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TOPOLOGICAL CONJUGACY OF THE SIMPLEST NONSINGULAR THREE-DIMENSIONAL FLOWS

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The simplest nonsingular flows on closed orientable 3*-manifolds are studied. We establish that each class of topological equivalence of the simplest nonsingular flow on a lens consists of an infinite set of topological conjugacy classes. We obtain necessary and sufficient conditions for the topological conjugacy of the flows under consideration. Bibliography*: 5 *titles. Illustrations*: 4 *figures.*

1 Introduction and Formulation of the Results

Flows $f^t, f'^t\colon M \to M$ on a manifold M are *topologically equivalent* if there exists a homeomorphism $h\colon M \to M$ mapping the trajectories of the flow f^t to trajectories of the flow f'^t with preservation of the motion direction along trajectories. Two flows are said to be *topologically conjugate* if $h \circ f^t = f'^t \circ h$, i.e., h maps trajectories to trajectories, preserving not only direction, but also the time of motion along the trajectory.

In this paper, we consider the so-called nonsingular Morse–Smale flow (NMS-flow) f^t given on closed orientable 3-manifolds M^3. The nonwandering set of such a flow consists of finitely many periodic hyperbolic orbits. In a neighborhood of a hyperbolic periodic orbit, the flow is topologically conjugate to a suspension over some linear diffeomorphism of the plane given by

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a matrix with positive determinant and real eigenvalues different from 1 in modulus (cf. for example, [3, Theorem 4.2, 4.3]). If both eigenvalues are larger (less) than 1 in modulus, then the corresponding periodic orbit is called *repelling* (*attracting*); in the opposite case, it is called a *saddle*.

Since the underlying manifold M^3 is the union of stable (unstable) manifolds of all its periodic orbits (cf., for example, [5, Theorem 2.2]), the NMS-flow f^t necessarily has at least one attracting periodic orbit and at least one repelling periodic orbit. The NMS-flow $f^t\colon M^3 \to M^3$ is the *simplest* if its wandering set consists exactly of two periodic orbits: attracting c_+ and repelling c_-. We denote by $\mathscr{S}$ the class of such flows. As established in [1], all manifolds admitting flows in the class $\mathscr{S}$ are lens spaces and, as proved in [4], on each lens space there are exactly two classes of topological conjugacy of the simplest flows (cf. Figure 1) except for the 3-sphere $\mathbb{S}^3$ and the projection spaces $\mathbb{R}P^3$ where the equivalence class is single. We recall that the *lens space* is a three-dimensional manifold $L_{p,q} = V_+ \cup_j V_-$ obtained as a result of gluing two copies of the solid torus $V_+ = \mathbb{V}$, $V_- = \mathbb{V}$ along some homeomorphism $j\colon \partial V_+ \to \partial V_-$ such that $j_*(\langle 0, 1\rangle) = \langle p, q\rangle$.

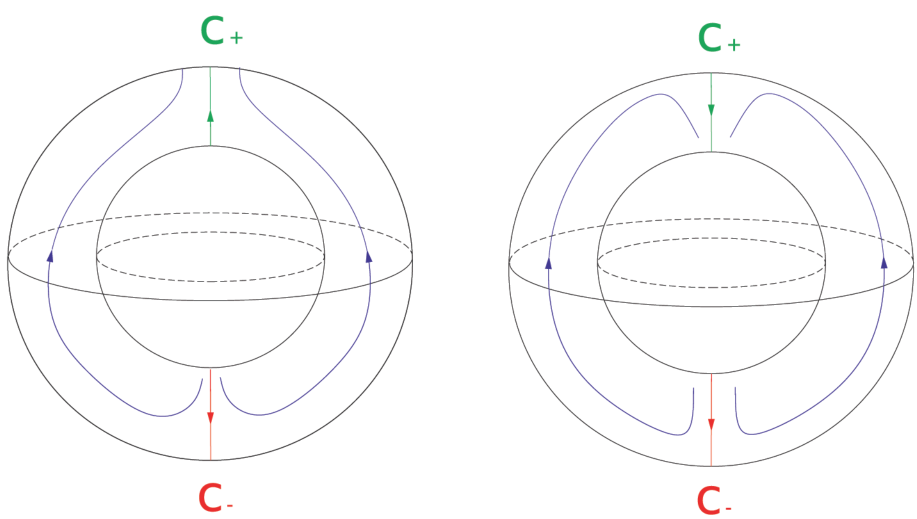

Figure 1. Examples of topologically nonequivalent flows on $\mathbb{S}^2 \times \mathbb{S}^1$.

