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On Kripke completeness of modal predicate logics around quantified **K5**

Valentin Shehtman [a,b,*]

[a] *Kharkevich Institute for Information Transmission Problems, B. Karetny 19, Moscow 127051, Russia*
[b] *National Research University Higher School of Economics, Russia*

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ABSTRACT

The paper studies completeness and incompleteness of modal predicate logics in Kripke semantics, especially for logics of the form $\mathbf{Q\Lambda}$, minimal predicate extensions of modal propositional logics. We show that $\mathbf{Q\Lambda}$ is incomplete for a continual family of logics $\mathbf{\Lambda}$ above $\mathbf{K} + \Box(\Box p \to p)$, in particular for well-known **K5** and **K45**. On the other hand, in some cases we find completions of $\mathbf{Q\Lambda}$; they are obtained by adding a single extra axiom. Completeness proofs use canonical models, with some modifications, and the case of **QK5** is the most interesting from the technical side. We also introduce the "boxing" operation for modal predicate logics and prove transfer results for Kripke and Kripke sheaf completeness with respect to this operation.

1. Introduction

It is well known that, unlike the propositional case, for modal first-order predicate logics standard Kripke semantics does not fit well. There are numerous examples of incompleteness; even, if a propositional modal logic $\mathbf{\Lambda}$ is Kripke complete, its minimal predicate extension $\mathbf{Q\Lambda}$ may lose completeness. In particular, Silvio Ghilardi proved that for normal $\mathbf{\Lambda}$ extending **S4**, the logic $\mathbf{Q\Lambda}$ can be Kripke complete only if $\mathbf{\Lambda} \supseteq \mathbf{S5}$ or $\mathbf{\Lambda} \subseteq \mathbf{S4.3}$, cf. [10].

Still, sometimes the predicate logics $\mathbf{Q\Lambda}$ are complete. E.g., this happens for the "one-way PTC logics" (in the monomodal case they are axiomatized by formulas of the form $\Box p \to \Box^n p$ and closed formulas, cf. [9], Definition 1.11.4) and for the logics axiomatized by formulas expressing density, confluence, non-branching

* Correspondence to: Kharkevich Institute for Information Transmission Problems, B. Karetny 19, Moscow 127051, Russia.
E-mail address: shehtman@netscape.net.

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(cf. [9], Theorem 6.1.29 and sections 6.6, 6.7; [15], [3]). There are also some nice completeness theorems for logics of the form (**QΛ**+Barcan formula) (by Yoshihito Tanaka, Hiroakira Ono, and Tatsuya Shimura; cf. [9], chapter 7).

However, in contrast to powerful completeness results in propositional logic (cf. [2]), these theorems look scanty. How can we improve the situation?

One option is to leave Kripke semantics and to deal with its generalizations. This strategy can restore completeness in many cases, but leads to rather complicated semantics, cf. [16].

In this paper we choose another option. If a logic is Kripke incomplete, it makes sense to describe its completion. In several cases this task was solved by Max Cresswell [7], and we return to it here.[1] We focus on certain logics of the form **QΛ**. On the one hand, we present a new continual family of incomplete logics; on the other hand, we find completions for certain incomplete logics from that family.

The crucial propositional logic in our paper is **K5**, as known as Euclidean modal logic. This is the simplest PTC-logic, which is not one-way PTC, and the question about Kripke completeness of **QK5** is quite natural. In [9] (Ch.6, p. 492) we claimed that **QK5** is incomplete, but the proposed proof was incorrect. Now we prove this result as a particular case of a general Theorem 5.11.

The plan of the paper is as follows. Section 2 gives a brief introduction to modal predicate logics; it contains basic material on syntax, Kripke sheaf and Kripke semantics.

In section 3 we introduce the boxing operation and prove some its properties. In particular, Theorem 3.21 gives an axiomatization of boxing for predicate logics.

Section 4 proves transfer results for boxing; we consider canonicity and strong completeness in Kripke and Kripke sheaf semantics.

In section 5 we prove Kripke (and Kripke sheaf) incompleteness for the family of logics **QΛ** for any **Λ** between $\Box \cdot \mathbf{T}$ and **SL4** (Theorem 5.11). Here we apply the Kripke bundle semantics. Together with results from sections 3, 4 this allows us to describe Kripke completions of certain logics of the form $\mathbf{Q}(\Box \cdot \mathbf{\Lambda})$ (Theorem 5.14).

