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ON EMBEDDING OF THE MORSE–SMALE DIFFEOMORPHISMS IN A TOPOLOGICAL FLOW

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Abstract. This review presents the results of recent years on solving of the Palis problem on finding necessary and sufficient conditions for the embedding of Morse–Smale cascades in topological flows. To date, the problem has been solved by Palis for Morse–Smale diffeomorphisms given on manifolds of dimension two. The result for the circle is a trivial exercise. In dimensions three and higher new effects arise related to the possibility of wild embeddings of closures of invariant manifolds of saddle periodic points that leads to additional obstacles for Morse–Smale diffeomorphisms to be embedded in topological flows. The progress achieved in solving of Palis's problem in dimension three is associated with the recently obtained complete topological classification of Morse–Smale diffeomorphisms on three-dimensional manifolds and the introduction of new invariants describing the embedding of separatrices of saddle periodic points in a supporting manifold. The transition to a higher dimension requires the latest results from the topology of manifolds. The necessary topological information, which plays key roles in the proofs, is also presented in the survey.

CONTENTS

1. Statement of the Problem and History of the Issue

The dynamical systems with continuous (flows) and discrete (cascades) time are closely connected. Thus, if the flow on the manifold M^n has a global secant, then its properties are mostly defined by the properties of Poincaré map on this secant. The numerical methods of solving the differential equations naturally give mappings with discrete time. One of the indicators of the adequacy of numerical modeling consists in the fact that the resulting cascade is topologically conjugated to a shift per time unit along the trajectories of the original flow. In the papers [19, 20] it is shown that the Runge–Kutta discretization of the system of $n \geq 2$ differential equations defining the Morse–Smale flow without periodic trajectories (structurally stable stream with finite nonwandering set) on a disc, for a sufficiently small value of the discretization step, defines a discrete dynamical system topologically conjugated to a shift per time unit along the trajectories of the original flow. This means that the resulting discrete dynamical system is embedded in the topological flow.

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Study of the connection between the cascades and streams leads to the classical problem of finding conditions for the embedding of diffeomorphisms (or homeomorphisms) in the flow. Let M^n be a smooth connected closed manifold of dimension n. Recall that a *C^m-flow* ($m \geq 0$) on the manifold M^n is a family of C^m-diffeomorphisms $X^t : M^n \to M^n$ continuously depending on $t \in \mathbb{R}$, and such that $X^0(x) = x$ and $X^t(X^s(x)) = X^{t+s}(x)$ for all $s, t \in \mathbb{R}$, $x \in M^n$. A C^0-flow is also called a *topological flow*.

We say that the diffeomorphism $f : M^n \to M^n$ of a closed manifold M^n is *embedded in C^m-flow* if f is the shift per time unit along the trajectories of some C^m-flow X^t ($f = X^1$) given on M^n.

Since the flow defines the isotopy connecting the shift per time unit along the trajectories and the identity mapping, then the diffeomorphisms nonisotopic to the identity are not embedded in any flows. Thus, the set of cascades is significantly richer than the set of shifts per time unit along the flows trajectories. In [40] it is proven that the set of C^r-diffeomorphisms ($r \geq 1$) embedded in C^1-flow is a subset of the first category in $\mathit{Diff}^r(M^n)$. By [9], the set of C^2-diffeomorphisms embedded in C^1-smooth flow is nowhere dense in the space of Morse–Smale diffeomorphisms.

At the same time, for any diffeomorphism M^n there exists an open in $\mathit{Diff}^1(M^n)$ set of diffeomorphisms embedded in the topological flow. This fact follows from the following reasoning. By [46] on each manifold there exists a Morse function, such that its gradient flow can be arbitrarily approximated by the Morse–Smale flow X^t without closed trajectories. The shift per time unit X^1 along the trajectories of such flow is a Morse–Smale diffeomorphism, which, by [39, 41], are structurally stable. Hence, there exists a neighborhood $U(X^1) \subset \mathit{Diff}^1(M^n)$ such that any diffeomorphism $f \in U(X^1)$ is topologically conjugated to X^1 by some homeomorphism h, thus f is embedded in the topological flow $h^{-1}X^th$.

Recall what the diffeomorphism f given on closed manifold M^n is called *Morse–Smale diffeomorphism* if its nonwandering set Ω_f is finite and consists of hyperbolic periodic points, and for any two points $p, q \in \Omega_f$ the intersection of the stable manifold W^s_p of the point p and nonstable manifold W^u_q of the point q is transversal. Next we everywhere consider the Morse–Smale diffeomorphisms class $G(M^n)$ preserving the orientation on orientable manifolds.

In the paper by Palis [39] there are stated the following necessary conditions for the embedding of a Morse–Smale diffeomorphism $f : M^n \to M^n$ in the topological flow:

(1) *the nonwandering set Ω_f coincides with the set of stable points*;
(2) *the restriction of the diffeomorphism f on every invariant manifold of every fixed point $p \in \Omega_f$ preserves its orientation*;
(3) *if for different saddle points $p, q \in \Omega_f$ the intersection $W^s_p \cap W^u_q$ is not empty, then it doesn't contain the compact components of the connection.*

In further consideration we call conditions (1)–(3) to be the *Palis conditions.*

In [39] it is also shown that for $n = 2$ these conditions are sufficient (see Theorem 5.1). Also there is stated the problem of generalization of this result onto the case of higher dimension (note that from [18] it follows that the necessary and sufficient conditions of the embedding of Morse–Smale diffeomorphism of circle in the flow coincides with the first Palis condition). The Palis problem was exhaustively solved in dimension three in works [26, 45]; for the higher dimension it was solved only partially, for a class of Morse–Smale diffeomorphisms without the heteroclinic intersections given on sphere, see [24]. The present survey is devoted to the presentation of these results and related topological problems.

2. Properties of the Morse–Smale Diffeomorphisms and the Related Notation

We recall some facts related to the dynamics of Morse–Smale diffeomorphisms, which will be multiply used in further sections.

Let $f : M^n \to M^n$ be a diffeomorphism. The point $x \in M^n$ is called the *nonwandering point* of the diffeomorphism f if for each its neighborhood U and any natural number N there is such $n_0 \in \mathbb{Z}$

that $|n_0| \geq N$ and $f^{n_0}(U) \cap U \neq \varnothing$. Obviously, the periodic point is nonwandering. According to the definition of a Morse–Smale diffeomorphism its nonwandering set coincides with the set of periodic points.

The periodic point p of period m of diffeomorphism f is called *hyperbolic* if the differential $Df^m(p): T_pM^n \to T_pM^n$ considered as the linear mapping of the tangent space T_pM^n into itself doesn't have the eigenvalues modulo one. Due to Grobman–Hartman theorem [28–30], in a neighborhood of a hyperbolic periodic point p the diffeomorphism f^m is topologically conjugated to a linear diffeomorphism defined by the Jacobi matrix $\left(\frac{\partial f^m}{\partial x}\right)\Big|_p$.

From here we obtain that for a hyperbolic periodic point p there exist so called *stable manifold* $W^s_p = \{x \in M^n : \lim\limits_{n\to+\infty} d(f^{km}(x), p) = 0\}$ and *nonstable manifold* $W^u_p = \{x \in M^n : \lim\limits_{n\to+\infty} d(f^{-km_p}(x), p) = 0\}$, where d is a metric on M^n. The nonstable and stable manifolds are called the *invariant manifolds*. The number j equal to the number of eigenvalues of Jacobi matrix modulo larger than two and, correspondingly, coinciding with the dimension of nonstable manifold $\dim W^u_p$, is called the *Morse index* of the hyperbolic point p. Then the dimension of the stable manifold can be calculated with the formula $\dim W^s_p = n - j$.

Everywhere further we denote as Ω^j_f, $j \in \{0, \dots, n\}$ the set of hyperbolic periodic points of the diffeomorphism f with Morse index j. The point with Morse index $0 < j < n$ is called *saddle*, the others are called *nodal*, while nodal point with the index 0 is called *sink*, and with the index n *source*.

Recall that *n-ball* (*n-disk*) is a manifold with boundary homeomorphic to the standard ball $\mathbb{B}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$. The *sphere* is the manifold S^n homeomorphic to the boundary $\mathbb{S}^{n-1}$ of ball $\mathbb{B}^n$.

Due to the adjointness with the linear contraction, in the neighborhood of stable sink point p there exists a closed n-ball $U_p \subset W^s_p$ such that $f(U_p) \subset \operatorname{int} U_p$ and $\bigcap\limits_{k\geq 0} f^k(U_p) = p$. Thus, the stable hyperbolic sink point is the attractor of the diffeomorphism f in the sense of the following definition.

The closed f-invariant set $A \subset M^n$ is called an *attractor* of f if it has a compact neighborhood U_A such that $f(U_A) \subset \operatorname{int} U_A$ and $A = \bigcap\limits_{k\geq 0} f^k(U_A)$. The neighborhood U_A is called *attracting*, and the union $\bigcup\limits_{k\geq 0} f^{-k}(U_A)$ is called the *basin* of the attractor. The *repeller* is defined as the attractor for f^{-1}.

For each periodic hyperbolic point p, the connection component ℓ^s_p (ℓ^u_p) of $W^s_p \setminus p$ ($W^u_p \setminus p$) is called the *separatrix of the point p*. For each subspace $P \subset \Omega_f$, we denote as W^u_P (W^s_P) the union of unstable (stable) manifolds of all points of P.

The connection between the topology of the carrier manifold and the dynamical properties of Morse–Smale diffeomorphisms in many ways is explained by the following fact (see [31, 46]).

Proposition 2.1. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism. Then W^u_p and W^s_p are smooth submanifolds of M^n diffeomorphic to $\mathbb{R}^j$ and $\mathbb{R}^{n-j}$, respectively, for any periodic point $p \in \Omega_f$, and $M^n = \bigcup\limits_{p\in\Omega_f} W^u_p = \bigcup\limits_{p\in\Omega_f} W^s_p$.*

However, the invariant manifolds of saddle periodic points of Morse–Smale diffeomorphism are the submanifolds of M^n, their closures might have a complicated topological structure. For instance, we have such behavior when the saddle point separatrix is embedded in heteroclinic intersections.

Let σ_1, $\sigma_2 \in \Omega_f$ be different saddle periodic points of a Morse–Smale diffeomorphism f. The intersection of the invariant manifolds $W^s_{\sigma_1} \cap W^u_{\sigma_2}$, in case of $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$, is called *heteroclinic*. Since the invariant manifolds intersect transversally and each of $W^s_{\sigma_1}$, $W^u_{\sigma_2}$ is a submanifold, then each connection component of heteroclinic intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is also a submanifold. If $\dim\left(W^s_{\sigma_1} \cap W^u_{\sigma_2}\right) \geq 1$, then the connection component of such intersection is called the *heteroclinic manifold*. In particular, if $\dim\left(W^s_{\sigma_1} \cap W^u_{\sigma_2}\right) = 1$, then the heteroclinic manifold is called the *heteroclinic curve*.