In this paper, we establish that each equivalence class of the simplest nonsingular flow on a lens consists of an infinite set of classes of topological conjugacy. In addition, we establish necessary and sufficient conditions for the topological conjugacy of the flows under consideration. We describe the results in more detail.

Let $f^t \in \mathscr{S}$ be a nonsingular flow with two limit cycles: stable c_+ and unstable c_- cycles of periods T_+ and T_- respectively. In Section 3, we show that with each flow f^t we can associate a continuous function $\widehat{\delta} : \mathbb{T}^2 \to \mathbb{S}^1$ such that $\widehat{\delta}(0,0) = 0$, whereas the induced homomorphism $\widehat{\delta}_* : \mathbb{Z}^2 \to \mathbb{Z}$ is surjective and its kernel is nontrivial. Then on the torus $\mathbb{T}^2$ there exist generators α and β with the intersection index 1 and such that $\widehat{\delta}_*(\langle\alpha\rangle) = 1$ and $\widehat{\delta}_*(\langle\beta\rangle) = 0$. Moreover, the homotopy type $\langle\beta\rangle = \langle q, p\rangle$ of the curve β is uniquely determined from the above conditions. We denote by $\mathscr{A}$ the set of functions with these properties.

Two functions $\widehat{\delta}, \widehat{\delta}' \in \mathscr{A}$ are *equivalent* if there exists a homeomorphism $\widehat{\theta}\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that the induced isomorphism is given by the matrix

$$\widehat{\theta}_* = \begin{pmatrix} \pm 1 & 0 \\ k & 1 \end{pmatrix}, \quad k \in \mathbb{Z},$$

preserving the homotopy class of the torus meridian, and $\widehat{\delta}'(\widehat{\theta}(s,\ell)) - \widehat{\delta}'(\widehat{\theta}(0,0)) = \widehat{\delta}(s,\ell)$ for $(s,l) \in \mathbb{S}^1 \times \mathbb{S}^1$. We denote by $[\widehat{\delta}]$ the equivalence class of $\widehat{\delta}$.

The main result of the paper is proved in Section 4 and is formulated as follows.

Theorem 1.1. *Flows $f^t, f'^t \in \mathscr{S}$ are topologically conjugated if and only if*

(1) $T_\pm = T'_\pm$,

(2) $[\widehat{\delta}] = [\widehat{\delta}']$,

(3) $p = p'$, $q \equiv \pm q' \pmod p$.

Necessary and sufficient conditions for the topological conjugacy of flows in the considered class on the lens $\mathbb{S}^2 \times \mathbb{S}^1$ were earlier obtained in [2].

The key point of the proof of Theorem 1.1 is the existence of a unique f^t-invariant two-dimensional foliation in a neighborhood of a periodic orbit of the flow established in Lemma 2.1. The proof of the existence of infinite set of topologically non-conjugate flows in one equivalence class of $f^t \in \mathscr{S}$ it follows from the following theorem proved in Section 5.

Theorem 1.2. *For any positive numbers T_+ and T_- and any function $\widehat{\delta} \in \mathscr{A}$ there exists a flow $f^t \in \mathscr{S}$ with the corresponding parameters on the lens space $L_{p,q}$.*

2 Existence of Unique f^t-Invariant Two-Dimensional Projection of Foliations in Neighborhood of Periodic Orbit

Let $f^t \in \mathscr{S}$ be a nonsingular flow with two limit cycles: stable c_+ and unstable c_- cycles of periods T_+ and T_- respectively. We set $K_+ = W^s_{c_+} = M^3 \setminus c_-$ and $K_- = W^u_{c_-} = M^3 \setminus c_+$. We define the flow $A^t : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula $A^t(x_1, x_2, x_3) = (x_1, x_2, x_3 + t)$ and the homeomorphism $g_\pm \colon \mathbb{R}^3 \to \mathbb{R}^3$ by the formula

$$g_\pm(x_1, x_2, x_3) = \left(\frac{x_1}{2}, \frac{x_2}{2}, x_3 - T_\pm\right).$$

