3.4}$$

By [11, Statement 10.35], the homeomorphism $\overline{\varphi}_{P,n,k,l}$ is projected by the natural projection $\pi_J : S_g \times \mathbb{R} \to M_J$ to a homeomorphism $\varphi_{P,J,n,k,l} = \pi_J \overline{\varphi}_{P,n,k,l} \pi_J^{-1}$ of the manifold M_J if and only if

$$\overline{\varphi}_{P,n,k,l}\gamma = \gamma^m \overline{\varphi}_{P,n,k,l}, \quad m \in \{-1, 1\} \tag{3.5}$$

Substituting (3.2) and (3.3) into (3.5), we obtain the equality

$$(PJ(z), \varphi_{n,k,l}\alpha(r)) = (J^m P(z), \alpha^m \varphi_{n,k,l}(r)) \tag{3.6}$$

which implies

$$\varphi_{n,k,l}\alpha = \alpha^m \varphi_{n,k,l}. \tag{3.7}$$

Substituting (3.1) and 3.4 into (3.7), we find that $m = 1$ and, consequently, $PJ = JP$ which implies $J \in Z(P)$. □

By construction, the model homeomorphism $\varphi_{P,J,n,k,l}$ belongs to the class $\mathscr{G}$. We list the main properties of this homeomorphism. These properties follow from the definition.

Proposition 3.1. *The model homeomorphism $\varphi_{P,J,n,k,l}\colon M_J \to M_J$ possesses the following properties.*

(1) *The nonwandering set* $NW(\varphi_{P,J,n,k,l})$ *of the homeomorphism* $\varphi_{P,J,n,k,l}$ *consists of* $2nk$ *connected components* $\mathscr{B}_1, \dots, \mathscr{B}_{2nk}$ *such that each of them has period* k.

(2) *Each connected component* $\mathscr{B}_i$ *of the nonwandering set* $NW(\varphi_{P,J,n,k,l})$ *is a tame embedding of the oriented surface* S_g *into* M^3.

(3) *The set*

$$M_J \setminus \bigcup_{i=1}^{2nk} \mathscr{B}_i$$

consists of $2nk$ *connected components* $V_1, \dots, V_{2nk}$ *such that* $\mathrm{cl}\,(V_i)$ *is homeomorphic to* $S_g \times [0,1]$ *and the boundary consists of the surfaces* $\mathscr{B}_i$ *and* $\mathscr{B}_{i+1}$ *one of which is an attractor and the other is a repeller of the homeomorphism* $\varphi^k_{P,J,n,k,l}$.

(4) *The set*

$$M_J \setminus \bigcup_{j=1}^{k} \varphi^{j-1}_{P,J,n,k,l}(\mathscr{B}_i)$$

consists of k *connected components* $W_i^1, \dots, W_i^k$ *such that* $\mathrm{cl}\,(W_i^j)$ *is homeomorphic to* $S_g \times [0,1]$ *and the boundary consists of the surfaces* $\varphi^{\ell(j-1)}_{P,J,n,k,l}(\mathscr{B}_i)$ *and* $\varphi^{\ell j}_{P,J,n,k,l}(\mathscr{B}_i)$, *where* $\ell \in \mathbb{N}$ *is such that* $1 \leqslant \ell \leqslant k$ *and* $\ell l \equiv 1 \pmod{k}$.

4 Classification of Model Homeomorphisms

Proof of Theorem 1.2. *Necessity.* Assume that the homeomorphisms $\varphi = \varphi_{P,J,n,k,l}$ and $\varphi' = \varphi_{P',J',n',k',l'}$ are topologically conjugate by a homeomorphism $h : M_J \to M_{J'}$. By Lemma 2.1, the nonwandering sets of the homeomorphisms φ and φ' consist of $2nk$ and $2n'k'$ connected components with periods k and k' respectively. Since the homeomorphism h sends orbits of the mapping φ to orbits of the mapping φ', we have $k = k'$ and $2nk = 2n'k'$. Hence $n = n'$. We set $\mathscr{B}'_i = h(\mathscr{B}_i)$.

If $k = 1$, then assertion (a) is proved. If $k > 1$, then, according to Proposition 3.1 (4), there exist two connected components W_i^1 and W_i^k of the set $M_J \setminus \bigcup\limits_{j=1}^{k} \varphi^{j-1}(\mathscr{B}_i)$ whose boundary contains $\mathscr{B}_i$. Moreover, $\partial W_i^1 = \mathscr{B}_i \sqcup \varphi^{\ell}(\mathscr{B}_i)$ and $\partial W_i^k = \mathscr{B}_i \sqcup \varphi^{\ell(k-1)}(\mathscr{B}_i)$, where $\ell l \equiv 1 \pmod{k}$. The same is valid for the surfaces $\mathscr{B}'_i$, the diffeomorphism φ', and a number ℓ' such that $\ell' l' \equiv 1 \pmod{k}$. Hence we have either $h(W_i^1) = {W'}_i^1$ or $h(W_i^1) = {W'}_i^k$. In the first case, $\ell' = \ell$, and, in the second case, $\ell'(k-1) \equiv \ell \pmod{k}$, which implies either $l = l'$ or $k - l = l'$.

Thus, we have proved the necessity of condition (a). Let us prove the necessity of condition (b). For this purpose we consider two cases: $l = l'$ and $k - l = l'$.

