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Lattice of definability (of reducts) for integers with successor

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Abstract. In this paper the lattice of definability for integers with a successor (the relation $y = x + 1$) is described. The lattice, whose elements are also knows as reducts, consists of three (naturally described) infinite series of relations. The proof uses a version of the Svenonius theorem for structures of special form.

Keywords: definability, reducts, Svenonius theorem.

§ 1. Introduction. Background of the problem

The results of this work are related to the theory of definability, namely to the question of whether or not it is possible to define one relation through other ones. Alfred Tarski said: "Mathematicians, in general, do not like to operate with the notion of definability; their attitude towards this notion is one of distrust and reserve" [1], p. 110. Indeed, there are much fewer results (and publications) in definability theory than in model theory or proof theory. In 1959 Lars Svenonius established a fundamental result in this field (see below), which can be viewed as an analogue of the completeness theorem. Tarski begins his paper "Some methodological investigations on the definiteness of concepts" [1], pp. 296–310, with a parallel between definability and provability.

In the early 1970s Albert Abramovich Muchnik (1934–2019), a student of P. S. Novikov and the scientific supervisor of A. L. Semenov, posed a number of tasks for his disciple concerning definability in structures related to finite automata. One of these tasks was to generalise to the multidimensional case the Cobham theorem on definability of relations definable by finite automata in two number systems. The solution of this problem was one of the results of A. L. Semenov's PhD thesis, see [2]. Later, Andrei Albertovich Muchnik (1958–2007), son of Albert Abramovich and a student of Semenov, found a new proof of the Cobham–Semenov theorem using the concept of self-definability, which was introduced by him. The results of Semenov and Andrei Muchnik have been used and discussed in a number of papers; among recent publications, see, for example, [3].

At the same time, in the early 1970s, Albert Muchnik, with reference to P. S. Novikov, formulated the problem of describing the lattice of definability spaces (the term appeared later) for the addition of integers. The task did not seem so difficult, and A. L. Semenov offered it to his students. Two of them, L. V. Kostyukov

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(later a famous writer) and O. V. Mitina (later a professor of psychology at the Moscow State University), proposed similar hypotheses about the composition of the elements of this lattice, but their proofs contained gaps. It became clear that the problem is not simple at all.

On the other hand, the authors of the present paper also became interested in general problems of definability lattices. In particular, in the classical work of Elgot and Rabin [4] the existence problem for maximal decidable definability spaces was raised in connection with extensions of finite automata definability. For the weak monadic case, this problem was solved by Soprunov in [5]. It remains open for first-order logic and monadic logic.

In the following years, the authors of the present paper together with Andrei Muchnik undertook a study of various issues related to definability lattices; see, for example, [6]. In particular, the authors of this paper have obtained a combinatorial analogue of the Svenonius theorem [7]. Examples of spaces of arbitrary width were constructed [8].

To illustrate the range of results in this area we point out a few examples. The first significant result, a description of the definability lattice of rational numbers with order, was obtained by Claude Fresnay [9] in 1965 (as indicated in [10]). Since then, this result has been repeatedly rediscovered (see [11] and [6]). As shown in these papers, the lattice of subspaces for the order of rational numbers contains five elements. If zero is added to the structure, then there are 116 elements [12]. The lattice for a random graph has five elements. For a random linear order, it has 42 elements [13]. Among the latest results, we note the proof [14] of the absence of spaces lying strictly between the space generated by the relation $+$ and all the constants on the carrier $\mathbb{Z}$ and the space generated by the relations $+$ and $<$ on the same carrier.

The authors of this paper together with V. Uspensky summed up some results in the review [15]. The main result of this paper was also announced therein.

In what follows we give the definitions which are needed, formulate the Svenonius theorem along with a corollary of this theorem, which is what will be used, describe the lattice under investigation, and, finally, formulate some conjectures and open problems.

§ 2. Definitions

Let S be a set of relations on a universe A and let R be the name of a relation on A. To define the *relation* R through S in a logical language L means

(1) to give names to the relations in a finite subset of S, and

(2) to write a formula in the language L that is equivalent to R (on A) using the given names as extra-logical symbols.

The *definability closure* of a set S (denoted by $[S]$) consists of all the relations that can be defined through S. This is a closure operation in the usual topological and algebraic sense (a Kuratowski closure). The set S is a base of the definability closure $[S]$. Closed sets of relations are called *definability spaces*.

The definability spaces of a given universe form the *definability lattice* with the intersection of sets and the closure of the union of sets as the lattice operations.

If S_1 and S_2 are sets of relations on the same universe, then $S_1 \succcurlyeq S_2$ if the definability space generated by S_2 is a subset of the that generated by S_1. If $S_1 \succcurlyeq S_2$ and $S_2 \succcurlyeq S_1$, then $S_1 \approx S_2$. The symbol $\succ$ is understood accordingly.

A definability space is said to be *countable* if it is countable or finite and its universe is countable. In this paper L is a first-order language with equality, and only countable definability spaces are considered. The *definability space* of a structure is the set of all relations definable in it. The *definability lattice* of the structure is formed by all subspaces of this space. Evidently, if two structures are elementary equivalent, then their definability lattices are isomorphic. Well known examples of definability spaces include: arithmetic relations, algebraic relations, automata-definable relations, and Presburger relations (that is, relations generated on $\mathbb{Z}$ by $+$ and $\leqslant$). The subspaces of a space are also called reducts of the original space or the original structure.

