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On the Topological Structure of Manifolds Supporting Axiom A Systems

Vyacheslav Z. Grines[1*], Vladislav S. Medvedev[1], and Evgeny V. Zhuzhoma[1***]**

[1] *National Research University Higher School of Economics, ul. Bolshaya Pecherskaya 25/12, 603005 Nizhny Novgorod, Russia*

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Abstract—Let M^n, $n \geqslant 3$, be a closed orientable n-manifold and $\mathbb{G}(M^n)$ the set of A-diffeomorphisms $f: M^n \to M^n$ whose nonwandering set satisfies the following conditions: (1) each nontrivial basic set of the nonwandering set is either an orientable codimension one expanding attractor or an orientable codimension one contracting repeller; (2) the invariant manifolds of isolated saddle periodic points intersect transversally and codimension one separatrices of such points can intersect only one-dimensional separatrices of other isolated periodic orbits. We prove that the ambient manifold M^n is homeomorphic to either the sphere $\mathbb{S}^n$ or the connected sum of $k_f \geqslant 0$ copies of the torus $\mathbb{T}^n$, $\eta_f \geqslant 0$ copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and $l_f \geqslant 0$ simply connected manifolds $N_1^n, \ldots, N_{l_f}^n$ which are not homeomorphic to the sphere. Here $k_f \geqslant 0$ is the number of connected components of all nontrivial basic sets, $\eta_f = \frac{\kappa_f}{2} - k_f + \frac{\nu_f - \mu_f + 2}{2}$, $\kappa_f \geqslant 0$ is the number of bunches of all nontrivial basic sets, $\mu_f \geqslant 0$ is the number of sinks and sources, $\nu_f \geqslant 0$ is the number of isolated saddle periodic points with Morse index 1 or $n-1$, $0 \leqslant l_f \leqslant \lambda_f$, $\lambda_f \geqslant 0$ is the number of all periodic points whose Morse index does not belong to the set $\{0, 1, n-1, n\}$ of diffeomorphism f. Similar statements hold for gradient-like flows on M^n. In this case there are no nontrivial basic sets in the nonwandering set of a flow. As an application, we get sufficient conditions for the existence of heteroclinic intersections and periodic trajectories for Morse–Smale flows.

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1. INTRODUCTION

Dynamical systems satisfying axiom A (in short, A-systems) were introduced by Smale [57]. By definition, a nonwandering set of an A-system has a hyperbolic structure and is the topological closure of the set of periodic orbits (see the basic notation of the theory of dynamical systems in [3, 16, 29, 53]). A-systems form a wide class containing Morse–Smale systems [44, 55] and Anosov systems [2]. According to Mane [33] and Robinson [52], A-systems contain all structurally stable dynamical systems.

One of the challenging problems in the theory of dynamical systems is the establishment of a relationship between the dynamical properties of a selected class of dynamical systems and the topological structures of supporting manifolds. A classical example is provided by the Morse inequalities obtained by Smale [55] for Morse–Smale dynamical systems. In the spirit of Morse inequalities [39], Smale established relations between the numbers of periodic points

*E-mail: vgrines@yandex.ru
**E-mail: medvedev-1942@mail.ru
***E-mail: zhuzhoma@mail.ru

and the Betty numbers $\beta_0(M^n)$, $\beta_1(M^n)$, $\dots$, $\beta_n(M^n)$ of the ambient n-manifolds M^n, where $\beta_i(M^n) = rank\ H_i(M^n, \mathbb{Z})$. After [55], many similar results were obtained for various classes of Morse–Smale systems, see the surveys [14, 34] and numerous references therein. Several papers have investigated the relationship for codimension one Anosov diffeomorphisms. Franks [11] and Newhouse [40] proved that any codimension one Anosov diffeomorphism is conjugate to a hyperbolic torus automorphism (as a consequence, a manifold admitting such diffeomorphisms is $\mathbb{T}^n$, $n \geqslant 2$).

According to Smale spectral decomposition theorem [53, 57], a nonwandering set of A-system splits into pairwise disjoint transitive closed and invariant pieces called *basic sets*. A basic set is called *trivial* if it is an isolated periodic orbit. Otherwise, a basic set is *nontrivial*. Good examples of nontrivial basic sets are expanding attractors and contracting repellers which were introduced by Williams [61, 62]. They are divided into orientable and nonorientable ones. An A-diffeomorphism of the n-torus $\mathbb{T}^n$ with an orientable codimension one expanding attractor can be obtained by the Smale surgery [57, pp. 788–789] of a codimension one Anosov diffeomorphism. Such diffeomorphisms are called *DA-diffeomorphisms*. The Plykin attractor is nonorientable [49]. It was proved in [18] that, if the nonwandering set of structurally stable diffeomorphism $f : M^n \to M^n$, $n \geqslant 3$, contains a codimension one orientable attractor or repeller Λ, then the manifold M^n is diffeomorphic to torus $\mathbb{T}^n$ and Λ is a unique nontrivial basic set in $NW(f)$.

Let us recall that the Morse index of a hyperbolic periodic point p is equal to the dimension of an unstable manifold $W^u(p)$. If the Morse index of the periodic point p is equal to 1 (or $n-1$), we say that the invariant manifold $W^s(p)$ (respectively $W^u(p)$) is codimension one.

Let M^n be a closed smooth (connected) orientable n-manifold, $n \geqslant 3$, and $\mathbb{G}(M^n)$ the set of A-diffeomorphisms $f : M^n \to M^n$ whose nonwandering set $NW(f)$ satisfies the following conditions (see Section 2 for details):

1) each nontrivial basic sets from $NW(f)$ is either an orientable codimension one expanding attractor or an orientable codimension one contracting repeller;

2) invariant manifolds of isolated saddle periodic points are intersected transversally and codimension one separatrices of such points can intersect only one-dimensional separatrices of other isolated saddle periodic orbits (there are no restrictions on heteroclinic intersections of separatrices whose codimensions do not equal one).

Let $\Lambda \subset NW(f)$ be a codimension one nontrivial basic set which is an expanding attractor (for a contracting repeller, all concepts below are similar since a repeller is an attractor for diffeomorphism f^{-1}). For each point $x \in \Lambda$, $W^u(x) \subset \Lambda$ and $W^s(x)\backslash x$ consists of two connected components. Moreover, according to [13], at least one of the connected components of $W^s(x)\backslash x$ has a nonempty intersection with Λ. A point $x \in \Lambda$ is called a *boundary* if one of the connected components of the set $W^s(x)\backslash x$ does not intersect with Λ. Such a component is denoted by $W^{s\varnothing}(x)$.

According to [13] (see also [16]), the set of boundary points of an expanding attractor Λ is nonempty and finite. Thus, any boundary point is periodic and is called a boundary periodic point of the expanding attractor Λ. The set $W^s(\Lambda) \setminus \Lambda$ consists of a finite number of linear connected components. The union of unstable manifolds $W^u(p_1), \dots, W^u(p_{r_{b^u}})$ of all boundary periodic points $p_1, \dots, p_{r_{b^u}}$ of the attractor Λ such that the stable components $W^{s\varnothing}(p_i)$ $(i = 1.\dots, r_{b^u})$ belong to the same path-connected component of the set $W^s(\Lambda) \setminus \Lambda$ is said to be a *bunch* b^u of the attractor Λ. The number r_{b^u} is called the *degree of the bunch* b^u. By virtue of [17], if Λ is an expanding orientable attractor, then each of its bunches has a degree that is equal to 2. Similarly, one can define the concept of bunch b^s for a codimension one contracting repeller.

According to [5, 57], the set Λ is uniquely expressed as the finite union of compact subsets

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_{r_\Lambda}, r_\Lambda \geqslant 1,$$

such that $f^{r_\Lambda}(\Lambda_j) = \Lambda_j, f(\Lambda_j) = \Lambda_{j+1},\ j \in \{1, \dots, r_\Lambda\}$ $(\Lambda_{r_\Lambda+1} = \Lambda_1)$. For every point x of Λ_j the set $W^s_x \cap \Lambda_j$ $(W^u_x \cap \Lambda_j)$ is dense in Λ_j. According to R. Bowen [5], the subset Λ_j is called a C-dense component. Let us notice that each C-dense component of the attractor Λ is a connected set.