We set $G_\pm = \{g^n_\pm : n \in \mathbb{Z}\}$ (cf. Figure 2) and $\Pi_\pm = \mathbb{R}^3/G_\pm$. We denote by $q_\pm \colon \mathbb{R}^3 \to \Pi_\pm$ the natural projection and by $a^t_\pm$ the flow induced by the flow $A^{\pm t}$ on $\Pi_\pm$. By construction, the flow $a^t_\pm$ has a unique periodic orbit $\widetilde{c}_\pm$.

Proposition 2.1 (cf. [3]). *There exists a homeomorphism $\eta_\pm \colon K_\pm \to \Pi_\pm$ conjugating the flows $f^t|_{K_\pm}$ and $a^t_\pm$.*

Lemma 2.1. *For any flow $f^t \in \mathscr{S}$ there exists a unique f^t-invariant two-dimensional foliation $\Xi_\pm$ on $K_\pm$ whose fibres $\xi_\pm$ are secant lines of trajectories of the flows $f^t|_{K_\pm}$ and*

$$f^{T_\pm}(x) \in \xi_\pm,\ f^t(x) \notin \xi_\pm, \quad 0 < t < T_\pm, \quad \text{if } x \in \xi_\pm.$$

Proof. For the sake of definiteness we consider a stable cycle. The case of unstable cycles is treated in the same way.

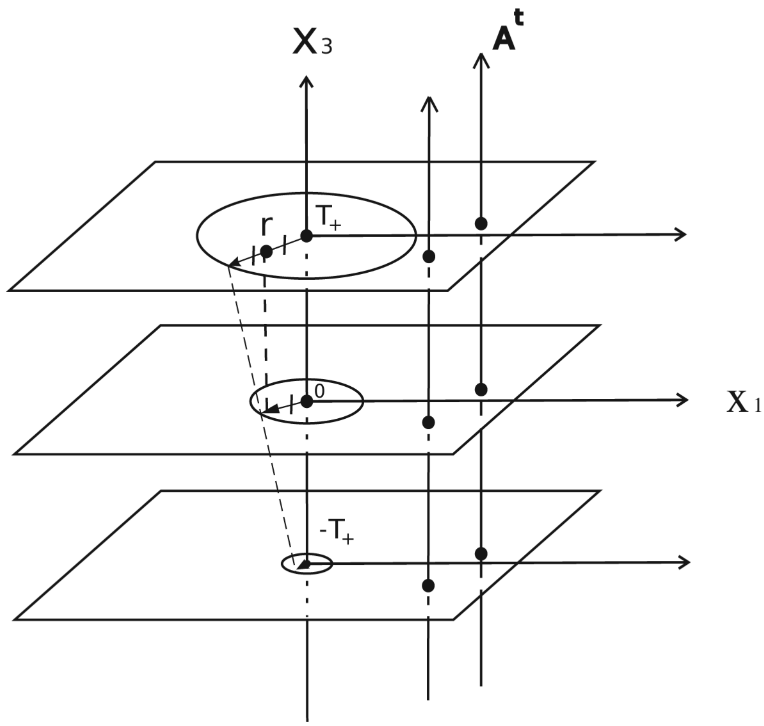

Figure 2. Flow A^t. Action of the homeomorphism g_+.

We first note that the existence of at least one foliation Ξ_+ with required properties follows from linearization. Indeed, let $\chi = \{x_3 = c, c \in \mathbb{R}\}$ be a foliation on $\mathbb{R}^3$ that is formed by horizontal planes $\widetilde{\chi}_+ = q_+(\chi)$, $\Xi_+ = \eta_+^{-1}(\widetilde{\chi}_+)$ (cf. Figure 3). Let us prove the uniqueness of the foliation Ξ_+.