The asymptotic behavior of the unstable separatrix in general case is described with the following proposition.

Proposition 2.2. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism. Then*

$$cl(\ell_p^u) \setminus (\ell_p^u \cup p) = \bigcup_{r \in \Omega_f : \ell_p^u \cap W_r^s \neq \varnothing} W_r^u$$

for each unstable separatrix ℓ_p^u of periodic point $p \in \Omega_f$. In particular, if ℓ_p^u is a saddle separatrix, not embedded in heteroclinic intersections, then $cl(\ell_\sigma^u) \setminus (\ell_\sigma^u \cup \sigma) = \{\omega\}$, where ω is a sink periodic point. With that, if $j = 1$, then $cl(\ell_\sigma^u)$ is the topologically embedded arc in M^n, and if $j \geq 2$, then $cl(\ell_\sigma^u)$ is the sphere $\mathbb{S}^j$ topologically embedded in M^n.

3. Palis Conditions

In this section we prove the necessity of the Palis conditions for the embedding of Morse–Smale diffeomorphism in the topological flow. This proof was proposed by Palis in [39].

Lemma 3.1 (necessary Palis conditions). *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism embedded in the topological flow X^t. Then*:

(1) *the nonwandering set Ω_f coincides with the set of fixed points*;

(2) *the restriction of the diffeomorphism f onto each invariant manifold of any fixed point $p \in \Omega_f$ preserves its orientation*;

(3) *if for different saddle points $p, q \in \Omega_f$ the intersection $W_p^s \cap W_q^u$ is not empty, then it doesn't contain compact connection components.*

Proof.

1. Suppose that the set Ω_f contains the periodic point p of period m_p greater than 1. Then the point p belongs to the closed trajectory of the flow X^t, and all the points of this trajectory are periodic of period m_p for the flow X^t. But then all these points are periodic for the diffeomorphism f, which is the shift per unit time along the flow X^t trajectories. That contradicts the finiteness of its nonwandering set.

2. Since Ω_f is hyperbolic, it follows that the invariant manifold W_p^u of an arbitrary fixed point $p \in \Omega_f$ either coincides with p, or is an open disc of dimension $\dim W_p^u \in \{1 \dots, n\}$ smoothly embedded in M^n. In case $W_p^u = p$, by the definition f preserves the orientation of W_p^u. Let $\dim W_p^u > 0$. Since f in embedded in X^t, then W_p^u is invariant relating to X^t. Thus the restriction $X^t|_{W_p^u}$ of X^t onto W_p^u is an isotopy from the identity mapping to $f|_{W_p^u}$, thus $f|_{W_p^u}$ is a mapping preserving the orientation.

3. Let the intersection $W_p^s \cap W_q^u$ be not empty for different saddle points $p, q \in \Omega_f$ and K be a compact connection component of this intersection. Then K is invariant with relation to X^t and, thus, invariant to the diffeomorphism f. Let $x \in K$, then the sequence $\{f^i(x)\}_{i \in \mathbb{Z}}$ contains a subsequence converging to some point $x^* \in K$. Thus x^* in nonwandering, which is impossible since $x^* \in W_p^s$. $\square$

The Fig. 1 shows the phase portraits of Morse–Smale diffeomorphisms. Their invariant manifolds of saddle points: a) intersect over a noncompact curve; b) intersect over a countable set of compact curves.

4. Embedding of Diffeomorphisms of Circle in the Flow

The unique closed one-dimensional manifold is the circle $\mathbb{S}^1$.

Due to [18] the homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ is embedded in the topological flow if and only if one of the following three conditions is satisfied: 1) h has a fixed point, 2) h is periodic, 3) h has a transitive orbit. From Proposition 2.1 it follows that if f is a Morse–Smale diffeomorphism, then its nonwandering set in nonempty (and finite, by definition) and consists of source and sink periodic points. For embedding in the flow it is necessary for f to preserve the orientation. Then if one of periodic points is fixed, then all points of diffeomorphism f are fixed. Thus the necessary and sufficient

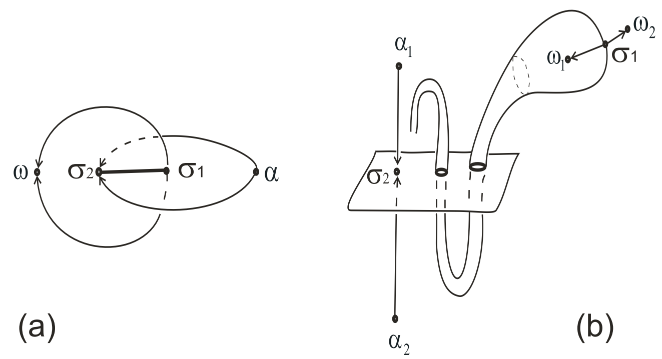

Fig. 1. Intersections of the invariant manifolds of saddle points

condition for embedding of Morse–Smale diffeomorphism of the circle in the topological flow is that at least one its periodic points is fixed. We give the independent proof of this fact.

Theorem 4.1. *The Morse–Smale diffeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is embedded in the topological flow if and only if its nonwandering set Ω_f consists of fixed points.*

Proof. The necessity follows from the Palis condition (1). Let us prove the sufficiency. Let the space Ω_f consist of fixed points. We construct the flow X^t on circle such that $f = X^1$.

The set of fixed points of the diffeomorphism f splits the circle $\mathbb{S}^1$ into a finite number of open arcs, each of which is f-invariant. Let $l \in \mathbb{S}^1$ be one of such arcs. We define the flow X_l^t on l such that f is the shift per time unit along the trajectories of the flow X_l^t. Let $c \subset l$ be a compact arc bounded by the points $x \in l$ and $f(x)$, then there exists a diffeomorphism $\varphi_c : [1,2] \to c$ such that $\varphi_c(1) = x, \varphi_c(2) = f(x)$. Note that $\bigcup\limits_{i\in\mathbb{Z}} f^i(c) = l$, thus for each point $y \in l$ there is an integer i_y such that $f^{i_y}(y) \in c$. We define the homeomorphism $\varphi_l : \mathbb{R}_+ \to l$ with the relation $\varphi_l(y) = 2^{-i_y}\varphi_c^{-1}(f^{i_y}(y))$. The homeomorphism φ_l conjugates the linear dilation $a_+(s) = 2s$, $s \in \mathbb{R}_+$ with the restriction $f|_l$ of f on l. The mapping a_+ is embedded in the flow $a_+^t(s) = 2^t s$. Let $X_l^t(y) = \varphi_l(a_+^t(\varphi_l^{-1}(y)))$, then $X_l^1(y) = f|_l$.

We similarly define the flow on all arcs of the circle bounded by the neighboring fixed points and define the flows in fixed points. As the result we obtain the flow X^t on the circle such that $X^1 = f$. □

5. Embedding of Morse–Smale Diffeomorphisms of Surfaces in the Flow

The following theorem is proved in [39] (see Theorem 4.2, p. 402).

Theorem 5.1 (Palis theorem)**.** *If diffeomorphism $f \in G(M^2)$ satisfies the Palis conditions, then it is embedded in a topological flow.*

Sketch of the proof. Note that for $n = 2$ the Palis condition (3) means that the invariant manifolds of different saddle fixed points of diffeomorphism $f : M^2 \to M^2$ do not intersect. In [39] for diffeomorphism f there was constructed a topological flow X^t, for which the shift per time unit f^1 along the trajectories coincides with f. The construction bases on the following steps.

1. From the hyperbolic conditions and the Palis conditions (1)-(2) it follows that for each saddle point $p \in \Omega_f$ there exists a neighborhood u_p and a homeomorphism $h_p : u_p \to \mathbb{R}^2$ such that $f|_{u_p} = h_p^{-1} b h_p|_{u_p}$, where $b(x,y) = (1/2x, 2y)$ is a linear homeomorphism of the plane. The mapping b is embedded in the flow $b^t(x,y) = ((1/2)^t x, 2^t y)$, thus the restriction of the diffeomorphism f onto u_p is embedded in the topological flow $g_p^t = h_p^{-1} b^t h_p|_{u_p}$. Suppose that $v = \{(x,y) \in \mathbb{R}^2 |\ x^2y^2 \le 1,\ |x| \le 1,\ |y| \le 2\}$, $v_p = h_p^{-1}(v)$, $V_p = \bigcup\limits_{i\in\mathbb{Z}} f^i(v_p)$. We assign to each point $M \in V_p$ the number $n \in \mathbb{Z}$ such that $f^n(M) \subset v_p$ and define the flow G_p^t on V_p with the relation $G_p^t(M) = f^{-n}(g_p^t(f^n(M)))$. We call a neighborhood V_p the *linearizing neighborhood of a saddle point* p. Obviously, one can choose such

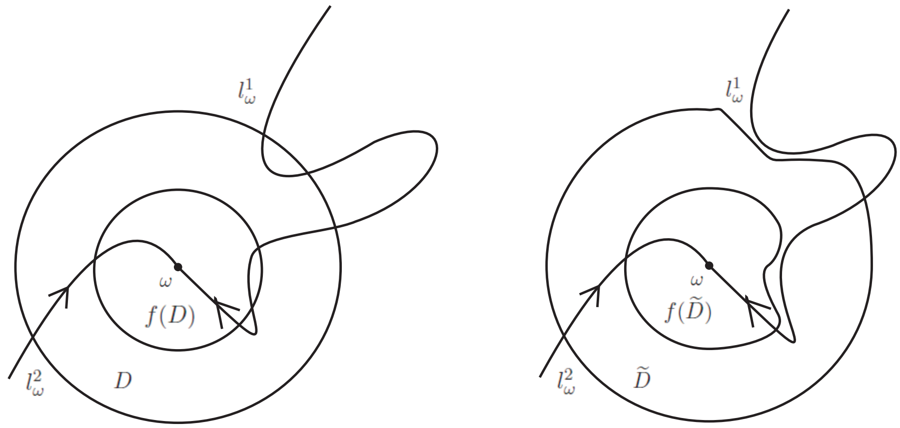

Fig. 2. Modification of disc D

linearizing neighborhoods that $V_p \cap V_q = \varnothing$ for each saddle points $p \neq q$. We denote as G^t the flow on the union of all linearizing neighborhoods, which coincides with G^t_p for each saddle point p.

