Action: respond to our copy-editing questions

Select each question and describe any changes we should make on the proof. Changes against journal style will not be made and proofs will not be sent back for further editing.

AQ1: Please check the spelling and accuracy of all author names and affiliations, particularly for any of your co-authors. Please also check that author surnames are correctly identified with a pink background. This is to ensure that forenames and surnames are tagged correctly for online indexing. **Incorrect names and affiliations may lead to an author not being credited for their work by funders, institutions, or other third parties.**

AQ2: If your manuscript has figures or text from other sources, please ensure you have permission from the copyright holder. For any questions about permissions contact jnls.author.support@oup.com.

AQ3: Please check that funding is recorded in a separate funding section if applicable. Use the full official names of any funding bodies, and include any grant numbers.

AQ5: Please note that, as per journal-specific instructions, in-text footnotes are not allowed. Hence, they were inserted to the main text instead. Please check if the edits made were appropriate and correct if necessary.

AQ6: Please check if the details in the Funding section were captured accurately.

AQ7: Please provide the page range for references 2 and 6.

AQ8: Please provide the publisher location for references 5, 14, 32, and 33.

AQ9: Please provide the publisher name and location for reference 16.

AQ10: Please provide an update for references 22, 29, 30, 37, and 43.

AQ11: Please provide the publisher name and location and year of publication for reference 35.

These proofs are for checking purposes only. They are not in final publication format. Please do not distribute them in print or online. Do not publish this article, or any excerpts from it, anywhere else until the final version has been published with OUP. For further information, see https://academic.oup.com/journals/pages/authors

Figure resolution may be reduced in PDF proofs and in the online PDF, to manage the size of the file. Full-resolution figures will be used for print publication.

Action: check your manuscript information

Please check that the information in the table is correct. We use this information in the online version of your article and for sharing with third party indexing sites, where applicable.

Full affiliations Each unique affiliation should be listed separately; affiliations must contain only the applicable department, institution, city, territory, and country.	1 Department of Mathematics, National Research University Higher School of Economics, Moscow, Russian Federation, and Independent University of Moscow, Russian Federation 2 Department of Mathematics, Dartmouth College, Hanover, NH, USA, and Department of Mathematics and International Laboratory of Algebraic Topology and Its Applications, National Research University Higher School of Economics, Moscow, Russian Federation
Group contributors The name of the group and individuals in this group should be given, if applicable (e.g. The BFG Working Group: Simon Mason, Jane Bloggs)	NA
Supplementary data files cited	NA
Funder Name(s) Please give the full name of the main funding body/agency. This should be the full name of the funding body without abbreviation or translation, if unsure, see https://search.crossref.org/funding	Russian Foundation for Basic Research Simons Foundation Russian Academic Excellence Project “5-100”

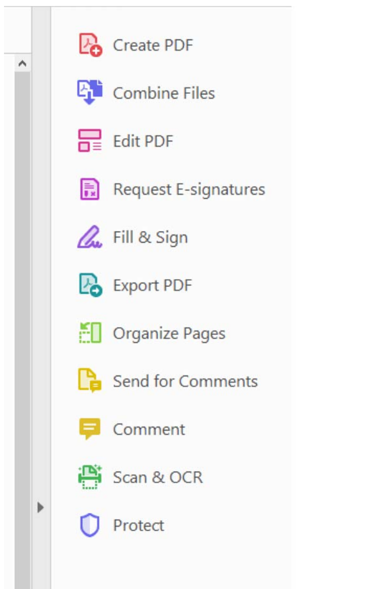
How to add your responses

These instructions show you how to add your responses to your proof using Adobe Acrobat Pro DC, version 7 onwards, or Adobe Reader DC. To check what version you are using, go to 'Help', then 'About'. The latest version of Adobe Reader is available for free from https://get.adobe.com/uk/reader/.

Displaying the Comment toolbars

Access toolbar in Adobe Reader DC

Access the Comment toolbar by clicking 'Comment' in the menu at the top right-hand side of the page (shown below).



The toolbar shown below will then be displayed at the top of the page.

Access toolbar in Adobe Acrobat Pro DC

Access the toolbar by clicking 'Comment' in the menu on the right-hand side of the page (shown at right). The toolbar shown below will be displayed.

Using Adobe Comment Tools

Most desired changes/corrections can be indicated with 3 tools. To indicate text to be deleted, use the 'Strikethrough text' tool (**1**). To indicate text to be added, use the 'Insert text at cursor' tool (**3**). To delete/replace text, use the 'Add note to replace text' tool (**2**).

Alternatively, the 'Add sticky note' tool (**4**) can be used to place a note describing the desired change (eg, "Change 7% to 8%"). For guidance on using commenting tools, visit Adobe's website at (https://helpx.adobe.com/acrobat/using/mark-text-edits.html).

Pop-up notes

In both Reader and Acrobat, when text is selected with the cursor and a Comment tool is clicked active (or vice versa), or when a sticky note is placed, a pop-up note will appear in the 'Comment' column at the right-hand side of the page. Type the text to be inserted or replaced (or type instructions for making the change in the sticky note).

****Important**:** When finished, if the 'Post' box

within a pop-up note is highlighted in blue, make sure to click 'Post'.

Saving comments

When you're finished working with the proof, simply close the document (you'll be prompted to save changes) or select 'Save as' from the 'File' menu to save a renamed version reflecting your changes.

****Important**:** Only use 'Comment' tools to indicate desired changes/corrections; **do not** access the 'Edit PDF' function or select 'Edit Text & Images' from the 'Edit' pull-down menu.

P. Pushkar and M. Temkin (2022) "Enhanced Bruhat Decomposition and Morse Theory,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–67
https://doi.org/10.1093/imrn/rnac326

Enhanced Bruhat Decomposition and Morse Theory

Petya Pushkar[1] and Misha Temkin[2,*]

[1]Department of Mathematics, National Research University Higher School of Economics, Moscow, Russian Federation, and Independent University of Moscow, Russian Federation and [2]Department of Mathematics, Dartmouth College, Hanover, NH, USA, and Department of Mathematics and International Laboratory of Algebraic Topology and Its Applications, National Research University Higher School of Economics, Moscow, Russian Federation

[*]*Correspondence to be sent to: e-mail: mikhail.temkin@dartmouth.edu*

Morse function is called strong if all its critical points have different critical values. Given such a function f and a field $\mathbb{F}$ Barannikov constructed a pairing of some of the critical points of f, which is now also known as barcode. With every Barannikov pair (a.k.a. bar in the barcode), we naturally associate (up to sign) an element of $\mathbb{F} \setminus \{0\}$; we call it Bruhat number. The paper is devoted to the study of these Bruhat numbers. We investigate several situations where the product of all the numbers (some being inversed) is independent of f and interpret it as a Reidemeister torsion. We apply our results in the setting of one-parameter Morse theory by proving that generic path of functions must satisfy a certain equation mod 2 (this was initially proven in [2] under additional assumptions).

On the linear-algebraic level, our constructions are served by the following variation of a classical Bruhat decomposition for $GL(\mathbb{F})$. A unitriangular matrix is an upper triangular one with 1s on the diagonal. Consider all rectangular matrices over $\mathbb{F}$ up to left and right multiplication by unitriangular ones. Enhanced Bruhat decomposition describes canonical representative in each equivalence class.

Received July 21, 2021; Revised March 28, 2022; Accepted May 26, 2022
Communicated by Prof. Dmitry Kaledin

© The Author(s) 2022. Published by Oxford University Press. All rights reserved. For permissions, please e-mail: journals.permission@oup.com.

Introduction

0.1 The context

In this paper, we study a certain invariant of a strong Morse function on a smooth compact manifold (which is supposed to have no boundary most of the time). Recall that a function is called Morse if all its critical points are nondegenerate. The function is called strong if all its critical points have different critical values. Morse theory is a classical branch of differential topology: one can extract a lot of topological information about the manifold in terms of a Morse function. On the other hand, Morse functions arise naturally in various situations and their properties are interesting in their own right.

Recall that each critical point of a Morse function carries a number—its index. The very 1st theorem of Morse theory states that one can find a CW-complex homotopy equivalent to the manifold, which k-cells correspond to critical points of index k.

Suppose now that one is given not only a strong Morse function f on a manifold M but also a field $\mathbb{F}$. To such a data, Barannikov [4] associated a powerful combinatorial structure on the set of critical points of f. Nowadays, it is also known as barcode and serves as a well-established tool in applied and symplectic topology; see [16] for a recent survey. This structure is a pairing of some critical points of neighboring indices. This pairing does depend on the field $\mathbb{F}$. For example, the number of unpaired critical points of index k equals $\dim_{\mathbb{F}} \mathsf{H}_k(M;\mathbb{F})$. Moreover, this pairing relies crucially on the fact the function is strong, that is, critical points are linearly ordered. If one starts to change a Morse function so that along the way it fails to be strong (i.e., two critical values collide), the pairing changes. We call these pairs Barannikov pairs. We will sketch their definition in the next subsection.

0.2 Bruhat numbers of a single function

To state our results concisely, we prefer to speak about oriented strong Morse functions. Roughly, the Morse function f is called oriented if one has chosen orientation on all the cells in the CW-complex constructed from it. This condition is both technical and minor: one can always orient a Morse function. As usual in topology, this choice only alters certain signs.

Our 1st result is not a theorem, but rather a construction. Namely, given an oriented strong Morse function f on a manifold M and a field $\mathbb{F}$, we naturally associate a nonzero number with each Barannikov pair. Here, by number, we mean an element of

$\mathbb{F}^* = \mathbb{F} \setminus \{0\}$. We call these numbers "Bruhat numbers" of f over $\mathbb{F}$. The reason is that their construction is closely related to the classical Bruhat decomposition for $GL(\mathbb{F})$. This paper is devoted to the study of these numbers from different perspectives. The 1st thing to mention is that changing the orientation of f may only change signs of some Bruhat numbers.

We will now sketch one of the possible constructions of both Barannikov pairs and Bruhat numbers. It will utilize the choice of a Riemannian metric on M (the output is independent of this choice). See Subsection 0.6 for links to alternative constructions. Recall that if one chooses a generic Riemannian metric g on M, then they can consider a Morse complex whose homology is isomorphic to the homology of M (this is a classical construction, it has nothing to do with a field). It is a chain complex of free abelian groups, formally spanned by critical points (points of index k are of degree k). Thus, the Morse differential is nothing but an integer matrix: differential of a critical point x is a linear combination of points of smaller index. The coefficient of the point y in this linear combination is the number of antigradient flowlines from x to y, counted with appropriate signs (thus, it is nonzero only if $f(x) > f(y)$). Since the function is strong, the set of critical points is ordered by increasing of critical values. Next, we note that choosing a different metric g' results in a different matrix of Morse differential. More precisely, these two matrices differ by a conjugation by unitriangular (i.e., triangular with 1s on the diagonal) one. We treat this unitriangular matrix as that of a change of basis of a complex. As a recollection, the class of a matrix of Morse differential up to conjugation by a unitriangular matrix is a well-defined invariant of f (i.e., does not depend on a metric). The corresponding classification problem is, however, very hard, so we consider the coefficients in $\mathbb{F}$. In such a case, we prove that every complex is isomorphic, up to unitriangular change of basis, to the direct sum of $0 \to \mathbb{F} \xrightarrow{\times\lambda} \mathbb{F} \to 0$ and $0 \to \mathbb{F} \to 0$. Generators of the former (which are themselves critical points) are Barannikov pairs. The corresponding number λ is a Bruhat number on a pair.

Weak Morse inequalities state that the number of critical points of index k is greater or equal to $\dim \mathsf{H}_k(M; \mathbb{Q})$. Let $\#\,\mathrm{Cr}(f)$ be the overall number of critical points of f. It is easy to show that if $\#\,\mathrm{Cr}(f) = \sum_k \dim \mathsf{H}_k(M; \mathbb{Q})$, then the Morse differential (w.r.t. any metric) must be identically zero. The next statement is applicable when this is not the case.

Corollary 0.1. Let f be an oriented strong Morse function on M. Suppose that $\#\,\mathrm{Cr}(f) > \sum_k \dim \mathsf{H}_k(M; \mathbb{Q})$. Then one can find two critical points x and y of neighboring indices s.t. the number of antigradient flowlines from x to y, counted with appropriate signs, is the same for any generic Riemannian metric. This number is nonzero and equals some Bruhat number of f over $\mathbb{Q}$.

In Subsection 3.7, this statement is derived from Theorem 3.23 that says that the matrix of a Morse differential of f w.r.t. any metric must obey certain restrictions. These restrictions, in turn, are expressed in terms of Bruhat numbers and Barannikov pairs. They are in the spirit of Bruhat decomposition; see the mentioned subsection for the precise statement and example. Note that, in particular, we claim that at least one Bruhat number over $\mathbb{Q}$ must be integer. For a general Bruhat number, this is false, however.

0.3 Interplay with the theory of torsions

In Proposition 3.25, we prove that if $\mathbb{F}$ is either $\mathbb{Q}$ or $\mathbb{F}_p$, then one can find a Morse function that has any prescribed number $\lambda \in \mathbb{F}^*$ as one of its Bruhat numbers; the manifold M may be any with $\dim M \geqslant 4$. Thus, one has no control over individual Bruhat number—it may turn out to be any number. We propose, however, to consider the alternating product of all the Bruhat numbers. The word alternating here means that each Bruhat number is raised to the power ± 1 depending on the parity of indices of critical points involved in the corresponding Barannikov pair. The term is used in analogy of Euler characteristic, which is the alternating sum of, say, cells in the CW-complex. In the following statement, this product is considered up to sign.

Theorem 0.2. Let f be a strong Morse function on M and $\mathbb{F}$ be a field. Suppose that $\mathsf{H}_k(M) = 0$ for all $0 < k < \dim M$. Then the alternating product of all Bruhat numbers (as an element from $\mathbb{F}^*/\pm 1$) is independent of f.

This is discussed in Subsections 4.1 and 5.5; in particular, we interpret this alternating product as a certain kind of torsion. For example, if $M = \mathbb{RP}^n$ and $\mathbb{F} = \mathbb{Q}$, then this product equals $\pm 2^{[n/2]}$, where brackets denote the integral part.

We shall now quickly recall the notion of Reidemeister torsion of a manifold. This is a purely algebro-topological invariant that itself has nothing to do with Morse theory. Let π denote the fundamental group of M. Suppose one is given a one-dimensional representation $\rho : \pi \to GL_1(\mathbb{F}) = \mathbb{F}^*$ over some field $\mathbb{F}$. Then they may be considered homology of M with coefficients twisted by ρ. If it vanishes, then they may furthermore define the Reidemeister torsion of M w.r.t. ρ, which is an element of the quotient group $\mathbb{F}^*/\pm\rho(\pi)$. This invariant is useful for distinguishing homotopy equivalent but non-homeomorphic manifolds, such as lens spaces.

We will now pour Morse theory into this setting. Suppose that now we are given not only oriented strong Morse function f and a field $\mathbb{F}$ but also a one-dimensional representation $\rho : \pi \to \mathbb{F}^*$. In this case, we construct twisted Barannikov pairs and Bruhat numbers. In general, the alternating product of twisted Bruhat numbers may well depend on f. The interesting case, however, is when one is able to define the Reidemeister torsion, that is, when the twisted homology vanishes.

Theorem 0.3. Let f be a strong Morse function on a manifold M, $\mathbb{F}$ be a field, and $\rho\colon \pi \to \mathbb{F}^*$ be a one-dimensional representation. Suppose that twisted homology vanishes. Then the alternating product of twisted Bruhat numbers of f equals the Reidemeister torsion of M. In particular, it is independent of f.

We also prove a non-acyclic analog of the above theorem (see Subsection 4.4).

0.4 One-parameter Morse theory

Let us now consider not a single strong Morse function but a generic path in the space of all functions on M. All but finitely points on this path are themselves strong Morse functions. However, after passing a function that fails to be strong and Morse Barannikov pairs and Bruhat numbers change. Thanks to the genericity assumption on the path it is possible to describe exactly the list of possible changes. For Barannikov pairs this was done already in [4] (see also [14] and pictures in the survey [16]). We do the same for Bruhat numbers on them; see Theorem 5.7 and 5.9. In particular, this gives a proof of Theorem 0.2. Moreover, it enables us to prove the theorem of Akhmetev–Cencelj–Repovs [2] in greater generality, which we shall now describe.

Let $\{f_t\}$ be a generic path on functions on M, that is, f_t is a function from M to $\mathbb{R}$ for each $t \in [-1, 1]$. To such a path one associates a Cerf diagram. It is a subset of $[-1, 1] \times \mathbb{R}$ consisting of points (t, x) s.t. x is a critical value of f_t. Practically, it is a set of plane arcs that start and end either at cusps or at vertical lines $t = \pm 1$. The only possible singularities of a Cerf diagram are cusps and simple transversal self-intersections. By orienting all the functions in a path, one may associate a sign with each cusp. The parity of negative cusps is a well-defined invariant of a path $\{f_t\}$, that is, it does not depend on the orientations. Another invariant of a path is a number of self-intersections of its Cerf diagram (i.e., the number points t_0 s.t. f_{t_0} is not strong). In the following statement, we consider functions on a cylinder $M = N \times [0, 1]$, which is a

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

manifold with boundary (here, N is a closed manifold). Adaptation of Morse theory to manifolds with boundary is well known. We stress out that by a function on a cylinder, we mean a function $g\colon N \times [0,1] \to [0,1]$ s.t. $g^{-1}(0) = N \times \{0\}$ and $g^{-1}(1) = N \times \{1\}$.

Corollary 0.4. Let $\{f_t\}$ be a generic path of functions on a cylinder $N \times [0,1]$ s.t. both f_{-1} and f_1 have no critical points. Let X be the number of self-intersections of its Cerf diagram and C be the number of negative cusps. Then one has

$$\mathsf{X} + \mathsf{C} = 0 \pmod 2.$$

In [2], this statement was proved under additional assumptions on N. In Subsection 5.6, we derive this corollary from a more general Theorem 5.14.

0.5 Related work

Barannikov pairs were introduced in [4] (see [16] for a recent survey). A close idea of construction of Bruhat numbers over $\mathbb{Q}$ appeared independently in [30]. In [43], the authors prove a theorem analogous to Barannikov's in the setting of complexes over the Novikov field, which is useful in symplectic topology. Translation of Bruhat numbers to this setting is a current work in progress. Reidemeister torsion in the setting of Morse–Novikov theory is studied in [25–27]. Morse theory plays an important role in the definition of higher Reidemeister torsion due to Igusa and Klein (see, e.g., [28]). See also the recent work [3] for the application of torsion-theoretic techniques in contact topology.

0.6 Organization of the paper

The 1st two sections provide an algebraic foundation for the further Morse-theoretical results. Namely, in Section 1, we do the necessary linear algebra and emphasize connection with the Bruhat decomposition. In Section 2, similar in spirit constructions are presented in the realm of homological algebra over a field. Section 3– 5 contain (but not exhausted by) results described in, respectively, Subsections 0.2– 0.4.

The paper contains many constructions and we therefore use a special environment for them. The text after the word construction describes input and output. The actual algorithm is placed between signs $\triangleright$ and $\triangleleft$. By default, construction does not involve any choices, that is, output depends only on the input. If it is not the case, we indicate this explicitly.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

Barannikov pairs together with Bruhat numbers are called B-data for brevity. These data can be extracted from an (oriented) strong Morse function in several ways. The thorough way is via the enhanced complexes, and in order to proceed with it, one has to understand the main results of Section 1 and 2 (proofs may be safely omitted). The quick way is more elementary, does not require any enhancements and is given in Remark 3.3. So the curious reader might use the following scheme:

1) read the formal definition of B-data in Subsection 2.3;
2) read Subsection 3.1 and Remark 3.3;
3) continue reading results in Section 3– 5.

The main results are formulated exclusively in terms of Bruhat numbers and Barannikov pairs (so again, no enhancements needed). However, in order to understand their proofs, one has to take the thorough way.

1 Enhanced Vector Spaces

In this section, we define and study enhanced vector spaces—a notion that we rely on in Section 2. All the constructions lie within the scope of linear algebra. Moreover, our main statement here (Lemma 1.7) may be formulated exclusively in terms of matrices, which is done right below (Lemma 1.2). Later, in Section 2, we proceed similarly in the setting of chain complexes over a field.

1.1 Formulation of results

In this subsection, we introduce main definitions of this section and formulate the main Lemma 1.7.

We start with the coordinate formulation. Let n and m be two natural numbers fixed once and for all throughout this section. Fix also a base field $\mathbb{F}$, all matrices are assumed to be over it.

Let T_n be the group of unitriangular matrices, that is, upper triangular $n \times n$ matrices with ones on the diagonal. The group $T_n \times T_m$ acts on the set $\mathrm{Mat}_{n,m}$ of all $n \times m$ matrices: $X \mapsto AXB^{-1}$. Since one has commuting actions of both T_n and T_m, the orbit space is usually denoted as $T_n \backslash \mathrm{Mat}_{n,m}/T_m$. Note that two $n \times m$ matrices lie in the same T_n orbit if and only if one can be obtained from another by a successive performing of the following elementary operation: add to one row a scalar multiple of another one, provided that the latter has higher index than the former. The analogous elementary operation for T_m is addition of a multiple of column with lower index to that with higher index.

Definition 1.1. An $n \times m$ matrix is called a *rook* matrix if in every row and in every column there is at most one nonzero entry.

Lemma 1.2. Every orbit in $T_n \backslash \mathrm{Mat}_{n,m}/T_m$ contains exactly one rook matrix.

The classical Bruhat decomposition is obtained from the above statement in two steps:

1) restrict to the case $n = m$, and consider only nondegenerate square matrices;
2) replace T_n by an upper triangular group.

Keeping in mind the slightly greater level of generality, we propose the term "enhanced Bruhat decomposition" (see, however, [6], which inspired this word choice). Note that rook matrix stores, in particular, the set of elements from $\mathbb{F}^*$, in contrast to the matrix of permutation in the classical case. We call these elements "Bruhat numbers". The proof of Lemma 1.2, however, goes along the classical lines. See [21] for an in-depth discussion.

Remark 1.3. Several intermediate decompositions between the classical Bruhat decomposition and enhanced Bruhat decomposition gathered attention in the literature. In [24], the condition on the square matrix to be invertible was dropped. In fact, the scope of this paper is broader in terms that the author considers other algebraic groups besides GL_n. Next, in [15], the authors consider rectangular matrices of arbitrary size.