For $f \in \mathbb{G}(M^n)$, we denote by $k_f \geqslant 0$ the number of all C-dense components of all nontrivial basic sets belonging to $NW(f)$, and by $\kappa_f \geqslant 0$ the number of all bunches belonging to the union of all nontrivial basic sets. Denote by $\mu_f \geqslant 0$ the number of all nodal periodic points (sinks and sources), by $\nu_f \geqslant 0$ the number of isolated saddle periodic points with Morse index 1 or $n-1$, and by $\lambda_f \geqslant 0$ the number of all periodic points whose Morse index does not belong to the set $\{0, 1, n-1, n\}$ of diffeomorphism f. Let $\mathbb{S}^n$ be an n-sphere, S^n be a manifold which is homeomorphic to $\mathbb{S}^n$ and $\mathbb{T}^n$ be an n-torus. For a nonnegative integer m, denote by $\mathcal{T}^n_m$ a manifold that is either an empty set if $m = 0$ or is a connected sum of $m \geqslant 1$ copies of the n-torus $\mathbb{T}^n$ if $m > 0$:

$$\mathcal{T}^n_m = \underbrace{\mathbb{T}^n \sharp \cdots \sharp \mathbb{T}^n}_{m \geqslant 1}.$$

Denote by $\mathcal{S}^n_m$ a manifold that is either the sphere $\mathbb{S}^n$ if $m = 0$ or is a connected sum of $m \geqslant 1$ copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ if $m > 0$:

$$\mathcal{S}^n_m = \underbrace{\left(\mathbb{S}^{n-1} \times \mathbb{S}^1\right) \sharp \cdots \sharp \left(\mathbb{S}^{n-1} \times \mathbb{S}^1\right)}_{m \geqslant 1}.$$

Denote by $\mathcal{N}^n_m$ a manifold that is either an empty set if $m = 0$ or a connected sum of simply connected manifolds N^n_i if $m > 0$:

$$\mathcal{N}^n_m = N^n_1 \sharp \cdots \sharp N^n_m,$$

where each manifold N_i is not homeomorphic to the sphere $\mathbb{S}^n$.

Theorem 1. *Let M^n be a closed orientable n-manifold, $n \geqslant 3$, supporting a diffeomorphism $f \in \mathbb{G}(M^n)$. Then there is an integer l_f, $0 \leqslant l_f \leqslant \lambda_f$, such that M^n is homeomorphic to the connected sum:*

$$\mathcal{T}^n_{k_f} \sharp \mathcal{S}^n_{\eta_f} \sharp \mathcal{N}_{l_f}$$

where

$$\eta_f = \frac{\kappa_f}{2} - k_f + \frac{\nu_f - \mu_f + 2}{2}.$$

Corollary 1. *Under the condition of Theorem 1, we have*

$$\pi_1(M^n) = \underbrace{\mathbb{Z}^n * \cdots * \mathbb{Z}^n}_{k_f} * \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{\eta_f}$$

where $*$ means the free product of groups.

The next remark follows from the main result of [17].

Remark 1. *If $f \in \mathbb{G}(M^n)$ is structurally stable and has a nontrivial orientable codimension one basic set, then the ambient manifold M^n is homeomorphic to the torus $\mathbb{T}^n$.*

Remark 2. *If for diffeomorphism $f \in \mathbb{G}(M^n)$ and the number $k_f = 0$ (κ_f also is equal to zero in this case), then f is a Morse–Smale diffeomorphism. In the particular case $n = 3$, $k_f = 0$ Theorem 1 is a generalization of the main result of [4], where it is proved that the ambient manifold M^3 is homeomorphic to $\mathcal{S}^3_{g_f}$, where $g_f = \frac{\nu_f - \mu_f + 2}{2}$. For $n > 3$, $k_f = 0$ Theorem 1 is a generalization of the main result from [18], where the authors required that stable and unstable manifolds of saddle periodic orbits have no intersections at all. In this case the ambient manifold M^n is homeomorphic to the connected sum: $\mathcal{S}^n_{g_f} \sharp \mathcal{N}_{l_f}$ where $g_f = \frac{\nu_f - \mu_f + 2}{2}$ and the integer l_f satisfies the inequality $0 \leqslant l_f \leqslant \lambda_f$.*

Projective-like manifolds were introduced in [37] (actually, the concept of projective-like manifolds appeared in [36] with no name).

Definition 1. *A closed smooth manifold M^n is called projective-like if*

- $n \in \{2, 4, 8, 16\}$;
- *there is a locally flat embedded $\frac{n}{2}$-sphere $S^{\frac{n}{2}} \subset M^n$ such that $M^n \setminus S^{\frac{n}{2}}$ is an open n-ball.*

Recall that a Morse–Smale diffeomorphism f is polar if the nonwandering set NW(f) contains exactly one sink periodic orbit and one source periodic orbit.

Theorem 2. *Let M^n be a closed orientable n-manifold, $n \geqslant 3$, supporting a diffeomorphism $f \in \mathbb{G}(M^n)$. Suppose that $l_f \neq 0$ and there is a manifold $N^n_{i_*}$ which belongs to the connected sum $\mathcal{N}^n_{l_f}$ and admits a polar diffeomorphism f_{i_*} whose nonwandering set contains exactly one saddle fixed point. Then*

a) $n \in \{4, 8, 16\}$;

b) if $n \in \{8, 16\}$, then $N^n_{i_*}$ is a projective-like manifold;

d) if $n = 4$, then $N^4_{i_*}$ is a disjoint union of an open ball B^4 and a 2-sphere S^2 which is not locally flat embedded in general[1]*;*

e) the homotopy groups $\pi_1(N^n_{i_*}) = \cdots = \pi_{\frac{n}{2}-1}(N^n_{i_*}) = 0$.

Remark 3. *The existence of such $N^n_{i_*}$ containing exactly one saddle fixed point whose Morse index differs from 1 and $n-1$ follows from [10] where the existence of closed manifolds admitting Morse functions with exactly three critical points was proved, and from [55] where it was proved that any gradient flow can be approximated by the Morse–Smale gradient flow.*

Let M^n be a closed smooth (connected) orientable n-manifold, $n \geqslant 3$. Denote by $\mathbb{G}^{flow}_{grad}(M^n)$ the set of Morse–Smale flows on M^n satisfying the following conditions (see Section 2 for details):

1) $f^t \in \mathbb{G}^{flow}_{grad}(M^n)$ is gradient-like, that is, the nonwandering set of f^t does not contain closed trajectories;

2) codimension one separatrices of saddle equilibrium states of $f^t \in \mathbb{G}^{flow}_{grad}(M^n)$ have no intersection with other separatrices of saddle equilibrium states.

For $f^t \in \mathbb{G}^{flow}_{grad}(M^n)$ denote by $\mu_{f^t} \geqslant 0$ the number of all nodal equilibrium states (sinks and sources), by $\nu_{f^t} \geqslant 0$ the number of isolated saddle equilibrium states with Morse index 1 or $n-1$, and by $\lambda_{f^t} \geqslant 0$ the number of all equilibrium states whose Morse index does not belong to the set $\{0, 1, n-1, n\}$.

The next theorem is a direct corollary of Theorem 1 in the case $k_f = 0$.

Theorem 3. *Let $f^t \in \mathbb{G}^{flow}_{grad}(M^n)$. Then there is an integer $0 \leqslant l_{f^t} \leqslant \lambda_{f^t}$ such that:*

M^n is homeomorphic to the connected sum:

$$\mathcal{S}^n_{g_{f^t}} \sharp \, \mathcal{N}_{l_{f^t}},$$

where $g_{f^t} = \frac{\nu_{f^t} - \mu_{f^t} + 2}{2}$.

Remark 4. *The statement similar to Theorem 2 holds for a manifold which belongs to the connected sum $\mathcal{N}_{l_{f^t}}$ and admits polar flow with a unique saddle equilibrium state. Moreover, this manifold is projective-like for $n = 4$.*

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[1] In [36], an example is given of a 4-manifold which is a union of an open ball B^4 and a 2-sphere S^2 such that it is not locally flat embedded.

The first application of Theorem 1 concerns the existence of heteroclinic intersections of codimension one separatrices that form codimension two submanifolds. In the particular case $n = 3$, the heteroclinic intersections of two-dimensional separatrices consist of heteroclinic curves. Note that heteroclinic curves are often the mathematical model of so-called separators considered in solar magnetohydrodynamics [15, 20, 50]. From the modern point of view, reconnections of the solar magnetic field along separators are responsible for solar flares [31, 32, 51].

Corollary 2. *Let $f : M^n \to M^n$ be an orientation-preserving Morse – Smale diffeomorphism of the closed orientable n-manifold M^n, $n \geqslant 3$. Suppose that the nonwandering set $NW(f)$ of f consists of μ_f nodal periodic points, ν_f codimension one saddle periodic points, and an arbitrary number of saddle periodic points that are not codimension one. If the fundamental group $\pi_1(M^n)$ does not contain a subgroup isomorphic to the free product $\mathbb{Z} * \cdots * \mathbb{Z}$ of $g_f = \frac{1}{2}(\nu_f - \mu_f + 2)$ copies of $\mathbb{Z}$, then there exist saddle periodic points $p, q \in NW(f)$ such that the Morse index of the point p equals 1, and the Morse index of the point q equals $n - 1$, and $W^s(p) \cap W^u(q) \neq \varnothing$.*

The second application is the following sufficient condition for the existence of a periodic trajectory for Morse – Smale flows.

