

RATIONAL DIFFERENTIAL FORMS ON THE VARIETY OF FLEXES OF PLANE CUBICS

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ABSTRACT. We prove that for every positive integer d , there are no nonzero regular differential d -forms on every smooth irreducible projective algebraic variety birationally isomorphic to the variety of flexes of plane cubics.

Below we use the standard notation from [3], [8].

Consider a three-dimensional complex vector space V and fix a basis x_1, x_2, x_3 of the dual space V^* . In the space $S^3(V^*)$ of degree three forms on the space V , all monomials of the form $x_1^{i_1} x_2^{i_2} x_3^{i_3}$, where $i_1 + i_2 + i_3 = 3$, constitute (for their fixed ordering) a basis. Let $\{a_{i_1, i_2, i_3} \mid i_1 + i_2 + i_3 = 3\}$ be the dual to it basis of the dual space $(S^3(V^*))^*$.

The sets of forms $\{x_j\}$ and $\{a_{i_1, i_2, i_3}\}$ are the projective coordinate systems on the projective spaces $\mathbf{P}^2 := PV$ and $\mathbf{P}^9 := PS^3(V^*)$ associated with V and $S^3(V^*)$ respectively. Let

$$\pi: S^3(V^*) \setminus \{0\} \rightarrow \mathbf{P}^9$$

be the natural projection. Consider the following forms on the product $\mathbf{P}^2 \times \mathbf{P}^9$:

$$F := \sum_{i_1+i_2+i_3=3} a_{i_1, i_2, i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}, \quad H := \det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right).$$

The closed subset X of $\mathbf{P}^2 \times \mathbf{P}^9$, defined by the system of equations $F = H = 0$ was explored in the papers [4], [6], [7]. It is irreducible, nine-dimensional and has singularities (as a matter of fact, X even is non-normal) [7]. Let

$$\mathbf{P}^2 \xleftarrow{p_2} X \xrightarrow{p_9} \mathbf{P}^9$$

be the natural projections. For every irreducible form $f \in S^3(V^*)$ such that the equation $f = 0$ defines on \mathbf{P}^2 a smooth cubic C , the fiber of the morphism p_9 over the point $\pi(f) \in \mathbf{P}^9$ is a set of nine points. The image of this set under the projection p_2 is exactly the set of all (nine) flexes (inflection points) of the cubic C . Therefore, the set of points of X in general position is identified with the set of all pairs (C, c) , where C is a smooth cubic on \mathbf{P}^2 , and c is its flex. For this reason, X is called (see. [7]) *the variety of flexes of plane cubics*.

Let Y be a smooth irreducible projective algebraic variety birationally isomorphic to the algebraic variety X . The main result (Theorem 4) of the paper [7] is the claim that the irregularity of the variety Y vanishes:

$$H^1(Y, \mathcal{O}_Y) = 0. \quad (1)$$

Below is proved a theorem, a special case of which is the equality (1):

THEOREM. *Maintain the above notation. Then for every positive integer d , the following properties hold:*

- (i) $H^d(Y, \mathcal{O}_Y) = 0$;
- (ii) *there are no nonzero regular differential d -forms on the variety Y .*

Proof. First, we note that properties (i) and (ii) are equivalent. Indeed, according to the Hodge decomposition, $H^q(Y, \Omega_Y^p) = \overline{H^p(Y, \Omega_Y^q)}$ for all integers $p, q \geq 0$ (see [3]). In view of $\Omega_Y^0 = \mathcal{O}_Y$, for $q = d, p = 0$, this gives $H^d(Y, \mathcal{O}_Y) = \overline{H^0(Y, \Omega_Y^d)}$, whence it follows that property (i) is equivalent to the equality $H^0(Y, \Omega_Y^d) = 0$, which, in turn, is a reformulation of property (ii).

We now prove that property (ii) holds.

The natural actions of the group $\mathrm{GL}(V)$ on V, V^* , and $S^3(V^*)$ induce its actions on $\mathbf{P}^2, \mathbf{P}^9$ and $\mathbf{P}^2 \times \mathbf{P}^9$, whose inefficiency kernels contain the group of scalar transformations $Z := \{\alpha \mathrm{Id}_V \mid \alpha \in \mathbf{C}^\times\}$. Therefore these latter actions induce the actions of the projective group $G := \mathrm{PGL}(V) = \mathrm{GL}(V)/Z$ on $\mathbf{P}^2, \mathbf{P}^9$, and $\mathbf{P}^2 \times \mathbf{P}^9$.

The variety X is invariant under the specified action of the group G on $\mathbf{P}^2 \times \mathbf{P}^9$. The classical Hesse's results ([5]; see also [1, pp. 291–299]) yield the following statements:

- (a) For every smooth cubic on \mathbf{P}^2 , there is a transformation from G , which maps this cubic to the cubic on \mathbf{P}^2 , defined by the equation

$$x_1^3 + x_2^3 + x_3^3 + \lambda x_1 x_2 x_3 = 0, \quad \lambda \in \mathbf{C}. \quad (2)$$

- (b) For every $\lambda \in \mathbf{C}$, distinct from $-3, -3\varepsilon, -3\varepsilon^{-2}$, where $\varepsilon = e^{2\pi i/3}$, the cubic on \mathbf{P}^2 defined by equation (2), is smooth and has exactly nine flexes, one of which is the point

$$q := (0 : -1 : 1).$$

- (c) For every smooth cubic C on \mathbf{P}^2 defined by equation (2), and every its flex c , there is an element $g \in G$ such that $g(C) = C$ and $g(c) = q$.

We now consider the morphism $\varphi: \mathbf{A}^1 \rightarrow \mathbf{P}^2 \times \mathbf{P}^9$ defined by the formula

$$\varphi(\lambda) := (q, \pi(x_1^3 + x_2^3 + x_3^3 + \lambda x_1 x_2 x_3)) \quad \text{for every } \lambda \in \mathbf{A}^1 = \mathbf{C}.$$

It follows from (b) that $\varphi(\mathbf{A}^1)$ lies in X , and from (a) and (c) that the morphism

$$G \times \mathbf{A}^1 \rightarrow X, \quad (g, \lambda) \mapsto g(\varphi(\lambda)), \quad (3)$$

is dominant. Since the underlying variety of every connected affine algebraic group is rational (see [2, Cor. 2]), it follows from the dominance of morphism (3) that the variety X , and hence Y , is unirational. Therefore there is a dominant rational map $\psi: \mathbf{P}^9 \dashrightarrow Y$. In view of the smoothness of \mathbf{P}^9 and Y and the projectivity of Y , the induced homomorphism of the spaces of rational differential d -forms

$$\psi^*: \Omega^d(Y) \rightarrow \Omega^d(\mathbf{P}^9)$$

defines an embedding of the spaces of regular differential d -forms

$$\Omega^d[Y] \hookrightarrow \Omega^d[\mathbf{P}^9]$$

(see [8, Chap. III, §6.1, Thm. 2]). Since $\Omega^r[\mathbf{P}^n] = 0$ for any positive r and n (see [3, Chap. 0, Sect. 7]), this implies that $\Omega^d[Y] = 0$, i.e., that statement (ii) holds. \square

REMARK. Along the way of proof, unirationality of X is proved.

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