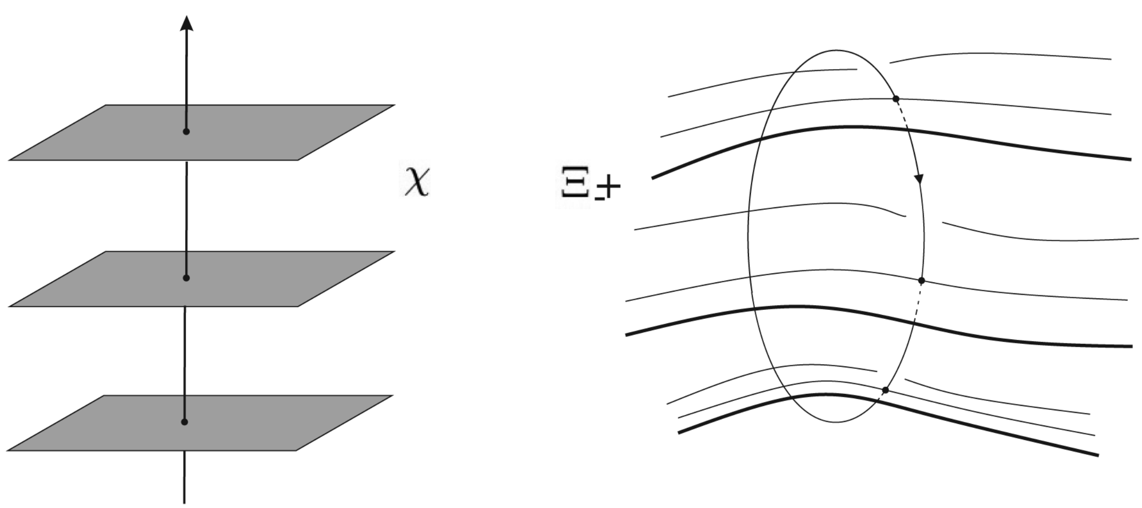

Figure 3. Invariant foliations χ and $\Xi_\pm$.

Let Ξ_+ and $\widehat{\Xi}_+$ be two foliations on K_+ possessing the required properties. Then $\chi = q_+^{-1}\eta_+(\Xi_+)$ and $\widehat{\chi} = q_+^{-1}\eta_+(\widehat{\Xi}_+)$ are two foliations in $\mathbb{R}^3$, invariant under the flow A^t. We consider the annulus $\varkappa = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid 1 \leqslant x_1^2 + x_2^2 \leqslant 2\}$ in $\mathbb{R}^3$. Let $\widehat{\varkappa}$ be an annulus on a fibre of the foliation $\widehat{\chi}$ passing through the origin and consisting of points of the intersection of this fibre with trajectories passing through points of the annulus $\varkappa$. We denote by $\theta : \varkappa \to \widehat{\varkappa}$ the homeomorphism sending a point $r \in \varkappa$ to a point of the intersection of trajectories of A^t passing through r and the annulus $\widehat{\varkappa}$. Since the foliations are invariant under the flow A^t and under the homeomorphism g_+, we conclude that the homeomorphism θ is extended to a homeomorphism $\theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\theta(\chi) = \widehat{\chi}$, $\theta A^t = A^t\theta$, $g_+\theta = \theta g_+$. Then the homeomorphism θ takes the form

$$\theta(x_1, x_2, x_3) = (\sigma(x_1, x_2, x_3), \varsigma(x_1, x_2, x_3), \widetilde{\psi}(x_1, x_2, x_3)),$$

where σ, ς, ψ are continuous mappings. Since the trajectories of the flow A^t are invariant under the mapping θ, we have $\sigma(x_1, x_2, x_3) = x_1$, $\varsigma(x_1, x_2, x_3) = x_2$, and the condition $\theta A^t = A^t\theta$ implies

$$\widetilde{\psi}(x_1, x_2, x_3 + t) = \widetilde{\psi}(x_1, x_2, x_3) + t.$$

We write $\widetilde{\psi}(x_1, x_2, x_3)$ in the form $\widetilde{\psi}(x_1, x_2, x_3) = x_3 + \psi(x_1, x_2, x_3)$. Then