Corollary 3. *Let f^t be a Morse – Smale flow without heteroclinic intersections on a closed orientable manifold M^n of dimension $n \geqslant 3$, and assume that the nonwandering set $NW(f^t)$ contains exactly μ_{f^t} nodal fixed points and $\nu_{f^t} \neq 0$ saddle fixed points with the Morse index 1 or $n - 1$. Then, if the fundamental group $\pi_1(M^n)$ does not contain a subgroup isomorphic to the free product $\mathbb{Z} * \cdots * \mathbb{Z}$ of $g_{f^t} = \frac{1}{2}\left(\nu_{f^t} - \mu_{f^t} + 2\right)$ copies of $\mathbb{Z}$, then the flow f^t has at least one periodic trajectory.*

Let us mention some topological obstructions to manifolds supporting codimension one basic sets. In [30], it was proved that, if a closed n-manifold M^n, $n \geqslant 3$, admits a codimension one expanding attractor (orientable or not), then M^n has a nontrivial fundamental group. In particular, there are no such diffeomorphisms of the n-sphere $\mathbb{S}^n$, $n \geqslant 3$. Plante [47] proved that an orientable codimension one expanding attractor defines a nontrivial element of the first homology group $H_1(M^n)$. Sullivan and Williams [59] showed that the real Čech homology of an orientable attractor (of any codimension) in its top dimension is nontrivial and finite-dimensional.

The structure of the paper is as follows. In Section 2, we formulate the main definitions and give some previous results. In Section 3, we prove the main results and their applications.

2. DEFINITIONS AND PREVIOUS RESULTS

Here, we recall basic definitions and formulate some results which we will need later. Many definitions for diffeomorphisms and flows are similar. We will give mainly the notation for diffeomorphisms providing the exact notation for flows if necessary. Let f be a C^∞ diffeomorphism of a closed manifold M^n endowed with some Riemannian metric d. A diffeomorphism f is said to be an *A-diffeomorphism* if its nonwandering set $NW(f)$ is hyperbolic[2)] and periodic points are dense in $NW(f)$ [57]. The *stable manifold* $W^s(x)$ of a point $x \in NW(f)$ is defined to be a set

[2)] An invariant set $\Lambda(f)$ of a diffeomorphism $f : M^n \to M^n$ (M^n is a closed smooth manifold) is hyperbolic if there is a continuous df-invariant splitting

$$T_{\Lambda(f)}M^n = E^s_{\Lambda(f)} \oplus E^u_{\Lambda(f)}$$

of tangent bundle $T_{\Lambda(f)}M^n$ in the sum of stable and unstable subbundles such that the following estimates hold:

$$\|df^k(v)\| \leqslant C\lambda^k\|v\|, \quad \|df^{-k}(w)\| \leqslant C\lambda^k\|w\|$$

for some real numbers $C > 0$ and $0 < \lambda < 1$, and for any $v \in E^s_{\Lambda(f)}, w \in E^u_{\Lambda(f)}$, $k \in \mathbb{N}$.

of points $y \in M^n$ such that $d\big(f^i(x), f^i(y)\big) \to 0$ as $i \to +\infty$[3]. The *unstable manifold* $W^u(x)$ of x is the stable manifold of x for the diffeomorphism f^{-1}. Stable and unstable manifolds are called *invariant manifolds*.

Smale's spectral decomposition theorem [57] says that the nonwandering set $NW(f)$ of an A-diffeomorphism f is a finite union of pairwise disjoint f-invariant closed sets $\Omega_1, \dots, \Omega_k$ such that every restriction $f|_{\Omega_i}$ is topologically transitive. These Ω_i are called the *basic sets* of f. For any $x \in \Omega_i$, $\dim W^u(x) + \dim W^s(x) = n$.

Both $W^u(x)$ and $W^s(x)$ are endowed with a normal and intrinsic orientation. Hence, one can define the index of intersection at each point of $W^u(x) \cap W^s(x)$ [26]. By definition, let $W^s_\varepsilon(x) \subset W^s(x)$ (resp. $W^u_\varepsilon(x) \subset W^u(x)$) be the ε-neighborhood of x in the intrinsic topology of the manifold $W^s(x)$ (resp. $W^u(x)$), where $\varepsilon > 0$. Following [13], we call a basic set Ω_i *orientable* if for any $\alpha > 0$ and $\beta > 0$ the index of $W^s_{x,\alpha} \cap W^u_{x,\beta}$ does not depend on a point of intersection.

A basic set Ω_i is called an *attractor* if there is a closed neighborhood U of the set Ω_i such that $f(U) \subset int\ U$, $\bigcap\limits_{j \geqslant 0} f^j(U) = \Omega_i$. An invariant set is called a *repeller* if it is an attractor for f^{-1}. If the basic set Ω_i is an attractor (repeller), then for any point $x \in \Omega_i$ the unstable (stable) manifold $W^u(x)$ ($W^s(x)$) belongs to Ω_i. According to [62], an attractor (repeller) Ω_i is called an *expanding attractor* (*contracting repeller*) if its topological dimension equals the dimension of unstable (stable) manifolds of any points of the attractor (repeller) [62]. If the topological dimension of an expanding attractor (contracting repeller) equals $n-1$, then we say that it is a codimension one attractor (repeller). According to R. Plykin [48], any expanding attractor and contracting repeller of codimension one is locally homeomorphic to the direct product of an $(n-1)$-disk and a Cantor set.

An A-diffeomorphism $f : M^n \to M^n$ is *Morse–Smale* if the nonwandering set $NW(f)$ consists of a finitely many periodic orbits (including fixed points that are periodic orbits of period 1) and stable and unstable manifolds of periodic orbits intersect transversally. Thus, all basic sets of a Morse–Smale diffeomorphism are trivial.

A periodic orbit p is called a *sink* (resp. *source*) *orbit* if $\dim W^s(p) = n$ and $\dim W^u(p) = 0$ (resp. $\dim W^s(p) = 0$ and $\dim W^u(p) = n$). A sink or source periodic orbit is called a *nodal periodic orbit*. A periodic point σ is called a *saddle point* if $1 \leqslant \dim W^u(\sigma) \leqslant n-1$, $1 \leqslant \dim W^s(\sigma) \leqslant n-1$. A component of $W^u(\sigma) \setminus \sigma$ denoted by $Sep^u(\sigma)$ is called an *unstable separatrix* of σ. If $\dim W^u(\sigma) \geqslant 2$, then $Sep^u(\sigma)$ is unique. Similar notation holds for a stable separatrix. A saddle periodic point σ is called *codimension one* if either $\dim W^u(\sigma) = 1$, $\dim W^s(\sigma) = n-1$ or $\dim W^u(\sigma) = n-1$, $\dim W^s(\sigma) = 1$. The separatrices $Sep^s(\sigma)$ and $Sep^u(\sigma)$ are codimension one separatrices, respectively.

Let p, q be saddle periodic points such that $W^u(p) \cap W^s(q) \neq \varnothing$. The intersection $W^u(p) \cap W^s(q)$ is called *heteroclinic*. Due to the transversality of $W^u(p) \pitchfork W^s(q)$, a heteroclinic intersection is either a union of countable sets of heteroclinic points or countable sets of submanifolds of dimension $m \geqslant 1$. If $\dim\big(W^u(p) \cap W^s(q)\big) \geqslant 1$, in this case a connected component of intersection $W^u(p) \cap W^s(q)$ is called a *heteroclinic manifold* (in the case $n = 3$ a heteroclinic manifold is called a *heteroclinic curve*).