The main aim of this study is to describe the definability lattice for integers with successor.

§ 3. Permutations and the Svenonius theorem

The bijections of a set A onto A are also called permutations of A. The group of all permutations of A is denoted by $\mathrm{Sym}(A)$. A permutation φ of a set A *preserves* a relation R on A if and only if $R(\overline{a}) \equiv R(\varphi(\overline{a}))$ for every tuple $\overline{a}$ of elements in A, where $\varphi(\overline{a})$ is the tuple of images of $\overline{a}$ under the permutation φ. A permutation preserves a set of relations S if it preserves all relations in S. A collection F of permutations preserves S if every permutation in F preserves S.

A group $G_S \subseteq \mathrm{Sym}(A)$ can be associated with every set of relations S. The group G_S consists of all permutations of A that preserve S. We have $S_1 \subseteq S_2 \Rightarrow G_{S_1} \supseteq G_{S_2}$ (an antimonotone Galois correspondence). However, it is usually difficult to recover a definability space from the corresponding subgroup of $\mathrm{Sym}(A)$. The group corresponding to a subspace is called a *supergroup* of the group of the original space.

Let S_1 be a definability space on the universe A, let $B \subseteq A$, and let S_2 be the set of restrictions to B of all relations in S_1. Let us name the relations in a finite subset F of S_1 and use the same names for the restrictions of the relations in F to B. Every formula using only these names defines two relations, one on A and one on B. The second relation does not have to be a restriction of the first one to B. However, if it is a restriction for every formula, then S_2 is called an *elementary restriction* of S_1 (and S_1 is an *elementary extension* of S_2). Every elementary restriction of a definability space is also a definability space.

The Svenonius theorem [16] (more precisely, a consequence of this theorem given below) is the main tool used in this study. It can be formulated as follows (see [17], p. 516).

Svenonius Theorem. *Let S^- and S be countable definability spaces on a universe A such that $S^- \subset S$. Then the following two statements are equivalent for every relation $R \in S$:*

(a) $R \in S^-$;

(b) *for every S' which is a countable elementary extension of S and every $S_0 \subset S'$ and $R_0 \in S'$ such that the restriction of S_0 to A coincides with S^-, the restriction of R_0 to A coincides with R, and the group of permutations on the universe of S' that preserves all the relations from S_0 also preserves R_0.*

Thus, the Svenonius theorem states that, if the permutations of elementary extensions of the original space are considered in addition to the permutations of the original space, then it is possible to distinguish the subspaces of the original space.

Examples of using permutation groups to describe definability lattices can be found, for instance, in [10].

A countable definability space S is *maximal* (or upper complete) if the structures of all its countable elementary extensions are isomorphic (with a proper choice of names for the relations).

For maximal spaces, the Svenonius theorem can be simplified as follows.

Corollary 3.1. *Let S^- and S be countable definability spaces on a universe A such that $S^- \subset S$ and the space S is maximal. Then the following two statements are equivalent for every relation $R \in S$:*

(a) $R \in S^-$;

(b) *the group of permutations on A that preserves all relations in S^- also preserves R.*

Proof. The proof is simple and not provided here. □

§ 4. Results

This section describes the lattice of definability spaces of the structure $\langle \mathbb{Z}, \{'\} \rangle$, that is, of the set of integers with the successor relation. For every positive integer n, we define:

$A_{0,n}(x_1, x_2) \leftrightharpoons |x_1 - x_2| = n$;

$A_{1,n}(x_1, x_2, x_3, x_4) \leftrightharpoons x_1 - x_2 = x_3 - x_4 = n \vee x_1 - x_2 = x_3 - x_4 = -n$;

$A_{2,n}(x_1, x_2) \leftrightharpoons x_1 - x_2 = n$.

Thus, for every positive integer n, we have

$$A_{0,n}(x_1, x_2) \equiv A_{1,n}(x_1, x_2, x_1, x_2)$$

and $A_{1,n}$ is explicitly defined by using $A_{2,n}$.

We can naturally extend $A_{0,n}$ to $A_{0,0}$ as an equality. Hence,

$$A_{2,n} \succcurlyeq A_{1,n} \succcurlyeq A_{0,n} \succcurlyeq A_{0,0}.$$

According to [6], a relation will be regarded as false whenever the values of two of its arguments are equal. Such relations will be referred to as *non-gluing*.

It is easy to see that, for every relation R, one can construct a finite class of non-gluing relations that generates the same definability space as R. From now on, all definable relations will be non-gluing.

Theorem 4.1. *If a relation R is definable in $\langle \mathbb{Z}, \{'\} \rangle$, then $R \approx A_{i,d}$ for some positive integer d and some i. Moreover,*

(i) *all the $[A_{i,d}]$ are different*;

(ii) $[A_{i,d}] \cup [A_{j,k}] = [A_{m,n}]$, *where $m = \max\{i, j\}$ and n is the greatest common divisor of d and k*;

(iii) $[A_{i,d}] \cap [A_{j,k}] = [A_{m,n}]$, *where $m = \min\{i, j\}$ and n is the least common multiple of d and k.*

The rest of this paper is devoted to the proof of Theorem 4.1. This proof is based on Corollary 3.1 to the Svenonius theorem.