In sections 6, 7 we axiomatize completions of three particular logics beyond Theorem 5.14. To prove completeness we apply canonical models, with some modifications. From the technical side, the logic **QK5** is the most interesting case (Theorem 7.2). Here we use a certain "swinging" procedure to construct simultaneous Henkin completions for a set of consistent theories.

Section 8 explains how to deal with iterated boxing operation; we prove the iterated versions of results from sections 3 and 5.

In the conclusive section 9 we discuss further possible directions and some related problems.

In our completeness proofs we essentially use the methods from [9]. Much of notation and terminology from that book is preserved.

Table of main results

Axiomatization of boxing	Theorem 3.21, Proposition 3.10
Transfer of canonicity for boxing	Theorem 4.1
Transfer of strong Kripke sheaf completeness for boxing	Theorem 4.4
Transfer of strong Kripke completeness for boxing of **QΛ**	Theorem 4.8
Transfer of Kripke completeness for propositional boxing	Theorem 4.9

[1] However, for many standard logics Kripke completions are not recursively axiomatizable, cf. [6].

Counterexamples to Kripke completeness transfer for boxing	Corollary 5.5
Kripke incompleteness of $\mathbf{Q\Lambda}$ for $\mathbf{\Lambda}$ between $\Box\cdot\mathbf{T}$ and $\mathbf{SL4}$	Theorem 5.11
Kripke completions of $\mathbf{QK5}$, $\mathbf{QK45}$, $\mathbf{QK4\Box S5}$	Theorems 7.2, 6.13, 6.12

2. Preliminaries

2.1. Propositional logics

We suppose that the reader is familiar with main properties of modal propositional logics. In this paper we consider only normal logics with a single modal connective $\Box$. Arbitrary logics are denoted by $\mathbf{\Lambda}$, $\mathbf{\Lambda}_1$ etc. We assume that all logics are consistent.

We use the main constructions of Kripke frames and models, such as generated subframes, p-morphisms, disjoint sums, canonical models; for the definitions cf. [2], [9].

$\mathbf{ML}(\mathcal{C})$ denotes the logic of a class of Kripke frames $\mathcal{C}$ (or the logic *determined by* $\mathcal{C}$); recall that logics of this form are called *Kripke complete* (or $\mathcal{K}$*-complete*). A modal logic $\mathbf{\Lambda}$ is called *strongly Kripke complete* if every $\mathbf{\Lambda}$-consistent set of formulas Γ is satisfied at a point of some Kripke model over a $\mathbf{\Lambda}$-frame.

$\mathbf{K}$ denotes the minimal modal logic; $\mathbf{K} + X$ is the smallest logic containing a set of formulas X.

$M_{\mathbf{\Lambda}}$ denotes the canonical model of $\mathbf{\Lambda}$, $F_{\mathbf{\Lambda}}$ its canonical frame.

2.2. Predicate logics

We deal with normal monomodal predicate logics without equality, as they are defined in [9]. So in the language there are countably many predicate letters of all arities (including 0), but no function symbols or individual constants. The *length* of a formula A is the number of occurrences of quantifiers and logical connectives (except $\bot$) in A. Closed formulas are also called *sentences*. $\overline{\forall}A$ denotes the universal closure of a formula A (with quantifiers in a fixed order).

A *modal predicate logic* is a set of formulas containing the basic axioms of classical logic and $\mathbf{K}$ and closed under Modus Ponens (MP), $\forall$-introduction, $\Box$-introduction, and predicate substitution. For a logic L, the notation $L \vdash A$ means the same as $A \in L$. Members of a logic are also called ‘theorems’.

$\overline{L}$ denotes the set of all sentences in a logic L. Thus

$$\overline{L} = \{\overline{\forall}A \mid A \in L\}.$$

$\mathbf{Q\Lambda}$ denotes the minimal predicate extension of a propositional logic $\mathbf{\Lambda}$.