For $1 \leqslant m \leqslant n$, we presume Euclidean space $\mathbb{R}^m$ to be included naturally in $\mathbb{R}^n$ as the subset each of whose last $(n-m)$ coordinates equals 0. Let $e : M^m \to N^n$ be an embedding of a closed

[3] Let $\Lambda \subset M^n$ be a hyperbolic set for a diffeomorphism f and let d be the metric on M^n induced by the Riemannian metric on TM^n. Then for each $x \in \Lambda$ there exists a stable manifold $W^s_x = J^s_x(E^s_x)$, where $J^s_x : E^s_x \to M^n$ is an injective immersion with the following properties:
1) $W^s_x = \{y \in M^n : d\big(f^k(x), f^k(y)\big) \to 0 \text{ for } k \to +\infty\}$;
2) if $x, y \in \Lambda$, then W^s_x and W^s_y either coincide or they are disjoint;
3) $f(W^s_x) = W^s_{f(x)}$;
4) the tangent space for W^s_x at every point $y \in \Lambda$ is E^s_y;
5) if $x, y \in \Lambda$ are close, then W^s_x and W^s_y are C^1-close on compact sets (that is, the images of a compact K by the immersions J^s_x, J^s_y are C^1-close if x, y are close). See [27, 57] for details.

m-manifold M^m in the interior of n-manifold N^n. One says that $e(M^m)$ is *locally flat at* $e(x)$, $x \in M^m$ if there exist a neighborhood $U(e(x)) = U$ and a homeomorphism $h : U \to \mathbb{R}^n$ such that $h(U \cap e(M^m)) = \mathbb{R}^m \subset \mathbb{R}^n$. Otherwise, $e(M^m)$ is *wild at* $e(x)$ [9]. Similar notation holds for a compact M^m, in particular, $M^m = [0; 1]$.

Note that a separatrix $Sep^\tau(\sigma)$ is a smooth manifold. Hence, $Sep^\tau(\sigma)$ is locally flat at every point [9]. However, a priori, $clos\ Sep^\tau(\sigma) = W^\tau(\sigma) \cup \{\beta\}$ could be wild at a unique point β.

One of the key statements to prove Theorem 1 is the following result proved in [4] for $n = 3$.

Proposition 1. *Let $e : \mathbb{S}^{n-1} \to M^n$ be a topological embedding of the $(n-1)$-sphere $\mathbb{S}^{n-1}$, $n \geqslant 3$, which is a smooth immersion everywhere except at one point, and let $\Sigma^{n-1} = e(\mathbb{S}^{n-1})$. Then any neighborhood of Σ^{n-1} contains a closed neighborhood of Σ^{n-1} diffeomorphic to $\mathbb{S}^{n-1} \times [0; 1]$.*

Proof. It is enough to prove the statement for $n \geqslant 4$. Let Σ^{n-1} be a topologically embedded $(n-1)$-sphere that is smooth everywhere except at one point, say $N \in \Sigma^{n-1}$. According to [6] (see also [7, 8]), a wildly embedded $(n-1)$-sphere has to contain infinitely many points where the local flatness fails provided that $n \geqslant 4$. Therefore, Σ^{n-1} is a locally flat embedded $(n-1)$-sphere including at the point N. Hence, Σ^{n-1} has an arbitrary small neighborhood diffeomorphic to $\mathbb{S}^{n-1} \times [0; 1]$. This completes the proof. □

The following propositions (Propositions 2 and 3) follow from Lemma 6 and Lemma 7 in [38], respectively. For the reader's convenience, we give a sketch of the proof.

Proposition 2. *Let B^n_1, $B^n_2 \subset S^n$ be disjoint open n-balls such that the boundaries $S^{n-1}_1 = \partial B^n_1$, $S^{n-1}_2 = \partial B^n_2$ are locally flat embedded $(n-1)$-spheres. Then, given any orientation-reversing homeomorphism $\psi : S^{n-1}_1 \to S^{n-1}_2$, the manifold N^n obtained from $S^n \setminus (B^n_1 \cup B^n_2)$ after the identification of S^{n-1}_1 with S^{n-1}_2 under ψ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Sketch of the proof. Since S^{n-1}_1, S^{n-1}_2 are locally flat embedded $(n-1)$-spheres, $S^n \setminus (B^n_1 \cup B^n_2)$ is a closed n-annulus homeomorphic $\mathbb{S}^{n-1} \times [0; 1]$ [60]. We see that the spheres $S^{n-1} \times \{0\}$, $S^{n-1} \times \{1\}$ are identified by ψ. Hence, N^n is a total space of the locally trivial bundle over S^1 that is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. □

Proposition 3. *Let M^n be a closed (topological) manifold and S^{n-1} an $(n-1)$-sphere topologically embedded in M^n. Suppose that S^{n-1} has an open neighborhood U homeomorphic to the direct product $\mathbb{S}^{n-1} \times (-1, 1)$. If the manifold $M^n \setminus U$ is connected, then there is a topological closed manifold M^n_1 such that M^n is homeomorphic to the connected sum $M^n_1 \sharp (\mathbb{S}^{n-1} \times \mathbb{S}^1)$.*

Proof. Denote by S^{n-1}_1, S^{n-1}_2 the components of the boundary ∂U. Clearly, S^{n-1}_1 and S^{n-1}_2 are $(n-1)$-spheres locally flat embedded in M^n. Since $M^n \setminus U$ is connected, the manifold $\widetilde{M} = (M^n \setminus U) \cup (B^n_1 \cup B^n_2)$ is closed and connected where B^n_1, B^n_2 are disjoint n-balls such that $\partial B^n_i = S^{n-1}_i$, $i = 1, 2$. The connectedness of $\widetilde{M}$ implies that there is a closed subset $\widetilde{D}^n \subset \widetilde{M}$ containing $B^n_1 \cup B^n_2$ and homeomorphic to a closed n-ball. Set $Q_n = \widetilde{D}^n \setminus (B^n_1 \cup B^n_2)$. The boundary ∂Q^n consists of three components S^{n-1}_0, S^{n-1}_1, S^{n-1}_2, each of which is homeomorphic to an $(n-1)$-sphere. Attaching a closed n-ball to Q^n along the component S^{n-1}_0, one gets the set homeomorphic to $S^n \setminus (B^n_1 \cup B^n_2)$.

To get the original manifold M^n, we must glue the spheres S^{n-1}_1, S^{n-1}_2 by the orientation-reversing homeomorphism. Using the result of Proposition 2, we find that the manifold M^n is homeomorphic to the connected sum $M^n_1 \sharp (\mathbb{S}^{n-1} \times \mathbb{S}^1)$ for some closed manifold M^n_1. □

The next proposition is a key sufficient condition for a closed manifold N^n to be projective-like. To prove it we will use the models of projective-like manifold by means of the well-known Hopf fiber bundle $(\mathbb{S}^{n-1}, \mathbb{S}^{\frac{n}{2}}, \mathbb{S}^{\frac{n}{2}-1})$ [10, 43], where $n \in \{4, 8, 16\}$. Such models are induced by the following construction of a projective plane. Let $\mathbb{B}^2$ be an open 2-ball with the boundary

$\mathbb{S}^1 = \partial\mathbb{B}^2$. The identification of opposite points $(a_1; a_2)$, $(-a_1; -a_2)$ of $\mathbb{S}^1$ gives the factor-space $\mathbb{S}^1/(a_1; a_2) \sim (-a_1; -a_2)$ that is homeomorphic to $\mathbb{S}^1$. The natural projection

$$\mathbb{S}^1 \to \mathbb{S}^1/\big((a_1; a_2) \sim (-a_1; -a_2)\big) \quad = \ \mathbb{S}^1$$

gives the locally trivial fiber bundle $(\partial\mathbb{B}^2 = \mathbb{S}^1, \mathbb{S}^1, \mathbb{S}^0)$ with the fiber being a zero-dimensional circle $\mathbb{S}^0$ that is a union of two points. The projective plane $\mathbb{P}^2$ is obtained from the closed 2-ball *clos* $\mathbb{B}^2 = \mathbb{B}^2 \cup \partial\mathbb{B}^2$ by the identification of each fiber of the locally trivial fiber bundle $(\partial\mathbb{B}^2 = \mathbb{S}^1, \mathbb{S}^1, \mathbb{S}^0)$ with a point. It is obvious that the projective plane is homeomorphic to the union of a disk B^2 and a locally flat embedded circle S^1.

We now consider a smooth Hopf fiber bundle $(\partial\mathbb{B}^n = \mathbb{S}^{n-1}, \mathbb{S}^{\frac{n}{2}}, \mathbb{S}^{\frac{n}{2}-1})$, where $n \in \{4, 8, 16\}$. Let N^n be a manifold obtained from the closed n-ball *clos* $\mathbb{B}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ after the identification of each fiber of the Hopf fiber bundle $(\partial\mathbb{B}^n = \mathbb{S}^{n-1}, \mathbb{S}^{\frac{n}{2}}, \mathbb{S}^{\frac{n}{2}-1})$ with a point. Denote by S the set obtained after this identification. Since the Hopf fiber bundle is locally trivial, S is homeomorphic to the base $S^{\frac{n}{2}}$ that is an $\frac{n}{2}$-sphere. Clearly, $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$ is a locally flat embedded $(n-1)$-sphere in the closed n-ball *clos* $\mathbb{B}^n$. Hence, S is also locally flat embedded in N^n. We see that N^n is a disjoint union of the open n-ball B^n and $\frac{n}{2}$-sphere S that is locally flat embedded in N^n. Thus, according to Definition 1, the manifold N^n is a projective-like manifold.