$$x_3 + t + \psi(x_1, x_2, x_3 + t) = x_3 + \psi(x_1, x_2, x_3) + t$$

and, consequently,

$$\psi(x_1, x_2, x_3 + t) = \psi(x_1, x_2, x_3), \quad t \in \mathbb{R}.$$

This means that the mapping ψ is independent of x_3. We set $\psi(x_1, x_2, x_3) = \psi(x_1, x_2)$. Thus, θ has the form $\theta(x_1, x_2, x_3) = (x_1, x_2, x_3 + \psi(x_1, x_2))$. Since

$$g_+\theta = \theta g_+, \quad g_+(x_1, x_2, x_3) = \Big(\frac{1}{2}x_1, \frac{1}{2}x_2, x_3 - T_+\Big),$$

we have

$$x_3 + \psi\Big(\frac{x_1}{2}, \frac{x_1}{2}\Big) - T_+ = x_3 + \psi(x_1, x_2) - T$$

and, consequently,

$$\psi\Big(\frac{x_1}{2}, \frac{x_2}{2}\Big) = \psi(x_1, x_2).$$

We show that, in this case, $\psi(x_1, x_2)$ is a constant. Indeed, assume that $(x_1, x_2) \in \mathbb{R}^2$ and $(x_1^n, x_2^n) = \psi(x_1/2^n, x_2/2^n), n \in \mathbb{N}$. Since $\psi(x_1/2, x_2/2) = \psi(x_1, x_2)$, we have $\psi(x_1^n, x_2^n) = \psi(x_1, x_2)$ for every natural n. Since the mapping ψ is continuous, the sequence of constants $\psi(x_1/2^n, x_2/2^n)$ converges to $\psi(0, 0)$ and $\psi(x_1, x_2) = \psi(0, 0)$ for every $(x_1, x_2) \in \mathbb{R}^2$. Thus, $\theta(x_1, x_2, x_3) = (x_1, x_2, x_3 + b)$, where b is a constant. Consequently, $\theta(\chi) = \chi$. Since $\theta(\chi) = \widehat{\chi}$, we have $\widehat{\chi} = \chi$. ☐

3 Function $\widehat{\delta} \in \mathscr{A}$ Corresponding to Flow $f^t \in \mathscr{S}$

For $x = (x_1, x_2, x_3) \in (\mathbb{R}^3 \setminus Ox_3)$ we denote

$$||x|| = \sqrt{x_1^2 + x_2^2}, \quad s = \Big(\frac{x_1}{||x||}, \frac{x_2}{||x||}\Big) \in \mathbb{S}^1, \quad l = T_+ \log_2 ||x|| \in \mathbb{R}, \quad r = \log_2 ||x|| - \frac{x_3}{T_+} \in \mathbb{R}$$

and define the mapping $\mu_+ : \mathbb{R}^3 \setminus Ox_3 \to \mathbb{S}^1 \times \mathbb{R}^2$ by $\mu_+(x_1, x_2, x_3) = (s, l, r)$. We introduce the homeomorphism $\gamma_+ : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{R}^2$ by the formula $\gamma_+(s, l, r) = (s, l - T_+, r)$ and the flow $B^t : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{R}^2$ by the formula

$$B^t(s, r, l) = \Big(s, l, r - \frac{t}{T_+}\Big)$$

(cf. Figure 4). A direct calculation shows that $\mu_+ g_+ = \gamma_+ \mu_+$ and $\mu_+ A^t = B^t \mu_+$.