If a structure M is elementary equivalent to $\langle \mathbb{Z}, \{'\} \rangle$, then M is a disjoint union of copies of $\mathbb{Z}$. These copies are called *galaxies*. All structures M consisting of a countable set of galaxies are isomorphic. So, they are maximal. This structure, which is unique up to isomorphism, is denoted by $\mathbb{Z}^\omega$.

As a result, Corollary 3.1 can be applied to $\mathbb{Z}^\omega$.

Let us define an ordered set $Z_\infty = \mathbb{Z} \cup \{\infty\}$ such that the order on $\mathbb{Z}$ is the standard one and $z < \infty$ for every $z \in \mathbb{Z}$. Then we define the function of absolute value ($||$) on Z_∞ such that it is the standard one on $\mathbb{Z}$ and $|\infty| = \infty$. The subtraction function ($-$) takes $\mathbb{Z}^\omega$ onto Z_∞, coincides with the subtraction inside every galaxy (which is a copy of $\mathbb{Z}$), and is equal to ∞ if the arguments are from different galaxies. The expression $a > b$ for any $a, b \in \mathbb{Z}^\omega$ is an abbreviation of $\infty > a - b > 0$.

Next, a *permutation* is a permutation on the domain of the structure $\mathbb{Z}^\omega$. A permutation γ is called a *shift* if $\gamma(a) - \gamma(b) = a - b$ for every $a, b \in \mathbb{Z}^\omega$. The group of all permutations preserving $'$ is the group of all shifts, which we denote by Γ.

For every $a, b \in \mathbb{Z}^\omega$, there is a shift s such that $s(a) = b$. So, every 1-ary relation definable in $\mathbb{Z}^\omega$ is constant.

Lemma 4.1. *Suppose that tuples $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in \mathbb{Z}^\omega$ are such that $a_i - a_j = b_i - b_j$ for all i, j. Let a partial map γ be also given by the equality $\gamma(a_i) = b_i$ for every $a_0, \ldots, a_{n-1}$. Then γ can be extended to a shift.*

Lemma 4.1 is simple and can be proved by considering elements from the same or different galaxies.

Let a group of permutations Γ' include the group of shifts Γ. Then $z_1, z_2 \in \mathbb{Z}$ are said to be *equivalent* (with respect to Γ') if the equality $\gamma(a + z_1) - \gamma(a) = z_2$ holds for some $\gamma \in \Gamma'$ and $a \in \mathbb{Z}^\omega$. The relation thus defined is an equivalence in view of the properties of the supergroup containing the group of shifts. Let us prove the transitivity (the proof of the other properties is even easier). By definition, if $z_1 \sim z_2$ and $z_2 \sim z_3$, then the equalities $\gamma_1(a_1 + z_1) - \gamma_1(a_1) = z_2$ and $\gamma_2(a_2 + z_2) - \gamma_2(a_2) = z_3$ hold for some γ_1, a_1, γ_2, a_2. Then $\gamma(a_1 + z_1) - \gamma(a_1) = z_3$, where $\gamma = \gamma_2 \circ s \circ \gamma_1$ and s is a shift such that $s(\gamma_1(a_1)) = a_2$, and so $z_1 \sim z_3$. The equivalence class of z (with respect to Γ') is denoted by K_z. A number $z \in \mathbb{Z}$, $z \neq 0$, is *regular* (with respect to Γ') if K_z is finite and $|\gamma(a + z) - \gamma(a)| < \infty$ for every $a \in \mathbb{Z}^\omega$ and $\gamma \in \Gamma'$.

The equivalence, even though it is simple, proves extremely useful for our considerations. Let us take a look at the main examples. If $\Gamma' = \Gamma$, then the equivalence is trivial. If the group Γ' is generated by the set Γ and the permutation $x \mapsto -x$ on each copy of $\mathbb{Z}$ in $\mathbb{Z}^\omega$, then every $z \in \mathbb{Z}$ such that $z \neq 0$ is regular and $K_z = \{z, -z\}$. A slightly more sophisticated case is typical in our context. Let Γ' be generated by Γ and a permutation γ such that $\gamma(x) = x$ for all elements of $\mathbb{Z}^\omega$, except for one copy of $\mathbb{Z}$, where $\gamma(x) = x$ for even values of x and $\gamma(x) = -x$ for odd values of x. Then each even $z \neq 0$ is regular with $K_z = \{z, -z\}$, and no odd z is regular. Finally, there is no regular number for the group of all permutations.

Lemma 4.2. (a) *If z_1 and z_2 are regular numbers, then the numbers $z_1 \pm z_2$ are also regular.*

(b) *The greatest common divisor of two regular numbers is regular.*

Proof. (a) Let z_1 and z_2 be regular and let $\gamma \in \Gamma'$. Then

$$\gamma(a + z_1 + z_2) - \gamma(a) = \gamma(a + z_1 + z_2) - \gamma(a + z_1) + \gamma(a + z_1) - \gamma(a)$$

and

$$\gamma(a + z_1 + z_2) - \gamma(a + z_1) \in K_{z_2}, \qquad \gamma(a + z_1) - \gamma(a) \in K_{z_1}.$$

So, the set $K_{z_1+z_2}$ is finite and does not contain ∞.

The case of $z_1 - z_2$ is completely similar.

