Are mathematicians, physicists and biologists irrational? Mathematical and natural science studies *vs*. the transitivity axiom

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Abstract. An important and interesting phenomenon of the last few decades is the increasing number of mathematical studies of so-called intransitive dice with non-standard numbers on their faces and the popularization of them. The dice beat one another like in the rock-paper-scissors game. They violate the transitivity law (or axiom): "if it were true that whenever *x* dominates *y* and *y* dominates *z*, then also *x* dominates *z*". Physicists and biologists study intransitivity in their areas too. Yet many texts on rational decision-making contain statements that a key component of rationality is the transitivity axiom and that violation of transitivity is irrational. Are all the physicists, biologists and mathematicians, authors of popular books and math lovers—fans of the intransitive dice irrational? It is not the case. The main difference of most cognitive studies of intransitivity of preferences from intransitivity studies in mathematics and biology is that the cognitivists work with transitive options. In this case, intransitive options are fallacies. In the case of objectively intransitive options, fallacies are transitive choices of the intransitive options. Not only an Euclidean metric but also topological and graph theoretic approaches to rational cognition of objectively intransitive objects is a possible way to overcome some cognitivists' belief in the universality of the transitivity axiom.

One can say about transitivity-oriented and intransitivity-oriented paradigms in different scientific areas. Supporters of the transitivity-oriented paradigm consider transitivity as one of axioms of rational decision making and state that its violations is a fallacy. Supporters of the intransitivity-oriented paradigm consider intransitivity as a part of objective relations in the real world which should be revealed and comprehended.

From the point of view of cognition of complex systems containing intransitive relations, it seems reasonable to distinguish between four types of situations. (1) Relations are objectively transitive and problem posers and (or) solvers make correct conclusions about their transitivity. (2) Relations are objectively transitive, but problem posers and (or) solvers wrongly consider them as intransitive. (3) Relations are objectively intransitive and problem posers and (or) solvers make correct conclusions about their intransitivity. (4) Relations are objectively intransitive, but problem posers and (or) solvers wrongly consider them as transitive (e.g. because of taking the transitivity axiom for granted). This type has been minimally studied in cognitive psychology.

Models of intransitive relations in biological studies are worthy of cognitivists' potential interest. Making rational ecological decisions should take intransitive competition of species into account.

> The transitivity assumption is not necessary for every theoretical concept of rationality, and is therefore not a "precept of logic".
> Schauenberg, B. (1981). The role of transitivity in decision theory.

1. Introduction. An opportunity of a letter from theorists of rational choices to mathematicians and physicists

An important and interesting phenomenon of the last few decades is the increasing number of mathematical studies of so-called intransitive dice with non-standard numbers on their faces and the popularization of them. In 1970, Martin Gardner, a most famous popularizer of mathematics, published the article titled *The paradox of the nontransitive*[1] *dice and the elusive principle of indifference* in his column, *Mathematical Games*, in *Scientific American*. He wrote: "The three sets of four dice … were designed by Bradley Efron, a statistician at Stanford University, to dramatize some recent discoveries about a general class of probability paradoxes that violate transitivity" (Gardner, 1970, p. 119). Efron's dice A, B, C, and D have the following numbers on their faces.

Die A: 4, 4, 4, 4, 0, 0
Die B: 3, 3, 3, 3, 3, 3
Die C: 6, 6, 2, 2, 2, 2
Die D: 5, 5, 5, 1, 1, 1

"The incredible truth is that, regardless of which die he (your opponent – *A.P.*) picks, you can always pick a die that has a 2/3 probability of winning, or two-to-one odds in your favor!" (Ibid.). The dice beat one another like in the rock-paper-scissors game—and this metaphor has gotten the main one. One can read, for example, *Mathematicians Roll Dice and Get Rock-Paper-Scissors* by E. Klarreich (2023)—a popular review of modern mathematical studies of intransitive dice. Explanations of work of the simplest versions of intransitive dice are given in many popular texts, including the initial one by Gardner. Yet, one can start with an explanation from a simpler example (Poddiakov, 2019).

Let us consider 3 sets of 3 pencils of different lengths.