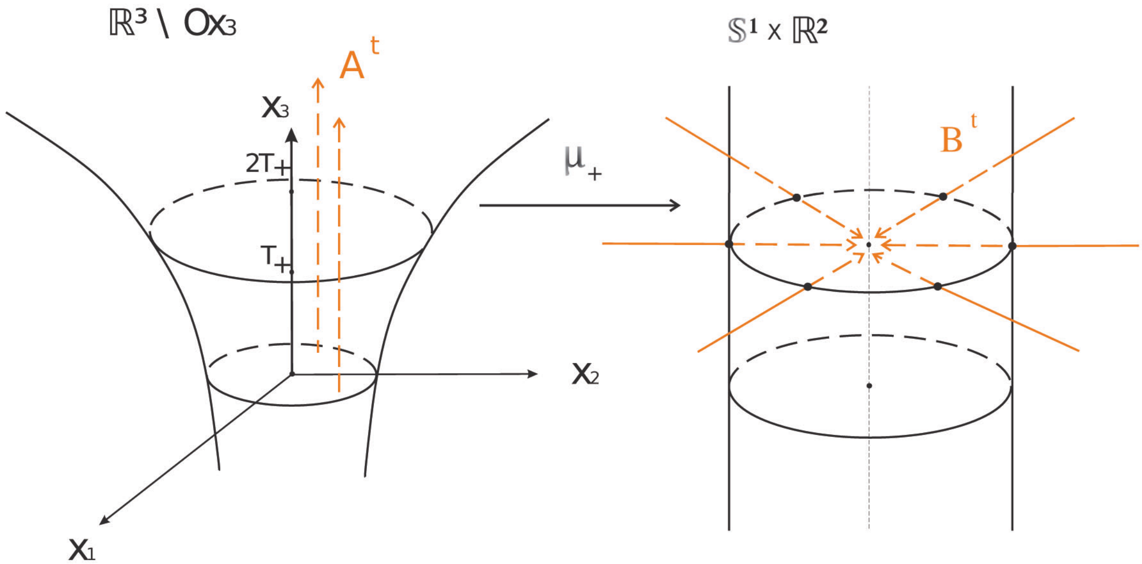

Figure 4. Action of the mapping μ_+, flows A^t and B^t.

We set $\Gamma_+ = \{\gamma_+^n, n \in \mathbb{Z}\}$. Then $(\mathbb{S}^1 \times \mathbb{R}^2)/\Gamma_+ = \mathbb{T}^2 \times \mathbb{R}$ and the natural projection $p_+ : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}$ induces the flow $b_+^t : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ by the formula

$$b_+^t(s, \ell, r) = \left(s, \ell, r - \frac{t}{T_+}\right), \quad s, \ell \in \mathbb{S}^1.$$

A direct calculation shows that the foliation $\mu_+(\chi)$ has the form

$$\mu_+(\chi) = \left\{(s, l, r) \in \mathbb{S}^1 \times \mathbb{R}^2 : r = \frac{l}{T_+} + r_0\right\}_{r_0 \in [0,1)}.$$

We set $\mathscr{F}_+ = p_+(\mu_+(\chi))$. By construction, the flows $f^t|_{K_+ \setminus c_+}$ and b_+^t are topologically conjugate via the homeomorphism $j_+ = p_+ q_+^{-1} \eta_+ : K_+ \setminus c_+ \to \mathbb{T}^2 \times \mathbb{R}$. Similarly, the flow $f^t|_{K_- \setminus c_-}$ is topologically conjugate to the flow $b_-^t : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ given by

$$b_-^t(s, \ell, r) = \left(s, \ell, r + \frac{t}{T_-}\right), \quad s, \ell \in \mathbb{S}^1,$$

via a homeomorphism $j_- : K_- \setminus c_- \to \mathbb{T}^2 \times \mathbb{R}$ similar to j_+. Moreover, the foliation $\mu_-(\chi)$ has the form

$$\mu_-(\chi) = \left\{(s, l, r) \in \mathbb{S}^1 \times \mathbb{R}^2 : r = \frac{l}{T_-} + r_0\right\}_{r_0 \in [0,1)}.$$

We set $\mathscr{F}_- = p_-(\mu_-(\chi))$ and introduce the mapping $\Phi \colon \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ by the formula $\Phi = j_- j_+^{-1}$. Without loss of generality we assume that $T_+ \geqslant T_-$. Otherwise, the further consideration can be performed for the mapping Φ^{-1}. Then the homeomorphism Φ topologically conjugates the flows b_+^t and b_-^t. Hence the fibres of $\Phi(\mathscr{F}_+)$ are invariant under shifts

$$(s, \ell, r) \mapsto \left(s, \ell, r + \frac{T_+}{T_-}\right).$$

We introduce the homeomorphism $\gamma : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ by the formula