2. Let ω be a sink fixed point of the diffeomorphism f. From the hyperbolicity of the point ω it follows that there exists a smoothly embedded disc $D \subset W^s_\omega$ such that $\omega \subset \operatorname{int} D$, $f(D) \subset \operatorname{int} D$.

Denote as $\ell^1_\omega, \dots, \ell^k_\omega$ the set of all separatrices of saddle points from W^s_ω, and as $V^1_\omega, \dots, V^k_\omega$ the components of the connection of linearizing neighborhoods from W^s_ω such that $\ell^i_\omega \subset V^i_\omega$ for all $i \in \{1, \dots, k\}$. Without loss of generality suppose that the boundary of the disc D is transversal to all separatrices of saddle points of f from W^s_ω (one can always achieve this with small movements), and transversal to the trajectories of the flow $G^t \cap W^s_\omega$ due to continuity. Then the intersection $\partial D \cap \bigcup\limits_{p \in \Omega^1_f} V_p$ consists of finite number of compact arcs. Then one can choose a disc $\widetilde{D} \subset W^s_\omega$ with the following properties:

(1) $\omega \subset \operatorname{int} \widetilde{D}$, $f(\widetilde{D}) \subset \operatorname{int} \widetilde{D}$,

(2) for all $i \in \{1, \dots, k\}$ the intersection $V^i_\omega \cap (\widetilde{D} \setminus f(\operatorname{int} \widetilde{D})$ consists of exactly one strip.

The Fig. 2 roughly shows the process of modification of disc D into disc $\widetilde{D}$, the boundary of which intersects with each separatrix from the set $\ell^1_\omega, \dots, \ell^k_\omega$ and only at one point.

Then the restriction of the flow G^t onto $\widetilde{D} \setminus f(\operatorname{int} \widetilde{D}) \cap \bigcup\limits_{p \in \Omega^1_f} V_p$ can be naturally completed to the flow g^t_ω in this set. The flow can be additionally defined on set $W^s_\omega \setminus \omega = \bigcup\limits_{i \in \mathbb{Z}} f^i(\widetilde{D} \setminus f(\operatorname{int} \widetilde{D}))$ as follows. For each point $M \in W^s_\omega \setminus \omega$, we put in correspondence an integer number n such that $f^n(M) \subset \widetilde{D} \setminus f(\operatorname{int} \widetilde{D})$ and set $g^t_\omega(M) = f^{-n}(G^t(f^n(M))$. Now for the construction of the required flow X^t we finally need to define the flow constructed from the flows G^t, G^t_ω, in fixed source and sink points. □

6. Embedding of Morse–Smale Diffeomorphisms of 3-Dimensional Manifolds in the Flow

6.1. 3-dimensional effects. As turned out, for dimension $n = 3$ there is an additional obstacle for Morse–Smale diffeomorphism to be embedded in topological flow. The obstacle is the possibility to wildly embed the separatrices of the saddle points, see Fig. 3. The first examples of such diffeomorphisms were constructed in [3, 43, 44].

Let us recall the definition of wild manifolds. Let M^n be a topological manifold of dimension $n \geq 3$ and $N^k \subset \operatorname{int} M^n$ be a compact topological manifold on dimension $k < n$ with nonempty boundary. By [10], the manifold N^k is called *locally flat at the point* $x \in N^k$ if there exist a neighborhood $U(x) \subset M^n$ of x and a homeomorphism $\varphi : U(x) \to \mathbb{R}^n$ such that $\varphi(N^k \cap U(x)) \subset \mathbb{R}^k$, where $\mathbb{R}^n$ is

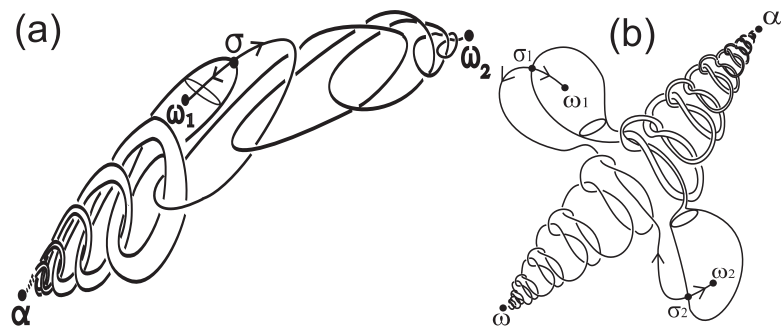

Fig. 3. Diffeomorphisms with wildly embedded separatrices

a Euclidean space, and $\mathbb{R}^k \subset \mathbb{R}^n$ is a hyperplane of dimension k. If the manifold N^k is locally flat at each point, then it is called *locally flat*. Note that in the latter case N^k is a submanifold of M^n. If N^k is not locally flat at least at one point $x \in N^k$, then it is called *wild* in M^n, and x is called a *wildness point*.

The right part of Fig. 3 shows a phase portrait of the diffeomorphism $f \in G(S^3)$, for which the closure of two-dimensional separatrix is a wild sphere. One of one-dimensional separatrices of fixed saddle point σ (containing a sink point ω_2 in its closure) is a wild arc.

The main obstacle to generalizing the proof of Theorem 5.1 for dimension $n = 3$ is that in the general case a neighborhood of a sink point ω doesn't contain a ball with the properties analogical to the properties of disc $\widetilde{D}$. Thus for a diffeomorphism with the phase portrait on the left part of Fig. 3, the boundary of each ball containing a sink point ω_2 intersects with the separatrix on the saddle σ containing the point ω_2 in its closure in at least three points. For the diffeomorphism f with a phase portrait on the right part of Fig. 3, in the neighborhood of a point ω there exists a ball D the boundary of which intersects with each separatrix from W^s_ω in one point only. But there doesn't exist a fibration of a ring $D \setminus \operatorname{int} f(D)$ into segments, which would contain the arcs of these one-dimensional separatrices as fibers. Thus the restriction of such diffeomorphisms onto stable manifolds of sink points is not embedded in the topological flows, for which the one-dimensional separatrices containing these sink points in closure would coincide with the flow trajectories. With that, by the results of Kuperberg [35], the wild arc might be a trajectory of a topological flow on 3-manifold.

For more precise understanding of embedding the diffeomorphisms from the class $G(M^3)$ into the topological flow we give several definitions.

We call the set $\mathbb{F} \subset \mathbb{R}^n$ the *standard one-dimensional pencil* if it consists of a finite number of rays with the initial point $O(0, \ldots, 0)$. We call the subset $F \subset \mathbb{R}^n$ endowed with the induced topology and homeomorphic to $\mathbb{F}$, a *one-dimensional pencil.* With that a pencil F is called *tame*, if there exists such homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ that $H(F) = \mathbb{F}$; otherwise a pencil F is called *wild.*

A particular case of a one-dimensional pencil in an arc. The first examples of wild arcs in $\mathbb{R}^3$ were constructed by Artin and Fox in 1948 (see [1]). Note that the tameness of each element of the pencil included into the pencil $F \subset \mathbb{R}^3$ does not guarantee that the whole pencil will be tame. For example, in [16] they construct an instance of so called *mildly wild one-dimensional pencil*, i.e., such wild pencil that any pencil in it with a lesser number of arcs is tame.

Let α be a source point of the diffeomorphism $f \in G(M^3)$. We denote as L_α the union of all one-dimensional stable separatrices of saddle points of the diffeomorphism f belonging to W^u_α. Suppose $F_\alpha = L_\alpha \cup \alpha$ and call F_α a *pencil of one-dimensional stable separatrices.*

We call a pencil of one-dimensional stable separatrices F_α *tame* if there exists a homeomorphism $h_\alpha : W^u_\alpha \to \mathbb{R}^3$, which maps F_α onto a standard tame pencil. Otherwise we say that the pencil of separatrices F_α is called *wild.* If a tame (wild) pencil F_α contains only one separatrix, then we call it *tame* (*wild*).

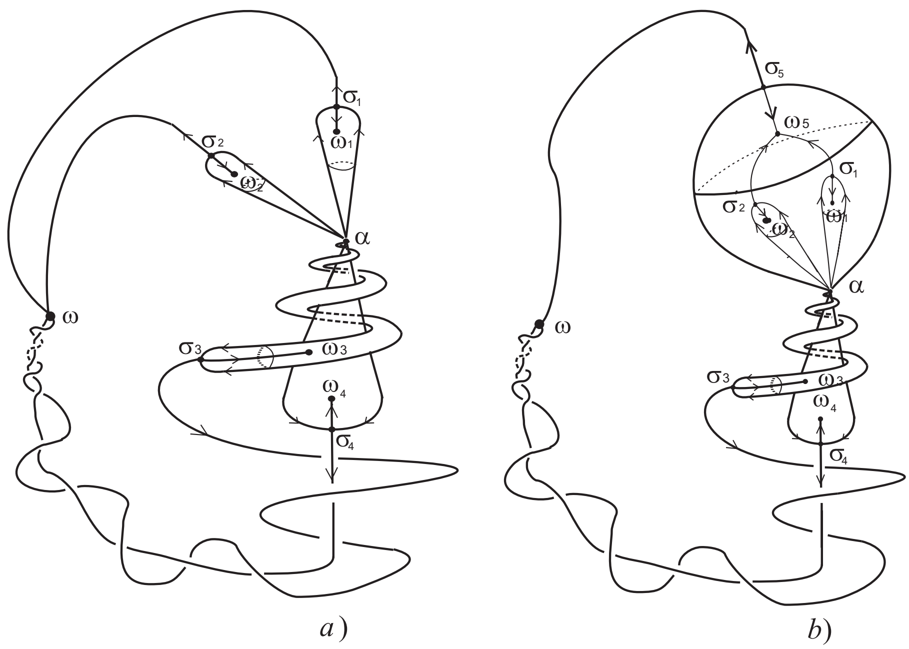

Fig. 4. Phase portraits of diffeomorphisms from class $G(\mathbb{S}^3)$ not embedded in any topological flows: a) a diffeomorphism with all tame pencils of one-dimensional separatrices, but F_ω is a nontrivial pencil; b) a diffeomorphism with all trivial pencils of one-dimensional separatrices.

Similarly one defines a *tame* (*wild*) *pencil* of one-dimensional nonstable separatrices F_ω consisting of a sink point ω and all one-dimensional nonstable separatrices L_ω of saddle points of diffeomorphism f from W^s_ω.

The Fig. 3, a) shows that the one-dimensional separatrix coming to a sink point ω_2 is a wild arc, and a pencil of separatrices coming to a sink ω in Fig. 3, b), is a mildly wild pencil.