Proposition 4. *Let a closed manifold N^n, $n \geqslant 4$, be a disjoint union of an open n-ball B^n and a k-sphere Σ^k, $1 \leqslant k \leqslant n-1$, locally flat embedded in N^n. Then N^n is projective-like.*

Proof. As k-sphere Σ^k, $1 \leqslant k \leqslant n-1$, is locally flat embedded in N^n, Σ^k has an open tubular neighborhood $T(\Sigma^k)$ such that its boundary $\partial T(\Sigma^k)$ is a submanifold of codimension one, and $T(\Sigma^k)$ is the total space of a locally trivial fiber bundle with the base Σ^k and a fiber B^{n-k} [26]. For convenience, we can assume that each fiber $\tilde{B}^{n-k}$ is an $(n-k)$-ball such that the boundary $\partial\tilde{B}^{n-k} = S^{n-k-1}$ belongs to $\partial T(\Sigma^k)$, and the center of $\tilde{B}^{n-k}$ belongs to Σ^k.

First, we have to show that $\partial T(\Sigma^k)$ is homeomorphic to S^{n-1}. Let us construct flows f_0^t and f_1^t on the sets B^n and *clos* $T(\Sigma^k) = T(\Sigma^k) \cup \partial T(\Sigma^k)$, respectively, as follows.

To construct f_0^t, take an arbitrary point $x_0 \in B^n$ that does not belong to *clos* $T(\Sigma^k)$. Since B^n is an open ball, there is a flow f_0^t on B^n such that f_0^t has a unique fixed point x_0 that is a hyperbolic source, and all one-dimensional trajectories leave any compact part of B^n in the positive direction (time increases).

To construct the flow f_1^t, we arrange the following:

a) each disk $\tilde{B}^{n-k}$ that is a fiber of the locally trivial bundle $\big(T(\Sigma^k), \Sigma^k, B^{n-k}\big)$ is invariant under f_1^t;

b) the restriction of f_1^t on $\tilde{B}^{n-k}$ has a sink at a point on Σ^k, corresponding to the center of the disk $\tilde{B}^{n-k}$, and has the set of equilibria that fills out the entire boundary of the disc $\tilde{B}^{n-k}$;

c) the one-dimensional trajectories on the set $(T(\Sigma^k) \setminus \Sigma^k) \cap \tilde{B}^{n-k}$ move in the positive direction to the sinks.

Let $\tilde{\Sigma}^{n-1}$ be an $(n-1)$-sphere smoothly embedded in N^n such that $\tilde{\Sigma}^{n-1}$ bounds the n-ball b_0^n with a point x_0 inside, and $\tilde{\Sigma}^{n-1}$ is transversal to the trajectories of the flow f_0^t. From the properties of this flow, and the equality $N^n = \Sigma^k \cup B^n$, and from the fact that $\partial T(\Sigma^k)$ is a compact subset of B^n, it follows that there is a number $\tau > 0$ such that the set $\tilde{\Sigma}_\tau^{n-1} = f_0^\tau(\tilde{\Sigma}^{n-1})$ belongs to $T(\Sigma^k)$. Moreover, the set $\partial T(\Sigma^k)$ belongs to $f_0^\tau(b_0^n)$. Clearly, $\tilde{\Sigma}_\tau^{n-1}$ is an $(n-1)$-sphere that is locally flat embedded in N^n. By construction, $\tilde{\Sigma}_\tau^{n-1}$ belongs to the wandering set of the flow f_1^t.

The intersection $T(\Sigma^k) \cap f_0^\tau(b_0^n)$ is an open set whose boundary contains $\partial T(\Sigma^k)$. Since $\partial T(\Sigma^k)$ is a submanifold of codimension one, it has a semi-neighborhood $\mathfrak{U} \subset \big(T(\Sigma^k) \cup \partial T(\Sigma^k)\big) \cap f_0^\tau(b_0^n)$

that is homeomorphic to $(0;1] \times \partial T(\Sigma^k)$. Let us take an open subset $int\,\mathfrak{U} \subset \mathfrak{U}$ homeomorphic to $(0;1) \times \partial T(\Sigma^k)$. Obviously,

$$\pi_i(int\,\mathfrak{U}) = \pi_i\left((0;1) \times \partial T(\Sigma^k)\right) = \pi_i\left(\partial T(\Sigma^k)\right), \quad i = 0, \ldots, n-2. \tag{2.1}$$

The set $A_0 = B^n \setminus clos\, f_0^\tau(b_0^n)$ is an open n-dimensional annulus homeomorphic to $(0;1) \times \mathbb{S}^{n-1}$. Therefore, its homotopy groups $\pi_i(A_0)$ are equal to zero for all $i = 0, \ldots, n-2$. Let us consider a representative $\gamma : S^i \to int\,\mathfrak{U}$ of the group $\pi_i(int\,\mathfrak{U}) = 0$, where S^i is an i-sphere. Since $\gamma(S^i) \cap \partial T(\Sigma^k) = \varnothing$, there exists a number $\tau_1 > 0$ such that $f_1^{\tau_1}(\gamma(S^i)) \subset A_0$. Hence, $\pi_i(int\,\mathfrak{U}) = 0$. Thus, (2.1) implies that $\pi_i(\partial T(\Sigma^k)) = 0$ for all $i = 0, \ldots, n-2$. It follows from the validity of the Poincaré conjecture for all dimensions $n \geqslant 3$ (see [12, 41, 45, 46, 56]) that the set $\partial T(\Sigma^k)$ is homeomorphic to an $(n-1)$-sphere.

Since $\partial T(\Sigma^k)$ is homeomorphic to S^{n-1}, the locally trivial bundle $\left(T(\Sigma^k), \Sigma^k, B^{n-k}\right)$ is (globally) nontrivial. The projection $\pi : T(\Sigma^k) \to \Sigma^k$ of this bundle induces a projection $\pi_* : \partial T(\Sigma^k) \to \Sigma^k$ such that $\pi_*^{-1}(x) = \partial B^{n-k} = S^{n-k-1}$ for any $x \in \Sigma^k$. Since the bundle $(T(\Sigma^k), \Sigma^k, B^{n-k})$ is locally trivial, π_* induces the locally trivial bundle $(\partial T(\Sigma^k), \Sigma^k, \partial B^{n-k}) = (S^{n-1}, S^k, S^{n-k-1})$. According to [1] (see also [43]), there are only the following bundles of this kind:

$$\mathbb{S}^3 \to \mathbb{S}^2, \text{ fiber } \mathbb{S}^1; \qquad \mathbb{S}^7 \to \mathbb{S}^4, \text{ fiber } \mathbb{S}^3; \qquad \mathbb{S}^{15} \to \mathbb{S}^8, \text{ fiber } \mathbb{S}^7.$$

It is easy to see that these bundles correspond to the following pairs (n,k): $(4,2)$, $(8,4)$, $(16,8)$.

We see that N^n is a disjoint union of an open n-ball B^n and an $\frac{n}{2}$-sphere $S^{\frac{n}{2}}$ locally flat embedded in N^n, where $n \in \{4, 8, 16\}$. One can consider the spheres $S^{\frac{n}{2}}$, $n = 4, 8, 16$, as bases of the Hopf bundles $(S^{n-1}, S^{\frac{n}{2}}, S^{\frac{n}{2}-1})$. Since $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$, N^n can be obtained from the closed n-ball $clos\,\mathbb{B}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ after the identification of every fiber of the Hopf bundle $(\partial\mathbb{B}^n = \mathbb{S}^{n-1}, \mathbb{S}^{\frac{n}{2}}, \mathbb{S}^{\frac{n}{2}-1})$ with a point. This completes the proof. □

For reference, we formulate the following statement proved in [22] (see also [19, 21, 23]).