$$\gamma(s, \ell, r) = \left(s, \ell, r + \frac{T_+}{T_-}\right)$$

and set $\Gamma = \{\gamma^n,\ n \in \mathbb{Z}\}$. Then $(\mathbb{T}^2 \times \mathbb{R})/\Gamma = \mathbb{T}^3$ and the natural projection $v : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^3$ induces a foliation $\widehat{\mathscr{F}_+} = v(\Phi(\mathscr{F}_+))$ such that each its fibre is a two-dimensional torus and

$$\widehat{\mathscr{F}_+} = \{(s, \ell, \rho) \in \mathbb{T}^3 :\ \rho = \widehat{\delta}(s, \ell) + \rho_0\}_{\rho_0 \in \mathbb{S}^1},$$

where $\widehat{\delta} : \mathbb{T}^2 \to \mathbb{S}^1$ is a continuous function such that $\widehat{\delta}(0, 0) = 0$, the induced homomorphism $\widehat{\delta}_* : \mathbb{Z}^2 \to \mathbb{Z}$ is surjective and its kernel is nontrivial, i.e., it is a subgroup isomorphic to $\mathbb{Z}$ with a unique up to a sign generator. Then, on the torus $\mathbb{T}^2$, there exist generators α and β with the intersection index 1 such that the coordinates of their homotopy classes form a unimodular matrix and $\widehat{\delta}_*(\langle\alpha\rangle) = 1$, $\widehat{\delta}_*(\langle\beta\rangle) = 0$. Therefore, the homotopy type $\langle\beta\rangle = \langle q, p\rangle$ of the curve β is uniquely found. We denote by $\mathscr{A}$ the set of functions possessing the above properties. Two functions $\widehat{\delta}, \widehat{\delta}' \in \mathscr{A}$ are *equivalent* if there exists a homeomorphism $\widehat{\theta} \colon \mathbb{T}^2 \to \mathbb{T}^2$ such that the induced isomorphism is given by the matrix

$$\widehat{\theta}_* = \begin{pmatrix} \pm 1 & 0 \\ k & 1 \end{pmatrix}, \quad k \in \mathbb{Z},$$

and $\widehat{\delta}'(\widehat{\theta}(s, \ell)) - \widehat{\delta}'(\widehat{\theta}(0, 0)) = \widehat{\delta}(s, \ell)$. We denote by $[\widehat{\delta}]$ the equivalence class of $\widehat{\delta}$.

4 Proof of Theorem 1.1

Necessity. Let $h \colon M^3 \to M^3$ be a homeomorphism conjugating flows f^t, $f'^t \in \mathscr{S}$. All objects related to the flow f'^t are marked by the prime, by analogy with objects of the flow f^t. Then $T_\pm = T'_\pm$ and the homeomorphism h induces a homeomorphism $h_- : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ conjugating the flows b^t_- and b'^t_-. Since the trajectories of the flows b^t_- and b'^t_- take the form $\{(s, \ell)\} \times \mathbb{R}$, $\{(s', \ell')\} \times \mathbb{R}$, the homeomorphism h_- determines the homeomorphism $\widehat{\theta} \colon \mathbb{T}^2 \to \mathbb{T}^2$ by the formula $\widehat{\theta}(s, \ell) = (s', \ell')$, where $h_-(\{(s, \ell)\} \times \mathbb{R}) = \{(s', \ell')\} \times \mathbb{R}$. Since h_- sends fibres of the foliation $\mathscr{F}_-$ to fibres of the foliation $\mathscr{F}'_-$, the induced isomorphism is determined by the matrix

$$\widehat{\theta}_* = \begin{pmatrix} \pm 1 & 0 \\ k & 1 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

In turn, the homeomorphism h_- induces a homeomorphism $\widehat{h}_- : \mathbb{T}^3 \to \mathbb{T}^3$ sending fibres of the foliation $\widehat{\mathscr{F}_+}$ to fibres of the foliation $\widehat{\mathscr{F}'_+}$, which implies

$$\widehat{\delta}'(\widehat{\theta}(s, \ell)) - \widehat{\delta}'(\widehat{\theta}(0, 0)) = \widehat{\delta}(s, \ell).$$

Hence $[\widehat{\delta}] = [\widehat{\delta}']$. Furthermore, $\widehat{\delta}_* \widehat{\theta}_* = \widehat{\delta}'_*$, consequently, $\widehat{\theta}_*(\langle\beta\rangle) = \langle\beta'\rangle$, which implies

$$\langle q', p'\rangle \begin{pmatrix} \pm 1 & 0 \\ k & 1 \end{pmatrix} = \langle q, p\rangle.$$

Hence $p = p'$ and $q \equiv \pm q' \pmod{p}$.