As turned out, the necessary condition of embedding of the diffeomorphism $f \in G(M^3)$ in the flow is even stronger than the tameness property. This condition uses the linear dilation of the Euclidean space $\mathbb{R}^3$ defined by the formula $A(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3)$.

A pencil of one-dimensional separatrices F_α is called *trivial* if there exists a homeomorphism $H_\alpha : W^u_\alpha \to \mathbb{R}^3$ such that $f|_{W^u_\alpha} = H^{-1}_\alpha A H_\alpha|_{W^u_\alpha}$ and $H_\alpha(F_\alpha)$ is a standard one-dimensional pencil. Similarly one can define a *trivial pencil* of one-dimensional separatrices F_ω.

From the reasoning above it is obvious that the triviality of all pencils of one-dimensional separatrices is a necessary condition for diffeomorphism $f \in G(M^3)$ to embed in topological flow (a strict proof of this fact is similar to the proof of Proposition 6.1, which we state below). It was a surprising fact that if we add the condition that all pencils of one-dimensional separatrices of saddle points of a diffeomorphism $f \in G(M^3)$ are trivial, it does not give sufficient conditions for its embedding in topological flow. An example illustrating this fact was constructed in [45], with the phase portrait shown in Fig. 4, b). The Fig. 4, a) shows a phase portrait of the diffeomorphism of class $G(S^3)$ with all pencils of one-dimensional separatrices being tame, although having a nontrivial pencil.

6.2. A scheme of a diffeomorphism. The solution to Palis problem in case $n \geq 3$ was possible due to the significant progress in solving the topological classification problem for Morse–Smale diffeomorphisms. In papers [3–8, 44] by Bonatti, Grines, Pochinka, Pécou, Medvedev and Laudenbach these was introduced a new complete topological invariant for Morse–Smale diffeomorphisms on 3-manifolds called a *scheme* of a diffeomorphism and the problem of realization of all classes of topological conjugation was solved. Due to this it was possible to state necessary and sufficient conditions for Morse–Smale diffeomorphism to be embedded in the topological flow. These conditions are expressed in a quite compact and natural condition on the scheme of a diffeomorphism. In order to state this conditions explicitly we firstly give the definition of a scheme of a diffeomorphism.

Recall that we denote as Ω^i_f a set of fixed points of the Morse–Smale diffeomorphism $f : M^3 \to M^3$ with nonstable manifolds of dimension $i \in \{0, 1, 2, 3\}$. We denote the class of such Morse–Smale diffeomorphisms preserving the orientation as $G(M^3)$.

Represent the manifold M^3 as the union of three sets $A_f = (\bigcup\limits_{\sigma\in\Omega^1_f} W^u_\sigma) \cup \Omega^0_f$, $R_f = (\bigcup\limits_{\sigma\in\Omega^2_f} W^s_\sigma) \cup \Omega^3_f$ and $V_f = M^n \setminus (A_f \cup R_f)$. From [27] it follows that the sets A_f, R_f, V_f are connected, A_f is an attractor, R_f is a repeller, and V_f consists of wandering orbits of the homeomorphism f coming from R_f to A_f.

Denote as $\widehat{V}_f = V_f/f$ the set of orbits of action f on V_f. It is known that $\widehat{V}_f$ is a manifold, and a natural projection $p_f : V_f \to \widehat{V}_f$ is a covering projection. With that a covering projection p_f induces an epimorphism $\eta_f : \pi_1(\widehat{V}_f) \to \mathbb{Z}$, which assigns an integer m to a homotopy class $[c] \in \pi_1(\hat{V}_f)$ such that a curve c lifted to V_f connects a point x with $f^m(x)$.

Let $\hat{L}^s_f = \bigcup\limits_{\sigma\in\Omega^1_f} p_f(W^s_\sigma \setminus \sigma)$, $\hat{L}^u_f = \bigcup\limits_{\sigma\in\Omega^2_f} p_f(W^u_\sigma \setminus \sigma)$.

Definition 6.1. The set $S_f = (\widehat{V}_f, \hat{L}^s_f, \hat{L}^u_f, \eta_f)$ is called a *scheme* of a diffeomorphism $f \in G(M^3)$.

Definition 6.2. The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in G(M^3)$ are called *equivalent* if there exists a homeomorphism $\hat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$ such that $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$, $\hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$ and $\eta_f = \eta_{f'}\hat{\varphi}_*$.

In the papers [4, 8] the following fact is proven.

Statement 6.1. *The diffeomorphisms $f, f' \in G(M^3)$ are topologically conjugated if and only if their schemes are equivalent.*

6.3. Necessary and sufficient conditions for Morse–Smale diffeomorphisms of 3-manifolds to embed in the flow. To state the conditions for the diffeomorphism $f \in G(M^3)$ to embed in topological flow we define the standard scheme.

Set $g_f = \frac{|\Omega^1_f \cup \Omega^2_f| - |\Omega^0_f \cup \Omega^3_f| + 2}{2}$, where $|P|$ stands for the cardinality of P. Denote by $\mathbb{S}_{g_f}$ an orientable closed surface of genus g_f and set $\mathbb{V}_{g_f} = \mathbb{S}_{g_f} \times \mathbb{R}$, $\widehat{\mathbb{V}}_{g_f} = \mathbb{S}_{g_f} \times \mathbb{S}^1$.

Define the flow $A^t_{g_f}$ on $\mathbb{V}_{g_f}$ with the relation $A^t_{g_f}(x, s) = (x, s + t)$, where $x \in \mathbb{S}_{g_f}, s \in \mathbb{R}$. By construction we have $\widehat{\mathbb{V}}_{g_f} = \mathbb{V}_{g_f}/_{A^1_{g_f}}$. Denote a natural projection by $p_{g_f} : \mathbb{V}_{g_f} = \widehat{\mathbb{V}}_{g_f}$.

Definition 6.3. The scheme S_f of a diffeomorphism $f \in G(M^3)$ is called *trivial* if there exists a homeomorphism $\hat{\psi}_f : \widehat{V}_f \to \widehat{\mathbb{V}}_{g_f}$ such that for each connected component $\hat{\lambda}$ of $\hat{L}^s_f \cup \hat{L}^u_f$ there exists a simple closed arc $c_{\hat{\lambda}} \subset \mathbb{S}_{g_f}$ such that $\hat{\psi}_f(\hat{\lambda}) = c_{\hat{\lambda}} \times \mathbb{S}^1$.

In [26, 45] the following fact is proven.

Theorem 6.1. *The diffeomorphism $f \in G(M^3)$ is embedded in a topological flow if and only if its scheme is trivial.*

Let us give a sketch of the proof of Theorem 6.1, splitting it into two statements.

Proposition 6.1. *Let the diffeomorphism $f \in G(M^3)$ be embedded in a topological flow. Then its scheme S_f is trivial.*

Sketch of the proof. If the diffeomorphism f is embedded in a topological flow X^t $(f = X^1)$, then the nonwandering set Ω_f of the diffeomorphism f coincides with the equilibrium state of the flow X^t. Here the stable (nonstable) manifold of any fixed point $p \in \Omega_f$ coincides with the stable (nonstable) manifold of the corresponding equilibrium state of the flow X^t.

Denote by X^t_f the restriction of the flow X^t onto V_f. From the construction of V_f it follows that for each point $x \in V_f$ the embeddings $\lim\limits_{t\to+\infty} X^t_f(x) \in A_f$ and $\lim\limits_{t\to-\infty} X^t_f(x) \in R_f$ hold. Thus, for each point $p, q \in V_f$ there exist neighborhoods $U_p, U_q \subset V_f$ and a constant $T > 0$ such that $X^t_f(U_p) \cap U_q = \varnothing$ for all $|t| > T$. Thus from [17, Theorem 3] it follows that the flow X^t_f is parallelizable, i.e., there exists a set $\Sigma_f \subset V_f$ and a homeomorphism $\xi_f : V_f \to \Sigma_f \times \mathbb{R}$ such that $\bigcup\limits_{t\in\mathbb{R}} X^t_f(\Sigma_f) = V_f$ and $\xi_f(X^t_f(z)) = (z, t)$ for all $z \in \Sigma_f, t \in \mathbb{R}$. This gives that Σ_f is a deformation retract of V_f. From [34, Theorem III.4, IV.3; p. 56, 69] it follows that the topological dimension of Σ_f equals 2. Then by [48, Theorem 2] Σ_f is a manifold without the boundary. Thus, Σ_f is a closed orientable surface. Denote as ρ_f the genus of this surface. We show now that $\rho_f = g_f$.

By the construction the surface Σ_f divides the manifold into two parts, with their closures denoted by P_{A_f}, P_{R_f}, and supposing that $A_f \subset \operatorname{int} P_{A_f}$, $R_f \subset \operatorname{int} P_{R_f}$. Moreover, the attractor A_f is a deformation retract of P_{A_f} and thus, A_f and P_{A_f} have the same homotopy type and the same Euler characteristic. With that $\chi(P_{A_f}) = 1 - \rho_f$, since P_{A_f} is a 3-manifold with the boundary Σ_f and $\chi(A_f) = |\Omega^0_f| - |\Omega^1_f|$, since A_f is a CW-complex consisting of $|\Omega^0_f|$ 0-cells and $|\Omega^1_f|$ 1-cells. Thus, $|\Omega^0_f| - |\Omega^1_f| = 1 - \rho_f$. From similar reasoning for the attractor obtain that $|\Omega^3_f| - |\Omega^1_f| = 1 - \rho_f$. Summing two last equalities, we obtain that $|\Omega^0_f| - |\Omega^1_f| + |\Omega^3_f| - |\Omega^2_f| = 2 - 2\rho_f$. From here, $\rho_f = \frac{|\Omega^1_f \cup \Omega^2_f| - |\Omega^0_f \cup \Omega^3_f| + 2}{2}$ and, hence, $\rho_f = g_f$.