We will now introduce the main notion of the present section. Several basic facts about it will be presented further in Subsections 1.2 and 1.4. Often, Bruhat decomposition is proven using inductive arguments. We tried to refrain from those during the course of this section (or, at least, hide them under the carpet of explicit constructions).

Definition 1.4. Let V be a vector space over $\mathbb{F}$. An *enhancement $\varkappa$ on a vector space V* is a choice of two structures:

1) a full flag on V, that is, a sequence of subspaces $0 = V^0 \subset V^1 \subset \ldots \subset V^{\dim V} = V$ s.t. $\dim(V^s/V^{s-1}) = 1$, $s \in \{1, \ldots, \dim V\}$;
2) a nonzero element $\varkappa_s$ in a one-dimensional vector space V^s/V^{s-1}, $s \in \{1, \ldots, \dim V\}$.

A vector space V with an enhancement will be called an *enhanced vector space* and denoted as $(V, \varkappa)$.

Enhanced vector spaces are called affine flags in [17].

Definition 1.5. Let $(V, \varkappa)$ and (W, μ) be two enhanced vector spaces, and let $\varphi\colon V \xrightarrow{\sim} W$ be an isomorphism of vector spaces. We say that φ is an *isomorphism of enhanced vector spaces* if

1) $\varphi(V^s) = W^s$;
2) $\widetilde{\varphi}_s(\varkappa_s) = \mu_s$, where

$$\widetilde{\varphi}_s\colon V^s/V^{s-1} \to W^s/W^{s-1}$$

is a map of quotient vector spaces induced by φ.

Remark 1.6. If $(V, \varkappa)$ and (W, μ) are two enhanced vector spaces of the same dimension, then one can always find an isomorphism (in the sense of Definition 1.5) between them.

By a basis of a finite-dimensional vector space V, we will mean a linearly ordered set of generators (zero vector space has empty set as its only basis). Given a basis $v = (v_1, \dots, v_{\dim V})$ of V, one constructs an enhanced vector space $(V, \varkappa(v))$ in the following straightforward way. For $s \in \{1, \dots, \dim V\}$, set $V^s := \langle v_1, \dots, v_s\rangle$ and $\varkappa(v)_s := p_s(v_s)$, where $p_s\colon V^s \to V^s/V^{s-1}$ is a standard projection. By a basis of an enhanced vector space $(V, \varkappa)$, we will mean a basis v of V s.t. the identity map is an isomorphism of enhanced vector spaces $(V, \varkappa)$ and $(V, \varkappa(v))$. Every enhanced vector space can be equipped with a basis.

The next lemma is equivalent to Lemma 1.2.

Lemma 1.7. Let $(V, \varkappa)$ and (W, μ) be two enhanced vector spaces and $A\colon V \to W$ be a linear map. There exists a basis v (resp. w) of an enhanced vector space $(V, \varkappa)$ (resp. (W, μ)) s.t. the matrix of A in these bases is a rook matrix. Moreover, this rook matrix is uniquely defined.

The change of basis in $(V, \varkappa)$ (resp. (W, μ)) results in multiplication of matrix of A by a matrix from $T_{\dim V}$ (resp. $T_{\dim W}$). So, Lemma 1.7 describes orbits in $T_{\dim W}\backslash \mathrm{Mat}_{\dim W, \dim V}/T_{\dim V}$.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

We will stick to the above formulation. It is possible to state the same without appealing to any bases whatsoever; this is done in Subsection 1.3.

Remark 1.8. Consider the rook matrix from Lemma 1.7. Remove all zero rows and zero columns from it to obtain a nondegenerate square matrix. Let $d \in \mathbb{F}^*$ be its determinant (by convention, determinant of an empty matrix is one). The number d will be important later since it will serve as a building block for the definition of torsion of strong Morse function (see Subsections 2.10 and 4.1).

Consider now the partial case when $(W, \mu) = (V, \varkappa)$ and A is an isomorphism. In this case, the number d is nothing but its determinant. Thus, d may be viewed as a generalization of determinant in the enhanced setting.

Remark 1.9. If the field $\mathbb{K}$ is an extension of $\mathbb{F}$, then one may consider A as a map between vector spaces over $\mathbb{K}$. Itis plain to see that rook matrix would not change after this operation. Indeed, multiplication of matrices only involves additions and multiplications.

Usually, what we are given in topological setup is a matrix over $\mathbb{Z}$ (note that this is not the case in Subsections 4.2 and 4.3). We then choose some field $\mathbb{F}$ and merely consider this matrix over this field. It follows from the previous paragraph that it is enough to consider only $\mathbb{Q}$ and $\mathbb{F}_p$.

Remark 1.10. In Lemma 1.7, bases v and w themselves need not be unique.

1.2 Construction of a rook matrix

In this subsection, we associate a rook matrix to a given map between enhanced vector spaces. In Subsection 1.4, we will show that this is the same matrix as the one addressed in Lemma 1.7.

We will need the following construction as a preliminary step.

Construction 1.11. Let $(V, \varkappa)$ be an enhanced vector space and $A\colon V \twoheadrightarrow W$ be a surjective map of vector spaces. We will now construct an induced enhancement on W.

▷ For $s \in \{1, \dots, \dim V\}$, define φ_s to be the composition

$$V^s \hookrightarrow V \xrightarrow{A} W.$$

Take any s s.t. $\dim \operatorname{Im} \varphi_s = \dim \operatorname{Im} \varphi_{s-1} + 1$, and denote this number by t. Set W^t to be $\operatorname{Im} \varphi_s$. This defines a full flag on W, that is, a vector space W^t for any $t \in \{1, \dots, \dim W\}$. We now need to produce an element μ_t in the vector space W^t/W^{t-1}; this vector space coincides with $\operatorname{Im} \varphi_s / \operatorname{Im} \varphi_{s-1}$. Define μ_t to be $\widetilde{\varphi_s}(\varkappa_s)$, where

$$\widetilde{\varphi_s} \colon V^s/V^{s-1} \xrightarrow{\sim} \operatorname{Im} \varphi_s / \operatorname{Im} \varphi_{s-1}$$

is an isomorphism of quotient vector spaces induced by φ_s. We have obtained an enhanced vector space (W, μ). $\lhd$

The following proposition will be used later in Subsection 1.4.

Proposition 1.12. Let $(V, \varkappa)$ be an enhanced vector space and $A \colon V \twoheadrightarrow W$ be a surjective map of vector spaces. Construction 1.11 produces an enhanced vector space (W, μ). We claim that for any basis of (W, μ), one can find a basis of $(V, \varkappa)$ s.t. A maps each basis element to either zero or another basis element.

Proof. We continue using notations from Construction 1.11. Let $w = (w_1, \dots, w_{\dim W})$ be the given basis of (W, μ). Take any $s \in \{1, \dots, \dim V\}$. The difference $\dim \operatorname{Im} \varphi_s - \dim \operatorname{Im} \varphi_{s-1}$ is either zero or one. In the former case, set v_s to be any vector from $(V^s \setminus V^{s-1}) \cap \operatorname{Ker} A$ s.t. its class in V^s/V^{s-1} coincides with $\varkappa_s$. In the latter case, set v_s to be any preimage of $w_{\dim \operatorname{Im} \varphi_s}$ under A. It is straightforward to check that the basis $(v_1, \dots, v_{\dim V})$ satisfies the desired property. ∎

Construction 1.19 and Proposition 1.20 are analogous statements for the case of injective map. In order to proceed to the construction of a rook matrix, we need two more definitions. For a nonzero element $v \in V$ of an enhanced vector space $(V, \varkappa)$, we define, first, a function $\operatorname{ht}(v)$ (stands for height) to be minimal s s.t. $v \in V^s$. Second, we define a function $\operatorname{cf}(v)$ (stands for coefficient) to be equal to $\lambda \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$ s.t. $p(v) = \lambda \varkappa_{\operatorname{ht}(v)}$, where p is a projection $V^{\operatorname{ht}(v)} \twoheadrightarrow V^{\operatorname{ht}(v)}/V^{\operatorname{ht}(v)-1}$.

Construction 1.13. Given a map $A \colon V \to W$ between enhanced vector spaces $(V, \varkappa)$ and (W, μ), we will now construct a rook matrix R of size $(\dim W) \times (\dim V)$.
$\rhd$ Fix any $s \in \{1, \dots, \dim V\}$. Consider a surjective map $L_s \colon W \twoheadrightarrow W/A(V^{s-1})$, and use Construction 1.11 to get an enhanced vector space $(W/A(V^{s-1}), \widetilde{\mu})$. Consider now an

element $\widetilde{A}(\varkappa_s) \in W/A(V^{s-1})$, where

$$\widetilde{A}\colon V^s/V^{s-1} \to W/A(V^{s-1})$$

is a map of quotient vector spaces induced by the restriction $A|_{V^s}$. If $\widetilde{A}(\varkappa_s) = 0$, then the s^{th} column of R is set to be zero. Otherwise, let λ and t' be respectively coefficient and height of $\widetilde{A}(\varkappa_s)$. Let $t \in \{1, \ldots, \dim W\}$ be the only number satisfying the condition $\dim L_s(W^t) = \dim L_s(W^{t-1}) + 1 = t'$. Finally, we set $R_{t,s}$ to be λ and all the other entries in the s^{th} column of R to be zero.

What is left to check is that in each row of R there is at most one nonzero entry. Suppose $R_{t,s} \neq 0$, then for any representative v of $\varkappa_s$, one has $A(v) \in W^t \setminus W^{t-1}$. This implies that for any $s' > s$ we have $\dim L_{s'}(W^t) = \dim L_{s'}(W^{t-1})$. Thus, $R_{t,s'} = 0$. $\lhd$

Remark 1.14. The fact that the entry $R_{t,s}$ of a rook matrix R is nonzero has the following characterization (we thank the anonymous referee for suggesting this AQ5 characterization), which may be easier to digest. Namely, $R_{t,s} \neq 0$ if and only if s is the smallest number satisfying the following condition: for any vector v from $V^s \setminus V^{s-1}$, the height $\mathrm{ht}(A(v))$ equals t.

1.3 Terminological digression

In this subsection, we introduce a bit of terminology that will be useful for understanding the content of Section 2.

Let X and Y be two sets and $\sim$ be an equivalence relation on X. If a map $g\colon X \to Y$ is constant on the equivalence classes, then we say that $g(x)$ is an invariant of some element $x \in X$. If, moreover, the induced map $\widetilde{g}\colon X/\!\sim\ \to Y$ is a bijection of sets, then we say that $g(x)$ is a full invariant.

Our next goal is to introduce a certain equivalence relation on the set of maps between fixed enhanced vector spaces. By an automorphism of an enhanced vector space $(V, \varkappa)$, we will mean an isomorphism from $(V, \varkappa)$ to itself (see Definition 1.5). We say that two maps A and B between enhanced vector spaces $(V, \varkappa)$ and (W, μ) are equivalent if there exists an automorphism C_1 (resp. C_2) of $(V, \varkappa)$ (resp. (W, μ)) s.t. $C_2 A C_1 = B$.

Remark 1.15. Note that A and B are equivalent if and only if there exist bases v_a and v_b of $(V, \varkappa)$ and bases w_a and w_b of (W, μ) s.t. the matrix of A in bases v_a and w_a coincides with the matrix of B in bases v_b and w_b.

Note that Construction 1.13 provides an invariant of a map between enhanced vector spaces considered up to equivalence. This invariant takes values in the set of rook matrices. It will follow from Subsection 1.4 that Lemma 1.7 can be reformulated as follows.

Lemma 1.16. Let $(V, \varkappa)$ and (W, μ) be two fixed enhanced vector spaces and $A\colon V \to W$ be some linear map. Then the corresponding rook matrix provided by Construction 1.13 is a full invariant of a map considered up to equivalence.

1.4 Proof of the main Lemma 1.7

In this subsection, we prove Lemma 1.7 (see Subsection 1.1 for a context). First, we prove the partial case when the map in question is an isomorphism. Second, we derive the general statement from it.

First of all, recall the lemma itself.

Lemma 1.7. Let $(V, \varkappa)$ and (W, μ) be two enhanced vector spaces and $A\colon V \to W$ be a linear map. There exists a basis v (resp. w) of an enhanced vector space $(V, \varkappa)$ (resp. (W, μ)) s.t. the matrix of A in these bases is a rook matrix. Moreover, this rook matrix is uniquely defined.

We will now deduce uniqueness from the existence. Suppose the map A is represented by a rook matrix R in some bases v and w. Then one checks straightforwardly that the Construction 1.13 produces the same matrix R as an output. Therefore, R is an invariant of a map A and we are done. The rest of this subsection is devoted to proving the existence part.

The next proposition is a partial case that will be used later.

Proposition 1.17. Let $(V, \varkappa)$ and (W, μ) be two enhanced vector spaces and $A\colon V \xrightarrow{\sim} W$ be an isomorphism. There exists a basis v (resp. w) of an enhanced vector space $(V, \varkappa)$ (resp. (W, μ)) s.t. the matrix of A in these bases is a rook matrix.

Proof. By a jump of a function $g\colon \{0, \dots, N\} \to \mathbb{Z}_{\geqslant 0}$, where $N \in \mathbb{Z}_{\geqslant 0}$, we will mean a number $x > 0$ s.t. $g(x) = g(x-1) + 1$. Fix any $s \in \{1, \dots, \dim V\}$. Consider now a function $x \mapsto \dim(A(V^s) \cap W^x)$, for $x \in \{0, \dots, \dim W\}$ (recall that $\dim V = \dim W$). It has exactly s jumps. Moreover, every jump of a function $x \mapsto \dim(A(V^{s-1}) \cap W^x)$ is also a jump of the function under consideration. Therefore, the latter function has exactly one "new" jump, call it t. It follows from the fact that t is a jump that $A(V^s) \cap (W^t \setminus W^{t-1}) \neq \varnothing$; take any element w from this set. It follows from the fact that t is actually a new jump that $A^{-1}(w) \in V^s \setminus V^{s-1}$.

By performing the above operation for all possible s, we construct a basis of W and, by taking a preimage, a basis of V. The end of the preceding paragraph implies that after appropriate reordering and rescaling these bases are bases of enhanced vector spaces $(V, \varkappa)$ and (W, μ). The statement follows. ■

Remark 1.18. This is a known proof of the Bruhat decomposition for GL_n adapted to our enhanced setting.

Construction 1.19. Let (W, μ) be an enhanced vector space and $A\colon V \hookrightarrow W$ be an injective map of vector spaces. We will now construct an induced enhancement on V.
▷ For $s \in \{1, \dots, \dim W\}$, define φ_s to be the composition

$$V \xhookrightarrow{A} W \twoheadrightarrow W/W^s.$$

Take any s s.t. $\dim \operatorname{Ker} \varphi_s = \dim \operatorname{Ker} \varphi_{s-1} + 1$ (call this number t). Set V^t to be $\operatorname{Ker} \varphi_s$. This defines a full flag on V, that is, a vector space V^t for any $t \in \{1, \dots, \dim V\}$. We now need to produce an element $\varkappa_t$ in the vector space V^t/V^{t-1}, which coincides with $\operatorname{Ker} \varphi_s / \operatorname{Ker} \varphi_{s-1}$. By identifying V with $\operatorname{Im} A$, we say that $\operatorname{Ker} \varphi_s \subset W^s$. Define $\varkappa_t$ to be $\alpha^{-1}(\mu_s)$, where

$$\alpha\colon \operatorname{Ker} \varphi_s / \operatorname{Ker} \varphi_{s-1} \xrightarrow{\sim} W^s/W^{s-1}$$

is an isomorphism of quotient vector spaces induced by the mentioned inclusion. We have obtained an enhanced vector space $(V, \varkappa)$. ◁

Proposition 1.20. Let (W, μ) be an enhanced vector space and $A\colon V \hookrightarrow W$ be an injective map of vector spaces. Construction 1.19 produces an enhanced vector space $(V, \varkappa)$. We claim that for any basis of $(V, \varkappa)$, one can find a basis of (W, μ) s.t. A maps each basis element to another basis element.

Proof. We continue using notations from Construction 1.19. Let $v = (v_1, \dots, v_{\dim V})$ be a given basis of $(V, \varkappa)$. Take any $s \in \{1, \dots, \dim W\}$. The difference $\dim \operatorname{Ker} \varphi_s - \dim \operatorname{Ker} \varphi_{s-1}$ is either zero or one. In the former case, set w_s to be any vector from $W^s \setminus W^{s-1}$ s.t. its class in W^s/W^{s-1} coincides with μ_s. In the latter case, set w_s to be $A(v_{\dim \operatorname{Ker} \varphi_s})$. It is straightforward to check that the basis $(w_1, \dots, w_{\dim W})$ satisfies the desired property. ■

Proof of Lemma 1.7. Uniqueness is shown in the beginning of the present subsection. To show the existence, consider the composition of three maps:

$$V \twoheadrightarrow V/\operatorname{Ker} A \xrightarrow{\sim} \operatorname{Im} A \hookrightarrow W.$$

Induce enhancement on $V/\operatorname{Ker} A$ from V via Construction 1.11 and on $\operatorname{Im} A$ from W via Construction 1.19. Apply Proposition 1.17 to the middle map to obtain bases $\widetilde{v}$ and $\widetilde{w}$ of its source and target, respectively. Apply now Proposition 1.12 to the basis $\widetilde{v}$ to get a basis v of V. Apply Proposition 1.20 to the basis $\widetilde{w}$ to get a basis w of W. By construction, bases v and w are the desired ones. ■

1.5 On a matrix of a map between enhanced vector spaces

In this subsection, we give several properties of a matrix of a map between enhanced vector spaces, written in appropriate basis.

Construction 1.21. Let R be a rook $n \times m$ matrix. We will now define a subset $\mathcal{T}(R)$ of a set $\mathrm{Mat}_{n,m}$ of all $n \times m$ matrices.

$\triangleright$Let $M \in \mathrm{Mat}_{n,m}$ be a matrix. We say that its entry $M_{i,j}$ is covered if there exists a pair of indices (i', j') s.t. the following two conditions hold:

1) $R_{i',j'} \neq 0$,
2) $(i < i'$ AND $j \geqslant j')$ OR $(i \leqslant i'$ AND $j > j')$.

The matrix M is said to be in $\mathcal{T}(R)$ if the following two conditions hold:

1) if the entry $M_{i,j}$ is not covered and $R_{i,j} = 0$, then it equals zero,
2) if the entry $M_{i,j}$ is not covered and $R_{i,j} \neq 0$, then $M_{i,j} = R_{i,j}$. $\triangleleft$

Here is an example, for $\mathbb{F} = \mathbb{Q}$, of the matrix R and the general form of a matrix M from the set $\mathcal{T}(R)$:

$$R = \begin{pmatrix} 0 & 0 & 4 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, M = \begin{pmatrix} * & * & * \\ 3 & * & * \\ 0 & * & * \\ 0 & 2 & * \end{pmatrix}.$$

In the case of classical Bruhat decomposition analogous set is nothing but a Bruhat cell. The next proposition is straightforward and well known in the classical case.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

Proposition 1.22. Let $(V,\varkappa)$ and (W,μ) be two enhanced vector spaces and $A\colon V \to W$ be some linear map. Let also v (resp. w) be some basis of enhanced vector space $(V,\varkappa)$ (resp. (W,μ)). Then the matrix of A in the bases v and w belongs to $\mathcal{T}(R)$, where R is a rook matrix from Lemma 1.7.

The next statement links properties of integral matrix (considered up to unitriangular change of basis) and its enhanced Bruhat decomposition over $\mathbb{Q}$. Let $\mathrm{Mat}_{n,m}(\mathbb{Z})$ be the set of all $n \times m$ matrices over $\mathbb{Z}$. In what follows, we will sometimes view it as a subset of matrices over $\mathbb{F} = \mathbb{Q}$ without mentioning this explicitly. Let also $T_n(\mathbb{Z})$ be the group of unitriangular matrices over $\mathbb{Z}$. The group $T_n(\mathbb{Z}) \times T_m(\mathbb{Z})$ acts on the set $\mathrm{Mat}_{n,m}(\mathbb{Z})$. The next proposition follows from Proposition 1.22.

Proposition 1.23. Consider $M \in \mathrm{Mat}_{n,m}(\mathbb{Z})$. Then any element M' from the orbit $T_n(\mathbb{Z}) \cdot M \cdot T_m(\mathbb{Z})$ lies in the set $\mathcal{T}(R)$, where R is a rook matrix over $\mathbb{F} = \mathbb{Q}$ associated with M.

As a corollary, one gets that at least one nonzero entry of R is integer. It also follows that there is at least one pair of indices (i,j) s.t. the entry $M'_{i,j}$ is the same for any M' from the mentioned orbit.

1.6 Geometric approach to enhancements

In this subsection, we briefly present geometric viewpoint on enhancements. All the proofs here are straightforward and left to the reader.

Recall that for an affine subspace A of a vector space V, the set $\{a - b \mid a, b \in A\}$ is a vector subspace of V, which is called the direction of A. We denote it by $\overrightarrow{A}$. The following definition is equivalent to Definition 1.4.

Definition 1.24. Let V be a vector space over $\mathbb{F}$. An *enhancement $\varkappa$ on a vector space V* is a choice of $\dim V$ affine subspaces $\{\varkappa_1, \dots, \varkappa_{\dim V}\}$ s.t.

1) $\dim \varkappa_s = s - 1$ for $s \in \{1, \dots, \dim V\}$,
2) $\mathrm{Span}(\varkappa_s) = \overrightarrow{\varkappa_{s+1}}$ (by convention, $\varkappa_{\dim V+1} = V$).

It follows that all the affine subspaces $\{\varkappa_1, \dots, \varkappa_{\dim V}\}$ are disjoint. In terms of Definition 1.4, each affine subspace is a full preimage of a distinguished element in the one-dimensional quotient space along the quotient map.