A *predicate theory* is a set of sentences with extra constants. We assume that the constants are taken from a fixed countable set (denoted by S^*, see below). Theories are denoted by capital Greek letters, theories without constants by capital Latin letters $(X, Y, \ldots)$. $\mathcal{L}(\Gamma)$ denotes the set of all sentences in the language of a theory Γ, D_Γ is the set of all individual constants occurring in Γ.

If L is a predicate logic, Γ a predicate theory, then in *L-derivations*[2] from Γ we can use the members of $\Gamma \cup L$, apply MP, $\forall$-introduction and also replace some free variables with constants. The notation $\Gamma \vdash_L A$ means that a formula (maybe, with constants) A is L-derivable from Γ. Γ is *L-consistent* if $\Gamma \nvdash_L \bot$.

[2] In [9] they are called ‘L-inferences’.

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$L + X$ denotes the smallest predicate logic containing a logic L and a theory X. We recall the following characterization of $L + X$ ([9], Theorem 2.8.4):

Proposition 2.1.

$$L + X \vdash A \ \ \textit{iff} \ \ \Box^\infty \overline{Sub}(X) \vdash_L A,$$

where $\overline{Sub}(X)$ *denotes the set of universal closures of substitution instances of* X *and* $\Box^\infty Y := \{\Box^n B \mid B \in Y,\ n \geq 0\}$.

2.3. Kripke sheaves

Let us recall some definitions and basic facts about Kripke sheaves from [9].

Definition 2.2. A *Kripke sheaf* over a propositional Kripke frame $F = (W, R)$ is a triple $\Phi = (F, D, \rho)$ where $D = (D_u)_{u \in W}$ is a family of nonempty domains, $\rho = (\rho_{uv})_{(u,v) \in R^*}$[3] is a family of *transition functions* $\rho_{uv} : D_u \longrightarrow D_v$ such that

- every ρ_{uu} is the identity function on D_u;
- uR^*vR^*w implies $\rho_{vw}\rho_{uv} = \rho_{uw}$.

F is called the *propositional base* of Φ.

Definition 2.3. A *valuation* on a Kripke sheaf Φ is a function ξ on predicate letters such that for every n-ary predicate letter P_k^n

$$\xi(P_k^n) = (\xi_u(P_k^n))_{u \in W},$$

where $\xi_u(P_k^n) \subseteq D_u^n$ (and D_u^0 is a fixed singleton $\{()\}$, where $()$ is the empty tuple).

The pair $M = (\mathbf{F}, \xi)$ is a *Kripke sheaf model* over Φ.

Given M, at every point $u \in W$ we evaluate *modal D_u-sentences*, i.e. sentences with constants from D_u:

$$\begin{array}{l}
M, u \vDash P_k^n(a_1, \dots, a_n) \text{ iff } (a_1, \dots, a_n) \in \xi_u(P_k^n),\\
M, u \vDash A \to B \text{ iff } (M, u \nvDash A \text{ or } M, u \vDash B),\\
M, u \nvDash \bot,\\
M, u \vDash \forall x A(x) \text{ iff } \forall a \in D_u\ M, u \vDash A(a),\\
M, u \vDash \Box A \text{ iff } \forall v \in R(u)\ M, v \vDash A|v,
\end{array}$$

where $A|v$ denotes the D_v-sentence obtained from A by replacing every individual $a \in D_u$ with $\rho_{uv}(a)$.

A D_u-sentence A is *valid at* Φ, u (in symbols, $\Phi, u \vDash A$), if $M, u \vDash A$ for any Kripke sheaf model M over Φ.

A modal formula A is *true in* M (in symbols, $M \vDash A$) if $M, u \vDash \overline{\forall} A$ for any $u \in W$. A is *valid* on a Kripke sheaf Φ (in symbols, $\Phi \vDash A$) if it is true in every Kripke sheaf model over Φ.

By Soundness theorem ([9], Theorem 3.6.17) $\mathbf{ML}(\Phi) := \{A \mid \Phi \vDash A\}$ is a modal predicate logic (the *modal logic of* Φ). The *modal logic of a class of Kripke sheaves* $\mathcal{C}$ is $\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML}(\Phi) \mid \Phi \in \mathcal{C}\}$. Logics $\mathbf{ML}(\mathcal{C})$ are called *Kripke sheaf complete* (or *$\mathcal{KE}$-complete*).