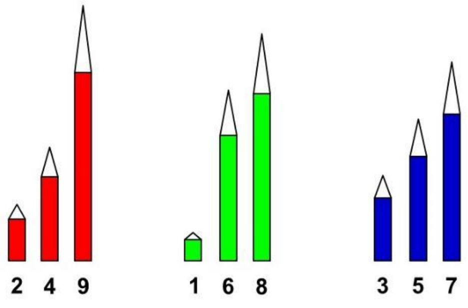

Fig. 1. Three sets of 3 pencils (Poddiakov, 2019).

Numbers to define the pencils' lengths are taken from the magic square presented by Gardner (1974, p. 120).

We compare the length of each pencil with the length of all the other pencils. One can see (Fig. 2) that:

- the red pencils beat the green ones 5 out of 9 times;
- the green pencils beat the blues ones 5 out of 9 times;
- the blue pencils beat the red ones 5 out of 9 times.

[1] Terms "intransitive dice" "nontransitive dice" are used as synonyms.

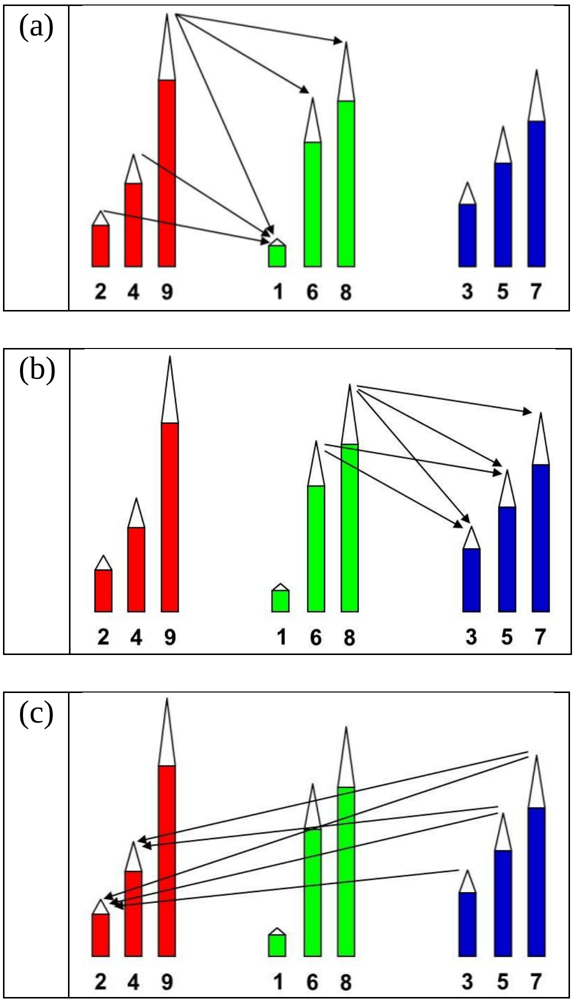

Fig. 2. Results of comparison.

(a) the red pencils beat the green ones 5 out of 9 times, (b) the green pencils beat the blues ones 5 out of 9 times, (c) the blue pencils beat the red ones 5 out of 9 times.

Now intransitive dice studies form an intensively developing area of mathematics, with participation of the Fields medalist, T. Gowers (Polymath, 2019), and mathematical awards for achievements in this area (Intransitive dice, 2017). A set of possible intransitive dice sets is infinite. New types of different intransitive dice (and other mathematical intransitive objects) with amazing properties are being designed and their characteristics are being studied. The practical consequences for applied statistics are discussed (Korneev & Krichevets, 2011; Thangavelu & Brunner, 2007). For example, it is possible that, in pair comparisons, you will find (all the results will be statistically significant):

- a bigger number of participants with higher scores of IQ from sample A than from sample B;
- a bigger number of participants with higher scores of IQ from sample B than from sample C;
- a bigger number of participants with higher scores of IQ from sample C than from sample A.

Let us imagine that the IQ scores are related to the pencils' lengths in a linear way, and the reasons for the results become clearer. This is the work of the intransitive dice paradox.

Aside works mentioned above, one can look through some other modern mathematical studies of intransitivity: (Akin, 2021; Akin & Saccamano, 2021; Buhler et al., 2018; Bozóki, 2014; Chamberland & Herman, 2015; Conrey et al., 2016; Gorbunova & Lebedev, 2022; Grime, 2017; Hązła et al., 2020; Hulko & Whitmeyer, 2019; Komisarski, 2021; West & Hankin, 2008).