Let $(V, \varkappa)$ be an enhanced vector space. Its dual V^* is naturally enhanced as well. By definition, the dual enhancement on V^* is a collection of affine subspaces

$$\mathrm{Uni}(\varkappa_s) = \{\xi \in V^* \mid \xi(\varkappa_s) = 1\}$$

for all s; we call them unificator spaces in analogy with annihilator spaces. Note that $\dim \mathrm{Uni}(\varkappa_s) = \dim V - s$. Note also that dual flag to

$$0 = \overrightarrow{\varkappa_1} \subset \ldots \subset \overrightarrow{\varkappa_{\dim V}} \subset V,$$

which by definition consists of annihilators

$$0 = \mathrm{Ann}(V) \subset \mathrm{Ann}(\overrightarrow{\varkappa_{\dim V}}) \subset \ldots \subset \mathrm{Ann}(\overrightarrow{\varkappa_1}) = V^*$$

coincides with the flag

$$0 = \overrightarrow{\mathrm{Uni}(\varkappa_{\dim V})} \subset \overrightarrow{\mathrm{Uni}(\varkappa_{\dim V-1})} \subset \ldots \subset \overrightarrow{\mathrm{Uni}(\varkappa_1)} \subset V^*.$$

We will now present a (geometric) reformulation of Construction 1.19. Namely, if $W \subset V$ is a linear subspace of an enhanced vector space $(V, \varkappa)$, then the collection of affine subspaces $\{\varkappa_s \cap W\}$ (for all possible s, but empty intersections must be excluded) is an enhancement on the vector space W.

We now turn to a reformulation of Construction 1.11. Namely, given an enhanced vector space $(V, \varkappa)$ and a surjective map $A\colon V \twoheadrightarrow W$, we will construct an induced enhancement on W. First, consider the dual map $A^*\colon W^* \hookrightarrow V^*$, which is an injection. Since V^* is naturally enhanced, its subspace W^* is enhanced as well. By identifying W with its double dual W^{**}, we get an induced enhancement on W.

2 Enhanced Complexes

In this section, we define and study enhanced complexes—an algebraic object that will carry a certain information about a strong Morse function (see Subsection 3.5). All the constructions lie within the scope of homological algebra of chain complexes over a field. They are similar in spirit to those in Section 1. The purpose is that enhanced complex is a useful algebraic container that stores some information about a strong Morse function; see Section 3.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

2.1 Definition of an enhanced complex

In this subsection, we define the object of study of this section.

Definition 2.1. Let $\mathcal{C}$ be a (chain) complex of vector spaces,

$$C_{n+1} = 0 \to C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \to 0 = C_{-1}.$$

An *enhancement* $\varkappa$ *on a complex* $\mathcal{C}$ is an enhancement $(\mathcal{C}_\bullet, \varkappa)$ on a vector space $\mathcal{C}_\bullet := \oplus_{j=0}^n C_j$ satisfying the condition that each $\mathcal{C}^s_\bullet$ is a subcomplex of $\mathcal{C}$ (we will therefore write $\mathcal{C}^s$ instead of $\mathcal{C}^s_\bullet$ in order to stress the structure of a complex). A complex with an enhancement will be called an *enhanced complex* and denoted as $(\mathcal{C}, \varkappa)$.

We call the number n the dimension of $\mathcal{C}$ and denote it by $\dim \mathcal{C}$.

Remark 2.2.

1. Recall that the aforementioned condition amounts to the following two:
    1. $\mathcal{C}^s$ is decomposed into the direct sum of graded components $\oplus_k C^s_k$ s.t. $C^s_k \subset C_k$ for $k \in \{0, \dots, n\}$;
    2. $\partial_k(C^s_k) \subset C^s_{k-1}$ for $k \in \{1, \dots, n\}$.
2. By Construction 1.19 the vector space C_k is also enhanced.

Remark 2.3. For an enhanced complex $(\mathcal{C}, \varkappa)$, the set $\{1, \dots, \dim \mathcal{C}_\bullet\}$ is $\mathbb{Z}_{\geqslant 0}$-graded: the degree $\deg s$ of s is given by the only degree in which the complex $\mathcal{C}^s/\mathcal{C}^{s-1}$ is nonzero.

Definition 2.4. Let $(\mathcal{C}, \varkappa)$ and $(\mathcal{D}, \mu)$ be two enhanced complexes, and let $\varphi \colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ be an isomorphism of complexes. We say that φ is an *isomorphism of enhanced complexes* if the induced map $\mathcal{C}_\bullet \to \mathcal{D}_\bullet$ is an isomorphism of enhanced vector spaces $(\mathcal{C}_\bullet, \varkappa)$ and $(\mathcal{D}_\bullet, \mu)$.

We say that two enhanced complexes are isomorphic if there is an isomorphism between them (compare Remark 1.6).

Remark 2.5. As we will show in Section 3, a strong Morse function (together with suitable orientations) gives rise to an enhanced complex (over any field $\mathbb{F}$), which is well-defined up to isomorphism.

2.2 Enhancement on $\mathbf{H}_\bullet(\mathcal{C})$

In this subsection, we construct enhancement on a homology of a certain class of complexes, which includes enhanced ones.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

For any complex $\mathcal{C}$ denote by $\mathsf{H}_\bullet(\mathcal{C})$ the direct sum $\oplus_j \mathsf{H}_j(\mathcal{C})$. Let $(\mathcal{C}, \varkappa)$ be an enhanced complex. Then, for any s, the vector space $\mathsf{H}_\bullet(\mathcal{C}^s, \mathcal{C}^{s-1})$ is one-dimensional with a preferred generator of degree $\deg s$ given by a class of relative chain $\varkappa_s \in \mathcal{C}^s/\mathcal{C}^{s-1}$.

Construction 2.6. Let $\mathcal{C}$ be a filtered (possibly infinite-dimensional) complex over a field $\mathbb{F}$, $0 = \mathcal{C}^0 \subset \ldots \subset \mathcal{C}^N = \mathcal{C}$. Suppose that for any s vector space $\mathsf{H}_\bullet(\mathcal{C}^s, \mathcal{C}^{s-1})$ is one-dimensional with a chosen generator h_s. We will now construct an enhancement on a homology vector space $\mathsf{H}_\bullet(\mathcal{C})$.
$\triangleright$ First, we will construct a filtration on $\mathsf{H}_\bullet(\mathcal{C})$. For $s \in \{0, \ldots, N\}$, let $\iota_s \colon \mathsf{H}_\bullet(\mathcal{C}^s) \to \mathsf{H}_\bullet(\mathcal{C})$ be a map induced by inclusion. Define the subset H (which stands for homology) of the set $\{1, \ldots, N\}$ to be the set of all s s.t. $\dim \operatorname{Im} \iota_s = \dim \operatorname{Im} \iota_{s-1} + 1$. Let s_i be the i^{th} element of H (counting from 1) and s_0 be zero. The sequence of subspaces $0 = \operatorname{Im} \iota_{s_0} \subset \operatorname{Im} \iota_{s_1} \subset \ldots \subset \operatorname{Im} \iota_{s_{\dim \mathsf{H}_\bullet(\mathcal{C})}} = \mathsf{H}_\bullet(\mathcal{C})$ is a full flag on $\mathsf{H}_\bullet(\mathcal{C})$ (follows from considering the exact sequence of a pair $(\mathcal{C}^s, \mathcal{C}^{s-1})$ for all s). To complete the construction of enhancement, we will now produce an element from $\operatorname{Im} \iota_{s_i}/\operatorname{Im} \iota_{s_{i-1}}$ for a given $i \in \{1, \ldots, \dim \mathsf{H}_\bullet(\mathcal{C})\}$. We denote s_i by s for convenience. Let $\deg s$ denote the degree in which the graded vector space $\mathsf{H}_\bullet(\mathcal{C}^s, \mathcal{C}^{s-1})$ is nonzero (this notation is coherent with the case when $\mathcal{C}$ is an enhanced complex).

Consider the following diagram, with horizontal line being a portion of a long exact sequence of a pair $(\mathcal{C}^s, \mathcal{C}^{s-1})$:

$$\begin{tikzcd}
\mathsf{H}_{\deg s}(\mathcal{C}^{s-1}) \arrow[r, "\theta"] \arrow[dr, "\iota_{s-1}"] & \mathsf{H}_{\deg s}(\mathcal{C}^s) \arrow[r, two heads, "p_*"] \arrow[d, "\iota_s"] & \mathsf{H}_{\deg s}(\mathcal{C}^s, \mathcal{C}^{s-1}) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
 & \mathsf{H}_\bullet(\mathcal{C}) & h_s
\end{tikzcd}$$

We write ι_s both for a map $\mathsf{H}_\bullet(\mathcal{C}^s) \to \mathsf{H}_\bullet(\mathcal{C})$ and for its restriction to $\mathsf{H}_{\deg s}(\mathcal{C}^s)$. It follows from the definition of H that $\dim \operatorname{Coker} \theta = 1$; therefore, $\operatorname{Ker} p_*$ is a proper subspace of $\mathsf{H}_{\deg s}(\mathcal{C}^s)$, which in turn implies that p_* is surjective. Denote by $p_*^{-1}(h_s)$ any preimage of h_s; it is defined up to elements from $\operatorname{Ker} p_* \simeq \operatorname{Im} \theta$. Finally, the desired element is a class of $\iota_s(p_*^{-1}(h_s))$ in the quotient space $\operatorname{Im} \iota_s/\operatorname{Im} \iota_{s-1} \simeq \operatorname{Im} \iota_s/\operatorname{Im} \iota_{s_{i-1}}$ (mind that $\operatorname{Im} \iota_{s_i - 1} = \operatorname{Im} \iota_{s_{i-1}}$); it is well defined. $\triangleleft$

Remark 2.7. Note that to a flag space of $\mathsf{H}_\bullet(\mathcal{C})$ of dimension d, one can associate a number $s \in \{1, \ldots, N\}$ as the unique solution of equation $\dim \operatorname{Im} \iota_s = \dim \operatorname{Im} \iota_{s-1} + 1 = d$. In other words, this is the smallest s such that given flag space is contained in the image of ι_s.

If one is given a nonzero vector v in the enhanced vector space $\mathsf{H}_\bullet(\mathcal{C})$, then they may consider the flag space of least possible dimension containing v (its dimension is $\mathrm{ht}(v)$). Combining this with the previous paragraph, one can associate a number s with a vector v. We will make use of this association in Subsection 2.4.

For a detailed treatment of the mentioned long exact sequence, see [29]. Without taking $\varkappa_s$ into account, it was first considered in [31]. This preferred generator appeared independently in [30]. We denote obtained enhancement as $(\mathsf{H}_\bullet(\mathcal{C}), \varkappa_{\mathsf{H}})$. Specializing the above discussion to a fixed degree k (and thus having $\deg s_i = k$), we get an enhanced vector space $(\mathsf{H}_k(\mathcal{C}), \varkappa_{\mathsf{H}_k})$; one may check that this is the same enhancement as the one induced by inclusion $\mathsf{H}_k(\mathcal{C}) \hookrightarrow \mathsf{H}_\bullet(\mathcal{C})$ via Construction 1.19. Note that the above procedure also gives enhancement on $\mathsf{H}_\bullet(\mathcal{C}^s)$ for any s (it will be crucial in what follows).

Remark 2.8. The reason for the chosen level of generality is that one may take the input to be a complex of singular chains on a manifold equipped with a Morse function. The filtration is then given by the sublevel sets. See Subsection 3.5, where B-data (described below in Subsection 2.4) is extracted from the function this way.

Remark 2.9. In the case when $\mathcal{C}$ is an enhanced complex, one may check that the following alternative construction produces the same enhancement on $\mathsf{H}_\bullet(\mathcal{C})$. Consider the map $\partial : \mathcal{C}_\bullet \to \mathcal{C}_\bullet$. Induce enhancements on $\operatorname{Ker}\partial$ and $\operatorname{Im}\partial$ via Construction 1.19. Induce enhancement on $\operatorname{Ker}\partial/\operatorname{Im}\partial = \mathsf{H}_\bullet(\mathcal{C})$ via Construction 1.11.

2.3 Definition of B-data

In this subsection, we introduce a certain data, called B-data, which will be used as a container that stores information about enhanced complex (and, eventually, about a Morse function). The actual process of extraction is given in Subsection 2.4. The letter B stands simultaneously for Barannikov, Bruhat, and barcode.

B-data consists of the following parts.

i) A nonnegative integer N along with a $\mathbb{Z}_{\geqslant 0}$-grading on a set $\{1, \ldots, N\}$, denoted by deg.

ii) Decomposition of $\{1, \ldots, N\}$ into the union of three disjoint sets U, L, H (these letters stand for upper, lower, and homological, for the reasons described below).

iii) Bijection $b\colon U \xrightarrow{1-1} L$ of degree -1 w.r.t. the grading. Map b must satisfy $b(s) < s$.

iv) A function $\lambda\colon U \to \mathbb{F}^*$. We write λ_s for its value on $s \in U$.

We call the image of λ "Bruhat numbers" of an enhanced complex (see the proof of Theorem 2.17 for the explanation). Two numbers s and $b(s)$ are said to form a Barannikov pair (or simply a pair). Itis convenient to think of each Bruhat number as being "written" on a Barannikov pair. Roughly speaking, B-data is a decomposition of some subset of $\{1, \ldots, N\}$ into Barannikov pairs (the rest of the elements are homological). Each pair consists of an upper element and a lower one and carries a Bruhat number. In other words, B-data is a grading on $\{1, \ldots, N\}$ together with a finite sequence of rook matrices $\{R_k\}$ over $\mathbb{F}$ (see Definition 1.1), where R_k is of size $(\#\{s| \deg s = k-1\}) \times (\#\{s| \deg s = k\})$ and $R_{k-1}R_k = 0$.

Figure 1 gives an example of B-data over $\mathbb{Q}$ and describes pictorial format, which we will use in future. Elements of the set $\{1, \ldots, N\}$ are drawn as dots, from bottom to top, pairs correspond to segments. Either to the left or to the right of a middle of a segment, we write a Bruhat number. The degree of an element is written either above or below this element, whatever is more convenient. In the example $N = 8$, degree of 1 is 0, degree of $2, 3, 4$ and 6 is 1 and degree of $5, 7, 8$ is 2. Next, $U = \{4, 5, 7, 8\}$, $L = \{1, 2, 3, 6\}$, $H = \varnothing$. Bruhat numbers are $6, 3, 2, 4$ (i.e., values of λ on 4,5,7,8, respectively). The map b is defined by the segments. Finally, two rook matrices are

$$R_1 = \begin{pmatrix} 0 & 0 & 6 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 4 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

2.4 Extraction of B-data

In this subsection, we will extract a B-data from an enhanced complex. These data are invariant under isomorphisms. In Subsection 2.5, we show that these data are in fact a full invariant of an enhanced complex considered up to isomorphism (in a sense of Subsection 1.3).

Remark 2.10. The extraction of B-data is done in the same way for the (more general) case of a complex $\mathcal{C}$ that satisfies conditions of Construction 2.6 (compare Remark 2.8). Indeed, one has to merely replace the symbol $[\varkappa_s]$ with h_s. In this case, however, we

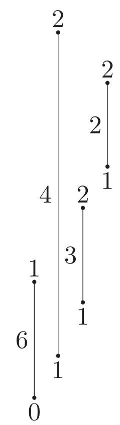


Fig. 1.

do not claim that that data are a full invariant; we do not even define any equivalence relation on the set of such complexes.

Let $(\mathcal{C}, \varkappa)$ be an enhanced complex. We will now construct a B-data. In the future, we will refer to it as a B-data of $(\mathcal{C}, \varkappa)$ and use the same letters N, U, L, H, b and λ for its ingredients when necessary. We continue using notations introduced in Subsections 2.1 and 2.2. Let $\delta\colon \mathsf{H}_{\deg s}(\mathcal{C}^s, \mathcal{C}^{s-1}) \to \mathsf{H}_{\deg s-1}(\mathcal{C}^{s-1})$ be the connecting homomorphism. To begin with, set N to be the number of filtration components (counting from zero), and set the grading on $\{1, \ldots, N\}$ to be the one defined in Remark 2.3. Now, for each $s \in \{1, \ldots, N\}$ s.t. $\delta([\varkappa_s]) \neq 0$, do the following.

1) Put s in U.
2) For any $t \in \{1, \ldots, s-1\}$, let $\iota_t^{s-1}\colon \mathsf{H}_{\deg s}(\mathcal{C}^t) \to \mathsf{H}_{\deg s}(\mathcal{C}^{s-1})$ be the map induced by inclusion. Now choose the only t s.t. $\dim \operatorname{Im} \iota_t^{s-1} = \dim \operatorname{Im} \iota_{t-1}^{s-1} + 1 = \operatorname{ht}(\delta([\varkappa_s]))$ (the function $\operatorname{ht}(\cdot)$ here is taken w.r.t. the enhancement on $\mathsf{H}_{\deg s-1}(\mathcal{C}^{s-1})$ constructed from $\mathcal{C}^{s-1}$). See Remark 2.7 for an informal meaning of such a t. Put this t in L.
3) Set $\lambda(s) := \operatorname{cf}(\delta([\varkappa_s]))$ w.r.t. the same enhancement.
4) Set $b(s) := t$.

Using diagram chasing similar to that in Construction 2.6, one verifies that such an operation is well defined in a sense that, first, each number will be put somewhere at most once and, second, that b enjoys desired properties; see [29]. Those numbers in

$\{1,\ldots,N\}$ that were not put anywhere by this operation are put in H. The extraction of B-data i)–iv) is over, itis plain to see that it is invariant under isomorphisms.

Remark 2.11. Here is another way to extract a B-data out of an enhanced complex. Apply Lemma 1.7 to a differential $\partial_k\colon C_k \to C_{k-1}$ viewed as a map between enhanced vector spaces (see Remark 2.2). This will give a sequence $\{R_k\}$ of rook matrices and thus weare done. Although this way is shorter, we find it less instructive.

Another approach is to use spectral sequence of a filtered (actually, enhanced) complex.

Remark 2.12. We will now give a yet another way to extract a B-data. This way is even quicker than the one described in Remark 2.11 and even less instructive. Let $s > t$ be two numbers from $\{1,\ldots,N\}$ s.t. $\deg s = \deg t + 1$. Let X be the image of $[\varkappa_s] \in \mathsf{H}_{\deg s}(\mathcal{C}^s, \mathcal{C}^{s-1})$ under the composition map

$$\mathsf{H}_{\deg s}(\mathcal{C}^s, \mathcal{C}^{s-1}) \xrightarrow{\delta} \mathsf{H}_{\deg t}(\mathcal{C}^{s-1}) \to \mathsf{H}_{\deg t}(\mathcal{C}^{s-1}, \mathcal{C}^{t-1}).$$

Let now Y be the image of $[\varkappa_t] \in \mathsf{H}_{\deg t}(\mathcal{C}^t, \mathcal{C}^{t-1})$ under the map $\mathsf{H}_{\deg t}(\mathcal{C}^t, \mathcal{C}^{t-1}) \to \mathsf{H}_{\deg t}(\mathcal{C}^{s-1}, \mathcal{C}^{t-1})$ induced by inclusion. If $X = \lambda Y \neq 0$, then s and t form a Barannikov pair and the corresponding Bruhat number is λ. The proof is a straightforward diagram chasing.

Again, arguing as in Construction 2.6, one can show that $\#\{s \in H | \deg s = k\} = \dim \mathsf{H}_k(\mathcal{C})$. We stress that everything except λ was essentially constructed in [4], while homological language was first used in [31]. A close idea of construction of Bruhat numbers over $\mathbb{Q}$ appeared independently in [30].

Remark 2.13. Consider once again the portion of a long exact sequence of a pair $(\mathcal{C}^s, \mathcal{C}^{s-1})$:

$$\mathsf{H}_{\deg s}(\mathcal{C}^s) \xrightarrow{p_*} \mathsf{H}_{\deg s}(\mathcal{C}^s, \mathcal{C}^{s-1}) \xrightarrow{\delta} \mathsf{H}_{\deg s-1}(\mathcal{C}^{s-1}).$$

Since the middle term is one-dimensional, there are two cases possible.

1) The map p_* is surjective while δ is zero. This case was used in Construction 2.6.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

2) The map δ is injective while p_* is zero. This case was used in the extraction of B-data.

Remark 2.14. Let s and t be some elements from $\{1, \ldots, N\}$. They form a Barannikov pair (i.e., $b(s) = t$) if and only if the following equalities hold:

$$\dim \mathsf{H}_\bullet(\mathcal{C}^{s-1}, \mathcal{C}^t) = \dim \mathsf{H}_\bullet(\mathcal{C}^s, \mathcal{C}^{t-1}) = \dim \mathsf{H}_\bullet(\mathcal{C}^{s-1}, \mathcal{C}^{t-1}) - 1 = \dim \mathsf{H}_\bullet(\mathcal{C}^s, \mathcal{C}^t) - 1.$$

This can be proven either by diagram chasing similar to that in Construction 2.6 or by Theorem 2.17 (see [13] and also [31]).

We will now take coordinate viewpoint, which will be useful in formulation of the classificational Theorem 2.17 in Subsection 2.5. By a basis of a chain complex $\mathcal{C}$, we will mean a basis $(c_1, \ldots, c_{\dim \mathcal{C}_\bullet})$ of a vector space $\mathcal{C}_\bullet$ s.t. each c_s belongs to some C_k, where k depends on s. By a basis of an enhanced complex $(\mathcal{C}, \varkappa)$, we will mean a basis c of a chain complex $(\mathcal{C}, \partial)$ s.t. identity map is an isomorphism of enhanced vector spaces $(\mathcal{C}_\bullet, \varkappa)$ and $(\mathcal{C}_\bullet, \varkappa(c))$. Every enhanced complex can be equipped with a basis.

Vice versa, given a basis c of a chain complex $\mathcal{C}$ s.t. the span $\langle c_1, \ldots, c_s \rangle$ is a subcomplex (for each $s \in \{1, \ldots, \dim \mathcal{C}_\bullet\}$), one may construct an enhanced complex $(\mathcal{C}, \varkappa(c))$ by declaring $(\mathcal{C}_\bullet, \varkappa) := (\mathcal{C}_\bullet, \varkappa(c))$.

Remark 2.15. Note that two enhanced complexes $(\mathcal{C}, \varkappa)$ and $(\mathcal{D}, \mu)$ are isomorphic if and only if the following holds:

1) $\dim \mathcal{C}_\bullet = \dim \mathcal{D}_\bullet$ and the gradings on $\{1, \ldots, \dim \mathcal{C}_\bullet\}$ constructed from $(\mathcal{C}, \varkappa)$ and $(\mathcal{D}, \mu)$ coincide (see Remark 2.3);
2) there exist bases c of $(\mathcal{C}, \varkappa)$ and d of $(\mathcal{D}, \mu)$ s.t. corresponding two matrices of differentials coincide (compare Remark 1.15).

Definition 2.16. Let $(\mathcal{C}, \varkappa)$ be an enhanced complex and c be its basis. We call c a *Barannikov basis* if the matrix of differential $\partial_k \colon C_k \to C_{k-1}$ in the basis c is a rook matrix (for any k).