[3] R^* denotes the reflexive transitive closure of R.

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A Kripke sheaf validating a modal predicate logic L is called an *L-sheaf*. So L is Kripke sheaf complete iff every sentence $A \notin L$ can be refuted in some L-sheaf. A formula A is a *logical consequence* of a logic L in Kripke sheaf semantics (in symbols, $L \vDash_{\mathcal{KE}} A$) if A is valid on all L-sheaves. The logic $C_{\mathcal{KE}}(L) := \{A \mid L \vDash_{\mathcal{KE}} A\}$ is the smallest $\mathcal{KE}$-complete extension of L, the *$\mathcal{KE}$-completion* of L.

From definitions we readily have

Lemma 2.4.

(1) *For a Kripke sheaf $\Phi = (W, R, D, \rho)$ and a sentence A*

$$\Phi, u \vDash \Box A \textit{ iff } \forall v \in R(u)\, \Phi, v \vDash A.$$

(2) *For a Kripke sheaf $\Phi = (F, D, \rho)$ and a propositional formula A,*

$$\Phi \vDash A \textit{ iff } F \vDash A.$$

Definition 2.5. Let u be a point in a Kripke sheaf model M, Γ a predicate theory. An *interpretation* of Γ at (M, u) is a map $\delta : D_\Gamma \longrightarrow D_u$. Given such an interpretation, we can transform every sentence $A \in \mathcal{L}(\Gamma)$ into a D_u-sentence $\delta \cdot A$ by replacing all occurrences of every constant $c \in D_\Gamma$ with $\delta(c)$.

Γ is *satisfiable* at (M, u) if there exists an interpretation $\delta : D_\Gamma \longrightarrow D_u$ such that $M, u \vDash \delta \cdot A$ for all $A \in \Gamma$. Γ is *satisfiable* in M if it is satisfiable at (M, u) for some point u.

Definition 2.6. A modal predicate logic L is called *strongly Kripke sheaf (or $\mathcal{KE}$-) complete*, if every L-consistent theory is satisfiable in some Kripke sheaf model over an L-sheaf.

2.4. Kripke semantics

The standard Kripke semantics for modal predicate logics can be regarded as a particular case of Kripke sheaf semantics. Recall that a *predicate Kripke frame* over a propositional frame $F = (W, R)$ is a pair $\mathbf{F} = (F, D)$, where $D = (D_u)_{u \in W}$, each D_u is nonempty and $D_u \subseteq D_v$ whenever uRv. So $\mathbf{F}$ is actually a Kripke sheaf over F, in which every ρ_{uv} is the inclusion map $D_u \hookrightarrow D_v$. F is called the *propositional base* of $\mathbf{F}$.

In this case Kripke sheaf models become *Kripke models*, and the truth definition for $\Box A$ transforms to the familiar one:

$$M, u \vDash \Box A \text{ iff } \forall v \in R(u)\ M, v \vDash A.$$

Other definitions (of the truth in a model, validity etc.) are the same as for arbitrary Kripke sheaves. In particular,

$$\mathbf{ML}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash A\},\ \mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$$

for a class of frames $\mathcal{C}$. Logics of the latter form are called *Kripke complete*.

Logical consequence in Kripke semantics is denoted by $\vDash_{\mathcal{K}}$. The set $\{A \mid L \vDash_{\mathcal{K}} A\}$ is the smallest Kripke complete extension of L, the *Kripke completion* of L; it is denoted by $C_{\mathcal{K}}(L)$, or (more often) by $\widehat{L}$.

Definition 2.7. A modal predicate logic L is called *strongly Kripke complete* if every L-consistent theory is satisfiable in some Kripke model over an L-frame.[4]

2.5. Some Kripke sheaf constructions

Definition 2.8. A *subsheaf* of a Kripke sheaf $\Phi = (W, R, D)$ obtained by restriction to $V \subseteq W$ is $\Phi \restriction V := (V, R \restriction V, D \restriction V, \rho \restriction V)$, where $R \restriction V = R \cap (V \times V)$, $D \restriction V = (D_u)_{u \in V}$, $(\rho \restriction V)_{uv} = \rho_{uv}$ for $u, v \in V$.