Popularization of the intransitive dice is intensive too. Popular books on mathematics contain paragraphs about them (Deulofeu, 2017; Codogno, 2014; Gardner, 2001; Scheinerman, 2012; Singh, 2013; Stewart, 2010), see also a column by Ben Orlin (2022), the author of *Math with Bad Drawings* (Orlin, 2018). Singh's popular article was published in *The Guardian* (Singh, 2004). Lots of videos with the dice and explanations how they beat one another are on different websites. A few years ago, one could buy Efron's dice in The National Museum of Mathematics, USA[2].

What about new types of intransitive objects? In 2016, I posed a new problem—to build a cycle of intransitively winning chess players' positions (Poddiakov, 2016, 2022a, b). I built four positions such that:

- position A of White is preferable (it should be chosen if the choice is possible) to position B of Black;
- position B of Black is preferable to position C of White;
- position C of White is preferable to position D of Black;
- position D of Black is preferable to position A of White.

Now, several certificated chess players develop this idea in their own directions (Filatov, 2017; Deeva, in print). (Both the design of intransitively winning chess players' positions by chess problem composers and subsequent solving chess problems "hidden" in the positions can be worthy of interest for a cognitive psychologist.)

The interest in intransitive paradoxes is a phenomenon of cognitive motivation awaiting explanation, yet here we will consider another aspect of the intransitive dice studies.

Why are the dice called intransitive (non-transitive)? They violate the transitivity law (or axiom): "if it were true that whenever x dominates y and y dominates z, then also x dominates z" (Von Neumann & Morgenstern, 1953, p. 38). Yet the dice are in intransitive relations of dominance "to roll more often a greater number than a number on an opponent's dice". For example, in a set of three intransitive dice A, B, and C,

Die A more often rolls a higher number than Die B in the pair A-B,
Die B more often rolls a higher number than Die C in the pair B-C,
but Die C more often rolls a higher number than Die A in the pair A-C.

The same concerns sets containing more intransitive dice.

Respectively, in playing games with any set of intransitive dice, one should make intransitive choices—e.g. to pick die A in pair A-B, to pick B in pair B-C, and C—in pair A-C. If it is a play for money and you wish to win, not lose your money, you, being a rational thinker who understands the intransitive relations between the intransitive dice, should make intransitive choices of them.

What about mathematicians themselves and their relationships with intransitive relations?

A mathematician inventing a new type of intransitive dice creates not only the dice but also a new sub-area of mathematically interesting intransitive choices necessary to win. A mathematician studying properties of different intransitive dice and their consequences (e.g. for probability theory) considers intransitive choices as a key component of winning strategies and builds conclusions about the probabilities to win by the intransitive choices.

"The notions of mathematical logic, such as transitivity and intransitivity relations, become crucial for understanding of behavior of physical systems. The Ramsey approach may be applied to the analysis of mechanical systems, when actual and virtual paths between the states in configurational space are considered" (Shvalb et al., 2022, p. 1). Klimenko (2013) has introduced ideas about competitive thermodynamics related to intransitivity. A model of mutating intransitive dice as a possible mechanism of genomes' evolution is published in *Physical Review E* (Kirkegaard & Sneppen, 2022). It means that the model can be of interests to readers—physicists. An advanced example of a complex problem based on the knowledge from different domains (biology, physics, and high school mathematics) is presented by Steven Strogatz in his book

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[2] https://web.archive.org/web/20200920084157/https://shop.momath.org/gato-non-transitive-dice-set-of-4.html, the item is not available now.

Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering. The beginning of the problem is formulated as an interdisciplinary introduction:

"In the children's hand game of rock-paper-scissors, rock beats scissors (by smashing it); scissors beat paper (by cutting it); and paper beats rock (by covering it). In a biological setting, analogs of this non-transitive competition occur among certain types of bacteria (Kirkup & Riley, 2004) and lizards (Sinervo & Lively, 1996). Consider the following idealized model for three competing species locked in a life-and-death game of rock-paper-scissors…" (Strogatz, 2015, pp. 191-192).