Proposition 5. *Let $f : M^n \to M^n$ be a Morse – Smale diffeomorphism, and $Sep^\tau(\sigma)$ a separatrix of dimension $1 \leqslant d \leqslant n-1$ of a saddle fixed point σ. Suppose that $Sep^\tau(\sigma)$ has no intersections with other separatrices. Then $Sep^\tau(\sigma)$ belongs to either an unstable (if $\tau = s$) or a stable (if $\tau = u$) manifold of some nodal fixed point (sink or source, respectively), say β, and the topological closure of $Sep^\tau(\sigma)$ is a topologically embedded d-sphere that equals $W^\tau(\sigma) \cup \{\beta\}$.*

Proposition 6. *Let $f : M^n \to M^n$ be a Morse – Smale diffeomorphism without heteroclinic manifolds on codimension one separatrices. Suppose that there is a codimension one saddle periodic point $\sigma \in NW(f)$ such that $\dim W^s(\sigma) = 1$, $\dim W^u(\sigma) = n-1$. Then there exists a codimension one saddle periodic point σ_* such that the unstable separatrix $Sep^u(\sigma_*)$ is codimension one and $Sep^u(\sigma_*)$ has no heteroclinic intersections.*

Proof. Without loss of generality, one can assume that all periodic points are fixed. Given any p, $q \in NW(f)$, we put $p \prec q$ provided that $W^s(p) \cap W^u(q) \neq \varnothing$ and there are no other $r \in NW(f)$ such that $W^s(p) \cap W^u(r) \neq \varnothing$, $W^s(r) \cap W^u(q) \neq \varnothing$. The relation $\prec$ is a partial ordering and this order is strict [55, 57].

Suppose the codimension one separatrix $Sep^u(\sigma)$ has heteroclinic intersections (otherwise, there is nothing to prove). The chain $\sigma \prec \sigma_1 \prec \cdots$ has a maximum point, say σ_*. Since f has no heteroclinic manifolds on codimension one separatrices, every saddle in this chain has a codimension one unstable separatrix. Since $Sep^u(\sigma_*)$ corresponds to the maximal point, $Sep^u(\sigma_*)$ does not intersect any separatrix of any other saddle point. □

Corollary 4. *Let $f : M^n \to M^n$ be a Morse – Smale diffeomorphism without heteroclinic manifolds on codimension one separatrices and let there be a codimension one saddle periodic point $\sigma \in NW(f)$ such that $\dim W^u(\sigma) = 1$, $\dim W^s(\sigma) = n - 1$. Then there exists a codimension one saddle periodic point σ_* such that the stable separatrix $Sep^s(\sigma_*)$ is codimension one and has no heteroclinic intersections.*

The following statement proved in [19] gives sufficient conditions for a Morse – Smale diffeomorphism to be polar.

Proposition 7. *Let $f : M^n \to M^n$ be a Morse – Smale diffeomorphism without codimension one saddle periodic orbits. Then f is a polar diffeomorphism, i. e., f has a unique source periodic orbit and unique sink periodic orbit.*

Proposition 8. *Let N^n be a closed n-manifold, $n \geqslant 3$. Suppose that N^n admits a polar Morse – Smale diffeomorphism $g : N^n \to N^n$ with no codimension one saddles. Then N^n is a simply connected manifold.*

Proof. Note that, if $n = 3$, then every saddle is codimension one. Hence, for the case $n = 3$, there are no saddles at all. This implies that N^3 is a 3-sphere. Thus, it remains to consider the case $n \geqslant 4$. Let $e : \mathbb{S}^1 \to N^n$ be a map representing an element of $\pi_1(N^n)$. Since $n \geqslant 4$, one can assume that e is a smooth immersion [24]. Without loss of generality, one can assume that $e(\mathbb{S}^1)$ does not contain periodic points of g. Take an unstable manifold $W^u(\sigma_1)$ of some saddle σ_1. Since $W^u(\sigma_1)$ is an image of $\mathbb{R}^k$ under a smooth immersion, one can slightly move $e(\mathbb{S}^1)$ to become transversal to $W^u(\sigma_1)$. By condition, $\dim W^u(\sigma_1) \leqslant n - 2$. The transversality gives that $e(\mathbb{S}^1) \cap W^u(\sigma_1) = \varnothing$. Continuing this procedure, one can obtain $e(\mathbb{S}^1)$ with no intersections with unstable manifolds of every saddle. Hence, $e(\mathbb{S}^1)$ belongs to the unstable manifold $W^u(\alpha)$ of unique source α of g. Since $W^u(\alpha)$ is homeomorphic to $\mathbb{R}^n$, $e(\mathbb{S}^1)$ is homotopic to zero. It follows that $\pi_1(N^n) = 0$. □

3. PROOFS OF THE MAIN RESULTS

Proof (*of Theorem* 1). Let M^n be a supporting manifold for $f \in \mathbb{G}(M^n)$. There are two cases: *a*) $k_f = 0$, *b*) $k_f \geqslant 1$.

Consider case a). It follows from item 1 of the definition of class $\mathbb{G}(M^n)$ that diffeomorphism f is a Morse – Smale diffeomorphism without heteroclinic manifolds on codimension one separatrices. Without loss of generality we can suppose that the nonwandering set of diffeomorphism f consists of fixed points. By virtue of Proposition 6, there exists a codimension one saddle σ whose codimension one separatrix has no heteroclinic intersections. For definiteness, let us assume that $\dim W^s(\sigma) = 1$, $\dim W^u(\sigma) = n - 1$. According to Proposition 5, the topological closure *clos* $W^u(\sigma)$ of $W^u(\sigma)$ is a topologically embedded $(n-1)$-sphere consisting of $W^u(\sigma)$ and a sink ω. By Proposition 1, there is an open neighborhood U of *clos* $W^u(\sigma)$ such that the topological closure *clos* U is diffeomorphic to $\mathbb{S}^{n-1} \times [0; 1]$. Since *clos* U contains the sink ω, $f^k(\mathit{clos}\ U) \subset U$ for a sufficiently large k. Passing to the iteration f^k if necessary, we can assume, without loss of generality, that $k = 1$.

Let us remove the neighborhood U from the manifold M^n. The manifold $M^n \setminus U$ has two boundary components Σ_1^{n-1}, Σ_2^{n-1}, each of which is homeomorphic to $\mathbb{S}^{n-1}$. Gluing to each Σ_i^{n-1} an n-ball B_i^n, $i = 1, 2$, we get a smooth closed manifold M_1^n. Since $f(\mathit{clos}\ U) \subset U$, we can continue the diffeomorphism f to the manifold M_1^n in such a way that inside each ball B_1^n, B_2^n the obtained diffeomorphism $f_1 : M_1^n \to M_1^n$ has exactly one hyperbolic sink, while all points except the sinks are wandering. Comparing the nonwandering sets of f_1 and f, one can see that f_1 has one saddle of codimension less and one node (the sink, in this case) more. We call the described procedure a *cutting along an unstable separatrix*. A similar cutting operation is considered along a stable separatrix (with adding sources).

After ν_f cuttings along codimension one separatrices of all saddles with the Morse indices $n - 1$ and 1, we obtain a Morse – Smale diffeomorphism $f_{\nu_f} : M^n_{\nu_f} \to M^n_{\nu_f}$ of a closed manifold $M^n_{\nu_f}$

consisting of finitely many connected components. The nonwandering set of f_{ν_f} contains exactly $\mu_f+\nu_f$ nodes, and does not contain codimension one saddles. By Proposition 7, each connected component of the manifold $M^n_{\nu_f}$ admits a polar diffeomorphism with exactly one source and exactly one sink. Hence, the number of connected components of $M^n_{\nu_f}$ is equal to $\gamma_f=\frac{1}{2}(\mu_f+\nu_f)$. Therefore, the number $\mu_f+\nu_f$ is even.

The cutting procedure above allows one to rebuild the original manifold M^n from the obtained connected components. If one gets a connected manifold after the cutting along a codimension one separatrix, the intermediate manifold is homeomorphic to the connected sum of some closed manifold and $\mathbb{S}^{n-1}\times\mathbb{S}^1$, according to Proposition 3. If after the cutting along a codimension one separatrix one gets a manifold consisting of two connected components, then the intermediate manifold is homeomorphic to the connected sum of two closed manifolds.

Denote by N^n_1, …, $N^n_{\gamma_f}$ the connected components of the manifold $M^n_{\nu_f}$. The number of cuttings that does not increase the number of the connected components is equal to

$$g_f=\nu_f-\frac{1}{2}\left(\mu_f+\nu_f\right)+1=\frac{1}{2}\left(\nu_f-\mu_f+2\right).$$

Each such cutting corresponds to the summand $\mathbb{S}^{n-1}\times\mathbb{S}^1$ in the resulting connected sum. Therefore, the connected sum of γ_f manifolds N^n_1, …, $N^n_{\gamma_f}$ and g_f copies of $S^{n-1}\times S^1$ is homeomorphic M^n.