The set of rook matrices $\{R_k\}$ of differentials ∂_k does not depend on a particular choice of a Barannikov basis. Indeed, this set is precisely the B-data, which is an invariant of an enhanced complex. The purpose of Subsection 2.5 is to show that every enhanced complex admits a Barannikov basis. This will imply that B-data is a full invariant of an enhanced complex considered up to isomorphism.

2.5 Classification of enhanced complexes

In this subsection, we prove the followingclassificational theorem.

Theorem 2.17. For every enhanced complex $(\mathcal{C}, \varkappa)$, there exists a Barannikov basis c. Moreover, the matrix of ∂ is the same for any Barannikov basis.

Remark 2.18.

1. Barannikov basis itself need not be unique (compare Remark 1.10).
2. Put differently, one may say that B-data is a full invariant of an enhanced complex considered up to isomorphism (see Subsection 1.3).
3. It is profitable to have a Barannikov basis at hand, since the complex takes the simplest form possible and becomes tractable.
4. The case when the complex is not enhanced, but only filtered, was proven in [4]. See also for an "ungraded" setting where a single upper triangular matrix is considered.
5. The existence part of the theorem was observed independently in [37].

For a matrix X, we denote by $X_{\bullet,j}$ its j^{th} column and by $X_{i,\bullet}$ it's i^{th} row.

Proof of Theorem 2.17. Uniqueness follows from the existence and the fact that B-data is invariant under isomorphisms (which is shown in Subsection 2.4). The rest is devoted to proving the existence part.

Fix any $k \in \{0, \ldots, \dim \mathcal{C}\}$. The differential $\partial_k : C_k \to C_{k-1}$ is a map of enhanced vector spaces $(C_k, \varkappa_k)$ and $(C_{k-1}, \varkappa_{k-1})$ (see Remark 2.2). Therefore, Lemma 1.7 produces, in particular, a rook matrix X and a basis x of $(C_k, \varkappa_k)$. Analogously, applying Lemma 1.7 to ∂_{k+1}, one obtains a rook matrix Y and another basis of $(C_k, \varkappa_k)$, call it y. Construct now a third basis v of $(C_k, \varkappa_k)$ as follows.

Fix $s \in \{1, \ldots, \dim C_k\}$. Obviously at least one of three cases listed below holds. On the other hand, it follows from $\partial^2 = 0$ and proof of Lemma 1.7 that all three are mutually exclusive. So, we define v_s depending on which of them holds.

1) $X_{\bullet,s} \neq 0$. Set $v_s := x_s$.
2) $Y_{s,\bullet} \neq 0$. Set $v_s := y_s$.

3) Both $X_{\bullet,s}$ and $Y_{s,\bullet}$ are zero. One is free to take either x_s or y_s as v_s.

We have constructed a basis of $(C_k, \varkappa_k)$ for each k.

The last step is to construct a basis c of $(\mathcal{C}, \varkappa)$. Take any $s \in \{1, \dots, \dim \mathcal{C}_\bullet\}$. Let v be a constructed basis of $(\mathcal{C}_{\deg s}, \varkappa_{\deg s})$. Define c_s to be $v_{\dim \mathcal{C}^s_{\deg s}}$ (see Remark 2.2). Finally, by construction c is a Barannikov basis. ∎

Remark 2.19. Three mentioned cases correspond respectively to the fact that s belongs to

1) U,
2) L,
3) H.

2.6 $\mathbb{Z}$-Enhanced complexes

In this subsection, we introduce a certain analogue of an enhanced complex, which is itself a complex of free abelian groups.

Definition 2.20. Let $\mathcal{C}$ be a (chain) complex of free abelian groups

$$C_{n+1} = 0 \to C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \to 0 = C_{-1}.$$

A $\mathbb{Z}$-*enhancement* $\varkappa$ *on a complex* $\mathcal{C}$ is a choice of the following two structures.

1) A filtration

$$0 = \mathcal{C}^0 \subset \dots \subset \mathcal{C}^{\operatorname{rk} \mathcal{C}_\bullet} = \mathcal{C}$$

of $\mathcal{C}$ by subcomplexes s.t. for each $s \in \{1, \dots, \operatorname{rk} \mathcal{C}_\bullet\}$ the quotient complex $\mathcal{C}^s/\mathcal{C}^{s-1}$ is isomorphic to $\mathbb{Z}$ concentrated in one degree.
2) A generator of $\mathcal{C}^s/\mathcal{C}^{s-1} \simeq \mathbb{Z}$.

A complex with a $\mathbb{Z}$-enhancement will be called a $\mathbb{Z}$-*enhanced complex* and denoted as $(\mathcal{C}, \varkappa)$.

The following notions and statements go in exactly the same manner as in the honest enhanced case.

1) The definition of an isomorphism between two $\mathbb{Z}$-enhanced complexes.
2) The definition of a basis of a $\mathbb{Z}$-enhanced complex (recall that by a basis, we always mean a linearly ordered set of generators). Matrix of differential ∂_k

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

in any basis is obviously integral, yet it will play important role in the end of this subsection.

3) Every $\mathbb{Z}$-enhanced complex can be equipped with a basis.
4) Let $c = (c_1, \ldots, c_{\mathrm{rk}\,\mathcal{C}_\bullet})$ be a basis of a complex of free abelian groups s.t.
   1) for each s, the span $\langle c_1, \ldots, c_s\rangle$ is a subcomplex;
   2) the induced filtration satisfies the condition 1) from Definition 2.1.
2) Then one can construct a $\mathbb{Z}$-enhanced complex $(\mathcal{C}, \varkappa(c))$.

It follows directly from the definitions that if $(\mathcal{C}, \varkappa)$ is a $\mathbb{Z}$-enhanced complex then $\mathcal{C} \otimes \mathbb{F}$ is an enhanced complex over $\mathbb{F}$. We denote it by $(\mathcal{C} \otimes \mathbb{F}, \varkappa)$.

Remark 2.21. An oriented strong Morse function on a manifold naturally gives rise to a $\mathbb{Z}$-enhanced complex. However, classifying such complexes up to isomorphism is a transcendentally hard problem. So, following [4], we proceed by tensoring the given complex by $\mathbb{F}$ for various fields. See Subsection 3.5.

The rest of this subsection is devoted to the interplay between properties of $\mathbb{Z}$-enhanced complex $(\mathcal{C}, \varkappa)$ and those of enhanced complex $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$. So we set $\mathbb{F} = \mathbb{Q}$ for the time being. Similar in spirit results are stated in Subsection 2.7 without proofs.

Recall that B-data may be viewed as a sequence of rook matrices $\{R_k\}$; see Subsection 2.3 and Remark 2.11. Recall also that in Subsection 1.5, we associated a subset $\mathcal{T}(R)$ of matrices with a rook matrix R.

Proposition 2.22. Let c be any basis of a $\mathbb{Z}$-enhanced complex $(\mathcal{C}, \varkappa)$. Then the matrix of differential ∂_k in this basis belongs to the set $\mathcal{T}(R_k)$, where R_k is a rook matrix from the B-data of $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$.

Proof. Let D be a matrix of differential ∂_k in some basis c of $(\mathcal{C}, \varkappa)$. After choosing another basis c' the matrix D gets multiplied by a unitriangular matrices (over $\mathbb{Z}$) from the left and from the right. The statement now follows from Proposition 1.23. ∎

See Subsection 1.5 for an example, which may be treated as a complex concentrated in two degrees. By a degree of a pair, we will mean degree of its lower point. A Barannikov pair is called short if there are no pairs of the same degree that lie inside it. Formally, (s, t) is a short pair if there is no pair (s', t') of the same degree s.t. $s > s' > t' > t$.

Corollary 2.23. Let $(\mathcal{C}, \varkappa)$ be a $\mathbb{Z}$-enhanced complex. Bruhat number of $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$ on any short pair is integer.

Proof. Take any short pair of degree, say, $k-1$. Let R_k be a rook matrix from the B-data of $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$. Short pairs correspond precisely to those nonzero entries of R_k that are not covered (in the terminology of Subsection 1.5). The statement now follows from Proposition 2.22. ■

The next statement is a mere combination of the previous two.

Corollary 2.24. Let $(\mathcal{C}, \varkappa)$ be $\mathbb{Z}$-enhanced complex, and let (s, t) be a short Barannikov pair of $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$. Let also c be any basis of $(\mathcal{C}, \varkappa)$. Then element c_s appears in the differential of c_t with the coefficient equal to Bruhat number on a pair (s, t). In particular, this coefficient does not depend on c.

2.7 Bruhat numbers over the rationals

In this subsection, we state several facts about interplay between $\mathbb{Z}$-enhanced complex $(\mathcal{C}, \varkappa)$ and enhanced complex $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$. The proofs will be given elsewhere.

For an abelian group G, we denote by $\#G$ its order (provided that G is finite) and by $\operatorname{Tors} G$ its torsion subgroup.

Proposition 2.25. Let $(\mathcal{C}, \varkappa)$ be a $\mathbb{Z}$-enhanced complex. Let also s and t ($s > t$, both from $\{1, \ldots, \operatorname{rk} \mathcal{C}_\bullet\}$) be a Barannikov pair of enhanced complex $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$ with Bruhat number $\lambda \in \mathbb{Q}^*$. One then has

$$\pm\lambda = \frac{\#\operatorname{Tors} \mathsf{H}_\bullet(\mathcal{C}^s, \mathcal{C}^{t-1})}{\#\operatorname{Tors} \mathsf{H}_\bullet(\mathcal{C}^{s-1}, \mathcal{C}^t)} = \frac{\#\operatorname{Tors} \mathsf{H}_{\deg t}(\mathcal{C}^s, \mathcal{C}^{t-1})}{\#\operatorname{Tors} \mathsf{H}_{\deg t}(\mathcal{C}^{s-1}, \mathcal{C}^t)}.$$

Section 4 will interpret Bruhat numbers as a certain kind of Reidemeister torsion. From this viewpoint, the given formula is of type "torsion=torsion". For its close relative, see [42, Theorem 4.7], proven in weaker generality by Milnor [34]. See also [12] for similar in spirit statement in symplectic topology. We stress out that we place no acyclicity condition on a complex $\mathcal{C}$.

Proposition 2.26. Let $(\mathcal{C}, \varkappa)$ be a $\mathbb{Z}$-enhanced complex. Then the following are equivalent:

1) $\operatorname{Tors} \mathsf{H}_\bullet(\mathcal{C}^s, \mathcal{C}^t) = 0$ for all $s > t$ (both from $\{1, \ldots, \operatorname{rk} \mathcal{C}_\bullet\}$);

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

2) all the Bruhat numbers of $(\mathcal{C} \otimes \mathbb{Q}, \varkappa)$ equal to ± 1;
3) the $\mathbb{Z}$-enhanced complex $(\mathcal{C}, \varkappa)$ is isomorphic (in the sense of item 1) in the list from Subsection 2.6) to the direct sum of complexes of two forms:
    1) $0 \to \mathbb{Z} \to 0$ and
    2) $0 \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \to 0$.

2.8 Taking B-data commutes with taking sub- and quotient complexes

Given an enhanced complex $(\mathcal{C}, \varkappa)$ and two integers $0 \leqslant l \leqslant m \leqslant \dim \mathcal{C}_\bullet$, consider a quotient complex $\mathcal{C}^m/\mathcal{C}^l$; it inherits an enhancement. The goal of this subsection is to provide a recipe on how to express B-data of this complex in terms of the initial one. To this aim we, first, describe this recipe and, second, prove that it is correct.

Let us fix the notations first. The aforementioned enhanced complex will be denoted as $(\mathcal{C}|_l^m, \varkappa|_l^m)$. Let (U, L, H, b, λ) be a B-data associated with $(\mathcal{C}, \varkappa)$. Given l and m, we will now define another B-data $(U', L', H', b', \lambda')$.

Set $U' := \{s \in U \mid l < s \leqslant m, b(s) > l\} - l$ (by convention, subtracting an integer l from a subset of integers yields another subset formed by differences with l of each element individually), $L' := \{s \in L \mid l < s \leqslant m, b^{-1}(s) \leqslant m\} - l$, $H' := \{1, \dots, m-l\} \setminus (U' \sqcup L')$. Define the grading on U' via its injection into U. Proceed similarly for L' and H'. For $s \in U'$, define

1) $b'(s) := b(s+l) - l$,
2) $\lambda'(s) := \lambda(s+l)$.

Proposition 2.27. Let $(\mathcal{C}, \varkappa)$ be an enhanced complex and (U, L, H, b, λ) be its B-data. For a given $0 \leqslant l \leqslant m \leqslant \dim \mathcal{C}_\bullet$, the B-data of $(\mathcal{C}|_l^m, \varkappa|_l^m)$ coincides with the data $(U', L', H', b', \lambda')$ constructed above.

Proof. Take Barannikov basis of $(\mathcal{C}, \varkappa)$ that exists by Theorem 2.17. Its elements with indices from $l+1$ to m, when mapped to $(\mathcal{C}|_l^m, \varkappa|_l^m)$, again form a Barannikov basis. The statement follows. ■

Remark 2.28. Informally, the B-data of $(\mathcal{C}|_l^m, \varkappa|_l^m)$ is obtained from that of $(\mathcal{C}, \varkappa)$ by the simplest procedure possible: one has to cut it from below and above at the given levels l and m.

Remark 2.29. Although usage of Theorem 2.17 makes the proof shorter, it is still possible to prove the above statement directly from the definitions.

2.9 Torsion of a chain complex

In this subsection, we recall Milnor's [34] definition of torsion of a based chain complex, closely following [42].

Let $v = (v_1, \ldots, v_{\dim V})$ and $v' = (v'_1, \ldots, v'_{\dim V})$ be two bases of a vector space V. Denote by $[v'/v] \in \mathbb{F}^*$ the determinant of a transition matrix from v to v'. We call two bases v and v' equivalent if $[v'/v] = 1$. Let us now be given an exact triple of vector spaces $0 \to U \xrightarrow{\iota} V \xrightarrow{\pi} W \to 0$ along with bases u and w of U and W, respectively. Construct a basis uw of V as follows. For a vector $w_i \in W$, set $\pi^{-1}(w_i) \in V$ to be any lift w.r.t. π. Now set $uw := (\iota(u_1), \ldots, \iota(u_{\dim U}), \pi^{-1}(w_1), \ldots, \pi^{-1}(w_{\dim W}))$. Equivalence class of uw is independent of chosen lifts of w_is.

Recall that for a chain complex $(\mathcal{C}, \partial)$, one defines boundaries B_k to be $\operatorname{Im} \partial_{k+1}$ and cycles Z_k to be $\operatorname{Ker} \partial_k$. One then has two exact triples:

$$0 \to B_k \to Z_k \to \mathsf{H}_k \to 0, \tag{1}$$

$$0 \to Z_k \to C_k \xrightarrow{\partial_k} B_{k-1} \to 0. \tag{2}$$

Let us now be given bases c_k of C_k and h_k of H_k (for all admissible k). Choose any basis b_k of B_k. Construct, first, a basis $b_k h_k$ of Z_k via triple (1), and, second, a basis $b_k h_k b_{k-1}$ of C_k via triple (2). Define the torsion of $\mathcal{C}$ to be

$$\tau(\mathcal{C}) := \prod_{k=0}^{\dim \mathcal{C}} [b_k h_k b_{k-1}/c_k]^{(-1)^{k+1}} \in \mathbb{F}^*.$$

It is straightforward to show that $\tau(\mathcal{C})$ depends only on $(\mathcal{C}, \partial)$, equivalence class of c_k and that of h_k (see [42]).

Remark 2.30. If one replaces basis c_k with c'_k for some particular k then the torsion gets multiplied by $[c'_k/c_k]^{(-1)^{k+1}}$. The same goes for h_k.

The rest of this subsection is devoted to the definition of torsion in terms of determinant lines. It is only needed in Subsection 4.4

First, a couple of definitions from linear algebra. For a nonzero vector space V, define its determinant line as $\det V := \Lambda^{\dim V} V$; the determinant line of a zero vector space is defined to be $\mathbb{F}$. For an exact triple of vector spaces $0 \to U \to V \to W \to 0$, one can show the existence of a canonical isomorphism $\det V \simeq \det U \otimes \det W$. A basis

$v = (v_1, \dots, v_{\dim V})$ of V gives rise to a nonzero vector $v_1 \wedge \dots \wedge v_{\dim} \in \det V$, which we denote by the same letter v, abusing the notation. Note that now the expression $[v'/v]$ can be viewed as a determinant of a transition matrix between two bases of $\det V$; this agrees with the definition given above.

Now, for a chain complex $(\mathcal{C}, \partial)$, define its determinant line as

$$\det \mathcal{C} := \bigotimes_{k=0}^{\dim \mathcal{C}} \det C_k^{(-1)^{k+1}},$$

where by C_k^{-1}, we mean the dual vector space C_k^*. One similarly defines the determinant line $\det \mathsf{H}_*(\mathcal{C})$ of homology by treating homology as a complex with trivial differential. By playing with exact triples (1) and (2), one can show the existence of a canonical isomorphism $\xi\colon \det \mathcal{C} \simeq \det \mathsf{H}_*(\mathcal{C})$.

We will now utilize the bases c_k of C_k and h_k of H_k. By taking dual basis when necessary, we obtain bases (i.e., nonzero vectors) of $\det C_k^{(-1)^{k+1}}$ for each k. Consequently, we get a basis c of $\det \mathcal{C}$. Similarly, we get a basis h of $\det \mathsf{H}_*(\mathcal{C})$. One can show that $\tau(\mathcal{C}) = [h/\xi(c)]$.

2.10 Torsion of an enhanced complex

In this subsection, we define and study torsion of an enhanced complex. In particular, we calculate it in terms of B-data.

We continue using notations introduced in Subsection 2.9. Any two bases of a given enhanced vector space $(V, \varkappa)$ are equivalent. Analogously, for any two bases c and c' of an enhanced complex $(\mathcal{C}, \varkappa)$, one has equivalence between $\underline{c}_k$ and $\underline{c'}_k$, where by $\underline{c}_k$ we mean (here and further) an ordered subset of c corresponding to a basis of $(C_k, \varkappa_k)$ (see Remark 2.2).

Let us now assemble all the pieces together. Let an enhanced complex $(\mathcal{C}, \varkappa)$ be given. Choose any basis c of $(\mathcal{C}, \varkappa)$. Recall that by Subsection 2.2, we have an enhanced vector space $(\mathsf{H}_k(\mathcal{C}), \varkappa_{\mathsf{H}_k})$ for each k. Choose any basis h_k of $(\mathsf{H}_k(\mathcal{C}), \varkappa_{\mathsf{H}_k})$. Define the torsion of an enhanced complex $(\mathcal{C}, \varkappa)$ to be the torsion of $\mathcal{C}$ w.r.t. bases $\underline{c}_k$ and h_k; denote it by $\tau(\mathcal{C}, \varkappa)$. This number is well defined since equivalence classes of both $\underline{c}_k$ and h_k are well defined.

Remark 2.31. We stress out that $\tau(\mathcal{C}, \varkappa)$ depends only on the enhancement on $\mathsf{H}_\bullet(\mathcal{C})$. Bruhat numbers, however, depend on enhancement on $\mathsf{H}_\bullet(\mathcal{C}^s)$ for various s.

Remark 2.32. Some kind of interplay between filtration and torsion also appears in [23, Appendix A].

Construction 2.33. Let (N, U, L, H, b) be a part of a B-data. We will now construct a permutation σ of N elements.

$\triangleright$Note that b does not have anything to do with b_i from the definition of torsion. For a fixed k, the set U determines a subset of a set $\{1, \dots, \#\{s \in \{1, \dots, N\} \mid \deg s = k\}\}$, call it U_k. Define L_k and H_k similarly; the map b determines a bijection $b_k : U_k \to L_{k-1}$. We will now define a permutation σ_k on $\#\{s \in \{1, \dots, N\} \mid \deg s = k\}$ elements by writing integers in a row. First, write down elements of L_k in increasing order. Second, write down elements of H_k also in increasing order. Third, write down elements of U_k, but this time in the order of increasing of $b_k(s)$, for $s \in U_k$.

For two permutations σ and π of length l and m, their (direct) sum $\sigma + \pi$ is defined as a permutation of $l + m$ elements acting as σ on the 1st l elements and as π on the last m elements. We define σ to be the sum $\sigma_0 + \ldots + \sigma_n$, where $n = \max_{s \in \{1, \dots, N\}} \deg s$. $\triangleleft$

The sign of a permutation σ will be denoted as $(-1)^\sigma$.

Proposition 2.34. Let $(\mathcal{C}, \varkappa)$ be an enhanced complex and (U, λ) be a part of its B-data. Let also σ be the permutation from Construction 2.33. We then have

$$\tau(\mathcal{C}, \varkappa) = (-1)^\sigma \prod_{s \in U} \lambda(s)^{(-1)^{\deg s}}.$$

Proof. Since any two bases of an enhanced vector space are equivalent, we may calculate $\tau(\mathcal{C}, \varkappa)$ in some Barannikov basis c, which exists by Theorem 2.17. We continue using notations introduced in the beginning of this subsection. Choose basis h_k to be c_s for all $s \in H$. Similarly, choose basis b_k to be c_s for all $s \in L$ (the linear order on both bases is induced from that on c). These choices yield a right-hand side by the very definitions. ∎

Remark 2.35. In Remark 1.8, we associated a number d to any map of enhanced vector spaces. It is straightforward to show that $\tau(\mathcal{C}, \varkappa)$ is equal (up to a sign) to the alternating product of such numbers for all maps $\partial_k \colon C_k \to C_{k-1}$ (see Remark 2.2). Here, the word "alternating" means that one has take d^{-1} instead of d if k is odd.

The rest of this subsection is devoted to discussing torsion of an enhanced complex in terms of determinant lines. It is only needed in Subsection 4.4. We continue using notations introduced in Subsection 2.9.

Choose any basis of enhanced complex $(\mathcal{C}, \varkappa)$, and consider the corresponding element $c \in \det \mathcal{C}$. Obviously, c does not depend on the chosen basis. Next, by Subsection 2.2, we have an enhanced vector space $(\mathsf{H}_k(\mathcal{C}), \varkappa_{\mathsf{H}_k})$ for each k. Choose any basis of $(\mathsf{H}_k(\mathcal{C}), \varkappa_{\mathsf{H}_k})$ for each k, and consider the corresponding element $h_\varkappa \in \det \mathsf{H}_*(\mathcal{C})$. Obviously, $h_\varkappa$ does not depend on the chosen bases (but it does depend on the enhancement $\varkappa$ on $\mathcal{C}$). The next proposition now follows straightforwardly from the discussion in Subsection 2.9.