Part (b) follows from (a). □

Lemma 4.3. *Let a group of permutations* Γ' *include* Γ *and let* d *be the greatest common divisor of all regular numbers with respect to* Γ'. *Then* $K_d = \{d\}$ *or* $K_d = \{d, -d\}$.

Moreover, if $K_d = \{d\}$, *then* $K_z = \{z\}$ *for every multiple* z *of* d. *And if* $K_d = \{d, -d\}$, *then* $K_z = \{z, -z\}$ *for every multiple* z *of* d.

Proof. Let D be a number equivalent to d with respect to Γ' with maximal absolute value. Then $D = N \cdot d$ or $D = -N \cdot d$ for some positive integer N.

Suppose that $N > 1$ and choose $\gamma \in \Gamma'$ and $a, b \in \mathbb{Z}^\omega$ such that $b - a = D$ and $\gamma(b) - \gamma(a) = d$. Then, for every k such that $0 \leqslant k < N$, we put

$$C_k = \{c_{k,i} \mid c_{k,i} = a + k \cdot d + i \cdot D,\ i \in \mathbb{Z}\}.$$

The collection $\{C_k\}$ is a partition of the set $\{a + z \cdot d \mid z \in \mathbb{Z}\}$. Since d is regular, the difference $\gamma(c) - \gamma(a)$ is a multiple of d for each $c \in C_k$, $0 \leqslant k < N$. Thus, the collection $\{\gamma(C_k)\}$ is a partition of the set $\{\gamma(a) + z \cdot d \mid z \in \mathbb{Z}\}$.

Let us consider the set

$$E = \{\gamma(a), \gamma(a) + d, \dots, \gamma(a) + (N-1) \cdot d\}.$$

Since $\gamma(a), \gamma(b) \in \gamma(C_0) \cap E$ and there are exactly N elements in E, there is a k' such that $0 \leqslant k' < N$ and $\gamma(C_{k'}) \cap E = \varnothing$. Since the absolute value of D is maximal in the equivalence class of d, it follows that all the elements of $\gamma(C_{k'})$ have to lie on one side of the segment E. Hence, either $\gamma(c) < \gamma(a)$ for each $c \in C_{k'}$ or $\gamma(a) + (N-1) \cdot d < \gamma(c)$ for each $c \in C_{k'}$. Otherwise there is a $c_{k',i}$ such that $|\gamma(c_{k',i+1}) - \gamma(c_{k',i})| > D$ and $c_{k',i+1} - c_{k',i} = D$, and D has maximal absolute value in its equivalence class. Suppose that $\gamma(c) < \gamma(a)$ for every $c \in C_{k'}$ (the other case is completely similar). Then there is a k' such that $0 \leqslant k'' < N$ and the set $\{c \in C_{k''} \mid \gamma(c) > \gamma(a)\}$ is infinite. Therefore, the absolute value of the difference $|\gamma(a + k' \cdot d + z \cdot D) - \gamma(a + k'' \cdot d + z \cdot D)|$ is unbounded for $z \in \mathbb{Z}$, which contradicts the regularity of $(k' - k'') \cdot d$.

So, $N = 1$ and $K_d = \{d\}$ or $K_d = \{d, -d\}$.

If $K_d = \{d\}$, then $K_z = \{z\}$ for every z that is a multiple of d. Suppose that $K_d = \{d, -d\}$ and $z = n \cdot d$, where n is a positive integer (the case when $z = -n \cdot d$ is similar). Then, for every i such that $0 \leqslant i < n$, every $\gamma \in \Gamma'$, and every $a \in \mathbb{Z}^\omega$, we have

$$\gamma(a + (i+1) \cdot d) - \gamma(a + i \cdot d) = d$$

or

$$\gamma(a + (i+1) \cdot d) - \gamma(a + i \cdot d) = -d.$$

Moreover, since

$$\gamma(a + (i+2) \cdot d) \neq \gamma(a + i \cdot d),$$

all the differences have the same sign. This means that

$$\gamma(a + n \cdot d) - \gamma(a) = n \cdot d$$

or

$$\gamma(a + n \cdot d) - \gamma(a) = -n \cdot d$$

for each $a \in \mathbb{Z}^\omega$, that is, $K_z \subset \{z, -z\}$. Since there are $\gamma \in \Gamma'$ and $a \in \mathbb{Z}^\omega$ such that $\gamma(a + d) - \gamma(a) = -d$, it follows that $\gamma(a + z) - \gamma(a) = -z$ and $-z \in K$. We can similarly obtain $z \in K$, that is, $K_z = \{z, -z\}$. □

Let a group of permutations Γ' include Γ and let d be the greatest common divisor of all the regular numbers with respect to Γ'. Then, for every $\gamma \in \Gamma'$, there are three possibilities:

(1) $\gamma(a + n \cdot d) - \gamma(a) = n \cdot d$
for every $a \in \mathbb{Z}^\omega$ and every positive integer n (such permutations γ are called permutations *of the first type*);

(2) $\gamma(a + n \cdot d) - \gamma(a) = -n \cdot d$
for every $a \in \mathbb{Z}^\omega$ and every positive integer n (such permutations γ are called permutations *of the second type*);

(3) $\gamma(a + n \cdot d) - \gamma(a) = n \cdot d$ or $\gamma(a + n \cdot d) - \gamma(a) = -n \cdot d$
for every $a \in \mathbb{Z}^\omega$ and every positive integer n, where each of the equalities is realised for some a and n (such permutations γ are called permutations *of the third type*).