Some works on intransitive ("rock-paper-scissors") competition in biology (e.g. between species or individuals) are the following: (Allesina & Levine, 2011; Feng et al., 2019; Friedman & Sinervo, 2016; Kerr et al., 2002; Laird & Schamp, 2018; Liow et al., 2019; Maynard et al., 2017; Permogorskiy, 2015; Precoda et al., 2017; Reichenbach et al., 2007; Soliveres & Allan, 2018; Zhang et al., 2012; Wu & Zhang, 2022).

Are all the researchers—biologists, physicists and mathematicians, authors of the popular books and math lovers—fans of the intransitive dice irrational? Is it a growing epidemic of irrationality? Why do I ask this question?

Many texts on rational decision-making contain statements that a key component of rationality is the transitivity axiom. Describing it as an integral part of a theory of rational choice, Steven Pinker (2021) writes in his book *Rationality*: "When you compare options two at a time, if you prefer A to B, and B to C, then you must prefer A to C. <…> …intransitivity is the epitome of irrationality…". See also: (Rational Choice Theory). I appreciate the book *Rationality* very much, but I cannot agree that intransitivity is the epitome of irrationality.

A quote from an interview with K. E. Stanovich, R. F. West, and M. E. Toplak, the authors of *The Rationality Quotient. Toward a Test of Rational Thinking*: "Transitivity says that if you prefer A to B and B to C, then you should prefer A to C. If, instead, you prefer A to B and prefer B to C and prefer C to A, you have violated the transitivity axiom and (it has been shown) you cannot be maximizing utility. You are not instrumentally rational. The *content* of A, B, and C do not matter to the axiom. It is relativistic in that sense. It doesn't care what the content of the options are, only that you stay consistent" (Five minutes.., 2016).

These quotes are from popular texts for lay people (which perhaps, have never been heard about intransitive dice). Can the next step be an open letter to the mathematicians engaged in intransitivity studies? "Dear mathematicians and physicists, you have violated the transitivity axiom and (it has been shown) you cannot be maximizing utility. You are not instrumentally rational. To get rational again, you should stop studies of objects and systems violating transitivity. You should better learn a theory of rational choice and formal logics". One cannot believe that such a letter could be possible.

Which reactions to intransitive systems' studies should be in cognitivists and economists considering the intransitivity axiom as a universal law?

Unfortunately, I cannot find any publications concerning this matter from them. A possible assumption is that the intransitive objects can seem to be rare "monsters" and "pathologies" to them, to use Lakatos' terms (Lakatos, Worrall, & Zahar, 1976; Novaes, 2022). Yet "pathological problems often provide interesting examples of counterintuitive behavior, as well as serving as an excellent illustration of why very detailed conditions of applicability are required in order for many mathematical statements to be universally true" (Weisstein, n.d.).

2. Do theorists of rational choices ignore objectively intransitive relations?

I am engaged both in cognitive studies of people's comprehension of intransitive relations and in mathematical studies of intransitivity and invention of intransitive objects. If I believed too much that the universality of transitive relations "is preferable than", "to dominate over' etc., I could not pose problems of building intransitively winning chess players' positions and intransitive mechanical constructions (e.g. such that construction A with double gears rotates faster than construction B with double gears in gear train A-B; construction B rotates faster than construction C with double gears in gear train B-C; and construction C rotates faster than construction A in gear train A-C). The constructions are geometrical versions of the Condorcet paradox of intransitive voting (Poddiakov, 2018)[3]. Also, the concept of meta-intransitivity of superiority has been introduced for some classes of objects (Poddiakov, Lebedev, in print). Meta-intransitivity is a property of systems, which are in intransitive relationships between one another, and, at the same time, each of them contains its own internal, nested intransitive cycles of superiority.

My position as a researcher of both: (a) intransitivity and (b) its understanding by people, has merits and demerits (for example, my belief, in the importance of the objective intransitivity studies can be a source of some biases). Yet I can compare these areas—in contrast with cognitivists who do not know about intransitivity studies in physics, mathematics, biology, and interdisciplinary studies of intransitivity (Abramova, 2009; Fisher, 2008; Klimenko, 2014, 2015; Poddiakov, 2019b; Poddiakov & Valsiner, 2013).