Each two-dimensional separatrix λ of diffeomorphism f is the union of the trajectories of the flow X^t_f homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Then there exists a simple closed curve $\gamma_\lambda \subset \Sigma_f$ such that $\xi_f(\lambda) = \gamma_\lambda \times \mathbb{R}$. And then there exists a homeomorphism $h_f : \Sigma_f \to \mathbb{S}_{g_f}$ such that $c_\lambda = h_f(\gamma_\lambda)$ is a simple smooth closed curve for each two-dimensional separatrix λ. Define a homeomorphism $\psi_f : V_f \to \mathbb{V}_{g_f}$ by the relation $\psi_f(X^t_f(z)) = A^t_{g_f}(h_f(z))$. By the construction the homeomorphism ψ_f conjugates the flows X^t_f and $A^t_{g_f}$, and its shifts per time unit. With that $\psi_f(\lambda) = c_\lambda \times \mathbb{R}$. By the construction $\hat{\mathbb{V}}_{g_f} = \mathbb{V}_{g_f}/A^1_{g_f}$. Then the homeomorphism $\hat{\psi}_f = p_{g_f}\psi_f p_f^{-1} : \hat{V}_f \to \hat{\mathbb{V}}_{g_f}$ satisfies the condition of Definition 6.3. Thus, a scheme S_f is trivial. This completes the proof of the proposition. ∎

Proposition 6.2. *Let a scheme S_f of a diffeomorphism $f \in G(M^3)$ be trivial. Then f is embedded in a topological flow.*

Scheme of the proof. Construct a topological flow $\tilde{X}^t$ on the manifold M^3. The shift per time unit of this flow is topologically conjugated with the diffeomorphism f by a homeomorphism $h : M^3 \to M^3$. From here it follows that the diffeomorphism f is embedded in a topological flow $X^t = h\tilde{X}^t h^{-1}$.

The construction of the topological flow is conducted similarly to [5] (see also [6] for more details) as the solution of the problem of realizing topological conjugacy classes of diffeomorphisms. We give the main steps in the construction.

Step 1. From the definition of the trivial scheme it follows that there exists such homeomorphism $\psi_f : V_f \to \mathbb{V}_{g_f}$ that:

(1) $f|_{V_f} = \psi_f^{-1} A^1_{g_f} \psi_f$, where $A^1_{g_f}$ is a time unit shift on the flow $A^t_{g_f}$;
(2) for each two-dimensional separatrix λ of the diffeomorphism f there exists a simple smooth closed arc c_λ on the surface $\mathbb{S}_{g_f}$ such that $\psi_f(\lambda) = c_\lambda \times \mathbb{R}$.

Recall that L^s_f, L^u_f is a union of all stable, (correspondingly, nonstable) two-dimensional separatrices of the diffeomorphism f. Set $\mathbb{L}^s = \psi_f(L^s_f)$ and $\mathbb{L}^u = \psi_f(L^u_f)$. For the set of cylinders $\mathbb{L}^\delta = \lambda^\delta_1 \cup \dots \cup \lambda^\delta_{l^\delta}$, $\delta \in \{s, u\}$, denote by $N(\mathbb{L}^\delta) = N(\lambda^\delta_1) \cup \dots \cup N(\lambda^\delta_{l^\delta})$ the set of their pairwise disjoint smooth tubular neighborhoods such that $N(\lambda^\delta_i) = K^\delta_i \times \mathbb{R}$, where $K^\delta_i \subset \mathbb{S}_{g_f}$ is a smooth two-dimensional ring for each $i = 1, \dots, l^\delta$.

Consider a subspace $N = \{(x_1, x_2, x_3) : (x_1^2+x_2^2)x_3^2 < 1\}$ in $\mathbb{R}^3$ and define a flow B^t by $B^t(x_1, x_2, x_3) = (2^{-t}x_1, 2^{-t}x_2, 2^t x_3)$. Set $\hat{N}^s = (N \setminus Ox_3)/B^1$. By the construction the manifold $\hat{N}^s$ is diffeomorphic to $K \times \mathbb{R}$, where K is a standard two-dimensional ring. Then there exists a diffeomorphism $\mu_i^s : N(\lambda_i^s) \to (N \setminus Ox_3)$ conjugating the flows $A^t_{g_f}|_{N(\lambda_i^s)}$ and $B^t|_{N\setminus Ox_3}$. Denote by $\mu^s : N(\mathbb{L}^s) \to (N \setminus Ox_3) \times \mathbb{Z}_{l^s}$ a diffeomorphism composed of the diffeomorphisms $\mu_1^s, \dots, \mu_{l^s}^s$. Set $Q^s = \mathbb{V}_{g_f} \bigcup\limits_{\mu^s} (N \times \mathbb{Z}_{l^s})$. Then the topological space Q^s is a smooth connected orientable 3-manifold without the boundary.

Set $\bar{Q}^s = (\mathbb{V}_{g_f}) \cup (N \times \mathbb{Z}_{l^s})$ and denote by $p_s : \bar{Q}^s \to Q^s$ a natural projection. Set $p_{s,1} = p_s|_{\mathbb{V}_{g_f}}$, $p_{s,2} = p_s|_{N\times\mathbb{Z}_{l^s}}$. Then the flow $\tilde{Y}_s^t$ on the manifold Q^s is defined by

$$\tilde{Y}_s^t(x) = \begin{cases} p_{s,1}(A^t_{g_f}(p_{s,1}^{-1}(x))), \ x \in p_{s,1}(\mathbb{V}_{g_f}); \\ p_{s,2}(B^t(p_{s,2}^{-1}(x))), \ x \in p_{s,2}(N \times \{i\}), \ i \in \mathbb{Z}_{l^s}. \end{cases}$$

By the construction the nonwandering set of the flow $\tilde{Y}_s^t$ consists of l^s fixed hyperbolic saddle point with Morse index equal to one.

Step 2. Denote again by $\mathbb{L}^u$, $N(\mathbb{L}^u)$ the images of these sets relating to the projection p_s. Set $\hat{N}^u = (N \setminus Ox_3)/(B^1)^{-1}$. Then there exists a diffeomorphism $\mu_i^u : N(\lambda_i^u) \to (N \setminus Ox_3)$ conjugating the flows $\tilde{Y}_s^t|_{N(\lambda_i^u)}$ and $B^{-t}|_{N\setminus Ox_3}$ for each $i = 1, \dots, l^u$. Denote by $\mu^u : N(\mathbb{L}^u) \to (N \setminus Ox_3) \times \mathbb{Z}_{l^u}$ the diffeomorphism composed of the diffeomorphisms $\mu_1^u, \dots, \mu_{l^u}^u$. Set $Q^u = Q^s \bigcup\limits_{\mu^u} (N \times \mathbb{Z}_{l^u})$. Then the topological space Q^u is a smooth connected orientable 3-manifold without the boundary.

Set $\bar{Q}^u = Q^s \cup (N \times \mathbb{Z}_{l^u})$ and denote by $p_u : \bar{Q}^u \to Q^u$ a natural projection. Set $p_{u,1} = p_u|_{Q^s}$, $p_{u,2} = p_u|_{N\times\mathbb{Z}_{l^u}}$. Then the flow $\tilde{Y}_u^t$ on the manifold Q^u is defined by the formula

$$\tilde{Y}_u^t(x) = \begin{cases} p_{u,1}(\tilde{Y}_s^t(p_{u,1}^{-1}(x))), \ x \in p_{u,1}(Q^s); \\ p_{u,2}(B^{-t}(p_{u,2}^{-1}(x))), \ x \in p_{u,2}(N \times \{i\}), \ i \in \mathbb{Z}_{l^u}. \end{cases}$$

By the construction the nonwandering set of the flow $\tilde{Y}_u^t$ consists of l^s fixed hyperbolic saddle point with Morse index equal to one and l^u fixed hyperbolic saddle point with Morse index equal to two.

Step 3. Set $R^s = Q^u \setminus W^s_{\Omega_{\tilde{Y}_u^t}}$ and denote by $\rho_1^s, \dots, \rho_{n^s}^s$ the connected components of R^s. Define the topological flow D^t on $\mathbb{R}^3$ by $D^t(x_1, x_2, x_3) = (2^{-t}x_1, 2^{-t}x_2, 2^{-t}x_3)$. Then each component of ρ_i^s is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and the flow $\tilde{Y}_u^t|_{\rho_i^s}$ is smoothly conjugated to the flow $D^t|_{\mathbb{R}^3\setminus O}$ via a diffeomorphism ν_i^s. Denote by $\nu^s : R^s \to (\mathbb{R}^3 \setminus Ox_3) \times \mathbb{Z}_{n^s}$ a diffeomorphism composed of diffeomorphisms $\nu_1^s, \dots, \nu_{n^s}^s$. Set $M^s = Q^u \bigcup\limits_{\nu^s} (\mathbb{R}^3 \times \mathbb{Z}_{n^s})$. Then the topological space M^s is a smooth connected orientable 3-manifold without the boundary.

Set $\bar{M}^s = Q^u \cup (\mathbb{R}^3 \times \mathbb{Z}_{n^s})$ and denote by $q_s : \bar{M}^s \to M^s$ a natural projection. Set $q_{s,1} = q_s|_{Q^u}$, $q_{s,2} = q_s|_{\mathbb{R}^3\times\mathbb{Z}_{n^s}}$. Then the flow $\tilde{X}_s^t$ on M^s is defined by

$$\tilde{X}_s^t(x) = \begin{cases} q_{s,1}(\tilde{Y}_u^t(q_{s,1}^{-1}(x))), \ x \in q_{s,1}(Q^u); \\ q_{s,2}(B^{-t}(q_{s,2}^{-1}(x))), \ x \in q_{s,2}(\mathbb{R}^3 \times \{i\}), \ i \in \mathbb{Z}_{n^s}. \end{cases}$$

By the construction the nonwandering set of the flow $\tilde{X}_s^t$ consists of l^s fixed hyperbolic stable points with Morse index equal to one, l^u fixed hyperbolic stable points with Morse index equal to two, and n^s fixed hyperbolic sink points.

Step 4. Set $R^u = M^s \setminus W^u_{\Omega_{\tilde{X}_s^t}}$ and denote by $\rho_1^u, \dots, \rho_{n^u}^u$ the connection components of set R^u. Then each component ρ_i^u is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and the flow $\tilde{X}_s^t|_{\rho_i^u}$ is smoothly conjugated with the flow $D^{-t}|_{\mathbb{R}^3\setminus O}$ by the diffeomorphism ν_i^u. Denote by $\nu^u : R^u \to (\mathbb{R}^3 \setminus Ox_3) \times \mathbb{Z}_{n^u}$ a diffeomorphism composed of diffeomorphisms $\nu_1^u, \dots, \nu_{n^u}^u$. Set $M^u = M^s \bigcup\limits_{\nu^u} (\mathbb{R}^3 \times \mathbb{Z}_{n^u})$. Then the topological set M^u is a smooth connected closed orientable 3-manifold.