If $K_d = \{d\}$, then every permutation $\gamma \in \Gamma'$ is of the first type. If $K_d = \{d, -d\}$, then a permutation $\gamma \in \Gamma'$ can be of the first, second, or third type.

From now on, Γ_R denotes the group of all permutations that preserve a relation R definable in $\langle \mathbb{Z}^\omega, \{'\} \rangle$.

For every $d \in \mathbb{Z}$ such that $d > 0$:

– $\Gamma_{A_{0,d}}$ is the group of all permutations γ such that $|\gamma(a + d) - \gamma(a)| = d$ for every $a \in \mathbb{Z}^\omega$; it is easy to notice that the regular numbers of this group are of the form $\{z \cdot d \mid z \text{ is an integer}\}$, and so this group consists of permutations of all three types;

– $\Gamma_{A_{1,d}}$ is the (proper) subgroup of $\Gamma_{A_{0,d}}$ consisting of all permutations of the first and second types;

– $\Gamma_{A_{2,d}}$ is the (proper) subgroup of $\Gamma_{A_{1,d}}$ consisting of all permutations of the first type.

Corollary 4.1. *For all positive integers d, k and $0 \leqslant i, j \leqslant 2$:*

(i) $i \neq j \vee d \neq k \rightarrow [A_{i,d}] \neq [A_{j,k}]$;

(ii) $[A_{i,d}] \succ [A_{i-1,d}]$;

(iii) $[A_{i,d}] \succ [A_{i,d \cdot k}]$;

(iv) $[A_{i,d}] \cup [A_{j,k}] = [A_{m,n}]$, *where $m = \max\{i, j\}$ and $n =$ the greatest common divisor of d and k;*

(v) $[A_{i,l}] \cap [A_{j,k}] = [A_{m,n}]$, *where $m = \min\{i, j\}$ and $n =$ the least common multiple of l and k.*

Proof. Parts (i)–(iv) follow directly from Corollary 3.1 and a comparison of the corresponding groups.

Let us give a sketch of the proof of (v). In view of Corollary 3.1, it should be shown that the group $\Gamma_{A_{m,n}}$ is generated by the union of $\Gamma_{A_{i,l}}$ and $\Gamma_{A_{j,k}}$. The implication in one direction is obvious since $\Gamma_{A_{i,l}}$ and $\Gamma_{A_{j,k}}$ are subgroups of $\Gamma_{A_{m,n}}$. Let $\gamma \in \Gamma_{A_{m,n}}$. We intend to show that γ is a composition of permutations in $\Gamma_{A_{i,l}}$ and $\Gamma_{A_{j,k}}$. For simplicity, we restrict ourselves to considering a single galaxy. Let us choose an arbitrary element a in this galaxy and assume that $l = l_1 \cdot d$, $k = k_1 \cdot d$, and $n = l_1 \cdot k_1 \cdot d$, where d is the greatest common divisor of l and k.

Since $\gamma(a+n) = \gamma(a) \pm n$, it follows that in one of the two groups (suppose that in $\Gamma_{A_{i,l}}$) there is a permutation γ' such that the sign of $\gamma(a+l) - \gamma(a)$ coincides with that of $\gamma'(a+n) - \gamma'(a)$. We put $|l'| = l$. The sign of l' coincides with that of $\gamma'(a+n) - \gamma'(a)$. We can assume that the permutation γ' transposes the pairs of elements $\langle a + z \cdot l,\, \gamma(a) + z \cdot l' \rangle$ and is identical on all other elements. We note that the permutations γ and γ' coincide on elements of the form $a + z \cdot n$. Let us now choose a permutation $\gamma'' \in \Gamma_{A_{j,k}}$ which transposes the pairs of elements $\langle a + z_1 \cdot l + z_2 \cdot k,\, \gamma(a) + z_1 \cdot l' + z_2 \cdot k \rangle$, where $0 < z_1 < k_1$, $z_1, z_2 \in \mathbb{Z}$, and is identical on all other elements (and, in particular, on elements of the form $\gamma(a + z \cdot n)$).

It is easy to see that the permutation $\gamma_0 = \gamma'' \circ \gamma'$ coincides with γ on all elements of the form $a + z \cdot n$ and is identical on all other elements.

Substituting elements of the form $a + c$ with $c < n$ in place of a, we can construct a set of n permutations whose composition coincides with γ on the given galaxy. □

If m is a positive integer, then two vectors $\overline{a}, \overline{b} \in \mathbb{Z}^\omega$ of the same length are said to be *m-indistinguishable* if $a_i - a_j = b_i - b_j$ for all i, j such that $|a_i - a_j| < m$ or $|b_i - b_j| < m$.