I agree with the statement: "The transitivity assumption is not necessary for every theoretical concept of rationality, and is therefore not a 'precept of logic'" (Schauenberg, 1981, p. 35). How could Bernd Schauenberg develop his arguments, if he had known about math studies of intransitivity? We do not know it.

The main difference of most cognitive studies of intransitivity of preferences from intransitivity studies in mathematics and biology is that the cognitivists work with options ordered in a linear way by themselves before a study. They design such tasks for participants that contain objectively transitive options, artfully provoking intransitive choices in the participants. It is really interesting and exciting. Yet, only a few cognitivists suggest participants choose between objectively intransitive options—for example, intransitive options provoking the wrong transitive choices of them. This disbalance seems not good from the point of view of unbiased experimental designs in studies of intransitivity cognition.

Let us consider a classical study by Amos Tversky (1969), published a year earlier than Martin Gardner's first article on the intransitive dice. In Experiment 1, Tversky suggested participants choose between the following gambles a, b, c, d, and e.

Table 1. The gambles employed in Experiment 1 (Tversky, 1969, p. 33).

Gamble	Probability of winning	Payoff (in $)	Expected value (in $)
a	7/24	5.00	1.46
b	8/24	4.75	1.58
c	9/24	4.50	1.69
d	10/24	4.25	1.77
e	11/24	4.00	1.83

One can see that probabilities of winning, payoffs and expected values are transitively ordered by Tversky in the table. It was important to him to show how participants violate this transitive order by intransitive choices of the options.

Probabilities of winning: 7/24 < 8/24 < 9/24 < 10/24 < 11/24.
Payoffs: 5.00 > 4.75 > 4.50 > 4.25 > 4.00.
Expected values: 1.46 < 1.58 < 1.69 < 1.77 < 1.83.

[3] A description of the Condorcet paradox which can be understood by secondary school students in the wider context of intransitivity is presented by Beardon (1999/2018), see also (Gehrlein, 2006).

Yet in the case of objectively transitive options, any intransitive preferences of them will be wrong. Tversky conducted a brilliant study of participants' fallacies in these conditions. Many cognitivists continue studies in this paradigm. This is the paradigm of consideration of wrong intransitive choices of objectively transitive options. Almost nobody says: "It's great but what about people's comprehension of objectively intransitive options?". I know only a few exceptions. I give three examples that seem to be most appropriate because the authors of the studies suggest participants to choose between intransitive options intransitivity of which is non-evident and should be revealed by the participants[4].

Howard suggested students choose between the intransitive dice. He wrote: "In the dice exercise, my students are impaled on the horns of a set of nontransitive dice… Because my students trusted that the transitive law held for probabilities, they assumed there must be a 'best' die to be found. It turned out they were chasing 'swamp gas'. The ironic part is that many students assume transitivity without even knowing what it is—let alone that they are wagering their hard-earned money on the law's appropriateness for this particular 'simple game of chance'" (Howard, 2003, p. 78-79).

Cason, Friedman, and Hopkins (2014) have conducted laboratory experiments that use visually oriented software to explore the dynamics of 3×3 games with intransitive best responses. "In the continuous slow adjustment treatment, we see distinct cycles in the population mix. The cycle amplitude, frequency and direction are consistent with the standard learning models" (Ibid., p. 112).

Butler and Pogrebna (2018) have designed pairs of lotteries inspired by the Steinhaus-Trybuła paradox (a paradox of intransitivity built before Efron's dice, see: [Trybuła, 1961; Steinhaus & Trybuła, 1959]). Using these objectively intransitive lotteries, Butler and Pogrebna have studied predictable (and rational) cyclic individual preferences (see also [Butler & Blavatskyy, 2020]).

It is of special interest how the authors and their research critic, Michael H. Birnbaum, polemicized (Birnbaum, 2020; Butler, 2020). One should note that Birnbaum follows the transitivity axiom: "if a person prefers A to B and B to C, then that person should prefer A to C" (Birnbaum et al., 2016, p. 75) and, as far as I know, he suggests only objectively transitive options for participants to choose and does not suggest objectively intransitive ones. Birnbaum writes: "Because the data analyses presented in Butler and Pogrebna (2018) were based on these older methods, a skeptic could remain unconvinced that their data actually contained any real evidence against transitivity" (Birnbaum, 2020, p. 1046). It is an interesting argument from the researcher following the older paradigm of the intransitivity axiom shaped in the 1940-50s (von Neumann & Morgenstern, 1944/1953; May, 1954) and not writing about the modern studies of objective intransitivity—in contrast with Butler and Pogrebna. Do Butler and Pogrebna use "older methods?" In their experimental methods, they do use new options for experimental economics—objectively intransitive ones.