Set $\bar{M}^u = M^s \cup (\mathbb{R}^3 \times \mathbb{Z}_{n^u})$ and denote by $q_u : \bar{M}^u \to M^u$ a natural projection. Set $q_{u,1} = q_u|_{M^s}$, $q_{u,2} = q_u|_{\mathbb{R}^3 \times \mathbb{Z}_{n^u}}$. Then the flow $\tilde{X}^t_u$ on the manifold M^u is defined by

$$\tilde{X}^t_u(x) = \begin{cases} q_{u,1}(\tilde{X}^t_s(q^{-1}_{u,1}(x))),\ x \in q_{u,1}(M^s); \\ q_{u,2}(B^{-t}(q^{-1}_{u,2}(x))),\ x \in q_{u,2}(\mathbb{R}^3 \times \{i\}),\ i \in \mathbb{Z}_{n^u}. \end{cases}$$

By the construction the nonwandering set of the flow $\tilde{X}^t_u$ consists of l^s fixed hyperbolic saddle points with Morse index equal to one, l^u fixed hyperbolic stable points with Morse index equal to two, n^s fixed hyperbolic sink points and n^u fixed hyperbolic source points.

Step 5. Set $\tilde{f} = \tilde{X}^1_u$. By the construction the diffeomorphism $\tilde{f}$ is a Morse–Smale diffeomorphism on the manifold M^u and its reduction $\tilde{f}|_{V_{\tilde{f}}}$ is topologically conjugated with the diffeomorphism $f|_{V_f}$ via a homeomorphism. This homeomorphism maps two-dimensional separatrices of the diffeomorphism $\tilde{f}$ into two-dimensional separatrices of the diffeomorphism f preserving stability. Thus the schemes of the diffeomorphisms $\tilde{f}$ and f are equivalent, and $\tilde{f}, f$ are topologically conjugated by Statement 6.1. Thus $M^u = M^3$ and $\tilde{X}^t = \tilde{X}^t_u$ is the desired flow. □

6.4. Connection between a scheme triviality condition and Palis condition. Let a scheme of the diffeomorphism $f \in G(M^3)$ be trivial. We show that it gives the Palis condition.

1. Show that all saddle periodic points of the diffeomorphism f have period 1. Suppose that $\sigma \in \Omega^2_f$ is a saddle point of period m_σ such that the diffeomorphism $f|_{W^u_\sigma}$ preserves the orientation. Then there exists such homeomorphism $h : W^u_\sigma \to \mathbb{R}^2$ that $hf^{m_\sigma}|_{W^u_\sigma} = a_+h$, where $a_+ : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear mapping of the plane given by formula $a_+(x_1, x_2) = (2x_1, 2x_2)$. Set $K = \{(x_1, x_2)|\ 1 \le x_1^2 + x_2^2 \le 4\}$. The ring K $(h^{-1}(K))$ is a fundamental domain of the action of a_+ (f) on $\mathbb{R}^2 \backslash \{O\}$ $(\bigcup\limits_{i=0}^{m_\sigma-1} W^u_{f^i(\sigma)} \backslash f^i(\sigma))$. The orbit space $\mathbb{R}^2 \setminus \{O\}/_{a_+}$ $(\hat{\lambda}^u_\sigma = (\bigcup\limits_{i=0}^{m_\sigma-1} W^u_{f^i(\sigma)} \setminus f^i(\sigma))/_f = p_f(\bigcup\limits_{i=0}^{m_\sigma-1} W^u_{f^i(\sigma)} \setminus f^i(\sigma))$ of this action is obtained by gluing the components of the boundary of the annulus K $(h^{-1}(K))$ by diffeomorphism a_+ (f). Since a_+ preserves the orientation, then the manifold $\mathbb{R}^2 \backslash \{O\}/_{a_+}$ and, hence, $\hat{\lambda}^u_\sigma$ is diffeomorphic to torus. Choose the arc $\tilde{l}$ on set $h^{-1}(K)$. This arc connects the points x and $f^{m_\sigma}(x)$, which belong to different connected components on the boundary of the annulus $h^{-1}(K)$. Then the closed arc $l = p_f(\tilde{l})$ is a loop on torus $\hat{\lambda}^u_\sigma$, nonhomotopic to zero, and $\eta_f([p_f(l)]) = m_\sigma$. Then from the condition on the existence of the homeomorphism $\hat{\psi}_f : \hat{V}_f \to \widehat{\mathbb{V}}_{g_f}$ such that $\hat{\psi}_f(\hat{\lambda}^u_\sigma) = c_{\hat{\lambda}^u_\sigma} \times \mathbb{S}^1$ it follows that $m_\sigma = 1$.

2. Show that the reduction of the diffeomorphism f on the invariant manifold of an arbitrary saddle point preserves the orientation. From this condition it follows that each separatrix of an arbitrary saddle point is invariant. It leads to the fact that each sink and source point are fixed, and each of these points, by Proposition 2.1 and 2.2, is in the closure of a separatrix of the saddle point.

Let $\sigma \in \Omega^2_f$ be a fixed saddle point such that a diffeomorphism $f|_{W^u_\sigma}$ changes the orientation of W^u_σ. Then there exists a homeomorphism $h : W^u_\sigma \to \mathbb{R}^2$ such that $hf|_{W^u_\sigma} = a_-h$, where $a_- : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear mapping of a plane given by formula $a_-(x_1, x_2) = (-2x_1, 2x_2)$. Then the orbit space $\mathbb{R}^2 \backslash \{O\}/_{a_-}$ $(\hat{\lambda}^u_\sigma = (W^u_\sigma \backslash \sigma)/_f = p_f(W^u_\sigma \backslash \sigma))$ is diffeomorphic to the Klein bottle, which contradicts to the triviality of the scheme.

Let $\sigma' \in \Omega^1_f$ and $f|_{W^u_{\sigma'}}$ change the orientation. Since f preserves the orientation, then $f|_{W^s_{\sigma'}}$ changes the orientation of $W^s_{\sigma'}$. Apply the same reasoning to the point σ' as to the point σ. As the result we obtain that all saddle points of the diffeomorphism $f \in G(M^3)$ with trivial scheme are fixed, and the reduction of the diffeomorphism f onto the invariant manifold of an arbitrary saddle point preserves the orientation.

3. Let p, q be such fixed saddle points of the diffeomorphism f that $W^u_p \cap W^s_q \ne \varnothing$. Show that the intersection $W^u_p \cap W^s_q$ doesn't contain the compact connected components.

Set $\hat{\lambda}^u_p = p_f(W^u_p \setminus p), \hat{\lambda}^s_q = p_f(W^s_q \setminus q)$. If $p \in \Omega^1_f, q \in \Omega^2_f$, then $W^u_p \subset A_f$, hence, the projection of the manifold $W^s_q \setminus q$ doesn't contain the points from $W^s_q \cap W^u_p$. Thus $\hat{\lambda}^s_p$ is non-compact, which contradicts the fact that this set is homeomorphic to the torus (in the trivial scheme). If $p \in \Omega^2_f$, and $q \in \Omega^2_f$, then by the condition there exist such closed arcs $c_p, c_q \subset \mathbb{S}_{g_f}$ that $\hat{\psi}_f(\hat{\lambda}^u_p) = c_p \times \mathbb{S}^1$ and $\hat{\psi}_f(\hat{\lambda}^s_q) = c_q \times \mathbb{S}^1$. Hence, the projection of each connected component of the intersection $W^u_p \cap W^s_q$ is a set of type $\{x\} \times \mathbb{S}^1$, where $x \in c_p \cap c_q$ is a point. From the construction it follows that $p_f^{-1}(\{x\} \times \mathbb{S}^1)$ is homeomorphic to the real line embed in V_f, hence, the intersection $W^u_p \cap W^s_q$ does not contain the compact components.

7. Sufficient Conditions of Morse–Smale Diffeomorphism to be Embedded in Flow on Sphere of Dimension 4 and Higher

Denote by $G_*(S^n)$ a class of Morse–Smale diffeomorphism preserving the orientation on sphere S^n of dimension $n \geq 4$ such that for each $f \in G_*(S^n)$ the invariant manifolds of different saddle points $p, q \in \Omega_f$ do not intersect. Since there are no intersections of the invariant manifolds of different periodic saddle points, the set of periodic saddle points of the diffeomorphism $f \in G_*(S^n)$ consists of points with invariant manifolds of dimension 1 and $(n-1)$ only (see [23, Theorem 1.3], [24, Proposition 4.2], and also [42, Lemma 2.2]).

Since we study the issue of the diffeomorphism f to embed in the topological flow, we suppose that all points from Ω_f are fixed (that implies that all Palis conditions for $f \in G_*(S^n)$ are satisfied).

By Proposition 2.1 the invariant manifolds of periodic saddle points of any Morse–Smale diffeomorphism $f : M^n \to M^n$ are smooth submanifolds. Moreover, by Proposition 2.2, if a nonstable separatrix ℓ^u_σ of a saddle point σ does not intersect with any stable manifolds of saddle points distinct from σ, then the closure $cl\, \ell^u_\sigma$ of this separatrix consists of the separatrix itself, a point σ, and a sink point ω. Thus the following statement holds.

Proposition 7.1. *Let $f \in G_*(S^n)$, and σ be its periodic saddle point. Then the set $cl\, \ell^u_\sigma$ is a sphere of dimension $n-1$ if $\sigma \in \Omega^{n-1}_f$, and a compact arc if $\sigma \in \Omega^1_f$.*

Unlike the dimension 3, the closures of the separatrices of saddle points of the diffeomorphism $f \in G_*(S^n)$ are topological submanifolds of the sphere S^n. This fact directly follows from the following proposition.

Proposition 7.2.

(1) *Let $N^{n-1} \subset \operatorname{int} M^n$ be a wild manifold, $n \geq 4$, and B be a set of points such that N^{n-1} is locally flat at each point of $N^{n-1} \setminus B$. Then B in noncountable.*
(2) *Let $l \in \mathbb{R}^n$ be a wild arc, $n \geq 4$. Then a set of its wildness points is more than countable.*
(3) *The pencil $F \subset \mathbb{R}^n$, $n \geq 4$, of tame arcs is tame*[1].

The first statement of Proposition 7.2 is the corollary of the results of Cantrell, Chernavskii and Kirby[2] (see [11], [15, Statement 3A.6]). The second and third statements of Proposition 7.2 follow from [12, 13]. Note that [2] proves the existence of the wild arcs in the Euclidean space $\mathbb{R}^n$ of dimension $n \geq 4$ (but then these arcs by [12, 13] have more than countable number of wildness points).

From Proposition 7.2, 7.1 it follows that the separatrices of saddle points of diffeomorphism $f \in G_*(S^n)$ of dimension $(n-1)$ are tame spheres, and one-dimensional separatrices form tame pencils. By the methods of [12] one can prove a stronger fact of triviality of the pencils of one-dimensional

[1]i.e., the Euclidean space $\mathbb{R}^n$ of dimension $n \geq 4$ doesn't contain mildly wild pencils.

