PICARD GROUP OF CONNECTED AFFINE ALGEBRAIC GROUP

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Abstract. We prove that the Picard group of a connected affine algebraic group G is isomorphic to the fundamental group of the derived subgroup of the reductive algebraic group $G/\mathscr{R}_u(G)$, where $\mathscr{R}_u(G)$ is the unipotent radical of G.

Below all algebraic varieties are defined over an algebraically closed field k. We follow the point of view on algebraic groups accepted in [1]. By the fundamental group of a connected semisimple algebraic group S we mean the kernel of the canonical isogeny $\widetilde{S} \to S$, where $\widetilde{S}$ is the universal covering of the group S. Recall that the derived subgroup of a connected reductive algebraic group is connected and semisimple (see [1, Sects. I.2.2 and II.14.2]). If G is a connected algebraic group, then by its Picard group $\mathrm{Pic}(G)$ we mean the Picard group of its underlying group variety. If the groups A and B are isomorphic, then we write $A \cong B$. By $\mathbb{A}^d$ we denote the d-dimensional affine space.

The purpose of this note is to prove the following theorem.

Theorem. *Let G be a connected affine algebraic group and let $\mathscr{R}_u(G)$ be its unipotent radical. Then the group* $\mathrm{Pic}(G)$ *is isomorphic to the fundamental group of the derived subgroup of the reductive algebraic group $G/\mathscr{R}_u(G)$.*

Example. Let $G = \mathrm{GL}_n$. Then the group $\mathscr{R}_u(G)$ is trivial, and the derived group of the group G is the semisimple group SL_n. The latter is simply connected, so its fundamental group is trivial. Therefore, $\mathrm{Pic}(G) = 0$ by Theorem 1. This agrees with the fact that the group variety of the group GL_n is isomorphic to an open subset of the affine space $\mathbb{A}^{n^2}$.

The following lemma is used in the proof of this theorem.

Lemma. *Let X be an irreducible smooth algebraic variety, and let U be a nonempty open subset of $\mathbb{A}^d$. Then* $\mathrm{Pic}(X \times U) \cong \mathrm{Pic}(X)$.

Proof of Lemma. We can assume that $d > 0$. Let $D_1, \ldots, D_m$ be all irreducible $(d-1)$-dimensional components of the variety $\mathbb{A}^d \setminus U$. From $\mathrm{Pic}(\mathbb{A}^d) = 0$ it follows that every divisor D_i in $\mathbb{A}^d$, and hence the divisor $B_i := X \times D_i$ in $X \times \mathbb{A}^d$ is principal. Therefore (see [4, Chap. II, Prop. 6.5(c)]),

$$\mathrm{Pic}(X \times \mathbb{A}^d) \cong \mathrm{Pic}\big((X \times \mathbb{A}^d) \setminus \textstyle\bigcup_{i=1}^m B_i\big). \tag{1}$$

If the complement of $X \times U$ in $(X \times \mathbb{A}^d) \setminus \bigcup_{i=1}^m B_i$ is not empty, then each of its irreducible components has in $(X\ times \mathbb{A}^d) \setminus \bigcup_{i=1}^m B_i$ codimension is at least 2 and therefore (see [4, Chap. II, Prop. 6.5(b)]),

$$\mathrm{Pic}\big((X \times \mathbb{A}^d) \setminus \textstyle\bigcup_{i=1}^m B_i\big). \cong \mathrm{Pic}(X \times U) \tag{2}$$

At the same time, as is known (see [4, Chap. II, Prop. 6.6]),

$$\mathrm{Pic}(X \times \mathbb{A}^d) \cong \mathrm{Pic}(X). \tag{3}$$

The assertion of the lemma is now a consequence of the formulas (1), (2), and (3).

Proof of Theorem. The connected affine algebraic group $\mathscr{R}_u(G)$ is unipotent. Therefore, it follows from [5, Props. 1, 2] that the group varieties of the groups G and $\big(G/\mathscr{R}_u(G)\big) \times \mathscr{R}_u(G)$ are isomorphic, and the group variety of the group $\mathscr{R}_u(G)$ is isomorphic to an affine space. From this and from the lemma we obtain

$$\mathrm{Pic}(G) \cong \mathrm{Pic}\big(G/\mathscr{R}_u(G)\big). \tag{4}$$

Let S and Z be, respectively, the derived group and the connected component of the identity of the center of the connected reductive algebraic group $G/\mathscr{R}_u(G)$. According to [3, Thm. 1], the group varieties of the groups $G/\mathscr{R}_u(G)$ and $S \times Z$ are isomorphic. Since the affine algebraic group Z is a torus, its group variety is isomorphic to an open subset of some affine space. From this and from the lemma we get

$$\mathrm{Pic}\big(G/\mathscr{R}_u(G)\big) \cong \mathrm{Pic}(S). \tag{5}$$

Consider the universal covering $\widetilde{S}$ of the connected semisimple algebraic group S, and let $\pi\colon \widetilde{S} \to S$ be the canonical isogeny. Since the algebraic group $\widetilde{S}$ is simply connected semisimple, it follows from [2, Prop. 1] that $\mathrm{Pic}(\widetilde{S}) = 0$. Since, in view of the semisimplicity of the group $\widetilde{S}$, the group $\mathrm{Hom}_{\mathrm{alg}}(S, \mathbb{G}_m)$ is trivial, from this and from [2, Col. of Thm. 4] we obtain

$$\mathrm{Pic}(S) \cong \mathrm{Hom}_{\mathrm{alg}}\big(\ker(\pi), \mathbb{G}_m\big). \tag{6}$$

Finally, since the group $\ker(\pi)$ is finite and abelian, we have

$$\mathrm{Hom}_{\mathrm{alg}}\big(\ker(\pi), \mathbb{G}_m\big) \cong \ker(\pi). \tag{7}$$

The assertion of the theorem now follows from the formulas (4), (5), (6), and (7).

REMARK. The above theorem corrects Theorem 6 of [2]. The latter asserts that the group $\mathrm{Pic}(G)$ (in the notation used above) is isomorphic to the fundamental group of the semisimple group $G/\mathscr{R}(G)$, where $\mathscr{R}(G)$ is the solvable radical of G. If the group extension $1 \to \mathscr{R}(G) \to G \to G/\mathscr{R}(G) \to 1$ splits, i.e., G is isomorphic to a semidirect product of $G/\mathscr{R}(G)$ and $\mathscr{R}(G)$, then $G/\mathscr{R}(G)$ is isomorphic to the derived group of $G/\mathscr{R}_u(G)$, and so the formulated assertion is true in view of the theorem proved above. But in general this is not the case, as the example above shows: in it, $G/\mathscr{R}(G)$ is the group PGL_n whose fundamental group is isomorphic to the group of all n-th roots of 1 in the field k. This latter group is nontrivial if n is not a power of the characteristic of k (however, $\mathrm{Pic}(G) = 0$ for every n).

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