Proposition 2.36. Let $(\mathcal{C}, \varkappa)$ be an enhanced complex. Let also $c \in \det \mathcal{C}$ and $h_\varkappa \in \det \mathsf{H}_*(\mathcal{C})$ be two elements constructed from it as above. Then,

$$\xi(c) = \frac{h_\varkappa}{\tau(\mathcal{C}, \varkappa)},$$

where $\xi\colon \det \mathcal{C} \simeq \det \mathsf{H}_*(\mathcal{C})$ is a canonical isomorphism.

3 Morse Theory

In the 1st part of this section, we introduce, after necessary preparations, a construction that associates an enhanced complex over $\mathbb{F}$ with a strong Morse function (see Subsection 3.5). This justifies a thorough study of enhanced complexes in the previous section. We then proceed to discuss various properties of Bruhat numbers of a given strong Morse function. The majority of results translates readily to the setting of discrete Morse theory in a sense of Forman [18]; as in the smooth case, the strongness assumption on a function is crucial and must be satisfied.

3.1 Setup

In this subsection, we recall basic notions of Morse theory and fix appropriate notations, setting the stage for our results.

Let M be a smooth closed manifold fixed once and for all throughout this section. Recall that a smooth function $f\colon M \to \mathbb{R}$ is called Morse if all its critical points are nondegenerate. A smooth function is called strong if all its critical points have different critical values. Fix a strong Morse function f on M once and for all throughout this section. For $a \in \mathbb{R}$, the subspace $M^a := \{x \in M \mid f(x) \leqslant a\}$ is called a sublevel set.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

Morse's idea was to track how the homotopy type of M^a changes while a grows from $-\infty$ to $+\infty$. This is performed by investigating the critical points of f, the set of which is denoted by $\mathrm{Cr}(f) \subset M$. Since f is strong those are in bijection with critical values of f (this set is finite because of the compactness of M). Keeping this bijection in mind, we will freely switch between points and values without mentioning this explicitly. The set $\mathrm{Cr}(f)$ is $\mathbb{Z}_{\geqslant 0}$-graded by index of a critical point, the degree of $c \in \mathrm{Cr}(f)$ is denoted by $\deg c$. Though itis more natural to say "index of critical point", we will mostly say "degree" in order to be consistent with Section 2. The set of all critical points of degree k is denoted by $\mathrm{Cr}_k(f)$. Note that the set $\mathrm{Cr}(f)$ is also naturally linearly ordered; we denote by $c_s \in \mathrm{Cr}(f)$ (for $s \in \{1, \ldots, \#\,\mathrm{Cr}(f)\}$) its s^{th} element w.r.t. this order. By ε, we will mean a sufficiently small positive real number.

It follows from foundational results of Morse theory, which we recall in Subsection 3.4 that for $c \in \mathrm{Cr}(f)$, one has $\mathsf{H}_{\deg c}(M^{f(c)+\varepsilon}, M^{f(c)-\varepsilon}; \mathbb{Z}) \simeq \mathbb{Z}$. We say that a critical point is oriented if the generator of this free abelian group of rank one is chosen. A strong Morse function is called oriented if all its critical points are oriented (see Subsection 3.3 for a discussion).

It will be convenient to fix the set of real numbers r_s (for $s \in \{0, \ldots, \#\,\mathrm{Cr}(f)\}$) s.t.

$$r_0 < f(c_1) < r_1 < \ldots < f(c_{\#\,\mathrm{Cr}(f)}) < r_{\#\,\mathrm{Cr}(f)}.$$

Such numbers are called regular values.

Fix a field $\mathbb{F}$ once and for all. All the chain complexes and homologies are assumed to be over $\mathbb{F}$ unless stated otherwise. If the group of coefficients is given explicitly, it goes after a semicolon, for example, $\mathsf{H}_2(M; \mathbb{Z})$.

3.2 B-Data associated with a strong Morse function

In this subsection, we present and discuss the following construction.

Construction 3.1. Let f be an oriented strong Morse function on M and $\mathbb{F}$ be a field. We will now construct a B-data.

$\triangleright$Since $\mathrm{Cr}(f)$ is linearly ordered, it is in natural bijection with $\{1, \ldots, N\}$; define the grading on the latter by that on the former. The manifold M is filtered, as a topological space, by subspaces

$$\varnothing = M^{r_0} \subset M^{r_1} \subset \ldots \subset M^{r_N} = M.$$

Moreover, for each $s \in \{1,\dots,N\}$, the space M^{r_s} is homotopy equivalent to $M^{r_{s-1}}$ with a single cell of dimension $\deg s$ attached. Since f is oriented, the one-dimensional vector space $\mathsf{H}_{\deg s}(M^{r_s}, M^{r_{s-1}}) \simeq \mathsf{H}_{\deg s}(M^{r_s}, M^{r_{s-1}}; \mathbb{Z}) \otimes \mathbb{F} \simeq \mathbb{F}$ has a preferred basis $o \otimes 1$, where o is a generator of $\mathsf{H}_{\deg s}(M^{r_s}, M^{r_{s-1}}; \mathbb{Z}) \simeq \mathbb{Z}$. Therefore, the complex of singular chains on M (with coefficients in $\mathbb{F}$) satisfies conditions of Construction 2.6 and we are able to extract a B-data as in Subsection 2.4 (see Remark 2.10). $\triangleleft$

In particular, we have just constructed enhancement on a homology vector space $\mathsf{H}_\bullet(M)$ as well as on $\mathsf{H}_\bullet(M^{r_s})$ for all s. We call the image of λ "Bruhat numbers" of oriented Morse function; the same goes for the Barannikov pairs (or, briefly, pairs). Informally, Construction 3.1 decomposes some critical points (equivalently, values) of f into Barannikov pairs. Moreover, it associates a Bruhat number (i.e., an element of $\mathbb{F}^*$) with each pair (see Subsection 2.4). Points $c_s \in \mathrm{Cr}(f)$ s.t. $s \in H$ are called homological critical points. The number of homological points of index k equals $\dim \mathsf{H}_k(M)$ (see Subsection 2.4). Analogously, points from U (resp. L) are called upper (resp. lower). We stress out that we have not yet considered any finite-dimensional approximation of filtered complex of singular chains on M; we will do so in Subsections 3.4 and 3.5.

Remark 3.2. Changing the orientation of some critical point $c_s \in \mathrm{Cr}(f)$ alters the B-data as follows. First, the decomposition into pairs stays the same. Second, if c_s is homological, then the whole B-data stays the same. Otherwise, if c_s belongs to some pair, then the Bruhat number on this pair gets multiplied by -1. Therefore, canonically we can associate Bruhat numbers to pairs only up to a sign.

Remark 3.3. We will now present an alternative way to associate B-data with an oriented strong Morse function (and a field). This way is quick and does not make use of any sort of enhancement whatsoever. Let x and y be two critical points s.t. $f(x) > f(y)$ and $\operatorname{ind} x - 1 = \operatorname{ind} y = k$. Consider the fundamental class of the attaching sphere for x, it lives in $\mathsf{H}_k(M^{f(x)-\varepsilon})$. Let X be its image under the natural map $\mathsf{H}_k(M^{f(x)-\varepsilon}) \to \mathsf{H}_k(M^{f(x)-\varepsilon}, M^{f(y)-\varepsilon})$. Consider now an attaching disk for y. It has a relative fundamental class, which lives in $\mathsf{H}_k(M^{f(y)+\varepsilon}, M^{f(y)-\varepsilon})$. Let Y be its image under the natural map $\mathsf{H}_k(M^{f(y)+\varepsilon}, M^{f(y)-\varepsilon}) \to \mathsf{H}_k(M^{f(x)-\varepsilon}, M^{f(y)-\varepsilon})$ induced by inclusion. Critical points x and y form a Barannikov pair with Bruhat number λ if and only if $X = \lambda Y \neq 0$. An illustration for $k = 1$ is given in Figure 2.

This is a reformulation of Remark 2.12 in the geometric setup.

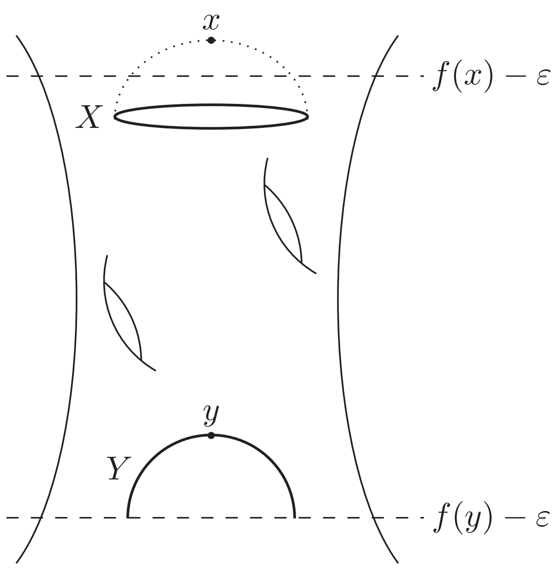


Fig. 2. To Remark 3.3. Classes X and Y are drawn in bold. Dotted dome depicts an attaching 2-disk for x.

A close idea of construction of Bruhat numbers over $\mathbb{Q}$ appeared independently in [30]; in particular, considerations from Remark 3.3 are present there.

Remark 3.4. The original Barannikov's construction [4] produces the same set of pairs. To see this, one should combine Remark 2.18 and results from Subsection 3.5. See also [16] for a topological data analysis perspective.

We conclude this subsection by several remarks. The construction of homological critical points goes back to Lyusternik and Shnirelman. Note that this construction implies weak Morse inequalities: $\#\operatorname{Cr}_k(f) \geqslant \dim \mathsf{H}_k(M;\mathbb{F})$ for any field $\mathbb{F}$. See also [44] for the innovative fruitful applications of similar ideas in symplectic topology.

Remark 2.14 translates to topological setting as follows. The condition for two critical values a and b to form a Barannikov pair is equivalent to

$$\begin{aligned}\dim \mathsf{H}_\bullet(M^{a-\varepsilon}, M^{b+\varepsilon}) = \dim \mathsf{H}_\bullet(M^{a+\varepsilon}, M^{b-\varepsilon}) &= \dim \mathsf{H}_\bullet(M^{a-\varepsilon}, M^{b-\varepsilon}) - 1\\ &= \dim \mathsf{H}_\bullet(M^{a+\varepsilon}, M^{b+\varepsilon}) - 1.\end{aligned}$$

See [13] and also [31].

3.3 A digression on oriented Morse functions

In this subsection, we give an alternative definition of an oriented Morse function (see Subsection 3.1).

Let Q be a quadratic form on a vector space V over $\mathbb{R}$. Consider a vector subspace $L \subset V$ of maximal dimension (this dimension is called negative inertia index of Q) such that the restriction $Q|_L$ is negative-definite. One can show that the space $\{L\}$ of all such vector subspaces is contractible. Therefore, since any covering of a contractible space is trivial, the space of all such oriented subspaces has two contractible components. Note that a particular choice of a component determines an orientation of L (as a vector space).

Now, given a Morse function f and any its critical point p, one may consider a vector space T_pM and a Hessian $\mathrm{Hess}_p(f)$ on it. Function f is called oriented if, for every critical point p, one of the two mentioned components is chosen. One may check that there is a canonical correspondence between this definition and the one given in Subsection 3.1.

3.4 CW-Complex associated with a strong Morse function

In this subsection, we briefly recall the classical results from Morse theory, following [33], in order to fix the notations needed further in Subsection 3.5.

Theorem 3.5. Let f be an oriented strong Morse function on a closed manifold M. Then there exists a CW-complex K s.t. the following holds:

1) M is simple homotopy equivalent to K;
2) cells of K are in bijection with critical points of f. Moreover, dimension of a cell equals the index of a critical point.

Remark 3.6. The CW-complex K need not be unique (see Construction 3.12), but its simple homotopy type obviously is. Roughly speaking, the purpose of Subsection 3.5 is to encode the information that can be extracted uniquely from f in algebraic terms. Note that Morse theory originated before CW-complexes were invented; see [36].

Remark 3.7. Orientations of cells in K may be naturally chosen by invoking orientation of f; see Construction 3.9.

Remark 3.8. The fact that the mentioned homotopy equivalence (as well as all the others in this subsection) is actually simple is folklore; see [38, Theorem 3.8] for a proof.

We will need this fact in Section 4 for statements involving (Reidemeister) torsion. We will denote general homotopy equivalence by $\simeq$ and, whenever we want to emphasize that it is simple, we write $\overset{s}{\simeq}$.

The key ingredient in the proof of Theorem 3.5 is the following construction (we continue using notations introduced in Subsection 3.1). (For a topological space X, by $X \cup_\varphi e^k$, we mean X with a k-cell attached along φ.)

Construction 3.9. For $s \in \{1, \dots, \#\operatorname{Cr}(f)\}$, let r_s and r_{s-1} be two corresponding regular values of f, and let $k = \deg s$. We will now recall the construction of a continuous map $\varphi : S^{k-1} \to M^{r_{s-1}}$ s.t. $M^{r_s} \overset{s}{\simeq} M^{r_{s-1}} \cup_\varphi e^k$. NB: this construction involves some choices.

▷We will only sketch the argument; for details, see [33]. The space $M^{r_{s-1}}$ is a smooth manifold with boundary $f^{-1}(r_{s-1})$. Choose an antigradient-like vector field on M. Its flow produces a smooth map $S^{k-1} \to f^{-1}(r_{s-1}) \subset M^{r_{s-1}}$, where the source is viewed as a small sphere around critical point c_s. For a different gradient-like vector field, the resulting map will differ by an isotopy. This way, one gets an embedding $M^{r_{s-1}} \cup_\varphi e^k \hookrightarrow M^{r_s}$. It is then shown to be a simple homotopy equivalence.

Note that since the function f is oriented, the sphere S^{k-1} is oriented as well. Thus, the cell e^k is oriented too. ◁

The next proposition is obvious.

Proposition 3.10. Although the map φ from Construction 3.9 depends on some choices made along the way, its homotopy class is uniquely defined.

Remark 3.11. We stress out that it is not claimed that the homotopy class of a map φ satisfying the property that $M^{r_s} \overset{s}{\simeq} M^{r_{s-1}} \cup_\varphi e^k$ is unique. This assertion is only true for the map φ constructed by the recipe given above.

In order to get to Theorem 3.5, one then proceeds with the following construction.

Construction 3.12. Let s, r_s, r_{s-1} and k be as in Construction 3.9. Suppose that $M^{r_{s-1}} \overset{s}{\simeq} K$, where K is some CW-complex. We will now recall the construction of a cellular map $\psi : S^{k-1} \to K$ s.t. $M^{r_s} \overset{s}{\simeq} K \cup_\psi e^k$ (note that r.h.s. is again a CW-complex). NB: this construction involves some choices.

▷Apply cellular approximation theorem to the map φ from Construction 3.9. ◁

Again, the next proposition is obvious.

Proposition 3.13. Although the map ψ from Construction 3.12 depends on some choices made along the way, its homotopy class is uniquely defined.

3.5 Enhanced complex associated with a strong Morse function

In this subsection, we discuss the following two statements.

Construction 3.14. Let f be an oriented strong Morse function on M and $\mathbb{F}$ be a field. We will now construct an enhanced complex $(\mathcal{C}, \varkappa)$. NB: this construction involves some choices.

Proposition 3.15. Although Construction 3.14 depends on some choices made along the way, the isomorphism class of an enhanced complex $(\mathcal{C}, \varkappa)$ is uniquely defined.

We will first describe Construction 3.14.

$\triangleright$ Take any CW-complex K constructed by the virtue of Theorem 3.5. Its cells are naturally linearly ordered by the order of critical values of f. Moreover, the 1st s cells form a CW-subcomplex, simple homotopy equivalent to M^{r_s}. Consider an algebraic complex $\mathcal{C}'$ (of free abelian groups) associated with K. It has a preferred (ordered) basis c given by cells (mind that they carry natural orientation since f is oriented; see Construction 3.9). Thus, one has a $\mathbb{Z}$-enhanced complex $(\mathcal{C}', \varkappa(c))$.

The desired enhanced complex is now taken to be $(\mathcal{C}' \otimes \mathbb{F}, \varkappa(c))$ (see Subsections 2.4 and 2.6). $\triangleleft$

Remark 3.16. Construction 3.14 produces not only an enhanced complex but also its basis. The matrix of differential, however, does depend on the choices made (practically, a cellular approximation from Construction 3.12). From this viewpoint, Proposition 3.15 says that for any two choices, the corresponding matrices of differential are conjugate by an unitriangular (i.e., triangular with ones on the diagonal) base change. This can be pushed to the full proof; see Remark 3.18.

Proof of Proposition 3.15. By the classificational Theorem 2.17, it suffices to prove that B-data associated with $(\mathcal{C}, \varkappa)$ is uniquely defined (see Remark 2.18). We will do that by identifying it with a B-data constructed from f in Subsection 3.2. We denote the complex of singular chains (over $\mathbb{F}$) mentioned there by $\mathcal{C}_{\text{sing}}$ and by $\mathcal{C}^s_{\text{sing}}$ we mean the subcomplex corresponding to subspace M^{r_s}. Recall that since f is oriented the generator of $\mathsf{H}_{\deg s}(\mathcal{C}^s_{\text{sing}}, \mathcal{C}^{s-1}_{\text{sing}}) \simeq \mathbb{F}$ is chosen (see Subsection 3.2).

First of all, the two gradings on the set $\{1, \ldots, \#\,\mathrm{Cr}(f)\}$ coincide since they are defined in terms of indices of critical points of f. Next, for each s homology vector spaces, $\mathsf{H}_k(\mathcal{C}^s)$ and $\mathsf{H}_k(\mathcal{C}^s_{\mathrm{sing}})$ are naturally isomorphic (for all k) since both of them compute $\mathsf{H}_k(M^{r_s})$. Moreover, their filtrations are the same (w.r.t. the mentioned isomorphism) since they are defined topologically in terms of inclusions $M^{r_t} \hookrightarrow M^{r_s}$ for various t. Finally, the chosen generators in $\mathsf{H}_{\deg s}(\mathcal{C}^s_{\mathrm{sing}}, \mathcal{C}^{s-1}_{\mathrm{sing}})$ and in $\mathsf{H}_{\deg s}(\mathcal{C}^s, \mathcal{C}^{s-1})$ are the same for the same reason: they are defined topologically as a certain element in $\mathsf{H}_{\deg s}(M^{r_s}, M^{r_{s-1}})$. This implies that enhancements on $\mathsf{H}_k(\mathcal{C}^s)$ and $\mathsf{H}_k(\mathcal{C}^s_{\mathrm{sing}})$ are the same. The statement now follows, since B-data was extracted in Subsection 2.4 in terms of enhancements on $\mathsf{H}_k(\mathcal{C}^s)$ for various k and s. ∎

Remark 3.17. It follows from Subsection 3.4 that B-data associated $(\mathcal{C}, \varkappa)$ coincides with the one constructed in Subsection 3.2. It follows from Remark 1.9 that it is enough to consider only fields $\mathbb{Q}$ and $\mathbb{F}_p$ (note that this is not the case in Subsections 4.2 and 4.3).

Remark 3.18. It is possible to prove Proposition 3.15 somewhat more explicitly without appealing to Theorem 2.17. We will now sketch the argument. Consider two approximations ψ_1 and ψ_2 from Construction 3.12. Since they are homotopic, corresponding algebraic complexes associated with CW-complexes differ by a change of basis. This change of basis is unitriangular by construction. Arguing inductively on the number of cells, one obtains a unitriangular change of basis that turns one chain complex into another. This means precisely that two corresponding enhanced complexes are isomorphic.

Remark 3.19. B-data stays unaltered if one replaces f with $\varphi \circ f \circ \psi$, where $\psi: M \to M$ is any diffeomorphism and $\varphi: \mathbb{R} \to \mathbb{R}$ is a diffeomorphism preserving an orientation. Consequently, B-data stays the same under the continuous deformation of f in the class of oriented strong Morse functions on M (see Subsection 5.1).

3.6 Invariant of a map between two manifolds equipped with Morse functions

In this subsection, we discuss the following construction.

Construction 3.20. Let M_1 and M_2 be two manifolds equipped with oriented strong Morse functions f_1 and f_2, respectively. Let also $l: M_1 \to M_2$ be a continuous map. We will now construct a rook matrix R_k of size $\dim \mathsf{H}_k(M_2) \times \dim \mathsf{H}_k(M_1)$ (for any k).

▷By Subsection 3.2, the function f_1 (resp. f_2) gives rise to an enhancement on a homology vector space $\mathsf{H}_k(M_1)$ (resp. $\mathsf{H}_k(M_2)$). Consider the induced map $l_k: \mathsf{H}_k(M_1) \to \mathsf{H}_k(M_2)$ and plug it into Lemma 1.7 to get the desired rook matrix. ◁

Without choosing a particular orientation of strong Morse functions the nonzero entries of R_k are defined only up to a sign. Note that we did not make use of any complexes whatsoever.

3.7 Morse complex

In this subsection, we describe how Morse complex fits into our setting of enhanced complexes. In particular, we give a certain description of a matrix of Morse differential (w.r.t. any Riemannian metric) in terms of B-data.

Let g be a generic Riemannian metric on M and f be an oriented strong Morse function. Then one can define a Morse complex $\mathcal{M}(f,g)$ whose integral homology is naturally isomorphic to that of M. Itis a complex of free abelian groups, formally generated by critical points. In this basis, the differential (mapping k-chains to $(k-1)$-chains) becomes a matrix $(\partial_{i,j})$. The matrix element $\partial_{i,j}$ equals the number of antigradient flowlines from j^{th} critical point of index k to i^{th} critical point of index $k-1$, counted with appropriate signs. For brevity, we say that i^{th} critical point "appears in the differential" of j^{th} critical point with coefficient $\partial_{i,j}$.

Remark 3.21. As we saw in Subsection 3.4, in order to construct a complex generated by critical points out of a function f, one has to make a choice of cellular approximation. On the other hand, in order to construct a Morse complex, one has to choose a Riemannian metric. These two choices are actually very close to each other in the following sense.