There are two possibilities: 1) $g_f=0$, 2) $g_f\neq 0$. For the first one, there are two subcases: 1a) all manifolds N^n_1, …, $N^n_{\gamma_f}$ are homeomorphic to the sphere $\mathbb{S}^n$. Then M^n is homeomorphic to $\mathbb{S}^n$. 1b) among N^n_1, …, $N^n_{\gamma_f}$, there exist exactly $1\leqslant l_f\leqslant\gamma_f$ manifolds that are not homeomorphic to $\mathbb{S}^n$. Then M^n is homeomorphic to

$$N^n_1\sharp\cdots\sharp N^n_{l_f}.$$

It is obvious that $l_f\leqslant\lambda_f$. If $g_f\neq 0$, we find in a similar way that either M^n is homeomorphic to the connected sum

$$\underbrace{\left(\mathbb{S}^{n-1}\times\mathbb{S}^1\right)\sharp\cdots\sharp\left(\mathbb{S}^{n-1}\times\mathbb{S}^1\right)}_{g_f}$$

of g_f copies of $\mathbb{S}^{n-1}\times\mathbb{S}^1$, or M^n is homeomorphic to the connected sum

$$\underbrace{\left(\mathbb{S}^{n-1}\times\mathbb{S}^1\right)\sharp\cdots\sharp\left(\mathbb{S}^{n-1}\times\mathbb{S}^1\right)}_{g_f}\sharp N^n_1\sharp\cdots\sharp N^n_{l_f}$$

for some $1\leqslant l_f\leqslant\lambda_f$. By construction, each manifold N^n_i $(i=1,\ldots,l_f)$ admits a polar diffeomorphism without codimension one saddle periodic orbits, so it is simply connected according to Proposition 8. Thus, the ambient manifold is homeomorphic to

$$\mathcal{T}^n_{k_f}\sharp\,\mathcal{S}^n_{g_f}\sharp\,\mathcal{N}^n_{l_f},$$

where $k_f=0$ and $\mathcal{T}^n_{k_f}=\varnothing$. This completes the proof of case a).

Consider case b), that is, $k_f\geqslant 1$.

Let Λ be a C-dense component of a nontrivial basic set of f and $l_\Lambda\geqslant 1$ be a minimal positive integer such that $f^{l_\Lambda}(\Lambda)=\Lambda$. It follows directly from Theorem 5.1 [17] (see also [49, 63]) that there is a neighborhood $U(\Lambda)$ of Λ homeomorphic to $\mathbb{T}^n\setminus(\cup_{i=1}^{\kappa_\Lambda}B_i)$, where B_i is an n-ball and κ_Λ is the number of all bunches belonging to Λ. Moreover, $clos\, f^{l_\Lambda}\big(U(\Lambda)\big)\subset U(\Lambda)$, $\cap_{j\geqslant 0}f^{jl_\Lambda}\big(U(\Lambda)\big)=\Lambda$ if Λ belongs to an attractor, and $clos\, f^{-l_\Lambda}\big(U(\Lambda)\big)\subset U(\Lambda)$, $\cap_{j\geqslant 0}f^{-jl_\Lambda}\big(U(\Lambda)\big)=\Lambda$ if Λ belongs to a repeller.

Let us recall for convenience a construction of a neighborhood $U(\Lambda)$ which was done in [17]. For definiteness, suppose that Λ is a C-dense component of an expanding attractor. First, one proves the existence of a periodic point $p\in\Lambda$ such that one of the components of $W^s(p)\setminus\{p\}$ does not intersect Λ. Following [13], such a $p\in\Omega$ is called a *boundary periodic point*. A periodic point is *interior* if it is not a boundary periodic point. According to [13], there are finitely many boundary periodic points. An unstable manifold $W^u(x) \subset \Lambda$, $x \in \Lambda$, is called a *boundary unstable manifold* if $W^u(x)$ contains a boundary periodic point. The boundary unstable manifolds split into a finite number of so-called bunches in the following way. The union of pairwise disjoint unstable manifolds $W^u(p_1), \ldots, W^u(p_k)$ containing the boundary points $p_1, \ldots, p_k$, respectively, is said to be a *k-bunch* if there are points $x_i \in W^u(p_i)$ and arcs $(x_i, y_i)^s_\varnothing$, $y_i \in W^u(p_{i+1})$, $1 \leqslant i \leqslant k$, where $p_{k+1} = p_1$, $y_k \in W^u(p_1)$. Here $(x_i, y_i)^s_\varnothing$ means an arc of $W^s(x_i) = W^s(y_i)$ such that $(x_i, y_i)^s_\varnothing \cap \Lambda = \varnothing$. Since Λ belongs to the orientable expanding attractor and the dimension $n \geqslant 3$, every bunch of Λ is a 2-bunch.

Let B_{pq} be a bunch consisting of two unstable manifolds $W^u(p)$ and $W^u(q)$, where p and q are boundary periodic points. Clearly, p and q have the same period, denoted by $m = m(p, q)$. Given any point $x \in W^u(p)$, there is a unique point $y \in W^u(q)$ such that $(x, y)^s = (x, y)^s_\varnothing$, and vice versa. Let the map $\varphi_{pq} : (W^u(p) \setminus \{p\}) \cup (W^u(q) \setminus \{q\}) \to (W^u(p) \setminus \{p\}) \cup (W^u(q) \setminus \{q\})$ be given by $\varphi_{pq}(x) = y$ whenever $(x, y)^s = (x, y)^s_\varnothing$. The restriction $f^m|_{W^u(p)}$ has the sole hyperbolic repelling fixed point p. Therefore, there is a closed C^1 embedded $(n-1)$-ball $p \in D_p \subset W^u(p)$ bounded by the C^1 embedded $(n-2)$-sphere $S^{n-2}_p = \partial D_p$ such that S^{n-2}_p and $f^m(S^{n-2}_p)$ bound a closed $(n-1)$-annulus. Since $\varphi_{pq}|_{(W^u(p)-p)}$ is a homeomorphism, $\varphi_{pq}(S^{n-2}_p)$ is a locally flat embedded $(n-2)$-sphere S^{n-2}_q bounding the closed $(n-1)$-ball $D_q \subset W^u(q)$. Set $C_{pq} = \bigcup_{x \in S^{n-2}_p} [x, \varphi_{pq}(x)]^s_\varnothing$. By construction, C_{pq} contains the spheres S^{n-2}_p, S^{n-2}_q and is homeomorphic to the closed $(n-1)$-cylinder $\Sigma^{n-2} \times [0, 1]$. Hence, the set $S_{pq} = D_p \cup D_q \cup C_{pq}$ is homeomorphic to the $(n-1)$-sphere. Continuing this way for every 2-bunch from Λ, we get the family of $(n-1)$-spheres $S_{p_j q_j}$, $j = 1, \ldots, \kappa_\Lambda$. Slightly moving $S_{p_j q_j}$ inside of $M^n \setminus \Lambda$, one gets smooth $(n-1)$-spheres $S'_{p_j q_j}$, $j = 1, \ldots, \kappa_\Lambda$, called *characteristic spheres*. The union of all these spheres bounds a neighborhood $U(\Lambda)$ of Λ such that *clos* $f^{l_\Lambda}\big(U(\Lambda)\big) \subset U(\Lambda)$, $\cap_{j \geqslant 0} f^{j l_\Lambda}\big(U(\Lambda)\big) = \Lambda$. Let us attach the n-balls $B^n_{p_j q_j}$, $j = 1, \ldots, \kappa_\Lambda$, to $U(\Lambda)$ along $S'_{p_j q_j}$, $j = 1, \ldots, l$, to get a closed manifold, say T^n. By construction, the manifold T^n is a closed connected manifold.

The lamination formed by the unstable manifolds $W^u(x)$, $x \in \Lambda$, can be extended to a codimension one foliation F on T^n such that every leaf of F is homeomorphic (in the interior topology) to $\mathbb{R}^{n-1}$. It follows from [42] that the manifold T^n is homotopy equivalent to an n-torus $\mathbb{T}^n$. It follows from [54, 58] that the manifold T^3 is homeomorphic to the 3-torus $\mathbb{T}^3$. By Theorem 8.1 [25], the manifold T^4 is homeomorphic to the 4-torus $\mathbb{T}^4$ and, according to [28], for any $n \geqslant 5$ the manifold T^n is homeomorphic to the n-torus $\mathbb{T}^n$.

Let $\Lambda_1, \ldots, \Lambda_{k_f}$ be C-dense components of all nontrivial basic sets belonging to $NW(f)$ endowed by pairwise disjoint neighborhoods $U(\Lambda_1), \ldots, U(\Lambda_{k_f})$ described above for the C-dense component Λ. Consider the manifold $M^n_1 = M^n \setminus \big(\cup_{i=1}^{\kappa_f} U(\Lambda_i)\big)$ (which is not connected in general).