A *submodel* $M \restriction V$ of a Kripke sheaf model $M = (\Phi, \xi)$ is $M \restriction V = (\Phi \restriction V, \xi \restriction V)$, where $(\xi \restriction V)_u = \xi_u$ for each $u \in V$. If V is stable (i.e. $R(V) \subseteq V$), the subsheaf $\Phi \restriction V$ and the submodel $M \restriction V$ are called *generated*.

Lemma 2.9 *(Generation lemma). For generated subsheaves and submodels:*

(1) *for any $u \in V$ and D_u-sentence A*

$$M, u \models A \text{ iff } M \restriction V, u \models A;$$

(2) *for any $u \in V$ and D_u-sentence A*

$$\Phi, u \models A \text{ iff } \Phi \restriction V, u \models A;$$

(3) *for any formula B*

$$M \models B \text{ implies } M \restriction V \models B;$$

(4) $\mathbf{ML}(\Phi) \subseteq \mathbf{ML}(\Phi \restriction V)$.

Recall that the disjoint sum of propositional Kripke frames is

$$\bigsqcup_{i \in I} (W_i, R_i) := (W, R), \text{ where } W = \bigcup_{i \in I} (W_i \times \{i\}), \ (x, i)R(y, j) \text{ iff } i = j \,\&\, xR_i y.$$

Definition 2.10. For a family of Kripke sheaves, $\Phi_i = (F_i, D_i, \rho_i)$, $i \in I$, the *disjoint sum*[5] is the Kripke sheaf

$$\bigsqcup_{i \in I} \Phi_i := \left(\bigsqcup_{i \in I} F_i, D, \rho \right),$$

where

$$D_{(u,i)} := (D_i)_u, \ \rho_{(u,i)(v,i)}(a) := \rho_{uv}(a).$$

Then the *disjoint sum* of Kripke sheaf models $M_i = (\Phi_i, \theta_i)$ is

[4] The definition of strong completeness given in [9] involves only theories without constants. That definition does not fit for our purposes.
[5] In [9] disjoint sums are defined in another way. Here we need a slightly different notion, for which we use the same terminology and notation. Two versions of disjoint sums are logically equivalent.

$$\bigsqcup_{i\in I} M_i := \left(\bigsqcup_{i\in I} \Phi_i, \theta\right),$$

where

$$\theta_{(u,i)} := (\theta_i)_u.$$

A particular case of a generated subsheaf (submodel) is a cone:

Definition 2.11. A *cone* in a Kripke sheaf Φ is a generated subsheaf of the form $\Phi{\uparrow}u := \Phi \restriction R^*(u)$, where R^* is the reflexive transitive closure of the accessibility relation R.

Similarly, a *cone* in a Kripke sheaf model M over Φ is $M{\uparrow}u := M \restriction R^*(u)$.

For the case of Kripke frames the above definitions are rather well-known, so we do not write them explicitly.

2.6. Canonical models

Now we recall the construction and some properties of canonical models for modal predicate logics ([9], section 6.1).

Let us fix a universal countable set of constants S^*. A subset $S \subseteq S^*$ is called *small* if $(S^* - S)$ is infinite. We will consider only theories Γ, for which D_Γ is a small subset of S^*.

Definition 2.12. A predicate theory is called *L-complete* if it is maximal among L-consistent theories in the same language.

Lemma 2.13. *If a predicate theory* Γ *is* L*-complete,* $A \in \mathcal{L}(\Gamma)$*, then*

$$\Gamma \vdash_L A \text{ iff } A \in \Gamma;$$

in particular, $\overline{L} \subseteq \Gamma$.

Definition 2.14. An theory Γ has the *Henkin property* if for any sentence $\exists x A(x) \in \mathcal{L}(\Gamma)$ there exists a constant $c \in D_\Gamma$ such that

$$(\exists x A(x) \to A(c)) \in \Gamma.$$

An L-complete theory with the Henkin property is called *L-Henkin*. An *L-place* is an L-Henkin theory with a small set of constants.