Given a function f, one can construct a handle decomposition s.t. each handle of index k corresponds to some critical point of index k. The metric g specifies the way to modify this decomposition, such that each handle of index k is attached to the union of handles of smaller index. This, in turn, allows one define the handle complex, which is precisely the Morse complex. On the other hand, one can collapse handles to cells and obtain a CW-complex.

In our terms, one has obtained a $\mathbb{Z}$-enhanced complex $(\mathcal{M}(f,g),\varkappa(c))$, where c is a basis consisting of critical points (see Subsection 2.6). It follows directly that B-data of $(\mathcal{M}(f,g)\otimes\mathbb{F},\varkappa(c))$ coincides with that of f constructed in Subsection 3.2. Therefore, by Theorem 2.17, enhanced complex $(\mathcal{M}(f,g)\otimes\mathbb{F},\varkappa(c))$ is isomorphic to $(\mathcal{C},\varkappa)$ from Subsection 3.5. So, practically, one may use any option to extract B-data from a given f: either a complex of singular chains, or a CW-complex or a Morse complex.

Remark 3.22. Analogously to Remark 3.16, the matrix of Morse differential does depend on Riemannian metric, while the isomorphism class of enhanced complex $(\mathcal{M}(f,g)\otimes\mathbb{F},\varkappa(c))$ does not. The same goes for $\mathbb{Z}$-enhanced complex $(\mathcal{M}(f,g),\varkappa(c))$.

For a B-data associated with f, let R_k be the corresponding rook matrix of size $\mathrm{Cr}_{k-1}(f)\times\mathrm{Cr}_k(f)$ (see Subsection 2.3). In other words, nonzero elements of R_k equal to the Bruhat numbers on Barannikov pairs of points of degrees k and $k-1$. The next theorem follows readily from Proposition 2.22.

Theorem 3.23. Let f be an oriented strong Morse function on a manifold M. Let also R_k be the rook matrix associated with f over $\mathbb{Q}$ (for $k\in\{1,\dots,\dim M\}$). Then the matrix of Morse differential ∂_k w.r.t. any Riemannian metric g belongs to the set $\mathcal{T}(R_k)$.

For example, suppose that f has a B-data as depicted in Figure 1 and $k=2$. Then the corresponding rook matrix and general form of a matrix of a 2nd Morse differential P are

$$R_2=\begin{pmatrix}0&0&4\\3&0&0\\0&0&0\\0&2&0\end{pmatrix},P=\begin{pmatrix}*&*&*\\3&*&*\\0&*&*\\0&2&*\end{pmatrix}.$$

Recall that weak Morse inequalities state that $\#\,\mathrm{Cr}_k(f)\geqslant\dim\mathsf{H}_k(M;\mathbb{F})$ (for any field $\mathbb{F}$). It is easy to see that if $\#\,\mathrm{Cr}(f)=\sum_k\dim\mathsf{H}_k(M;\mathbb{Q})$, then the Morse differential (w.r.t. any metric) must be identically zero. The next corollary is applicable when this is not the case.

Corollary 0.1. Let f be an oriented strong Morse function on M. Suppose that $\#\,\mathrm{Cr}(f)>\sum_k\dim\mathsf{H}_k(M;\mathbb{Q})$. Then one can find two critical points x and y of neighboring indices s.t. the number of antigradient flowlines from x to y, counted with appropriate signs, is the same for any generic Riemannian metric. This number is nonzero and equals some Bruhat number of f over $\mathbb{Q}$.

Proof. By assumption, there is at least one Barannikov pair of f over $\mathbb{Q}$. Take any short one. The statement now follows from Corollary 2.24. ■

Remark 3.24. The 1st part of Corollary 0.1 can be proven without appealing to any Barannikov pairs and Bruhat numbers whatsoever. Indeed, if one unwraps all the

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

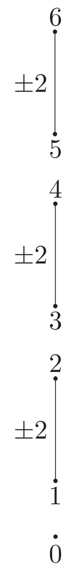


Fig. 3.

definitions and constructions involved, they arrive at a proof that is based on techniques like exact sequences of a pair.

3.8 A few examples and properties

In this subsection, we quickly give several introductory examples and properties of Bruhat numbers.

1. Let f be a function on $\mathbb{RP}^n$ that descends from the function $x_1^2 + 2x_2^2 + \ldots + (n+1)x_{n+1}^2$ defined on a unit sphere $S^n \subset \mathbb{R}^{n+1}$. It has $(n+1)$ critical points of all possible indices from 0 to n (ordered by increasing of index). If $\operatorname{char} \mathbb{F} = 2$, then all of them are homological. Otherwise, $(2k)^{\text{th}}$ and $(2k-1)^{\text{th}}$ critical points form a Barannikov pair with Bruhat number ± 2 (for any $k \in \{1, \ldots, [n/2]\}$, where brackets denote the integral part). This is seen readily from any description of B-data, given either in Subsection 3.2 or in Subsection 3.5. See Figure 3 for an example for $n = 6$.

2. The proof of the next statement uses one-parameter Morse theory and therefore postponed until Subsection 5.2. See also [30] for other results concerning realizabilty of Bruhat numbers.

Proposition 3.25. Let $\mathbb{F}$ be either $\mathbb{Q}$ or $\mathbb{F}_p$ and $\lambda \in \mathbb{F}^*$ be any nonzero number. Let also M be any closed manifold s.t. $\dim M \geqslant 4$. Then one can find an oriented strong Morse function f on M that has λ as one of its Bruhat numbers.

In particular, Bruhat number over $\mathbb{F} = \mathbb{Q}$ may well be noninteger.

3. Recall that a Barannikov pair is called short if there are no pairs of the same degree lying inside it; see Subsection 2.6. The next statement follows directly from Corollary 2.23.

Proposition 3.26. Over $\mathbb{F} = \mathbb{Q}$, Bruhat number on any short pair is integer.

In particular, if there are Barannikov pairs over $\mathbb{Q}$ at all, at least one of them must carry integer Bruhat number.

4. The 1st statement from Subsection 2.7 translates straightforwardly to the topological setting via Subsection 3.5. In this setting, it expresses Bruhat numbers of oriented strong Morse function over $\mathbb{Q}$ in terms of torsion in (relative) integral homology of various sublevel sets.

3.9 Poincare duality

In this subsection, prove the following proposition.

Proposition 3.27. Let M be closed and orientable and f be an oriented strong Morse function on it. Let also $\mathbb{F}$ be a field. Then B-data for $-f$ is B-data for f turned upside down. Bruhat numbers on pairs remain the same.

We need to first make some comments on the formulation. Since M is orientable and f is oriented, the strong Morse function $-f$ is also naturally oriented. Each critical point of f of index k is also a critical point of $-f$ of index $n-k$, where $n = \dim M$. By turning the B-data upside down, we mean, formally speaking, precomposing all its ingredients with the automorphism of the set $\{1, \ldots, \#\mathrm{Cr}(f)\}$ given by $s \mapsto \#\mathrm{Cr}(f) - s$. Under this operation, upper and lower critical points swap their roles (while the pairing remains the same). Note that the classical Poincare duality over the field $\dim \mathsf{H}_k(M; \mathbb{F}) = \dim \mathsf{H}_{n-k}(M; \mathbb{F})$ follows immediately from Proposition 3.27. Indeed, homological points remain homological after the involution, but their indices change to complementary ones. The proof, however, goes along the classical lines.

Proof of Proposition 3.27. Choose a generic Riemannian metric g on M. Matrix of differential in a Morse complex $\mathcal{M}(-f, g)$ is obtained from that in $\mathcal{M}(f, g)$ by transposing. Therefore, the same goes for the matrices of differential in the two corresponding based enhanced complexes. Consider now any unitriangular matrix P_k

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

that maps the basis c (of critical points) of the 1st complex to a Barannikov basis. If follows from the formula $(P_{k-1}D_kP_k)^T = P_k^T D_k^T P_{k-1}^T$ that transposed matrix P_k^T (which is unitriangular w.r.t. reversed order) maps the initial basis of the 2nd complex to the Barannikov one. Moreover, the matrices of differentials after these two changes of bases still differ by taking the transpose. The statement follows. ■

3.10 On pairs of extremal degrees

Recall that by a degree of a pair we mean degree of its lower point. In this subsection, we prove the following proposition.

Proposition 3.28. Let f be a strong Morse function on M and $\mathbb{F}$ be a field.

1) The set of pairs of degree 0 is independent of $\mathbb{F}$. Bruhat number on any such pair is ± 1.
2) Suppose that M is orientable. Then the set of pairs of degree $\dim M - 1$ is again independent of $\mathbb{F}$. Bruhat number on any such pair is again ± 1.

Proof. Fix any $s \in \{1, \ldots, \#\operatorname{Cr}(f)\}$ s.t. $c_s \in U$ (i.e., c_s is an upper point in a pair) and $\deg s = 1$. We need to prove that $\lambda(s) = \pm 1$. Recall that $\{r_s\}$ is a set of regular values of f, and let $d = \dim \mathsf{H}_0(M^{r_{s-1}})$. Let also $i_1 < \ldots < i_d$ be the ordered sequence of numbers that comprise the set $\{t \in H^{s-1} | \deg t = 0\}$, where H^{s-1} is the set H from the B-data for the function $f|_{M^{r_{s-1}}}$. In other words, $(c_{i_1}, \ldots, c_{i_d})$ is the ordered set of homological critical points of degree zero for $f|_{M^{r_{s-1}}}$. It follows from Construction 2.6 that the sequence $([c_{i_1}], \ldots, [c_{i_d}])$ is a basis (up to signs, which are irrelevant to the statement) of an enhanced vector space $\mathsf{H}_0(M^{r_{s-1}})$ (square brackets denote taking the homology class).

On the other hand, the connecting homomorphism $\delta\colon \mathsf{H}_1(M^{r_s}, M^{r_{s-1}}) \to \mathsf{H}_0(M^{r_{s-1}})$ maps the chosen generator, represented by an oriented segment, to $[a] - [b]$, where a and b are the endpoints. The 1st statement now follows. The 2nd one is obtained from it via the Poincare duality (Proposition 3.27). ■

Remark 3.29. Orientability assumption in the 2nd statement of Proposition 3.28 is crucial. Indeed, the conclusion fails already for $M = \mathbb{RP}^2$ and $\mathbb{F} = \mathbb{Q}$; see Subsection 3.8.

Remark 3.30. Since the set of pairs from Proposition 3.28 is independent of the field $\mathbb{F}$, one is tempted to find an alternative definition that does not involve $\mathbb{F}$. Indeed, it can be proven that the mentioned set may be recovered from the Kronrod–Reeb [1, 39] graph of f (see, e.g., [5]). Moreover, the Kronrod–Reeb graph "sees more" than the set of pairs of extremal indices in a sense that one can find two functions on the same manifold with different Kronrod–Reeb graphs but identical mentioned sets of pairs. The reason is, roughly, that B-data keeps track of filtration on homology induced by f, while Kronrod–Reeb graph keeps track of a canonical basis in H_0 (or $\mathsf{H}_{\dim M}$) of the sublevel set.

4 Bruhat Numbers and the Theory of Torsions

As we saw in Proposition 3.25, any number may appear as a Bruhat number of some function; in a sense, there is no control over the individual Bruhat number. However, sometimes the alternating product of all these numbers turns out to be independent of f. Thus, this product depends only on the manifold M. In the present section, we make this statement precise (in Subsection 4.1) and provide a framework where the mentioned product of Bruhat numbers equals the Reidemeister torsion of M (in Subsection 4.3).

4.1 Torsion of a Morse function

In this subsection, we set the stage for the further results.

Definition 4.1. Let f be an oriented strong Morse function on M and $\mathbb{F}$ be a field. Let also σ be a permutation from Construction 2.33 (and $(-1)^\sigma$ its sign). The number

$$\tau(f, \mathbb{F}) = (-1)^\sigma \prod_{s \in U} \lambda(s)^{(-1)^{\deg s}} \in \mathbb{F}^*$$

is called the *torsion* of f over $\mathbb{F}$.

We refer to the r.h.s. as "alternating product" of all Bruhat numbers, in analogy with alternating sum, which is used to define Euler characteristic. Actually, torsion is very much similar to the Euler characteristic; see [35]. We will simply write τ when its ingredients are understood. We will now link Definition 4.1 with the classical notion of torsion (see a broad yet concise book [42] for an in-depth discussion).

By results from Subsection 3.4 starting from f, one can construct a (nonunique) CW-complex X which is simple-homotopy equivalent to M. So, from the viewpoint of torsion theory X and M are the same (see Subsection 4.2). Next, in Subsection 3.5, we used X to construct a (unique up to isomorphism) enhanced complex $(\mathcal{C}, \varkappa)$. Further, in Subsection 2.10, we studied the torsion (defined in a classical way) of any enhanced complex. By the very definitions, it is a topological torsion of X (and, therefore, M) w.r.t.

a certain basis in homology (see Remark 4.2). Finally, Proposition 2.34 identified the torsion (defined in a classical way) with the formula from Definition 4.1.

Remark 4.2. Here is a more concise way of saying the same. Take enhancement on a homology $\mathsf{H}_k(M)$ given by f. Take any basis of this enhancement. Any two such bases differ by a unitriangular matrix. Therefore, they are equivalent (in the sense of Subsection 2.9). Thus, the torsion of M w.r.t. any chosen basis of enhancement on $\mathsf{H}_k(M)$ is the same. This is precisely τ.

Remark 4.3. In this section, we will actually be interested only in the number $\pm\tau \in \mathbb{F}^*/\pm 1$. So the reader may temporarily disregard the permutation σ and orientation of f. The sign will be important later in Section 5.

Theorem 0.2. Let f be a strong Morse function on M and $\mathbb{F}$ be a field. Suppose that $\mathsf{H}_k(M) = 0$ for all $0 < k < \dim M$. Then the alternating product of all Bruhat numbers (as an element from $\mathbb{F}^*/\pm 1$) is independent of f.

The proof requires bifurcation analysis and therefore postponed until Subsection 5.5. For example, taking M to be $\mathbb{RP}^n$, one sees that $\tau(f, \mathbb{Q}) = \pm 2^{[n/2]}$, where brackets denote integral part. Indeed, one has to calculate such a τ for some particular Morse function on $\mathbb{RP}^n$. They do so for a standard one from Subsection 3.8.

Remark 4.4. Since in Theorem 0.2 the number $\pm\tau(f, \mathbb{F})$ turns out to be independent of f, one is tempted to give an alternative definition that does not involve f. We will now sketch the construction. Recall that the torsion is defined whenever there is a chosen basis in homology (or, at least, an equivalence class of such a basis). Usually, in topology, there is no canonical choice of such a basis, so one studies torsion in the setting where there is no homology at all. However, in $\mathsf{H}_0(M)$ and $\mathsf{H}_{\dim M}(M)$, there is an obvious distinguished basis. The number $\pm\tau(f, \mathbb{F})$ is the torsion w.r.t. it.

Remark 4.5. We will now provide the formula for the number $\pm\tau(f, \mathbb{F})$ from Theorem 0.2 in terms of homology of M over $\mathbb{Z}$. For an integer l, denote by $[l]$ its class in $\mathbb{F}$. For a finite group G, we denote its order by $\#G$. The formula now reads

$$\pm\, \tau(f, \mathbb{F}) = \pm \prod_{k=1}^{\dim M - 1} [\#\mathsf{H}_k(M; \mathbb{Z})]^{(-1)^{k+1}}.$$

So, as an invariant of a manifold $\pm\tau(f, \mathbb{F})$ is not very interesting; the whole point of Theorem 0.2 is that this number does not depend on f. The proof of the formula will appear elsewhere. For now, we will only provide sanity check. First, it follows from the assumption of the theorem that $\mathsf{H}_k(M; \mathbb{Z})$ is finite (for $1 \leqslant k \leqslant \dim M - 1$), so the order makes sense. Second, if $\mathbb{F} = \mathbb{F}_p$, then assumption implies that $\#\mathsf{H}_k(M; \mathbb{Z})$ is not divisible by p, so taking inverses on the r.h.s. makes sense.

As stated in Remark 4.4, the number $\pm\tau(f, \mathbb{F})$ can be interpreted as a certain kind of torsion. In light of this interpretation, expressions similar to the r.h.s. of the above formula appear in the works [8] and [7].

4.2 Reidemeister torsion: recollection

In this subsection, we briefly recall the notion of Reidemeister torsion (see [42] for details).

For a topological space X, let π be its fundamental group, and let $\widetilde{X} \to X$ be a universal covering. Choose a CW-decomposition of X, and denote the corresponding algebraic complex of free abelian groups by $\mathcal{C}$. Consider the lift of the CW-structure to $\widetilde{X}$. By the virtue of an action of π on its cells, one constructs a complex of free (right) $\mathbb{Z}[\pi]$-modules $\widetilde{\mathcal{C}}$ (recall that $\mathbb{Z}[\pi]$ is the integral group ring of π). The rank of k-chains still equals the number of k-cells in X. Moreover, one can choose a basis in $\widetilde{\mathcal{C}}$ by choosing any lift of each cell. Therefore, each element of this basis is defined up to multiplication by elements of π.

Let now $\rho\colon \pi \to GL_1(\mathbb{F}) = \mathbb{F}^*$ be a one-dimensional representation over $\mathbb{F}$. This map extends to the homomorphism of rings $\mathbb{Z}[\pi] \to \mathbb{F}$ by linearity. Conversely, any such homomorphism of rings arises from some representation since it is enough to define it on generators of $\mathbb{Z}[\pi]$. By notation abuse, we denote this homomorphism by the same letter ρ. Further, $\mathbb{F}$ acquires the structure of a left $\mathbb{Z}[\pi]$-module by the formula $r \cdot x = \rho(r)x$, where $r \in \mathbb{Z}[\pi], x \in \mathbb{F}$. On the other hand, $\mathbb{F}$ is also a right module over itself. Putting all this together, one now considers $\widetilde{\mathcal{C}} \otimes_\rho \mathbb{F}$, which is a complex of vector spaces over $\mathbb{F}$. Moreover, this complex carries a basis, each element of which is defined up to multiplication by elements of $\rho(\pi)$ (which is a multiplicative subgroup of $\mathbb{F}^*$). The homology of $\widetilde{\mathcal{C}} \otimes_\rho \mathbb{F}$ is called ρ-twisted homology of X and denoted here by $\mathsf{H}_*(X; \rho)$. If one is given an equivalence class of its basis (in a sense of Subsection 2.9), then they can consider torsion $\tau(X, \rho)$ of the complex of twisted chains, which lives in $\mathbb{F}^*/\rho(\pm\pi)$. The sign ambiguity is due to several things:

1) there is no canonical orientation of cells of X;

2) there is also no canonical linear order on them;
3) there is no canonical CW-decomposition after all.

If twisted homology vanishes, $\tau(X, \rho)$ is called the Reidemeister torsion of X w.r.t. ρ. It is known to be independent of CW-decomposition and to be stable under simple homotopy equivalences. The theorem of Chapman [11] states that homeomorphism is a simple homotopy equivalence.

Remark 4.6. Practically, passing from $\widetilde{C}$ to $\widetilde{C} \otimes_\rho \mathbb{F}$ means the following. Write the differential of $\widetilde{C}$ in some basis (e.g., in the basis of lifts of cells) to get a matrix with coefficients in $\mathbb{Z}[\pi]$. Replace each coefficient x with $\rho(x)$, and consider the resulting matrix as a map between vector spaces over $\mathbb{F}$.

Remark 4.7. Traditionally torsion theory is said to be born in the beginning of 20$^{\text{th}}$ century by the virtue of works of Reidemeister [40] and Franz [19]. We note that, quite surprisingly, the work of Cayley [9] contains some of the 1st torsion-theoretic ideas while staying in the purely algebraic setting.

4.3 Reidemeister torsion and Bruhat numbers

In this subsection, we introduce the notion of a twisted B-data and show that the alternating product of its Bruhat numbers equals the Reidemeister torsion, whenever the latter is defined (see Theorem 0.3). In particular, this product is independent of a function.

Let G be a multiplicative subgroup of $\mathbb{F}^*$ and V be a vector space over $\mathbb{F}$. By V/G we will denote a set, which is a quotient of V by the natural action of G.

Definition 4.8. An *enhancement up to G on a vector space V* is a choice of two structures:

1) a full flag on V, that is, a sequence of subspaces $0 = V^0 \subset V^1 \subset \ldots \subset V^{\dim V} = V$ s.t. $\dim(V^s/V^{s-1}) = 1$, $s \in \{1, \ldots, \dim V\}$;
2) a nonzero element $\varkappa_s$ in a quotient set $(V^s/V^{s-1})/G$, $s \in \{1, \ldots, \dim V\}$.

Enhancement up to G is still denoted by $(V, \varkappa)$.

Definitions of isomorphism between two such vector spaces, as well of the complex enhanced up to G go exactly in the same manner as in the usual case. Moreover, all the major statements from Section 1 and 2 translate readily to this new setting, with the only following exception. The nonzero elements of rook matrix from Section 1 are only defined up to multiplication by elements from G. Consequently, Bruhat numbers of a complex enhanced up to G now live in the quotient set $\mathbb{F}^*/G$.

If f is a strong Morse function and $\mathbb{F}$ is a field of characteristic not two, then f defines a complex enhanced up to $\mathbb{Z}_2 = \{\pm 1\} \subset \mathbb{F}^*$ (see Remark 3.2).

Construction 4.9. Let f be an oriented strong Morse function on a manifold M, $\mathbb{F}$ be a field, and $\rho\colon \mathbb{Z}[\pi] \to \mathbb{F}$ be a homomorphism of rings. We will now construct an isomorphism class of a complex enhanced up to $\rho(\pi)$.

$\triangleright$Apply Construction 3.12 to f to get a CW-complex X, simple homotopy equivalent to M. Lift the CW-structure to the universal cover $\widetilde{X}$, choose any preimage of each cell, and consider the corresponding algebraic complex $\widetilde{\mathcal{C}}$ of free $\mathbb{Z}[\pi]$-modules. It is basis is defined up to action of π and naturally linearly ordered. Moreover, the matrix of differential is upper triangular w.r.t. this order, since so is differential in $\mathcal{C}$. In particular, the span of the first s basis elements is a subcomplex (for $s \in \{1, \dots, \#\operatorname{Cr}(f)\}$).

Consider now the complex $\widetilde{\mathcal{C}} \otimes_\rho \mathbb{F}$. It inherits a linearly ordered basis that is defined up to an action of $\rho(\pi)$. The desired complex enhanced up to $\rho(\pi)$ is now taken to be the one associated with this basis (see the end of the Subsection 2.4).