Case 1: $l = l'$. We set $\overline{\varphi} = \overline{\varphi}_{P,J,n,k,l}$ and $\overline{\varphi}' = \overline{\varphi}_{P',J',n',k',l'}$. By [11, Statement 10.35], there exists a lifting $\overline{h} : S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ of the homeomorphsim h that is a homeomorphism conjugating the homeomorphisms $\overline{\varphi}$ and $\overline{\varphi}'$

$$\overline{h}\overline{\varphi} = \varphi'\overline{h}. \tag{4.1}$$

Without loss of generality we can assume that $\overline{h}(S_g \times \{0\}) = S_g \times \{0\}$ (otherwise, we can choose another covering possessing the same property). We set $R = \pi_J(S_g \times \{0\})$. Then

$$\pi_J^{-1}(R) = S_g \times \Big\{ \bigcup_{i \in \mathbb{Z}} \frac{i}{k} \Big\}.$$

We set $\overline{R}_i = S_g \times \{i/k\}$, $i \in \mathbb{Z}$. The same notation with prime will be used for the homeomorphism φ'. Then $\overline{h}(\overline{R}_i) = \overline{R}'_i$. We define the homeomorphism $H_i : S_g \to S_g$ by the formula $(H_i(z), i/k) = \overline{h}(z, i/k)$. We recall that the homeomorphism $\overline{\varphi} : S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ is written as

$$\overline{\varphi}(z, r) = \Big(P(z), \psi(r) + \frac{l}{k} \Big), \tag{4.2}$$

where $P : S_g \to S_g$ is a pseudo-Anosov homeomorphism and $\psi : \mathbb{R} \to \mathbb{R}$ is the diffeomorphism given by

$$\psi(r) = r + \frac{1}{4\pi nk} \sin(2\pi nkr). \tag{4.3}$$

The mapping $\overline{\varphi}'$ has a similar form. Then the equality (4.1) for points $(z, 0) \in S_g \times \{0\}$ can be written as

$$\Big(H_l P(z), \frac{l}{k} \Big) = \Big(P' H_0(z), \frac{l'}{k} \Big)$$

which implies $H_l P = P' H_0$ and, consequently, $H_0^{-1} H_l P = H_0^{-1} P' H_0$. Since $\overline{h}(S_g \times [0, l/k]) = S_g \times [0, l/k]$, the homeomorphisms H_0 and H_l are homotopic (cf., for example, [12, Theorem 5.15.3]). Then the pseudo-Anosov homeomorphisms P and $H_0^{-1} P' H_0$ are homotopic. Hence they are topologically conjugate by a homeomorphism $\widetilde{H}$ homotopic to the identity mapping (see, for example, [7]), i.e., $\widetilde{H} P = H_0^{-1} P' H_0 \widetilde{H}$ which implies $HP = P'H$ for $H = H_0 \widetilde{H}$.

Since the homeomorphism $\overline{h}$ is projected by the homeomorphism h to the space of orbits $h : M_J \to M_{J'}$, arguing as in Lemma 1.1, we can show that $H_0 J = J' H_k$. Then $H_0 J H_0^{-1} = J' H_k H_0^{-1}$ and, consequently, homeomorphisms $H_0 J H_0^{-1}$ and J', as well as homeomorphisms J and $H_0^{-1} J' H_0$ are homotopic. Since $J \in Z(P), J' \in Z(P')$ in view of Lemma 1.1, from Proposition 1.1 it follows that the homeomorphisms J and $H_0^{-1} J' H_0$ are topologically conjugate by the homeomorphism $\widetilde{H}$, i.e., $\widetilde{H} J = H_0^{-1} J' H_0 \widetilde{H}$ which implies $HJ = J'H$.

Case 2: $k - l = l'$. We set $\overline{\varphi} = \overline{\varphi}_{P,J,n,k,l}$. For $\overline{\varphi}'$ we consider the covering of the homeomorphism φ' given by the formula $\overline{\varphi}'(z, r) = (P'(z), \psi'(r) - l/k)$. Introducing the notation as in Case 1, we have $\overline{h}(\overline{R}_i) = \overline{R}'_{-i}$ and $H_l P = P' H_0$. Since $\overline{h}(S_g \times [0, l/k]) = S_g \times [-l/k, 0]$, we get $HP = P'H$ for some homeomorphism $H : S_g \to S_g$. As in the proof of Lemma 1.1, we can show that $H_0 J = J'^{-1} H_k$. Further, as above, we can show that homeomorphisms J and $H_0^{-1} J'^{-1} H_0$ are homotopic. Since $J \in Z(P), J'^{-1} \in Z(P')$ in view of Lemma 1.1, from Proposition 1.1 if follows that homeomorphisms J and $H_0^{-1} J'^{-1} H_0$ are topologically conjugate by the homeomorphism $\widetilde{H}$, i.e., $\widetilde{H} J = H_0^{-1} J'^{-1} H_0 \widetilde{H}$ which implies $HJ = J'^{-1} H$.

Sufficiency. Assume that $k = k'$, $n = n'$, and $H \colon M^2 \to M^2$ is a homeomorphism such that $PH = HP'$. We set $\overline{h}(z, r) = (H(z), r)$ if $HJ = J'H$ and $\overline{h}(z, r) = (H(z), -r)$ if $HJ = J'^{-1} H$. Then $\overline{h}\gamma = \gamma' \overline{h}$ if $HJ = J'H$ and $\overline{h}\gamma = \gamma'^{-1} \overline{h}$ if $HJ = J'^{-1} H$. Since $(P(H(z)), \varphi(r)) = (H(P'(z)), \varphi(r))$ and $(P(H(z)), \varphi(-r)) = (H(P'(z)), -\varphi(r))$, in both cases, the homeomorphism $\overline{h}$ conjugate homeomorphisms $\overline{\varphi}$ and $\overline{\varphi}'$. By [11, Statement 10.35], $\overline{h}$ is projected to the homeomorphism $h = p_{J'} \overline{h} p_J^{-1}$ conjugating the homeomorphisms φ and φ'. □

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