Sufficiency. Let the flows f^t, $f'^t \in \mathscr{S}$ satisfy Conditions (1)–(3) of the theorem. We construct a homeomorphism $h \colon M^3 \to M^3$ conjugating the flows f^t and f'^t. By Condition (2) there exists a homeomorphism $\widehat{\theta}\colon \mathbb{T}^2 \to \mathbb{T}^2$ realizing the equivalence of the functions $\widehat{\delta}$ and $\widehat{\delta}'$. We set $\widehat{\theta}(s,\ell) = (s',\ell')$ and introduce the homeomorphism $h_- : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ by the formula $h_-(s,\ell,r) = (s',\ell',r+\ell'-\ell)$. Then h_- transforms fibres of the foliations $\mathscr{F}_-$ and $\Phi(\mathscr{F}_+)$ to fibres of the foliations $\mathscr{F}'_-$ and $\Phi'(\mathscr{F}'_+)$ respectively. We set $h = j_-^{-1} h_- j_- : K_- \setminus c_- \to K_- \setminus c_-$. By construction, the homeomorphism h_- transforms fibres of the foliation $\Xi_\pm$ to fibres of the foliation $\Xi'_\pm$ respectively. Since $T_\pm = T'_\pm$, the homeomorphism h is uniquely extended on M^3 to the required homeomorphism conjugating the flows f^t and f'^t.

5 Proof of Theorem 1.2

For given periods $T_\pm$ we construct the flows $a^t_\pm$ on $\Pi_\pm$. For a given function $\widehat{\delta} \in \mathscr{A}$ we define the homeomorphism $\overline{Q} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ by the formula $Q(s,\ell,r) = (s,\ell,r+T_-\widehat{\delta}(s,\ell)-\ell)$. We set $Q = q_- p_-^{-1} \overline{Q} p_+ q_+^{-1} : \Pi_+ \backslash \widetilde{c}_+ \to \Pi_- \backslash \widetilde{c}_-$ and $M^3 = \Pi_+ \cup_Q \Pi_-$ and denote by $\nu : \Pi_+ \sqcup \Pi_- \to M^3$ the natural projection. Since $\langle \beta \rangle = \langle q,p \rangle$, we have $M^3 \cong L_{p,q}$. Then the flow $f^t \in \mathscr{S}$ is well defined on M^3 by the formulas $f^t_{\nu(\Pi_+)} = \nu a^t_+ (\nu|_{\Pi_+})^{-1}$ and $f^t_{\nu(\Pi_-)} = \nu a^t_- (\nu|_{\Pi_-})^{-1}$. The theorem is proved.

Acknowledgments

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References

1. B. Campos, A. Cordero, J. Martinez-Alfaro, and P. Vindel, "NMS flows on three-dimensional manifolds with one saddle periodic orbit," *Acta Math. Sin., Engl. Ser.* **20**, No. 1, 47–56 (2004).

2. A. L. Dobrolyubova and V. E. Kruglov, "Topological conjugacy of non-singular flows with two limit cycles on $S^2 \times S^1$" [in Russian], *Zh. Sredn. Mat. Obshch.* **24**, No. 1, 40–53 (2022).

3. M. C. Irwin, "A classification of elementary cycles," *Topology* **9**, 35–47 (1970).

4. O. V. Pochinka and D. D. Shubin, "Non-singular Morse–Smale flows on n-manifolds with attractor-repeller dynamics," *Nonlinearity* **35**, No. 3, 1485–1499 (2022).

5. S. Smale, "Differentiable dynamical systems," *Bull. Am. Math. Soc.* **73**, No. 6, 747–817 (1967).

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