Lemma 4.4. *For every formula R in the signature $\{'\}$, there is a positive integer w such that $R(\overline{a}) \equiv R(\overline{b})$ for every two w-indistinguishable tuples $\overline{a}, \overline{b} \in \mathbb{Z}^\omega$.*

Proof. Let a formula $Q(w, \overline{x}, \overline{y})$ in the signature $\{+, <\}$ express the fact that the tuples $\overline{x}$ and $\overline{y}$ are w-indistinguishable. Then the assertion of Lemma 4.4 can be written in the form

$$(\exists\, w)(\forall\, \overline{x})(\forall\, \overline{y})(Q(w, \overline{x}, \overline{y}) \to (R(\overline{x}) \equiv R(\overline{y}))).$$

Let us consider a countable non-standard elementary extension M_0 of the structure $\langle \mathbb{Z}, \{+, <\} \rangle$ and an arbitrary non-standard number $w_0 > 0$ in M_0. For all tuples $\overline{a}, \overline{b} \in M_0$, it follows from $Q(w_0, \overline{a}, \overline{b})$ that, for every standard k, the equation $a_i - a_j = k$ holds if and only if $b_i - b_j = k$. The structures M_0 and $\mathbb{Z}^\omega$ are isomorphic (as structures with the only relation $'$), and therefore Lemma 4.1 can be used to extend the map $f(a_i) = b_i$ to a shift. Thus

$$(\forall\, \overline{a})(\forall\, \overline{b})(Q(w_0, \overline{a}, \overline{b}) \to (R(\overline{a}) \equiv R(\overline{b})))$$

is true in M_0. Then

$$(\forall\, \overline{a})(\forall\, \overline{b})(Q(m, \overline{a}, \overline{b}) \to (R(\overline{a}) \equiv R(\overline{b})))$$

is true for any standard number m as well. Evidently, M_0 is the enrichment of the maximal structure having the only relation $'$, and therefore the corresponding depletion of M_0 is isomorphic to Z^ω. So the assertion of the lemma holds in $\mathbb{Z}^\omega$. □

A positive integer w is called a *boundary* of a definable relation R if and only if $R(\overline{a}) \equiv R(\overline{b})$ for every two w-indistinguishable vectors $\overline{a}, \overline{b} \in \mathbb{Z}^\omega$.

The following lemma states that, if the differences between some elements of a vector $\overline{a}$ are non-regular (with respect to the group Γ_R), then they can be replaced by elements for which the corresponding differences are infinite, without changing the values of R.

Lemma 4.5. *Let an n-ary relation R be definable in $\langle \mathbb{Z}^\omega, \{'\} \rangle$ and let*

$$\overline{a} = \langle a_0, \ldots, a_{n-1} \rangle \in \mathbb{Z}^\omega.$$

Then a tuple

$$\overline{b} = \langle b_0, \ldots, b_{n-1} \rangle \in \mathbb{Z}^\omega$$

can be constructed such that

(i) $R(\overline{a}) \equiv R(\overline{b})$;

(ii) *if the difference* $a_i - a_j \neq 0$ *is non-regular with respect to* Γ_R, *then* $b_i - b_j = \infty$;

(iii) *if the difference* $a_i - a_j$ *is regular, then* $|a_i - a_j| = |b_i - b_j|$; *moreover, if* Γ_R *contains permutations of the first type only, then* $a_i - a_j = b_i - b_j$;

(iv) *if* Γ_R *contains no permutation of the third type, then*

– *either* $a_i - a_j = b_i - b_j$ *for every* i *and* j *with a regular difference* $a_i - a_j$,

– *or* $a_i - a_j = b_j - b_i$ *for every* i *and* j *with a regular difference* $a_i - a_j$.

Proof. The lemma can be proved by induction on the number of pairs i, j such that $a_i - a_j$ is a finite non-zero non-regular number. Suppose that $a_0 - a_1 \neq 0$ is finite and non-regular. It is possible to construct a vector $\overline{b}$ and a permutation $\gamma \in \Gamma_R$ such that

(a) $R(\overline{a}) \equiv R(\overline{b})$;

(b) $b_0 - b_1 = \infty$;

(c) for every $i, j < n$, if $a_i - a_j = \infty$, then $b_i - b_j = \infty$;

(d) for every $i, j < n$, if $b_i - b_j = \infty$, then $a_i - a_j$ is non-regular;

(e) for every $i, j < n$, if $b_i - b_j < \infty$, then $b_i - b_j = \gamma(a_i) - \gamma(a_j)$.

Let w be a boundary of R and let w' be the maximal absolute value among the elements of the set $\bigcup\{K_{a_i - a_j} \mid \text{the difference } a_i - a_j \text{ is regular}\}$.

Assume that the non-zero finite difference $a_0 - a_1$ is non-regular and take a permutation $\gamma' \in \Gamma_R$ for which $|\gamma'(a_0) - \gamma'(a_1)| > n \cdot \max(w, w')$. Let us choose a shift s such that $s(a_0) = a_0$, $s(a_1) = a_1$, and if $a_i - a_j = \infty$, then $|\gamma'(s(a_i)) - \gamma'(s(a_j))| > n \cdot \max(w, w')$.

We put $\gamma = \gamma' \circ s$. Then $|\gamma(a_0) - \gamma(a_1)| > n \cdot \max(w, w')$ and $|\gamma(a_i) - \gamma(a_j)| > n \cdot \max(w, w')$ if $a_i - a_j = \infty$.

We put $\overline{a}' = \gamma(\overline{a})$. In this case

– if $a_i - a_j < \infty$, then $a'_i - a'_j = \gamma(a_i) - \gamma(a_j)$.

Now we want to transform a' into a vector b such that

– if $|a'_i - a'_j| < n \cdot \max(w, w')$, then $b_i - b_j = a'_i - a'_j$;

– if $|a'_i - a'_j| > n \cdot \max(w, w')$, then $b_i - b_j = \infty$ (in particular, $b_1 - b_0 = \infty$).