[2]In [15] it is shown that Proposition 7.2 is the corollary of the results of Chernavskii and Kirby obtained independently in 1968. Earlier in 1963, Cantrell obtained a less general statement, which can be stated as follows: *if the sphere $S^{n-1} \subset S^n$, $n \geq 4$, is wild and B is a set of points such that S^{n-1} is locally flat at each point of the set $S^{n-1} \setminus B$, then B consists of more than one point* (see [11]).

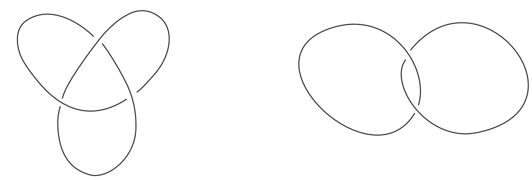

Fig. 5. a) nontrivial arc; b) nontrivial link.

separatrices (see [21, Corollary 4.1]). But from here doesn't follow the fact that all pencils of the separatrices of dimension $(n-1)$ are tame and all diffeomorphisms from the class $G_*(S^n)$ as $n \geq 4$ embed in the topological flows. However, in [24] there is stated a duality between the embeddings of the separatrices of dimension 1 and $(n-1)$ and the following theorem is proven.

Theorem 7.1. *Any diffeomorphism $f \in G_*(S^n)$, $n \geq 4$, is embedded in topological flow.*

To prove Theorem 7.1 we use the diffeomorphism scheme introduced below similarly to dimension 3. We introduce the triviality of the scheme and give the main ideas to prove the fact that the scheme of any diffeomorphism $f \in G_*(S^n)$ is trivial. After we proved the triviality of the scheme, the proof of the diffeomorphism f to be embedded in the topological flow is conducted completely similar to the proof of Theorem 6.1.

We represent a sphere S^n as the union of sets $A_f = (\bigcup\limits_{\sigma \in \Omega_f^1} W_\sigma^u) \cup \Omega_f^0$, $R_f = (\bigcup\limits_{\sigma \in \Omega_f^{n-1}} W_\sigma^s) \cup \Omega_f^n$, and $V_f = M^n \setminus (A_f \cup R_f)$.

Denote by $\widehat{V}_f = V_f/f$ the orbit space of the action of f on V_f and by $p_f : V_f \to \widehat{V}_f$ a natural projection. Set $\hat{L}_f^s = \bigcup\limits_{\sigma \in \Omega_f^1} p_f(W_\sigma^s \setminus \sigma)$, $\hat{L}_f^u = \bigcup\limits_{\sigma \in \Omega_f^{n-1}} p_f(W_\sigma^u \setminus \sigma)$.

Definition 7.1. A set $S_f = (\widehat{V}_f, \hat{L}_f^s, \hat{L}_f^u)$ is called the *scheme* of a diffeomorphism $f \in G_*(S^n)$.

Definition 7.2. Schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in G_*(S^n)$ are called *equivalent* if there exists such homeomorphism $\hat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$, that $\hat{\varphi}(\hat{L}_f^s) = \hat{L}_{f'}^s$ and $\hat{\varphi}(\hat{L}_f^u) = \hat{L}_{f'}^u$.

In [23], in particular, the following statement is proven.

Statement 7.1. *The diffeomorphisms $f, f' \in G_*(S^n)$ are topologically conjugated if and only if their schemes are equivalent.*

Definition 7.3. The scheme S_f of diffeomorphism $f \in G_*(S^n)$ is called *trivial* if there exists such homeomorphism $\hat{\psi}_f : \widehat{V}_f \to \mathbb{S}^{n-1} \times \mathbb{S}^1$, that for each connection component $\hat{\lambda}$ of the set $\hat{L}_f^s \cup \hat{L}_f^u$ there is a smoothly embedded sphere $S_{\hat{\lambda}}^{n-2} \subset \mathbb{S}^{n-1}$ of dimension $(n-2)$ such that $\hat{\psi}_f(\hat{\lambda}) = S_{\hat{\lambda}}^{n-2} \times \mathbb{S}^1$.

7.1. Auxiliary results. The following statement proven in [24] (see also elaborations in [25]), summarizes the results obtained in [14, 32, 38, 47] related to the embeddings of the trivial codimension (more than 3). Particularly, from these results follows that all locally flatly embedded closed arcs and links (unions of the closed arcs) in $\mathbb{R}^n$ of dimension $n \geq 4$, are trivial, i.e., are mapped by the homeomorphism on the space onto arcs (unions of arcs), in the coordinate plane. The examples of a nontrivial closed arc and a nontrivial link in $\mathbb{R}^3$ are shown on the Fig. 5.

A simple closed arc $\beta \in M^n$ is called a *knot*, and the image of the topological embedding $e : S^1 \times B^{n-1} \to M^n$ such that $e(S^1 \times \{O\}) = \beta$, is called a *tubular neighborhood* of knot β.

Proposition 7.3. *Let M^n be a topological neighborhood, with, probably, nonempty boundary ∂M^n, and $\{\beta_i\}_{i=1}^k$, $\{\beta_i'\}_{i=1}^k$ be the families of pairwise disjoint simple closed arcs locally flatly embedded in*

$\operatorname{int} M^n$ *such that for each* $i \in \{1, \dots, k\}$ *the arcs* β_i, β'_i *are homotopical. Let* $\{N_{\beta_i}\}_{i=1}^k, \{N_{\beta'_i}\}_{i=1}^k$ *be pairwise disjoint tubular neighborhoods of these arcs in* $\operatorname{int} M^n$.

Then there exists a homeomorphism $h : M_n \to M^n$ *such that* $h(\beta_i) = \beta'_i, h(N_{\beta_i}) = N_{\beta'_i}$, $i \in \{1, \dots, k\}$, *and* $h|_{\partial M^n} = id$.

The main instrument to prove the triviality of the diffeomorphism scheme of the considered class is surgery along the knots. As shown in Proposition 7.4, in dimensions 4 and higher such surgery doesn't change the topology of the manifold (which, as known, is wrong in 3-dimensional case).

Let M^n be a topological manifold with, probably, nonempty boundary, $\beta \in \operatorname{int} M^n$ be a knot and $N_\beta \subset \operatorname{int} M^n$ be its tubular neighborhood. We glue the manifolds $M^n \setminus \operatorname{int} N_\beta$ and $\mathbb{B}^{n-1} \times \mathbb{S}^1$ by an arbitrary homeomorphism reversing the orientation $\varphi : \partial N_\beta \to \mathbb{S}^{n-2} \times \mathbb{S}^1$ and denote the obtained manifold by Q^n. We say that Q^n is obtained from M^n by *surgery along the knot* β.

Proposition 7.4. Q^n *is homeomorphic to* M^n.

Proof. Set $N' = M^n \setminus \operatorname{int} N_\beta$, then $Q^n = N' \bigcup\limits_{\varphi} \mathbb{B}^{n-1} \times \mathbb{S}^1$ and for all $X \subset N' \cup \mathbb{B}^{n-1} \times \mathbb{S}^1$ we have a natural projection $\pi : X \to Q^n$.

Let $\psi = \varphi^{-1}\pi^{-1}|_{\pi(\mathbb{S}^{n-2}\times\mathbb{S}^1)}$. By [36] the homeomorphism ψ is continued to the homeomorphism $\Psi : \pi(\mathbb{B}^{n-1} \times \mathbb{S}^1) \to N_\beta$. Then the mapping $H : Q^n \to M^n$, defined by the relations

$$H(x) = \begin{cases} \pi^{-1}(x) = x, x \in \pi(\operatorname{int} N'), \\ \Psi(x), x \in \pi(\mathbb{B}^{n-1} \times \mathbb{S}^1), \end{cases}$$

is the desired homeomorphism. ☐

Let the fundamental group $\pi_1(M^n)$ of the manifold M^n be isomorphic to $\mathbb{Z}$. We call the knot $\beta \in M^n$ *trivial* if the homeomorphism $e_* : \pi_1(\beta) \to \pi_1(M^n)$ induced by the embedding is an isomorphism.

From Proposition 7.3 we directly have the following statement.

Corollary 7.1. *Let* $\beta \in \mathbb{S}^{n-1} \times \mathbb{S}^1$ *be a trivial knot and* N_β *be its tubular neighborhood. Then* $(\mathbb{S}^{n-1} \times \mathbb{S}^1) \setminus \operatorname{int} N_\beta$ *is homeomorphic to* $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

The latter corollary together with Proposition 7.4 gives the following statement.

Corollary 7.2. *Let* $Q_1^n, \dots, Q_{k+1}^n$, $k \geq 0$, *be pairwise disjoint manifolds homeomorphic to* $\mathbb{S}^{n-1} \times \mathbb{S}^1$; $\beta_1, \dots, \beta_{2k} \subset \bigcup\limits_{i=1}^{k+1} Q_i$ *be locally flat trivial knots such that*:

(1) *each manifold* Q_i^n *contains at least one knot from the set* $\beta_1, \dots, \beta_{2k}$;
(2) *for each* $j \in \{1, \dots, k\}$ *the knots* β_{2j-1}, β_{2j} *belong to different manifolds from* $Q_1^n, \dots, Q_{k+1}^n$.

Let $\psi_j : \partial N_{\beta_{2j-1}} \to \partial N_{\beta_{2j}}$ *be a homeomorphism inverting the natural orientation,* $j \in \{1, \dots, k+1\}$, *and* Q^n *be a manifold obtained from* $(\bigcup\limits_{j=1}^{k+1} Q_j^n) \setminus (\bigcup\limits_{i=1}^{2k} \operatorname{int} N_{\beta_i})$ *by gluing the components on the boundary along the homeomorphisms* $\psi_1, \dots, \psi_{k+1}$.

Then Q^n *is homeomorphic* $\mathbb{S}^{n-1} \times \mathbb{S}^1$, *and the projection of each manifold* ∂N_β *divides* Q^n *into two connection components, with each closure homeomorphic to* $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

7.2. Proof of the triviality of scheme of the diffeomorphism $f \in G_*(S^n)$**.** Let $f \in G_*(S^n)$. Prove that the scheme S_f is trivial. By Proposition 7.3 it is sufficient to prove that the manifold $\widehat{V}_f$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and each connection component of $\hat{L}_f^u \cup \hat{L}_f^s$ divides $\widehat{V}_f$ into two connection components, the closures of which are homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Now we give the main idea of the proof.

Set $k_i = |\Omega^i_f|$, $i \in \{0, 1, n-1, n\}$. Since the closures of all stable (nonstable) separatrices of dimension $(n-1)$ divide the support sphere S^n into disjoint sets, each containing exactly one sink (source) point, then $k_0 = k_1 + 1$, $k_n = k_{n-1} + 1$.