3. On the "money pump" argument

Traditional theorists of rational choice are right when they refer to the "money pump" argument: intransitive choices of objectively transitive options lead to a loss of money, if a person is ready to sell an option which they have and pay an additional sum of money to buy a preferable option. This cycle will work till the loss of all the money. It is explained very well by different researchers, from Tversky to Pinker. Yet, they do not write that, in the case of objectively intransitive options, the money pump works against those ones who make transitive choices. The people lose money and "chase 'swamp gas'"—like in Howard's experiment.

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[4] I do not consider numerous studies of decision making in the rock-paper-scissors game itself because its rules are declared for participants in experimenters' instructions as preliminary information.

In general, in the case of objectively transitive options, intransitive options are fallacies. In the case of objectively intransitive options, fallacies are transitive choices of the intransitive options.

To save the transitivity axiom, Bar-Hillel and Margalit (1988) offered the following principle, based on distinguishing between preferences and choices. Holding rational preferences is a requirement of a higher order than holding rational choices. Rational preferences should be transitive (option “$5 win” is preferable to a “$0 win” which is preferable to “losing $5”—without any cycling between them). Yet, in games of intransitive dice or lotteries, the only way to hold the transitivity of preferences is to make intransitive choices of the objectively intransitive options—with the hope that opponents will make transitive choices of them. Bar-Hillel and Margalit have written that “the classical money-pump argument against intransitive choice cycles is inapplicable to these contexts. We conclude that the requirement for transitivity, though powerful, is not always overriding” (Ibid., p. 119). Fishburn wrote that money pump reasoning “is a clever device, but one that applies transitive thinking to intransitive world” (Fishburn, 1991, p. 118).

Let us consider “an advanced version of the intransitive dice game, rules of which allow players to negotiate swaps of their dice by buying/selling (‘I wish to buy your dice for 5 dollars’/ ‘I wish to sell my dice for 7 dollars’). It can be profitable to buy and swap each following, ‘more advantageous die’ in intransitive way. In that case, the money pump will paradoxically pump money not out, but into a bank of the player making the intransitive choices” (Poddiakov & Valsiner, 2013, p. 349).

Distinguishing between preferences and choices could be a lifebuoy (or “protective belt”, to use Lakatos’ [1978] term) for the transitivity axiom (transitivity should be held for preferences not choices). Yet this argument is not accepted by those who insist on the necessity of keeping transitivity of options. An example of non-distinguishing between preferences and options seems the following statement: “To have transitive preferences, a person, group, or society that prefers choice option x to y and y to z must prefer x to z. Any claim of empirical violations of transitivity by individual decision makers requires evidence beyond a reasonable doubt” (Regenwetter, Dana, & Davis-Stober, 2011, p. 42). It seems a useful addition to the open letter to the mathematicians violating transitivity persistently and constantly by inventing more and more intransitive objects and suggesting people solve math problems on intransitivity and correct intransitive choices.

4. Not only “metric rationality” but also “topological” and “graph theoretic” ones

Peter Fishburn, the author of a theory of preferences under risk without the transitivity axiom (Fishburn, 1982), wrote about rejection of intransitivity: “An analogous rejection of non-Euclidean geometry in physics would have kept the familiar and simpler Newtonian mechanics in place, but that was not to be” (Fishburn, 1991, p. 117).

One can ask: can the transitivity axiom in making rational decision theory be an analogue of the fifth postulate in Euclidean geometry with subsequent problematizations of the postulate in non-Euclidean geometries? It is an open question. Fishburn introduced topological ideas into his reasoning about intransitivity (Ibid.). It seems not a coincidence that now, reading literature on intransitive objects, one can notice that many illustrations of relations between intransitive dice are given not as tables (like the table of gambles from Tversky’s experiment) but as diagrams. These diagrams present structures of directed graphs—arrows showing directions of dominance in intransitive sets of dice. Different intransitive objects can be presented via different graph topologies—see e.g. Grime’s webpage[5] and Pegg’s webpage[6]. A proof that a tournament is transitive if and only if it has no cycles is held by means of graph theory (Wrath of Math, 2021).