Lemma 2.15. *Every* L*-consistent theory with a small set of constants can be extended to an* L*-place.*

For any modal predicate logic L there exists a canonical frame $PF_L = (PW_L, R_L, D_L)$ and a canonical model $PM_L = (PF_L, \xi_L)$,[6] where

- PW_L is the set of all L-places,
- $\Gamma R_L \Delta$ iff for any A, $\Box A \in \Gamma$ implies $A \in \Delta$,

[6] [9] uses a different notation for PF_L and PM_L, and they were called 'V-canonical'.

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- $(D_L)_\Gamma = D_\Gamma$, the set of all constants occurring in Γ,
- $PM_L, \Gamma \vDash A$ iff $A \in \Gamma$ for any atomic D_Γ-sentence A.

Theorem 2.16 *(Canonical model theorem).*

$$PM_L, \Gamma \vDash A \textit{ iff } A \in \Gamma$$

for any L-place Γ and $A \in \mathcal{L}(\Gamma)$.

Definition 2.17. A modal predicate logic L is called *canonical* if $PF_L \vDash L$ (or, equivalently, $PF_L \vDash \overline{L}$).

Theorem 2.16 implies

Corollary 2.18. *Every canonical logic is strongly Kripke complete.*

Proof. Assume that L is canonical, Γ_0 is an L-consistent theory. By our general assumption, D_{Γ_0} is a small subset of S^*. So there exists an L-place $\Gamma \supseteq \Gamma_0$ (Lemma 2.15). By Theorem 2.16, $PM_L, \Gamma \vDash \Gamma$, thus Γ_0 is satisfiable at (PM_L, Γ) under the trivial interpretation $\delta : D_{\Gamma_0} \longrightarrow D_\Gamma$ sending each constant into itself. By canonicity, $PF_L \vDash L$, therefore L is strongly Kripke complete. □

Lemma 2.19. *(Cf. [9], Lemma 6.1.25.) Let $L,\ L_1$ be modal predicate logics such that $L \subseteq L_1$. Then*

- $PW_{L_1} = \{\Gamma \in PW_L \mid \overline{L_1} \subseteq \Gamma\}$.
- *PM_{L_1} is a generated submodel of PM_L.*

3. Boxing

3.1. Propositional boxing

For a set of modal formulas X, put

$$\Box X := \{\Box A \mid A \in X\}.$$

Definition 3.1. For a propositional modal logic $\mathbf{\Lambda}$, we define its *boxing* as $\Box{\cdot}\mathbf{\Lambda} := \mathbf{K} + \Box\mathbf{\Lambda}$.

Note that $\Box{\cdot}\mathbf{\Lambda} \subseteq \mathbf{\Lambda}$.

Proposition 3.2. *Let X be a set of propositional modal formulas. Then*

$$\Box \cdot (\mathbf{K} + X) = \mathbf{K} + \Box X.$$

Proof. $\Box X \subseteq \Box(\mathbf{K} + X)$, so $\mathbf{K} + \Box X \subseteq \Box{\cdot}(\mathbf{K} + X)$.

To check the converse inclusion $\Box{\cdot}(\mathbf{K} + X) \subseteq \mathbf{K} + \Box X$, note that theorems of $\mathbf{K} + X$ can be derived from $\mathbf{K}$ and substitution instances of X by applying (MP) and $\Box$-introduction. So by induction on the derivation of $A \in \mathbf{K} + X$ we show that $\Box A \in \mathbf{K} + \Box X$.

Let $\mathbf{\Lambda} := \mathbf{K} + \Box X$.

If $A \in \mathbf{K}$, the claim is trivial.

If $A = B(C_1, \dots, C_n)$ is a substitution instance of $B(p_1, \dots, p_n) \in X$, then $\Box A = \Box B(C_1, \dots, C_n)$ is a substitution instance of $\Box B$, so $A \in \mathbf{\Lambda}$.

If A is obtained by (MP), from B and $B \to A$, then by IH

$$\Box B,\ \Box(B \to A) \in \mathbf{\Lambda}.$$

Since $\Box A$ is derivable from $\Box B,\ \Box(B \to A)$ in $\mathbf{K}$ using (MP), we have $\Box A \in \mathbf{\Lambda}$.