Set $\widehat{V}_\omega = (W^s_\omega \setminus \omega)/_f$ and $\widehat{V} = \bigcup\limits_{\omega \in \Omega^0_f} \widehat{V}_\omega$. Since the sink points are hyperbolic, the manifold $\widehat{V}_\omega$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Denote by $\beta_1, \ldots, \beta_{2k_1}$ the projections of one-dimensional separatrices into $\widehat{V}$. Since all separatrices are fixed, their projections are essential knots. Without loss of generality suppose that the set of knots is numerated such that knots β_{2j-1}, β_{2j} are the projections of the one-dimensional separatrices of the same saddle point $\sigma_j \in \Omega^1_f$, $j \in \{1, \ldots, k_1\}$.

From [46, Theorem 2.3, p. 753] it follows that each manifold $\widehat{V}^s_\omega$ contains at least one knot from $\beta_1, \ldots, \beta_{2k_1}$. Show that for each $j \in \{1, \ldots, k_1\}$ knots β_{2j-1}, β_{2j} belong to different connected components of $\widehat{V}$. Indeed, if $\beta_{2j-1}, \beta_{2j} \subset \widehat{V}^s_\omega$ for j, ω, then $cl\, W^u_{\sigma_j} = W^u_{\sigma_j} \cup \omega$ is homeomorphic to a circle. Since $cl\, W^s_{\sigma_j}$ divides the sphere S^n into two connection components and intersects the circle $cl\, W^u_{\sigma_j}$ in a point σ_j, when there is at least one point in $cl\, W^s_{\sigma_j} \cap cl\, W^u_{\sigma_j}$ distinct from σ_j, which leads to infinite number of nonwandering points and, hence, contradicts the definition of the diffeomorphism f.

Set $\mathbb{U} = \{(x_1, \ldots, x_n)|\ x_1^2(x_2^2 + \cdots + x_n^2) \leq 1\}$ and define the diffeomorphism $b : \mathbb{R}^n \to \mathbb{R}^n$ by $b(x_1, x_2, \ldots, x_n) = (2x_1, \frac{1}{2}x_2, \ldots, \frac{1}{2}x_n)$.

From hyperbolicity of the points $\sigma \in \Omega^1_f$ it follows that there exist pairwise disjoint neighborhoods $\{N_\sigma\}_{\sigma \in \Omega^1_f}$ of these points and the homeomorphism $\chi_\sigma : N_\sigma \to \mathbb{U}$ such that $f|_{N_\sigma} = \chi^{-1}_\sigma b \chi_\sigma$. It is easy to see that $\widehat{N}^u_\sigma = N_\sigma \setminus W^s_\sigma)/_f$ consists of two connection components, each being homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$, and $\widehat{N}^s_\sigma = N_\sigma \setminus W^u_\sigma)/_f$ being homeomorphic to the direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [-1, 1]$. With that the projection of a stable separatrix of σ coincides with the middle layer $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{0\}$. Denote by $\pi^u_\sigma : N_\sigma \setminus W^s_\sigma \to \widehat{N}^u_\sigma$ and $\pi^s_\sigma : N_\sigma \setminus W^u_\sigma \to \widehat{N}^s_\sigma$ the natural projections.

Denote by N_{2j-1}, N_{2j} the connection components of $\widehat{N}^u_{\sigma_j}$ containing the knots β_{2j-1}, β_{2j}, respectively. Set $K_j = \widehat{N}^s_{\sigma_j}$, $T_j = \widehat{V}^s_{\sigma_j}$, define the homeomorphism $\psi_j : \partial N_{2j-1} \cup \partial N_{2j} \to \partial K_j$ by $\psi_j = \pi^s_\sigma (\pi^u_\sigma)^{-1}$ and denote by $\Psi : \bigcup\limits_{j=1}^{k_1} \partial N_{2j-1} \cup \partial N_{2j} \to \bigcup\limits_{j=1}^{k_1} \partial K_j$ such homeomorphism that $\Psi|_{\partial N_{2j-1} \cup \partial N_{2j}} = \psi_j|_{\partial N_{2j-1} \cup \partial N_{2j}}$.

Since

$$V_f = \Big(\bigcup_{\omega \in \Omega^0_f} V^s_\omega \setminus \Big(\bigcup_{\sigma \in \Omega^1_f} V^u_\sigma \Big)\Big) \bigcup \Big(\bigcup_{\sigma \in \Omega^1_f} V^s_\sigma \Big) = \Big(V_f \setminus \Big(\bigcup_{\sigma \in \Omega^1_f} N^u_\sigma \Big)\Big) \bigcup \Big(\bigcup_{\sigma \in \Omega^1_f} N^s_\sigma \Big),$$

then

$$\widehat{V}_f = \Big(\widehat{V}_f \setminus \Big(\bigcup_{\sigma \in \Omega^1_f} \widehat{N}^u_\sigma \Big)\Big) \bigcup_\Psi \Big(\bigcup_{\sigma \in \Omega^1_f} \widehat{N}^s_\sigma \Big) = \Big(\widehat{V}_f \setminus \Big(\bigcup_{j=1}^{2k_1} N_j \Big)\Big) \bigcup_\Psi \Big(\bigcup_{j=1}^{k_1} K_j \Big).$$

Thus, the manifold $\widehat{V}_f$ is obtained from $\bigcup\limits_{\omega \in \Omega^0_f} \widehat{V}^s_\omega$ by surgery along the knots $\beta_1, \ldots, \beta_{2k_1}$. By Corollary 7.2 $\widehat{V}_f$ is homeomorphic to the direct product $\mathbb{S}^{n-1} \times \mathbb{S}^1$, and the projection of each connection component of ∂K_j divides $\widehat{V}_f$ into two connection components, with each closure being homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$. Since the projection of a stable separatrix of point σ_j at ∂K_j and any connected component of the boundary K_j reduce in K_j a direct product $\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]$, then the projection of a stable separatrix of point σ_j at $\widehat{V}_f$ also divides $\widehat{V}_f$ into two connected components, with each closure being homeomorphic to $\mathbb{B}^{n-1} \times \mathbb{S}^1$.

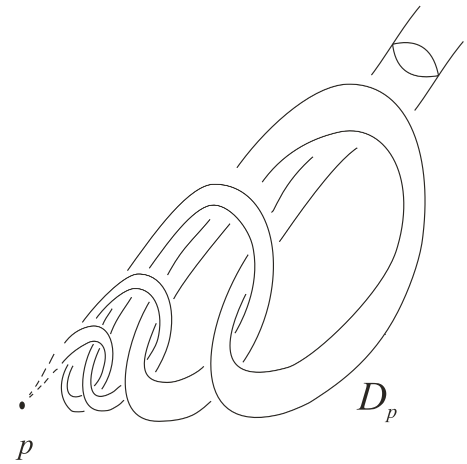

Fig. 6. Disc $D_p \subset W^s_p$

On the other hand,

$$V_f = \Big(\bigcup_{\alpha\in\Omega^n_f} V^u_\alpha \setminus \Big(\bigcup_{\sigma\in\Omega^{n-1}_f} V^s_\sigma\Big)\Big)\bigcup\Big(\bigcup_{\sigma\in\Omega^{n-1}_f} V^u_\sigma\Big) = \Big(V_f \setminus \Big(\bigcup_{\sigma\in\Omega^{n-1}_f} N^s_\sigma\Big)\Big)\bigcup\Big(\bigcup_{\sigma\in\Omega^{n-1}_f} N^u_\sigma\Big).$$

Similarly to the previous reasoning we obtain that $\widehat{V}_f$ is obtained from $\bigcup\limits_{\alpha\in\Omega^n_f} \widehat{V}^u_\alpha$ by surgery along the projections of stable one-dimensional separatrices of saddle points of diffeomorphism f. Each component of $\hat{L}^u_f$ divides $\widehat{V}_f$ into two connected components, with closure of each component being homeomorphic to $\mathbb{B}^{n-1}\times\mathbb{S}^1$.

7.3. Discussion of the conditions of Theorem 7.1. If any of the conditions of Theorem 7.1 is not satisfied, it allows to construct a counterexample to the statement of the theorem. The necessity of the conditions i), ii) in Theorem 7.1 was shown in Lemma 3.1.

The condition that the support manifold is a sphere is not necessary, however in [49] there is an instance of Morse–Smale diffeomorphism $f_0 : M^4 \to M^4$ on manifold M^4 different from sphere S^4 and satisfying the conditions i)–iii), but not embedded in the topological flow. The nonwandering set of the diffeomorphism f_0 consists of exactly three fixed points: a source, a sink and a saddle. The invariant manifolds of this diffeomorphism have the dimension two, and each its closure is a wild sphere (see [49, Theorem 4, it. 2]). Supposing that the diffeomorphism f_0 is embedded in the topological flow X^t_0, then the nonwandering set of this flow consists of three equilibrium states coinciding with the fixed points of the diffeomorphism f_0. Each state has a neighborhood in which the flow X^t_0 is locally topologically equivalent to the linear flow with the eigenvalues, whose real part differs from zero.

By [50, Theorem 3] all such flows are topologically equivalent, and in [49] there is an example of Morse–Smale flow from the considered class with all closures of invariant manifolds of saddle equilibrium states being tame spheres. Thus, the closures of invariant manifolds of the equilibrium states of flow X^t_0 being the saddle of the diffeomorphism f_0, are tame spheres. Thus we obtain a contradiction with the construction of diffeomorphism f_0. From [23, Theorem 1.3] it follows that if the invariant manifolds of different saddle points of Morse–Smale diffeomorphism $f : S^n \to S^n$ do not intersect, then its nonwandering set Ω_f consists of points with the dimension of the nonstable manifold for each point being in $\{0, 1, n-1, n\}$. In particular, this fact clarifies why is M^4 not homeomorphic to a sphere.

In [37] there is an example of Morse–Smale diffeomorphism $f_1 : S^4 \to S^4$ satisfying the conditions i)-ii) of Theorem, but not embedded into topological flow. The nonwandering set of the diffeomorphism f_1 consists of two sources, two sinks and two saddles p, q such that $\dim W^s_p = \dim W^u_q = 3$. With that the intersection $W_p^s \cap W_q^u$ is not empty and its closure in W_p^s is a wildly embedded open disc D_p with a wildness point p. More precisely, for each ball $B^3 \subset W_p^s$, for which p is an internal point, the intersection of the boundary of this ball with the disc D_p consists of no less than three connection components (see Fig. 6). The diffeomorphism f_1 satisfies all conditions of Theorem 7.1 but iii). Similarly to the proof of Proposition 6.1 it can be proved that there is no such topological flow in W_p^s, for which disc D_p is invariant, and the reduction of f_1 onto W_p^s is a shift per unit time. From here it follows that f_1 is not embedded in the topological flow.

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