It remains to prove that obtained complex is well defined up to an isomorphism. Indeed, let X' be another CW-complex obtained by the virtue of Construction 3.12. Matrices of cellular differentials (in any degree) in these two complexes are conjugated by a unitriangular matrix over $\mathbb{Z}$ (see Remark 3.18). Therefore, the matrices of differentials in $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}'$ are conjugated by a triangular matrix with elements from π on the diagonal. The desired statement follows. $\triangleleft$

Consequently, given the data as in the Construction 4.9, one can construct Barannikov pairs and Bruhat numbers, which are elements of $\mathbb{F}^*/\rho(\pi)$ (without choosing a particular orientation of f these numbers live in $\mathbb{F}^*/\pm\rho(\pi)$). To emphasize the presence of ρ, we say "twisted Barannikov pairs" and "twisted Bruhat numbers".

Remark 4.10. One can also adapt construction from Remark 3.3 to the case of twisted coefficients. This provides a quick way of defining twisted Barannikov pairs and Bruhat numbers.

One then defines torsion $\tau(f, \rho)$ of f exactly as in Definition 4.1. Again, as in Subsection 4.2, Proposition 2.34 justifies the name. In short, the alternating product of twisted Bruhat numbers of f equals the torsion of M w.r.t. a certain basis of the vector space $\mathsf{H}_\bullet(M; \rho)$ defined by f. Generally, this basis and, consequently, $\tau(f, \rho)$ may well

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

depend on f; this basis is even nonuniquely defined, but this arbitrariness does not affect $\tau(f, \rho)$.

However, combining all of the above with Subsection 4.2, one gets the following theorem.

Theorem 0.3. Let f be a strong Morse function on a manifold M, $\mathbb{F}$ be a field, and $\rho\colon \pi \to \mathbb{F}^*$ be a one-dimensional representation. Suppose that twisted homology vanishes. Then the alternating product of twisted Bruhat numbers of f equals the Reidemeister torsion of M. In particular, it is independent of f.

4.4 Non-acyclic Reidemeister torsion and Bruhat numbers

In this subsection, prove the analog of Theorem 0.3 in the case where twisted homology is not assumed to vanish.

We start with generalities. We continue using notations introduced in Subsections 4.2, 4.3, 2.9, and 2.10. For a topological space X and a map $\rho\colon \pi \to \mathbb{F}^*$, choose a CW-decomposition of X, and consider the complex of vector spaces $\widetilde{\mathcal{C}} \otimes_\rho \mathbb{F}$. It carries a cellular basis, defined up to $\rho(\pi)$. By Subsection 2.9, this basis gives rise to the element $c \in \det(\widetilde{\mathcal{C}} \otimes_\rho \mathbb{F})/\rho(\pi)$. Consider now the canonical isomorphism $\xi\colon \det(\widetilde{\mathcal{C}} \otimes_\rho \mathbb{F})/\pm\rho(\pi) \xrightarrow{\sim} \det \mathsf{H}_*(X; \rho)/\pm\rho(\pi)$. One can show that the element $\xi(\pm c)$ does not depend on the CW-decomposition of X. We call it a non-acyclic Reidemeister torsion and denote as $\tau_{na}(X, \rho)$.

We will now pour Morse theory into this setting. Fix a strong Morse function f on M. By Subsection 2.10, it gives rise to an element $h_f \in \det \mathsf{H}_*(M; \rho)/\pm\rho(\pi)$ (as usual, via the enhanced complex construction). Recall that by $\tau(f, \rho)$, we denote the alternating product of twisted Bruhat numbers of f. We now have the following theorem.

Theorem 4.11. Let f be a strong Morse function on a manifold M, $\mathbb{F}$ be a field, and $\rho\colon \pi \to \mathbb{F}^*$ be a one-dimensional representation. In the notations given above, one has

$$\tau_{na}(M, \rho) = \frac{h_f}{\tau(f, \rho)}.$$

Proof. Follows readily from Proposition 2.34 and 2.36. ■

In words, this theorem says that the non-acyclic Reidemeister torsion of M is equal to a certain element in the twisted determinant line of homology of M divided by the alternating product of twisted Bruhat numbers. Note that both ingredients, as suggested by notation, do depend on f. But the l.h.s. of the equality, obviously, does not.

Remark 4.12. Theorem 4.11 indeed implies Theorem 0.3. The former, however, requires introducing additional notions, making the result less accessible. This is the reason why we decided to prove Theorem 0.3 separately by more elementary means.

5 One-Parameter Morse Theory

One-parameter Morse theory deals with generic paths (in other words, one-parameter families) in the space of all smooth functions on M. The endpoints of a generic path are strong Morse functions—this is essentially the statement that strong Morse functions form an open dense subspace. However, finitely many points of such a path may fail to be either strong or Morse functions. It is exactly at these points where the B-data associated with a strong Morse function changes. In this section, we describe how exactly these changes look like (see Subsection 5.3). This allows to prove some statements from the previous sections (see Subsection 5.5). On the other hand, this also enables us to reprove a theorem of Akhmetev–Cencelj–Repovs [2] in greater generality (see Subsection 5.6).

5.1 Generalities on one-parameter Morse theory

In this subsection, we recall foundations of one-parameter Morse theory, initiated by Cerf [10].

A path in the space of functions on M is practically a map $F: M \times [-1, 1] \to \mathbb{R}$. Let t be a coordinate along $[-1, 1]$, which we occasionally refer to as "time". Define the function $f_t: M \to \mathbb{R}$ by $f_t(x) := F(x, t)$. For convenience, we will write $\{f_t\}$ instead of F. By a point of a path $\{f_t\}$, we will mean a function f_{t_0} for some particular $t_0 \in [-1, 1]$. Fix a generic path $\{f_t\}$ once and for all throughout this section (see [10] for the precise definition of genericity). Its endpoints f_{-1} and f_1 are strong Morse functions on M. Moreover, the same holds for all but finitely many points of $\{f_t\}$. The rest of this subsection is devoted to describing what changes may occur to a function at these points.

We first introduce a couple of definitions. One says that two strong Morse functions f and g are isotopic if there exists a diffeomorphism φ (resp. ψ) of $\mathbb{R}$ (resp. M) isotopic to the identity s.t. $g = \varphi \circ f \circ \psi$. Roughly, isotopic functions represent the same object from the viewpoint of Morse theory (see Remark 3.19). Analogously, two paths $\{f_t\}$ and $\{g_t\}$ are said to be equivalent if there exists an isotopy $\{\varphi_t\} : \mathbb{R} \times [-1, 1] \to \mathbb{R}$ (resp. $\{\psi_t\} : M \times [-1, 1] \to M$) s.t. $g_t = \varphi_t \circ f_t \circ \psi_t$ ($\varphi_0 = \mathrm{id}, \psi_0 = \mathrm{id}$).

It is a folklore result that if path $\{f_t\}$ consists only of strong Morse functions, then it is equivalent to a constant path. See [22] for a rigorous proof. The following description of changes of a strong Morse function along a generic path is to be understood as description of a certain explicit representative in the equivalence class of a path in question.

We will depict paths of functions in the following manner. The Cerf diagram of a path $\{f_t\}$ is a subset of $[-1,1] \times \mathbb{R}$ consisting of points (t,x) s.t. x is a critical value of f_t. Topologically, it is a set of (possibly self-intersecting and non-closed) curves in the plane.

As proven in [10], in a generic one-parameter family, there are two possible changes of isotopy class of a strong Morse function, which we call events. (Since there are only finitely many of them anyway, we assume for convenience that f_t is strong Morse for all t except for a single value $t = 0$.)

1) The function f_0 is strong, but non-Morse. This case is given by the local formula

$$f_t(x_1,\dots,x_n) = x_1^3 \pm tx_1 + Q(x_2,\dots,x_n),$$

where $(x_1,\dots,x_n)$ is some small coordinate neighborhood around the non-Morse critical point of f_0 (outside of a bit bigger neighborhood the function does not change at all) and Q is a nondegenerate quadratic form. At the moment, $t = 0$ the birth/death (depending on the sign) of two points of neighboring indices happens (the lower index is the index of Q). On a Cerf diagram, this corresponds to a (left or right) cusp. This event is called birth/death event.

2) The function f_0 is Morse, but not strong. This happens when two critical values collide. Outside of small neighborhoods around two corresponding critical points the function does not change at all; moreover, critical points themselves do not move along the path. On the Cerf diagram, this corresponds to a transversal self-intersection; in a sense, a pair of critical values is swapped. The space on non-strong functions is sometimes called Maxwell stratum. So, we call this event a Maxwell event.

Remark 5.1. Note that in both cases there are exactly two distinguished critical points: either two newborn/about to die points or a couple with swapping critical values. We will refer to them as “critical points involved in event”. They go one straight after another in the natural linear order. All the other critical points of f_{-1} and of f_1 are in natural bijection. We will always keep this bijection in mind without mentioning it explicitly.

Now we may describe a Cerf diagram a bit more precisely: it is a set of plane arcs (smooth in the interior) whose endpoints are either at cusps or has t coordinate equal to ± 1. These arcs do not have vertical tangencies and may self-intersect. To each arc, an integer number is assigned, namely the index of any critical point on it (mind that, for a generic point in a path, critical points are in bijection with critical values). From the viewpoint of Theorem 3.5, each arc corresponds to a cell in the CW-complex obtained via f_t. Birth/death event translates to birth/death of two cells of neighboring dimensions s.t. one appears in the cellular differential of another one with coefficient ± 1. Note that this is exactly the building block of the simple homotopy equivalence.

5.2 Any Bruhat number is realizable

In this subsection, we prove the following.

Proposition 3.25. Let $\mathbb{F}$ be either $\mathbb{Q}$ or $\mathbb{F}_p$ and $\lambda \in \mathbb{F}^*$ be any nonzero number. Let also M be any closed manifold s.t. $\dim M \geqslant 4$. Then one can find an oriented strong Morse function f on M that has λ as one of its Bruhat numbers.

The plan would be to construct a generic path of functions on M that starts with any strong Morse function and ends with the one satisfying the desired property. The tools for constructing such a path were essentially developed by Smale [41] and restated in Morse-theoretical terms by Milnor [32], which is our main reference here. In the following three statements, one is given a strong Morse function f_{-1} on a manifold M, equipped with a generic Riemannian metric g. Recall from Subsection 3.7 that g gives rise to a Morse complex, formally spanned by critical points. We write ∂_k for its differential, which maps k-chains to $(k-1)$-chains. The following two operations alter the function, but does not change the metric.

Proposition 5.2. Given any $k \in \{0, \dots, \dim M - 1\}$, one can find a generic path $\{f_t\}$ which contains a single event, namely a birth event. The indices of newborn points c_{s-1} and c_s are k and $k+1$, respectively. Both lie in the small neighborhood of any regular point of f_{-1} chosen in advance. Moreover, if c_t and $c_{t'}$ are two critical points not involved in the event, then one appears in the Morse differential of another with the same coefficient for f_{-1} and f_1.

Proposition 5.3. Suppose that c_s and c_{s-1} are two neighboring critical points of f_{-1} satisfying $\operatorname{ind} c_s \leqslant \operatorname{ind} c_{s-1}$. Then one can find a generic path $\{f_t\}$ that contains a single event, namely a Maxwell event; its swapping points are c_s and c_{s-1}. Moreover, the matrix of Morse differential ∂_k (resp. ∂_{k+1}) for f_1 equals that for f_{-1} multiplied on the right (resp. left) by a transposition $(s, s-1)$.

The next operation is called handle sliding. It alters the metric but does not change the function.

Proposition 5.4. Suppose that c_s and c_{s-1} are two neighboring critical points of f_{-1} of the same index k, $k \in \{2, \dots, \dim M - 2\}$. Suppose also that points c_s and c_{s-1} lie in the same connected component of $f_{-1}^{-1}([f_{-1}(c_{s-1}) - \varepsilon, f_{-1}(c_s) + \varepsilon])$. Then one can find a new metric $g'_{\pm}$ s.t. new matrix of Morse differential ∂'_k (resp. ∂'_{k+1}) equals the old one ∂_k (resp. ∂_{k+1}) multiplied on the right (resp. left) by the base change mapping c_s to $c_s \pm c_{s-1}$.

We shall specify the place in [32] where one can find the proof of Proposition 5.4. Shortly after the beginning of the proof of Theorem 7.6 (basis theorem), the author writes: "The steps involved are roughly as follows: increase f in the neighborhood of p_1, alter the vector field so that the left-hand disk of p_1 "sweeps across" p_2 with positive sign, and then readjust the function so that there is only one critical value.". The required argument is contained in the 2nd step.

Proof Proposition 3.25. Take any oriented strong Morse function f_0 on M and any generic Riemannian metric g_0. We will be modifying f_0 (in the small neighborhood of its regular point) and g_0. For clarity, we write B (birth) for usage of Proposition 5.2, S (swap) for Proposition 5.3, and HS (handle slide) for Proposition 5.4. Fix any $k \in \{2, \dots, \dim M - 2\}$. After any birth event, we orient two newborn points s.t. the coefficient of one in the differential of another would be $+1$ (there are two such choices); indices of newborn points will always be k and $k-1$. Fix any integers n and m s.t. $n/m = \lambda$ (by the notation abuse, we identify integer and its class in $\mathbb{F}$). Note that if $\mathbb{F} = \mathbb{F}_p$, one may take m to be 1.

We present necessary modifications in the Table 1. In the 2nd column, we write which modification should be applied. In the 3rd one, we list pairs of points affected by modification (if there are several pairs listed, then modifications should be applied consecutively in the given order). If modification is of type B, then we simply fix the notations in the 2nd column (and specify the level at which points are being born). If modification is of type HS, then it should be applied sufficiently many times as to obtain matrix in the 5th column (see below).

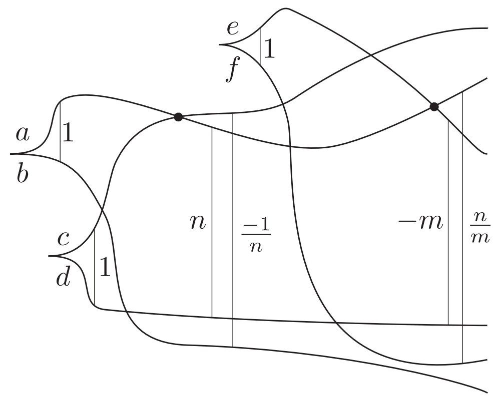


Fig. 4. To the proof of Proposition 3.25. Moments of nontrivial bifurcation are marked by black dots (see Subsection 5.4).

In the last three columns, we describe the function obtained by modifications. Namely, in the 4th column, we give the linear order on critical points. In the 5th one, we write (integral) submatrix of Morse differential that takes into account only mentioned critical points. The star denotes some integer number. In the last column, we write the corresponding rook matrix (recall that it is a matrix over $\mathbb{F}$). For rook matrices, we write only nonzero entries. If some table entry remains unchanged, we do not write it.

As seen, after all the modifications, points a and f form a Barannikov pair with Bruhat number $n/m = \lambda$. The Cerf diagram of the resulting path is given in Figure 4. See Subsections 5.3 and 5.4 for a description of bifurcations of B-data in one-parameter families. ∎

Remark 5.5. Note that after performing modifications from the proof of Proposition 3.25, the alternating product of Bruhat numbers does not change (up to sign; see Subsection 4.1). It is not surprising since indeed it sometimes does not depend on the function at all. However, we will now sketch the construction of a strong Morse function on $\mathbb{CP}^2$ s.t. it has only one Barannikov pair and the corresponding Bruhat number is λ for any λ from $\mathbb{F}_p$. Fix any integer n s.t. its class in $\mathbb{F}_p$ is λ.

Consider first a standard strong Morse function on $\mathbb{CP}^2$ that has three critical points, a, b, and c (of indices 0, 2, and 4, respectively). Apply Proposition 5.2 to introduce a pair of points d and e (of indices 2 and 1) between (in the sense of natural linear order) a and b. Apply Proposition 5.4 enough many times s.t. e would appear in the differential of b with coefficient n. Finally, apply Proposition 5.3 to points b and d (the bifurcation

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

Table 1 To the proof of Proposition 3.25.

#	Type	(c_s, c_{s-1})	Order	Morse diff.	Rook matrix
1	B	(a,b), (c,d) $(f_1(c) < f_1(b))$	dcba	$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$
2	HS	(a,c)		$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	
3	S	(b,c), (b,d), (a,c)	bdac	$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$	$\begin{pmatrix} & \frac{-1}{n} \\ n & \end{pmatrix}$
4	B	(e,f) $(f_1(c) < f_1(f))$	bdacfe	$\begin{pmatrix} 1 & 0 & * \\ n & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} & \frac{-1}{n} & \\ n & & \\ & & 1 \end{pmatrix}$
5	HS	(e,a), (e,c)		$\begin{pmatrix} 1 & 0 & 0 \\ n & 1 & -m \\ 0 & 0 & 1 \end{pmatrix}$	
6	S	(f,c), (f,a), (f,d), (e,c), (e,a)	bfdeac	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -m & n & 1 \end{pmatrix}$	$\begin{pmatrix} & & \frac{-1}{n} \\ & \frac{n}{m} & \\ -m & & \end{pmatrix}$

will necessarily be nontrivial). The resulting function will have points b and e paired with Bruhat number λ.

Remark 5.6. During the course of proof of Proposition 3.25, we introduced six new critical points in total. One can show that using the same technique, it is impossible to get by with only four of them; we will only sketch the argument. Indeed, Bruhat number on any short Barannikov pair over $\mathbb{Q}$ is integer (see Proposition 3.26). Since the product of two must be ± 1, the 2nd Bruhat number may only be the reciprocal of an integer.

5.3 B-Data in families

In this subsection, we start describing how B-data behaves along the generic path of functions. In Subsection 5.4, we finish this description.

First of all, we will orient all the functions in the path in the following way. Pick a generic point on some arc of the Cerf diagram. It corresponds to a critical point of some f_t; orient it. Extend this orientation by continuity to all the critical points lying the same arc (excluding the cusps). Apply this procedure to all the arcs. This recipe allows us to orient all the points in the path $\{f_t\}$ by making only finite number of binary choices, namely 2^l where l is the number of arcs. We use the term "orientation of an arc" for short.

Recall that we have to fix a field $\mathbb{F}$ in order to define B-data. Next, if the path $\{f_t\}$ consists of only strong Morse functions, then these data stay the same for all the time; see Remark 3.19 and Subsection 5.1.

We use the term "bifurcations" for the description of the way B-data changes after two events from Subsection 5.1. Disregarding the Bruhat numbers, this description was presented already in [4] (see [29] for a different proof). See also the paper [14] and pictures in the survey [16]. Thus, our job is to determine how Bruhat numbers change along the way (see Remark 3.4). In the case of birth/death event, we restrict ourselves to birth one for brevity (death one is obtained from birth one by reversing the time).

Before turning to formal statements, let us describe the basic general properties of bifurcations. Recall that in the path $\{f_t\}$ all functions are strong Morse except for f_0. Consider the following subset of critical points of f_1:

1) two points involved in event (i.e., either two newborn points or a couple with swapping critical values; see Remark 5.1);
2) the points paired with those, if any.

We will refer to points from this subset as ones "involved in the bifurcation". The reason behind this name is that all the other points retain their original pairs (in a sense of bijection from Remark 5.1). Moreover, the Bruhat numbers on these pairs remain the same. Thanks to all these facts, we are able to use pictorial format for describing the bifurcations. Namely, we will only depict points that are involved in the bifurcation (there are at most four of them).

Theorem 5.7. After the birth event a Barannikov pair of two newborn critical points appears; its Bruhat number is ± 1. All the other pairs and Bruhat numbers remain unaltered. See Figure 5.

Proof. Recall that by the preceding discussion in the present subsection, it suffices to track down only the Bruhat numbers. The 1st statement follows from the description given in Subsection 5.1.

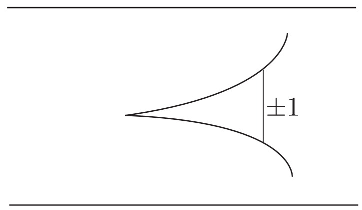


Fig. 5. Birth of two critical points.

Note that $\#\,\mathrm{Cr}(f_1) = \#\,\mathrm{Cr}(f_{-1}) + 2$; denote the non-Morse critical point of f_0 by p. Recall that outside of a small neighborhood of p the function f_t does not change along the path. Thus, all critical points of f_{-1} are also critical for f_1. Let c_{s+1} and c_s be two newborn critical points of f_1. Let $r_0 < \ldots < r_{\#\,\mathrm{Cr}(f_1)}$ be regular values of f_1. Take elements of the sequence $r_0 < \ldots < r_{s-1} < r_{s+2} < \ldots < r_{\#\,\mathrm{Cr}(f_1)}$ as sample regular values of f_{-1}. This way any regular sublevel set of f_{-1} is also that of f_1. One then arrives at the following diagram.

$$\begin{tikzcd}
M^{r_0} \arrow[r, phantom, "\subset"] \arrow[d, equal] & \ldots \arrow[r, phantom, "\subset"] & M^{r_{s-1}} \arrow[rrr, phantom, "\subset"] \arrow[d, equal] \arrow[drr, "\cong"] & & & M^{r_{s+2}} \arrow[r, phantom, "\subset"] \arrow[d, equal] & \ldots \arrow[r, phantom, "\subset"] & M^{r_{\#\,\mathrm{Cr}(f_1)}} \arrow[d, equal] \\
M^{r_0} \arrow[r, phantom, "\subset"] & \ldots \arrow[r, phantom, "\subset"] & M^{r_{s-1}} \arrow[r, phantom, "\subset"] & M^{r_s} \arrow[r, phantom, "\subset"] & M^{r_{s+1}} \arrow[r, phantom, "\subset"] & M^{r_{s+2}} \arrow[r, phantom, "\subset"] & \ldots \arrow[r, phantom, "\subset"] & M^{r_{\#\,\mathrm{Cr}(f_1)}}
\end{tikzcd}$$

Here, equality sign denotes set-theoretical equality of two subspaces of M and $\cong$ denotes a homeomorphism. The latter takes place since attaching two cells as in Subsection 5.1 does not change the homeomorphism type of a space. It now follows that Construction 3.1 produces the same B-data for f_{-1} and f_1 except for the above-mentioned newborn pair. ∎

Remark 5.8. The sign in ± 1 depends on the chosen orientations of arcs (see beginning of this subsection). See Subsection 5.6 for a theorem where it plays important role.