If $A = \Box B$ and $B \in (\mathbf{K} + X)$, then by IH, $\Box B \in \mathbf{\Lambda}$, i.e. $A \in \mathbf{\Lambda}$. □

Lemma 3.3. $\Box A \in \Box{\cdot}\mathbf{\Lambda}$ *only if* $A \in \mathbf{\Lambda}$.

Proof. Suppose $A \notin \mathbf{\Lambda}$. Then for some Kripke model M (the canonical model of $\mathbf{\Lambda}$) we have $M \vDash \mathbf{\Lambda}$, $M \nvDash A$.

Consider a model M^+ obtained by adding the root 0 below M, so that every point of M is accessible from 0. The truth values of proposition letters at 0 can be arbitrary.

Then M is a generated submodel of M^+, so by Generation lemma, $M^+, u \vDash \mathbf{\Lambda}$ for any $u \in M$. Hence $M^+, 0 \vDash \Box\mathbf{\Lambda}$, and also $M^+, u \vDash \Box\mathbf{\Lambda}$ for $u \in M$. Thus $M^+ \vDash \Box\mathbf{\Lambda}$.

Now note that $\Box\mathbf{\Lambda}$ is substitution closed, so every member of $\Box \cdot \mathbf{\Lambda}$ is derivable from $\Box\mathbf{\Lambda}$ and $\mathbf{K}$ using (MP) and $\Box$-introduction. Both these rules preserve the truth in M^+, thus $M^+ \vDash \Box{\cdot}\mathbf{\Lambda}$.

At the same time $M, u \nvDash A$ for some $u \in M$, hence $M^+, u \nvDash A$ by Generation lemma, and thus $M^+, 0 \nvDash \Box A$.

Since M^+ is a model of $\Box{\cdot}\mathbf{\Lambda}$ refuting $\Box A$, it follows that $\Box A \notin \Box{\cdot}\mathbf{\Lambda}$. □

Proposition 3.4.

(1) *Boxing embeds the poset of modal propositional logics in itself:*

$$\mathbf{\Lambda}_1 \subseteq \mathbf{\Lambda}_2 \text{ iff } \Box \cdot \mathbf{\Lambda}_1 \subseteq \Box \cdot \mathbf{\Lambda}_2.$$

(2) *Boxing is a complete embedding of the upper semilattice of modal propositional logics in itself.*

Proof. (1) 'Only if' is obvious. For 'if', suppose $\mathbf{\Lambda}_1 \not\subseteq \mathbf{\Lambda}_2,\ A \in \mathbf{\Lambda}_1 - \mathbf{\Lambda}_2$. Then $\Box A \in \Box{\cdot}\mathbf{\Lambda}_1$ by Definition 3.1, $\Box A \notin \Box \cdot \mathbf{\Lambda}_2$, by Lemma 3.3. Thus $\Box \cdot \mathbf{\Lambda}_1 \not\subseteq \Box \cdot \mathbf{\Lambda}_2$.

(2) Consider logics $\mathbf{\Lambda}_i = \mathbf{K} + X_i$ for $i \in I$; their join is

$$\sum_{i\in I} \mathbf{\Lambda}_i = \mathbf{K} + \bigcup_{i\in I} X_i.$$

By Proposition 3.2

$$\Box \cdot \mathbf{\Lambda}_i = \mathbf{K} + \Box X_i,$$

hence

$$\sum_{i\in I} \Box \cdot \mathbf{\Lambda}_i = \mathbf{K} + \bigcup_{i\in I} \Box X_i = \mathbf{K} + \Box(\bigcup_{i\in I} X_i),$$

which is $\Box \cdot \sum_{i\in I} \mathbf{\Lambda}_i$ by Proposition 3.2 again. □

Remark 3.5. Boxing does not preserve meets. To see this, consider the logics

$$\mathbf{\Lambda}_1 = \mathbf{Ver} = \mathbf{K} + \Box\bot,\ \ \mathbf{\Lambda}_2 = \mathbf{Triv} = \mathbf{K} + (\Box p \leftrightarrow p)$$

and the formula

$$A_0 := \Box^2\bot \vee \Box(\Box p \leftrightarrow p).$$

We claim that $A_0 \in ((\Box \cdot \mathbf{\Lambda}_1 \cap \Box \cdot \mathbf{\Lambda}_2) - \Box \cdot (\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2))$.