In the process of transformation, we consider in succession the pairs of elements of a' such that $|a'_i - a'_j| > n \cdot \max(w, w')$ and $|a'_i - a'_j| \neq \infty$. Thus, the elements a'_i and a'_j lie in the same galaxy U. Suppose that $a'_i < a'_j$. Let $c_0 < \cdots < c_k$ be all the elements of $\overline{a}'$ in U. There is an element c_m such that $a'_0 \leqslant c_m < c_{m+1} \leqslant a'_1$ and $c_{m+1} - c_m > \max(w, w')$. Let us choose a vector $\overline{b}$ in such a way that

– $b_i = a'_i$ if $a'_i \notin U$ or $a'_i \leqslant c_m$;

– all elements of $\{b_i \mid a'_i \geqslant c_{m+1},\ a'_i \in U\}$ lie in a separate galaxy that does not contain elements of $\overline{a}'$.

Due to Lemma 4.4, $R(\overline{a}') \equiv R(\overline{b})$ and $R(\overline{a}) \equiv R(\overline{a}')$.

Since $c_{m+1} - c_m > w'$, it follows that the difference $b_i - b_j$ is regular if and only if $a_i - a_j$ is a regular number. Thus, it is clear that conditions (a)–(e) hold.

Evidently, (i) follows from (a), and (iii)–(iv) follow from (d) and (e). It follows from (b)–(e) that the number of finite non-zero non-regular differences between the elements of $\overline{b}$ is less than that of $\overline{a}$.

It can be assumed that there are no pairs i, j for which $a_i - a_j$ is finite and non-regular. This gives us $\overline{b}$ as the required vector. □

The following statement claims that, for every non-trivial relation R definable in $\langle \mathbb{Z}^\omega, \{'\}\rangle$, the corresponding group of permutations Γ_R coincides with $\Gamma_{A_{i,j}}$ for some i, j. Thus, according to Corollary 3.1, $R \approx A_{i,j}$.

Proposition 4.1. *Every non-gluing relation R is either trivial or there is the greatest common divisor d of all regular numbers with respect to Γ_R. In this case*

(i) *if Γ_R contains no second- or third-type permutations, then $R \approx A_{2,d}$;*

(ii) *if Γ_R contains no third-type permutations and does contain a second-type permutation, then $R \approx A_{1,d}$;*

(iii) *if Γ_R contains a third-type permutation, then $R \approx A_{0,d}$.*

Proof. First of all, we note that, according to Lemma 4.5, if there are no regular numbers with respect to Γ_R, then $R(\overline{a}) \equiv R(\overline{b})$ for all tuples $\overline{a}$ and $\overline{b}$ such that $a_i \neq a_j$ and $b_i - b_j = \infty$, and thus a non-gluing relation without regular numbers is trivially definable through equality.

Secondly, it can readily be seen that: if the group Γ_R contains no second- or third-type permutations, then $R \succcurlyeq A_{2,d}$; if Γ_R contains no third-type permutations and contains second-type permutations, then $R \succcurlyeq A_{1,d}$; and if Γ_R contains a third-type permutation, then $R \succcurlyeq A_{0,d}$.

To prove the converse, we need to show that every permutation that preserves $A_{2,d}$ ($A_{1,d}$, $A_{0,d}$) belongs to Γ_R if the corresponding conditions hold.

Let Γ' denote the set of permutations γ such that $|\gamma(a+d) - \gamma(a)| = d$ for every $a \in \mathbb{Z}^\omega$. As mentioned above, $\Gamma' = \Gamma_{A_{0,d}}$. The subgroup of Γ' consisting of the first- and second-type permutations forms the group $\Gamma_{A_{1,d}}$. The subgroup of Γ' consisting of the first-type permutations is the group $\Gamma_{A_{2,d}}$.

The proofs of all three statements (i)–(iii) follow the same scheme: we suppose that there is a tuple $\overline{a} \in \mathbb{Z}^\omega$ and a permutation $\gamma \in \Gamma_{A_{i,d}} \setminus \Gamma_R$ for which $R(\overline{a}) \not\equiv R(\gamma(\overline{a}))$. Next, Lemma 4.5 is used to construct vectors $\overline{b}$ and $\overline{c}$ such that $R(\overline{a}) \equiv R(\overline{b})$; then $R(\gamma(\overline{a})) \equiv R(\overline{c})$. With the aid of Lemma 4.4, we can find a boundary w of R. Then, if necessary, w-indistinguishable vectors $\overline{v_1}$ and $\overline{v_2}$ can be constructed with $R(\overline{b}) \equiv R(\overline{v_1})$ and $R(\overline{c}) \equiv R(\overline{v_2})$. This contradicts Lemma 4.4.

To prove (i), suppose that there is a first-type permutation γ in $\Gamma' \setminus \Gamma_R$. Then there is a vector $\overline{a} = \langle a_0, \ldots, a_{n-1}\rangle \in \mathbb{Z}^\omega$ such that $R(\overline{a}) \not\equiv R(\gamma(\overline{a}))$. By the definition of the group Γ', if $a_i - a_j$ is non-regular with respect to Γ_R, then the difference $\gamma(a_i) - \gamma(a_j)$ is also non-regular. By the definition of a first-type permutation, if $a_i - a_j$ is regular with respect to Γ_R, then $\gamma(a_i) - \gamma(a_j) = a_i - a_j$. Lemma 4.5 can be used to choose vectors $\overline{b}$ and $\overline{c}$ that correspond to $\overline{a}$ and $\gamma(\overline{a})$. Since Γ_R contains first-type permutations only, it follows that the regular differences of the vectors $\overline{b}$ and $\overline{c}$ are the same, and $\overline{b}$ and $\overline{c}$ are w-indistinguishable for every finite w. Thus, $R(\overline{b}) \equiv R(\overline{c})$, a contradiction.