Since every neighborhood $U(\Lambda_i)$ is homeomorphic to $\mathbb{T}^n \setminus \Big(\cup_{i=1}^{\kappa_{\Lambda_i}} B_i\Big)$, where κ_{Λ_i} is the number of bunches of the basic set which contains the C-dense component Λ_i, each boundary component of M^n_1 is an $(n-1)$-sphere. Let us attach the n-balls B to the manifold M^n_1 along each boundary component belonging to ∂M^n_1 to get a closed manifold denoted by M^n_2. It follows from the inclusions *clos* $f^{l_\Lambda}\big(U(\Lambda)\big) \subset U(\Lambda)$, *clos* $f^{-l_\Lambda}\big(U(\Lambda)\big) \subset U(\Lambda)$ that one can extend $f|_{M^n_1}$ to M^n_2 to get a diffeomorphism $f_2 : M^n_2 \to M^n_2$ such that every attached ball contains either a unique hyperbolic sink periodic point or a unique hyperbolic source periodic point. Thus, the diffeomorphism f_2 belongs to the class $\mathbb{G}$, moreover, f_2 satisfies case a) considered above. Let us prove that M^n is homeomorphic to

$$\underbrace{\mathbb{T}^n \sharp \cdots \sharp \mathbb{T}^n}_{k_f \geqslant 1} \sharp \underbrace{\left(\mathbb{S}^{n-1} \times \mathbb{S}^1\right) \sharp \cdots \sharp \left(\mathbb{S}^{n-1} \times \mathbb{S}^1\right)}_{\beta_f \geqslant 0} \sharp M^n_2. \tag{3.1}$$

The appearance of copies $\mathbb{S}^{n-1} \times \mathbb{S}^1$, calculation of the integer β_f in the connected sum and the structure of a manifold M^n_2 can be explained in the following way. Let us consider a C-dense component Λ_i, $i \in \{1, \dots, k_f\}$, and any characteristic sphere S' for Λ_i. Take an open neighborhood V of S' such that the topological closure *clos* V is diffeomorphic to $\mathbb{S}^{n-1} \times [0;1]$. Let us remove the neighborhood V from the manifold M^n.

There are two possibilities: 1) $M^n \setminus V$ is connected; 2) $M^n \setminus V$ is not connected. In the first case, according to Proposition 3, there is a topological closed manifold Q^n such that M^n is homeomorphic to the connected sum $Q^n \sharp (S^{n-1} \times S^1)$. In the second case, the manifold M^n is homeomorphic to the connected sum of some manifolds Q_1 and Q_2. In the first case, difffeomorphism f can be extended to the manifold Q^n to diffeomorphism f_{Q^n} with two additional nodes (a sink and a source) and, in the second case, f can be extended to Q_1^n to a diffeomorphism $f_{Q_1^n}$ with an additional sink and to Q_2^n to a diffeomorphism $f_{Q_2^n}$ with an additional source. Continuing the process of cutting alone the characteristic spheres of all C-dense components $\Lambda_1, \dots, \Lambda_{k_f}$, we get k_f closed connected manifolds each of which is homeomorphic to $\mathbb{T}^n$ and closed manifolds $Q_1^n, \dots, Q_{\chi_f}^n$ for some number χ_f which satisfies the inequality $1 \leqslant \chi_f \leqslant \kappa_f$.

Thus, the manifold M^n is homeomorphic to the connected sum

$$\underbrace{\mathbb{T}^n \sharp \cdots \sharp \mathbb{T}^n}_{k_f \geqslant 1} \sharp \underbrace{\left(\mathbb{S}^{n-1} \times \mathbb{S}^1\right) \sharp \cdots \sharp \left(\mathbb{S}^{n-1} \times \mathbb{S}^1\right)}_{\beta_f \geqslant 0} \sharp Q_1^n \sharp, \dots, \sharp Q_{\chi_f}^n. \tag{3.2}$$

The integer β_f equals the number of cuttings along characteristic spheres that does not increase the number of the connected components. So β_f can be calculated by the formula

$$\beta_f = \kappa_f - k_f - \chi_f + 1.$$

Set $M_2^n = Q_1^n \sharp, \dots, \sharp Q_{\chi_f}^n$ and get the sum (3.1) from (3.2).

The nonwandering set of $f_{M_2^n}$ consists of trivial basic sets and has the same number of saddle periodic orbits as the nonwandering set of diffeomorphism f. Thus, the nonwandering set of the diffeomorphism $f_{M_2^n}$ contains exactly $\mu_f + \kappa_f - (2\chi_f - 2)$ nodes, ν_f saddles with Morse index 1 or $n-1$ and λ_f saddles with Morse index other than 1 or $n-1$. By the definition of the class $\mathbb{G}(M^n)$, the invariant manifolds of saddle periodic orbits of $f_{M_2^n}$ intersect transversally. Hence, $f_{M_2^n}$ is a Morse–Smale diffeomorphism. Applying to diffeomorphism $f_{M_2^n}$ the arguments of case a), we find that the manifold M_2^n is homeomorphic to the connected sum

$$\mathcal{S}^n_{\tilde{g}_f} \sharp \, \mathcal{N}_{l_f},$$

where $\tilde{g}_f = \frac{1}{2}\left(\nu_f - \mu_f - \kappa_f + 2\chi_f\right)$.

It follows from consideration of cases a) and b) that for diffeomorhism $f \in \mathbb{G}(M^n)$ the ambient manifold M^n is homeomorphic to

$$\mathcal{T}^n_{k_f} \sharp \mathcal{S}^n_{\beta_f} \sharp \mathcal{S}^n_{\tilde{g}_f} \sharp \, \mathcal{N}^n_{l_f},$$

where $\beta_f = \kappa_f - k_f - \chi_f + 1$, $\tilde{g}_f = \frac{1}{2}\left(\nu_f - \mu_f - \kappa_f + 2\chi_f\right)$ if $k_f \neq 0$ (that is $1 \leqslant \chi_f \leqslant \kappa_f$) and $\chi_f = 1$ if $k_f = 0$.

If $\nu_f = \mu_f = \lambda_f = 0$, then it follows from construction that $\chi_f = \frac{\kappa_f}{2}$ and $\beta_f = \frac{\kappa_f}{2} - k_f + 1$.

Set $\eta_f = \beta_f + \tilde{g}_f$. It is directly calculated that the number $\eta_f = \frac{\kappa_f}{2} - k_f + \frac{\nu_f - \mu_f + 2}{2}$. Thus, the proof is finished. $\square$

Proof (*of Theorem* 2). Since $N^n_{i_*}$ contains exactly one saddle fixed point, $N^n_{i_*}$ admits a polar diffeomorphism $f_{i_*}: N^n_{i_*} \to N^n_{i_*}$ whose nonwandering set consists of a sink ω_{i_*}, a source α_{i_*}, and a saddle σ_{i_*}. It follows from Proposition 5 that $\Sigma^k = W^u(\sigma_{i_*}) \cup \{\omega_{i_*}\}$ is a k-sphere, where $2 \leqslant k = \dim W^u(\sigma_{i_*}) \leqslant n-2$. It is well known [16, 57] that a manifold is the disjoint union of unstable manifolds of nonwandering orbits. Hence, the manifold

$$N^n_{i_*} = W^u(\sigma_{i_*}) \cup W^u(\omega_{i_*}) \cup W^u(\alpha) = W^u(\sigma_{i_*}) \cup \{\omega_{i_*}\} \cup W^u(\alpha_{i_*}) = \Sigma^k \cup W^u(\alpha_{i_*})$$

is the disjoint union of the open n-ball $B^n = W^u(\alpha_{i_*})$ and the k-sphere Σ^k topologically embedded in $N^n_{i_*}$.

It was shown in [35] that n is even and Σ^k is locally flat embedded provided that $n \geqslant 6$. Thus, it follows from Proposition 4 that $N^n_{i_*}$ is a projective-like manifold for $n \geqslant 6$. If $n = 4$, the manifold $N^4_{i_*}$ is the disjoint union of a topologically embedded 2-sphere and an open 4-dimensional ball. The theorem is proved. □

Proof (*of Theorem* 3). If a Morse – Smale flow has no periodic trajectories, then the time one map (shift along the trajectories) is a Morse – Smale diffeomorphism. Now, the result follows from Theorem 1. □

Proof (*of Corollaries* 2 *and* 3). The outline of the proof of the corollaries is the same: if we assume the contrary, then there exist decompositions of the ambient manifold M^n, according to Theorems 1 and 3. It follows from the Van Kampen theorem (see, for example, [43]) that $\pi_1(M^n)$ contains a subgroup $\mathbb{Z} * \cdots * \mathbb{Z}$. This contradiction proves the required assertions. □

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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