In contrast with it, objectively transitive options (e.g. the options from Tversky’s experiment) can be placed not in a topological but in metric space. Metric space is “an abstract set

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5 http://singingbanana.com/dice/article.htm

6 http://www.mathpuzzle.com/MAA/39-Tournament%20Dice/mathgames_07_11_05.html

with a distance function, called a metric, that specifies a non-negative distance between any two of its points" (Carlson). Indeed, we can measure distances between any two options in Tversky's experiment and present them in Euclidean space.

"Topological space, in mathematics, generalization of Euclidean spaces in which the idea of closeness, or limits, is described in terms of relationships between sets rather than in terms of distance" (Britannica). There are no "perfect values" of intransitive dice in any set of them, because, as many authors emphasize, there are no "the best die" and "the worst die" in the intransitive cycle: each die beats the second one and is beaten by the third, and they are equivalent in this sense. The value of an intransitive die is relative and depends on the context of comparison with another die. Kalenscher and Pennartz (2011) used graph theory to detect intransitive cycles in their participants' choices of objectively transitive lotteries. It seems a good start for the further study of context-dependent preferences of objectively transitive lotteries.

We have shown that any Euclidean metric of chess players' positions across their whole set is impossible because some chess players' positions form intransitive cycles. The space of mutual relations between winningness of chess players' positions is non-Euclidean (Poddiakov, 2022a). There are no perfect values (numbers in any absolute rating) of all the positions of White and Black in Euclidean space. The same applies for intransitive dice. One can conclude that intransitive objects are not "inhabitants" of universal metric space. Violations of real intransitivity are necessary to present the intransitive objects in metric space—in contrast with the topological and graph presentations. Not only the Euclidean metric but also the topological and the graph theoretic approaches to rational cognition of objectively intransitive objects are possible ways to overcome some cognitivists' belief in the universality of the transitivity axiom.

Conclusion

One can say about transitivity-oriented and intransitivity-oriented paradigms in different scientific areas. Supporters of the transitivity-oriented paradigm consider transitivity as one of axioms of rational decision making and state that its violations is a fallacy. Supporters of the intransitivity-oriented paradigm consider intransitivity as a part of objective relations in the real world which should be revealed and comprehended.

From the point of view of cognition of complex systems containing intransitive relations and cognition of whole "intransitive environments" ("intransitive worlds", to use Fishburn's term), it seems reasonable to distinguish between four types of cognitive situations (Poddiakov, 2006, 2010).

(1) Relations are objectively transitive and problem posers and (or) solvers make correct conclusions about their transitivity.

(2) Relations are objectively transitive, but problem posers and (or) solvers wrongly consider them as intransitive. Most studies are conducted in this paradigm (see e.g. May, 1954; Tversky, 1969; Regenwetter, Dana, & Davis-Stober, 2011).

(3) Relations are objectively intransitive and problem posers and (or) solvers make correct conclusions about their intransitivity (e.g. intransitivity of intransitive dice, lotteries, chess players' positions etc.)

(4) Relations are objectively intransitive, but problem posers and (or) solvers wrongly consider them as transitive (e.g. because of taking the transitivity axiom for granted). This type has been minimally studied in cognitive psychology.

Mathematical intransitive objects are not the only type of objects worthy of cognitivists' potential interest. There are many studies of intransitive competition between species in biology. They use mathematical models of intransitive relations. One should emphasize that making rational ecological decisions should take biological intransitivity into account. Here, people deal with the most complex topology of relations. Comprehension of, to metaphorically say, rhizomatic intransitivity in complex systems and a cognitive study of models of complexity containing intransitive relations in biological studies can be an interesting area for cognitive psychology.

What about an article titled *Visualizations of intransitive relations in biological studies as cognitive tools of exploring complexity?* What about *Cognitive requirements for comprehension of biological models of intransitivity*? It seems that a modern cognitivist studying intransitivity should be able, at least, to explain the main content of the Kirkegaard-Sneppen model of mutating intransitive dice as a possible mechanism of genomes' evolution to students.

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