To prove (ii), suppose that there is a first- or second-type permutation γ in $\Gamma' \setminus \Gamma_R$. Vectors $\overline{a}$, $\overline{b}$, and $\overline{c}$ can be selected as in the proof of (i).

If $b_i - b_j = c_i - c_j$ with a regular difference $b_i - b_j$, then the same equality holds for every regular difference, so this case is the same as (i).

If $b_i - b_j = c_j - c_i$ with a regular difference $b_i - b_j$, then the same equality holds for all regular differences of $\overline{b}$. There is a second-type permutation γ' in Γ_R, that is, $\gamma'(t + z \cdot d) - \gamma'(t) = -z \cdot d$ for every $z \in \mathbb{Z}$ and $t \in \mathbb{Z}^\omega$. Let us fix some $t \in \mathbb{Z}^\omega$ and consider the set $S = \{t + d \cdot z \mid z \in \mathbb{Z}\}$.

Let w be a boundary of R.

There is a vector $\overline{s}$ in S such that

– if $|c_i - c_j| < \infty$, then $s_i - s_j = c_i - c_j$;

– if $|c_i - c_j| = \infty$, then $|s_i - s_j| > w$.

By Lemma 4.4, $R(\overline{c}) \equiv R(\overline{s})$ holds. Since $s_i - s_j = \gamma'(s_j) - \gamma'(s_i)$ for all $i, j < n$, it follows that the vectors $\overline{b}$ and $\gamma'(\overline{s})$ are w-indistinguishable. This is a contradiction.

To prove (iii), suppose that there is a permutation γ in $\Gamma' \setminus \Gamma_R$. The vectors $\overline{a}$, $\overline{b}$, and $\overline{c}$ can be selected as in case (i).

If the permutation γ is of the first type, then the proof coincides with that in case (i).

If there is a permutation of the second or third type, then we take a permutation γ' of the third type in the group Γ_R and $t_1, t_2 \in \mathbb{Z}^\omega$ such that, for every $z \in \mathbb{Z}$, we have

$$\gamma'(t_1 + z \cdot d) - \gamma'(t_1) = z \cdot d, \qquad \gamma'(t_2 + z \cdot d) - \gamma'(t_2) = -z \cdot d.$$

Let us consider the sets $S_1 = \{t_1 + z \cdot d \mid z \in \mathbb{Z}\}$ and $S_2 = \{t_2 + z \cdot d \mid z \in \mathbb{Z}\}$.

If the permutation γ is of the second type, then the proof coincides with that of case (ii) if we take S_2 for S.

Finally, if γ is a permutation of the third type, then the following fact holds: if the difference $b_i - b_j$ is regular, then $b_i - b_j = c_i - c_j$ or $b_i - b_j = c_j - c_i$.

If $b_i - b_j = c_i - c_j$ and $b_k - b_l = c_l - c_k$, then the difference $b_k - b_i$ is non-regular, and so c_i and c_k lie in different galaxies.

Let w be a boundary of R. A collection $\{s_0, \dots, s_{n-1}\} \subset S_1 \cup S_2$ can be chosen in such a way that

– if $|c_i - c_j| < \infty$, then $s_i - s_j = c_i - c_j$;

– if $|c_i - c_j| = \infty$, then $|s_i - s_j| > w$ and $|\gamma'(s_i) - \gamma'(s_j)| > w$;

– if the difference $b_i - b_j$ is regular and $b_i - b_j = c_i - c_j$, then $s_i, s_j \in S_1$;

– if the difference $b_i - b_j$ is regular and $b_i - b_j = c_j - c_i$, then $s_i, s_j \in S_2$.

By Lemma 4.4, $R(\overline{s}) \equiv R(\overline{c})$. The equality $\gamma'(s_i) - \gamma'(s_j) = b_i - b_j$ holds for every regular difference $b_i - b_j$, so the vectors $\overline{b}$ and $\gamma'(\overline{s})$ are w-indistinguishable. We have arrived at a contradiction. □

Proposition 4.1 together with Corollary 4.1 form Theorem 4.1.

§ 5. Open problems and conjectures

Specific structures.

1. To obtain an explicit description of the definability lattice for the order on non-negative rational numbers based on the general description in [12] or independently of this description. It is clear that such a lattice includes the spaces with bases:

– a relation that highlights the zero;

– an order on the whole universe;

– the usual generators for the rational order restricted to positive rationals and false if one of the arguments is zero.

What else?

2. The order on integers. Can the definability lattice be obtained for this order by a natural combination of relations from this work and analogues for integers of relations from the lattice for the order of rational numbers?

Searches for structures that are elementary equivalent to maximal ones.

What can be said about the existence of maximal structures that are elementary equivalent to the following ones:

3. The order on integers.

4. The successor on positive integers.

5. An infinite undirected connected graph without cycles, all of whose vertices have degree three.

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