5.4 Maxwell event

In this subsection, we consider the 2nd type of event, namely self-intersection of a Cerf diagram (in other words, Maxwell event). This finishes the description of bifurcations of B-data in families started in Subsection 5.3.

Let us fix the notations first. Let c_{s+1} and c_s be two critical points of f_{-1} participating in the bifurcation. Recall from Subsection 5.1 that $\mathrm{Cr}(f_1)$ coincides, as an ordered subset of M, with $\mathrm{Cr}(f_{-1})$ with the order of c_{s+1} and c_s reversed. As we will see

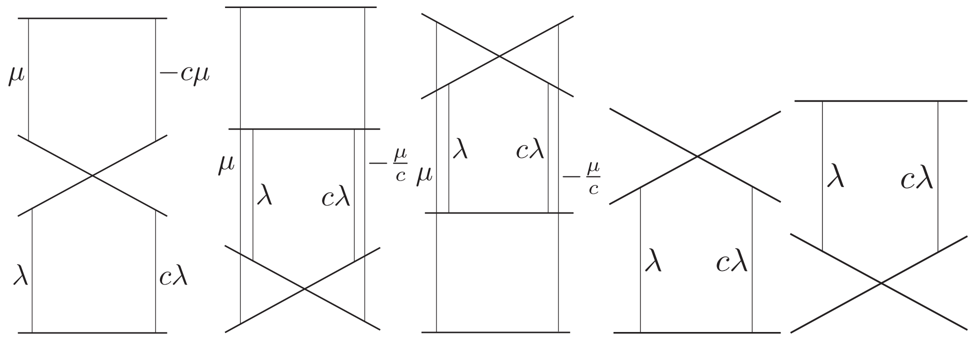


Fig. 6. Nontrivial bifurcations at the self-intersection of a Cerf diagram.

in Theorem 5.9 some bifurcations can only happen provided that certain restrictions on the linear order of involved critical points are satisfied. These restrictions depend on types of critical points (upper, lower, or homological); see also Remark 5.10. See the proof of Proposition 3.25 for an example.

Theorem 5.9. After the Maxwell event two types of bifurcations possible.

1) Trivial bifurcation. After it, points c_{s+1} and c_s keep their initial pairs (if any) and Bruhat numbers on them. No restrictions on the linear order of points are placed. The values $\deg c_{s+1}$ and $\deg c_s$ may be any.
2) Nontrivial bifurcation. The necessary condition is $\deg c_{s+1} = \deg c_s$. The list of five possible variants is given in Figure 6. Restrictions on the linear order can be deduced from the pictures; see Remark 5.10.

All the points not participating in the bifurcation keep their initial pairs (if any) and Bruhat numbers on them.

Remark 5.10. As seen on Figure 6, pairing and Bruhat numbers may well change after the nontrivial bifurcation. As for the restrictions on the linear order, suppose, for example, that both c_s and c_{s+1} are of upper type (picture 3). Then the restriction says that $b(s+1) < b(s)$ (where b is a pairing from the definition of B-data). Note that the same restrictions are involved in the definition of the ruling of a Legendrian knot [13, 20]. See [13] for discussion.

Remark 5.11. Suppose that (twisted) homology of M vanishes in degree k. Then there are no homological critical points of index k. Therefore, nontrivial bifurcation of such points can only be one of the 1st three types on Figure 6. In turn, this implies that the

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

alternating product of (twisted) Bruhat numbers stays the same after the bifurcation. This provides an alternative proof of the fact that alternating product of twisted Bruhat numbers does not depend on the function (assuming homology vanishes).

Proof of Theorem 5.9. As in Theorem 5.7, it suffices to track down only Bruhat numbers. Let $r_0 < \dots < r_{\#\,\mathrm{Cr}(f_1)}$ be regular values of f_{-1}. These values are also regular for f_1. Moreover, all but one sublevel sets of f_{-1} and f_1 coincide. The only exception is the sublevel set for the regular value r_s, so we write $M^{r_s} = \{x \in M | f_{-1}(x) \leqslant r_s\}$ and $\widetilde{M^{r_s}} = \{x \in M | f_1(x) \leqslant r_s\}$. One arrives at the following diagram.

$$
\begin{tikzcd}
 & & & M^{r_s} \arrow[dr, hook] & & & \\
M^{r_0} \arrow[r, phantom, "\subset"] & \dots \arrow[r, phantom, "\subset"] & M^{r_{s-1}} \arrow[ur, hook] \arrow[dr, hook] & & M^{r_{s+1}} \arrow[r, phantom, "\subset"] & \dots \arrow[r, phantom, "\subset"] & M^{r_{\#\,\mathrm{Cr}(f_1)}} \\
 & & & \widetilde{M^{r_s}} \arrow[ur, hook] & & &
\end{tikzcd}
$$

The case of trivial bifurcation and the very last statement of Theorem 5.9 now follow directly by unwrapping Construction 3.1. We now need to show how Bruhat numbers change after cases 1–3 of nontrivial bifurcation (see Figure 6; two other do not place any restriction on these numbers). By the very last statement, it suffices to prove that $\tau(f_{-1}) = -\tau(f_1)$ (see Subsection 4.1). Since the number τ admits torsion-theoretic interpretation, it depends only on two things: equivalence class basis c of an enhanced complex and equivalence class of basis h of homology. The latter is uniquely determined by the enhancement on $\mathsf{H}_\bullet(M)$ induced by a function. Since both points involved in the event are not homological, this enhancement is the same for f_{-1} and f_1.

We will now investigate how basis c changes after the bifurcation. Choose any CW-approximation (or any Riemannian metric) for f_{-1} in the sense of Subsection 3.4 (or, respectively, Subsection 3.7). The resulting CW-complex (or, respectively, Morse complex) also serves as a CW-complex associated with f_1 with the only difference that s^{th} and $(s+1)^{\text{th}}$ generators got swapped in the linear order. The determinant of a transposition matrix is -1 and the formula $\tau(f_{-1}) = -\tau(f_1)$ now follows from Remark 2.30. ∎

5.5 Manifolds with almost no $\mathbb{F}$-homology

In this subsection, we prove the following theorem; see Subsection 4.1 for a context.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

Theorem 0.2. Let f be a strong Morse function on M and $\mathbb{F}$ be a field. Suppose that $\mathsf{H}_k(M) = 0$ for all $0 < k < \dim M$. Then the alternating product of all Bruhat numbers (as an element from $\mathbb{F}^*/\pm 1$) is independent of f.

Proof. Take two strong Morse functions on M, and connect them by a generic path. The plan is to use Theorem 5.7 and 5.9 to prove that $\pm\tau$ does not change after any bifurcation along the path. The case of birth/death event obviously follows from Theorem 5.7. The rest is devoted to dealing with the Maxwell event.

It follows from Theorem 5.9 that if both points involved in Maxwell event are paired (i.e., not homological) then the number $\pm\tau$ stays the same after the bifurcation. On the other hand, by assumption, any homological point is of degree either zero or $\dim M$. In the former case, the 1st part of Proposition 3.28 implies that $\pm\tau$ again stays the same (regardless of whether the bifurcation is trivial or not). The rest is devoted to the latter case.

If M is non-orientable, then f may have a homological point of degree $\dim M$ only if $\operatorname{char}\mathbb{F} = 2$, but this case makes the initial statement trivial (see Remark 3.17). If M is orientable, we make use of the 2nd part of Proposition 3.28 in the same way as earlier. ■

5.6 A theorem of Akhmetev–Cencelj–Repovs

In this subsection, we apply our methods to reprove the theorem of Akhmetev–Cencelj–Repovs [2] in a greater generality. Roughly, it says that several numerical invariants of a generic path in the space of strong Morse functions satisfy a certain equation mod 2.

First of all, we need to pass to a bit more general setting. Cobordism is a manifold M with boundary $\partial_0 M \sqcup \partial_1 M$. By a function f on a cobordism $(M, \partial_0 M, \partial_1 M)$, we will mean a function $f\colon M \to [0,1]$ s.t. $f^{-1}(0) = \partial_0 M$ and $f^{-1}(1) = \partial_1 M$. The function f on cobordism is called Morse if all its critical points are nondegenerate and lie in the interior of M. Strongness property is defined in the same manner as in the closed case. All the classical results from Subsections 3.4 and 3.7 generalize readily to this setting; see [32]. The only thing to mention here is that now construction from Subsection 3.4 attaches cells one-by-one starting from $\partial_0 M$. As a consequence, one obtains a CW-decomposition of $M/\partial_0 M$, not M itself. Thus, all the various constructed complexes calculate relative homology $\mathsf{H}_*(M, \partial_0 M)$. All the results about enhanced complexes also translate readily. Finally, all of this allows us to define B-data in the setting of manifolds with boundary. Trivial cobordism is a cylinder $(N \times [0,1], N \times \{0\}, N \times \{1\})$, where N is a closed manifold.

We will now introduce mentioned invariants of a generic path $\{f_t\}$. The 1st one is the number of self-intersections of the Cerf diagram (or, in our terminology, the number of Maxwell events), call it X. To get to the 2nd one, recall that in Subsection 5.3, we described the procedure of orienting the arcs of a Cerf diagram, which outputs an orientation of each strong Morse function in a path. After orienting the arcs, somehow, one can assign a sign to each cusp of a Cerf diagram (i.e., birth/death event) as follows. Let t_0 be a point of birth (resp. death) event. Pick any value $t_1 > t_0$ (resp. $t_1 < t_0$) s.t. all functions between t_1 and t_0 (t_0 excluded) are strong Morse. Denote by c_{s+1} and c_s two newborn (resp. about to die) critical points of f_{t_1}. It follows from classical results recalled in Subsection 3.4 that cellular differential of c_{s+1} contains c_s with coefficient either 1 or -1 regardless of choices made (essentially, cellular approximation). Using another language, one may say the same about the Morse differential w.r.t. any Riemannian metric (see Subsection 3.7). The sign of a cusp is now defined as the sign of this number. Let C be the number of negative cusps. Changing orientation of an arc changes the sign of each cusp that serves as this arc's endpoint (obviously, there are at most two such cusps). Therefore, if both f_{-1} and f_1 have no critical points, then the parity of C is a well-defined invariant of a path $\{f_t\}$. We are now turning to a corollary that is easier to state compared with the main theorem. For example, it does not appeal to any field or Bruhat numbers whatsoever, thus remaining entirely in the realm of Cerf theory. This corollary asserts a certain relation between two introduced invariants of a path (the number X and the parity of C).

Corollary 0.4 ([check and update corollary number).] Let $\{f_t\}$ be a generic path of functions on a cylinder $N \times [0, 1]$ s.t. both f_{-1} and f_1 have no critical points. Let X be the number of self-intersections of its Cerf diagram and C be the number of negative cusps. Then one has

$$\mathsf{X} + \mathsf{C} = 0 \pmod 2.$$

Remark 5.12. In [2], Corollary 0.4 was proved using two different methods, both requiring additional assumptions on N. The 1st method is based on h-principle and requires N to be stably parallelizable and simply connected. The 2nd one is based on parametric Morse theory and requires the dimension of N to be at least five. This method is geometrical and relies on strong results of Smale akin to those in Subsection 5.2. These results, in turn, only valid if the dimension of N is big enough and lead, for example, to the famous h-cobordism theorem. On the other hand, our approach of using

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

enhanced complexes, after deducing all the necessary generalities in Subsection 5.3, is entirely combinatorial.

We will now give one more definition in order to state the main theorem of this subsection. Fix a field $\mathbb{F}$ s.t. char $\mathbb{F} \neq 2$, and consider the set of Barannikov pairs of some oriented strong Morse function f. We say that two pairs overlap if they overlap when viewed as segments on the real line. Formally, two pairs (s_1, s_2) and (t_1, t_2) overlap if either $s_1 < t_1 < s_2 < t_2$ or $t_1 < s_1 < t_2 < s_2$. Let O be the number of all overlapping (unordered) pairs of Barannikov pairs (we stress out that we place no restrictions on the degrees of points). Define now

$$\tau'(f, \mathbb{F}) := (-1)^{\mathsf{O}} \prod_{s \in U} \lambda(s)^{(-1)^{\deg s}} \in \mathbb{F}^*.$$

Remark 5.13. In words, this is the alternating product of all the Bruhat numbers times the sign depending on the parity of O. This sign is different from the one from Definition 4.1. As usual, we drop the ingredients of τ' when they are understood.

Let now $(M, \partial_0 M, \partial_1 M)$ be any cobordism s.t. relative homology $\mathsf{H}_*(M, \partial_0 M)$ vanishes (recall that we take coefficients to be in $\mathbb{F}$ by default). (For instance, one may take an h-cobordism.) Then Theorem 0.2 (in the aforementioned setting) implies that $\pm\tau'(f)$ in independent of f.

We are now ready to state the main theorem of this subsection, which we do in multiplicative notation. Recall that X is the number of self-intersections of the Cerf diagram and C is the number of its negative cusps. It is easy to check that the number $\frac{\tau'(f_1)}{\tau'(f_{-1})}(-1)^{\mathsf{C}} \in \{\pm 1\}$ is a well-defined invariant of a path, that is, it does not depend on orientations of arcs. The next theorem asserts a certain relation between this invariant and a number X. In the conclusion of the theorem, multiplication takes place in $\mathbb{F}$.

Theorem 5.14. Let $\mathbb{F}$ be a field and $(M, \partial_0 M, \partial_1 M)$ be a cobordism s.t. $\mathsf{H}_*(M, \partial_0 M) = 0$. Let also $\{f_t\}$ be a (somehow oriented) generic path of functions on it. Then one has

$$\frac{\tau'(f_1)}{\tau'(f_{-1})}(-1)^{\mathsf{C}}(-1)^{\mathsf{X}} = 1.$$

Proof. The plan is to track down when the number $\tau'(f_t)$ changes its sign as t varies from -1 to 1. It suffices to prove that it does so exactly after t passes

1) a self-intersection of the Cerf diagram,

2) a negative cusp (either left or right).

Note that by assumption there are no homological points, that is, all of the points are paired.

We will first sort out the 2nd case. The number O does not change after a birth/death event, since the newborn/recently dead Barannikov pair does not overlap with any other pair. The statement now follows directly from Theorem 5.7.

We will now turn to the 1st case. Suppose that the bifurcation is nontrivial. Then the decomposition of critical points into pairs remains the same; thus, the number O remains unaltered. It then follows directly from Theorem 5.7 that the alternating product of Bruhat numbers changes its sign. Suppose now that the bifurcation is trivial. This time the set of Bruhat numbers remains the same, but the number O increases or decreases by one (depending on the juxtaposition of pairs involved in the bifurcation). ■

Since $\tau(f, \mathbb{F}) = 1$ for any f without critical points at all (and for any $\mathbb{F}$), the Corollary 0.4 follows straightforwardly by taking any $\mathbb{F}$ s.t. $\operatorname{char}\mathbb{F} \neq 2$.

Funding

This work was supported by the Russian Foundation for Basic Research [RFBR 18-01-00461 to P.P.]; the Simons Foundation; and the Russian Academic Excellence Project "5-100".

Acknowledgments

We are grateful to Leonid Rybnikov for fruitful discussions. We thank Andrei Ionov for bringing our attention to the fact that the word enhanced has been used since [6] for distinguishing; in particular, unitriangular group from upper triangular one. The 2nd author thanks Sergey Melikhov and Ed Miller for comments on the previous versions. He also thanks Petr Akhmetev for numerous explanations of the paper [2]. He thanks Anton Ayzenberg for [9]. Last, but not the least, we would like to express our gratitude to anonymous referee whose efforts greatly increased the readability of the paper. Apart from that, we thank them for bringing the oversight in the proof of Proposition 3.25 to our attention. AQ6

References

[1] Adelson-Velskii, G. M. and A. S. Kronrod. "About level sets of continuous functions with partial derivatives." *Dokl. Akad. Nauk SSSR* 49 (1945): 239–41.

[2] Akhmetev, P. M., M. Cencelj, and D. Repovs. "Some algebraic properties of Cerf diagrams of one-parameter function families." *Funct. Anal. Appl.* 39 (2005). AQ7

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

[3] Álvarez-Gavela, D. and K. Igusa. "A Legendrian Turaev torsion via generating families." *J. Éc. polytech. Math.* 8 (2021): 57–119.

[4] Barannikov, S. "The framed Morse complex and its invariants." *Adv. Sov. Math.* 22 (1994): 93–115.

AQ8 [5] Bauer, U., X. Ge, and Y. Wang. "Measuring Distance between Reeb Graphs." In *Proceedings of the Thirtieth Annual Symposium on Computational Geometry, SOCG'14*. 464–73. Association for Computing Machinery.

[6] Beilinson, A. and J. Bernstein. "A proof of Jantzen conjectures." *Adv. Sov. Math.* 16 (1993).

[7] Cappell, S. E. and E. Y. Miller. "Enumerative combinatorics of simplicial and cell complexes: Kirchhoff and trent type theorems." *Discrete Comput. Geom.* 61 (2019): 1–41.

[8] Catanzaro, M. J., V. Y. Chernyak, and J. R. Klein. "Kirchhoff's theorems in higher dimensions and Reidemeister torsion." *Homology Homotopy Appl.* 17 (2015): 165–89.

[9] Cayley, A. "On the theory of elimination." *Cambridge and Dublin Math. J.* 3 (1848): 116–20.

[10] Cerf, J. "La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie." *Publ. Math. Inst. Hautes Études Sci.* 39 (1970): 5–173.

[11] Chapman, T. A. "Topological invariance of whitehead torsion." *Amer. J. Math.* 96, no. 3 (1974): 488–97.

[12] Charette, F. "Quantum Reidemeister torsion, open Gromov–Witten invariants and a spectral sequence of Oh." *Int. Math. Res. Not. IMRN* 2019 (2017): 2483–518.

[13] Chekanov, Y. V. and P. E. Pushkar. "Combinatorics of fronts of Legendrian links and the Arnold 4-conjectures." *Russian Math. Surveys* 60, no. 1 (2005): 95–149.

[14] Cohen-Steiner, D., H. Edelsbrunner, and D. Morozov. "Vines and Vineyards by Updating Persistence in Linear Time." In *Symposium on Computational Geometry*. 119–26. ACM, 2006.

[15] Dumas, J.-G., C. Pernet, and Z. Sultan. "Fast computation of the rank profile matrix and the generalized Bruhat decomposition." *J. Symbolic Comput.* 83 (2017): 187–210.

AQ9 [16] Edelsbrunner, H. and J. Harer. "Persistent Homology—A Survey." In *Surveys on Discrete and Computational Geometry*, vol. 453. 453, 2008.

[17] Fock, V. and A. Goncharov. "Moduli spaces of local systems and higher Teichmüller theory." *Publ. Math. Inst. Hautes Études Sci.* 103 (2006): 1–211.

[18] Forman, R. "Morse theory for cell complexes." *Adv. Math.* 134 (1998): 90–145.

[19] Franz, W. "Über die torsion einer überdeckung." *J. Reine Angew. Math.* 173 (1935): 245–54.

[20] Fuchs, D. "Chekanov–Eliashberg invariant of Legendrian knots: existence of augmentations." *J. Geom. Phys.* 47, no. 1 (2003): 43–65.

[21] Fulton, W. and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. New York, NY: Springer.

AQ10 [22] Gay, D., K. Wehrheim, and C. Woodward. "Connected Cerf theory."

[23] Gelfand, I., M. Kapranov, and A. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. Lectures in Mathematics. Zürich/Boston: ETH/Birkhäuser.

[24] Grigoriev, D. "An analogue of the Bruhat decomposition for the closure of the cone of a Chevalley group of the classical series." *Dokl. Math.* 23 (1981): 393–7.

OUP UNCORRECTED PROOF – FIRST PROOF, 18/11/2022, SPi

[25] Hutchings, M. "Reidemeister torsion in generalized Morse theory." *Forum Math.* 14 (2002): 209–44.

[26] Hutchings, M. and Y.-J. Lee. "Circle-valued Morse theory and Reidemeister torsion." *Geom. Topol.* 3 (1999): 369–96.

[27] Hutchings, M. and Y.-J. Lee. "Circle-valued Morse theory, Reidemeister torsion, and Seiberg–Witten invariants of 3-manifolds." *Topology* 38 (1999): 861–88.

[28] Igusa, K. and J. Klein. "The Borel regulator map on pictures. II. An example from Morse theory." *K-Theory* 7, no. 3 (1993): 225–67.

[29] Laudenbach, F. "On an article by S. A. Barannikov."

[30] Le Peutrec, D., F. Nier, and C. Viterbo. "Bar codes of persistent cohomology and Arrhenius law for P-forms."

[31] Le Peutrec, D., F. Nier, and C. Viterbo. "Precise Arrhenius law for P-forms. The Witten Laplacian and Morse–Barannikov complex." *Ann. Henri Poincaré* 14 (2013): 567–610.

[32] Milnor, J. *Lectures on the H-Cobordism Theorem*, vol. 2258. Princeton University Press.

[33] Milnor, J. *Morse Theory*, vol. 51. Princeton University Press.

[34] Milnor, J. "Whitehead torsion." *Bull. Amer. Math. Soc. (N.S.)* 72, no. 3 (1966): 358–426. AQ11

[35] Mnev, P. *Lecture Notes on Torsions*.

[36] Morse, M. and G. B. Van Schaack. "The critical point theory under general boundary conditions." *Ann. of Math. (2)* 35, no. 3 (1934): 545–71.

[37] Pan, Y. and D. Rutherford. "Augmented legendrian cobordism in J^1 s^1."

[38] Qin, L. "On moduli spaces and CW structures arising from Morse theory on Hilbert manifolds." *J. Topol. Anal.* 2 (2010): 469–526.

[39] Reeb, G. "Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique." *C. R. Math. Acad. Sci. Paris* 222 (1946): 847–9.

[40] Reidemeister, K. "Homotopieringe und linsenräume." *Hamburger Abhandl.* 11 (1935): 102–9.

[41] Smale, S. "On the structure of manifolds." *Amer. J. Math.* 84 (1962): 387–99.

[42] Turaev, V. *Introduction to Combinatorial Torsions*. Lectures in Mathematics. Zürich/Basel: ETH/Birkhäuser.

[43] Usher, M. and J. Zhang. "Persistent homology and Floer–Novikov theory." 20.

[44] Viterbo, C. "Symplectic topology as the geometry of generating functions." *Math. Ann.* 292 (1992): 685–710.