Indeed, $A_0 \in \Box \cdot \mathbf{\Lambda}_1$, since $\Box^2\bot \in \Box \cdot \mathbf{\Lambda}_1$; $A_0 \in \Box \cdot \mathbf{\Lambda}_2$, since $\Box(\Box p \leftrightarrow p) \in \Box \cdot \mathbf{\Lambda}_2$.

To show that $A_0 \notin \Box \cdot (\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2)$, consider the frame $F_0 = (W_0, R_0)$, where $W_0 = \{0, 1, 2\}$, $R_0(0) = \{1, 2\}$, $R_0(1) = \varnothing$, $R_0(2) = \{2\}$. Then $F_0, 1 \vDash \mathbf{\Lambda}_1$ and $F_0, 2 \vDash \mathbf{\Lambda}_2$. Hence $F_0, 0 \vDash \Box(\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2)$ (by Lemma 2.4(1)). Also $F_0, 1 \vDash \Box\mathbf{\Lambda}_1$, $F_0, 2 \vDash \Box\mathbf{\Lambda}_2$, and thus $F_0, 1 \vDash \Box(\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2)$, $F_0, 2 \vDash \Box(\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2)$.

Thus $F_0 \vDash \Box(\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2)$ implying $F_0 \vDash \Box \cdot (\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2)$. On the other hand, $F_0, 0 \nvDash A_0$, since $F_0, 1 \nvDash \Box p \leftrightarrow p$ and $F_0, 2 \nvDash \Box\bot$. Therefore $A_0 \notin \Box \cdot (\mathbf{\Lambda}_1 \cap \mathbf{\Lambda}_2)$.

3.2. Predicate boxing

We define boxing for predicate logics similarly to the propositional case.

Definition 3.6. For a modal predicate logic L put $\Box \bullet L := \mathbf{QK} + \Box L$.

Lemma 3.7.

(1) $\Box \bullet L \subseteq L$.
(2) *For a predicate theory X without constants,*

$$\mathbf{QK} + \Box X \subseteq \Box \bullet (\mathbf{QK} + X).$$

(3) *For a propositional logic $\mathbf{\Lambda}$,*

$$\mathbf{Q}(\Box \cdot \mathbf{\Lambda}) \subseteq \Box \bullet \mathbf{Q\Lambda}.$$

Proof. (1), (2) are obvious. (3) follows from (2) for $X = \mathbf{\Lambda}$; notice that

$$\mathbf{QK} + \Box\mathbf{\Lambda} = \mathbf{QK} + (\mathbf{K} + \Box\mathbf{\Lambda}) = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}). \quad \Box$$

In general, the inclusions in (2) and (3) cannot be replaced with equality, as we shall see later on, so boxing does not commute with minimal quantifier extensions.[7] That is why we use different notation for propositional and predicate boxing.

Thus axiomatization of boxing in the predicate case makes some problem. However, the problem disappears after adding the Barcan axiom:

$$Ba := \forall x \Box P(x) \to \Box \forall x P(x).$$

Lemma 3.8. *For any set of modal sentences X,*

$$\Box \bullet (\mathbf{QK} + X) + Ba = \mathbf{QK} + \Box X + Ba.$$

In particular, for a propositional logic $\mathbf{\Lambda}$,

$$\Box \bullet \mathbf{Q\Lambda} + Ba = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}) + Ba.$$

[7] For example, for the logic $\mathbf{T} = \mathbf{K} + \Box p \to p$ we have $\mathbf{Q}(\Box \cdot \mathbf{T}) \neq \Box \bullet \mathbf{QT}$, since obviously, $\Box \bullet \mathbf{QT} \vdash \Box \forall x(\Box P(x) \to P(x))$, while $\mathbf{Q}(\Box \cdot \mathbf{T}) \nvdash \Box \forall x(\Box P(x) \to P(x))$ (see Lemma 5.12).

Proof. The inclusion ($\supseteq$) holds due to Lemma 3.7, so let us prove ($\subseteq$). It suffices to show that

$$\Box \bullet (\mathbf{QK} + X) \subseteq \mathbf{QK} + \Box X + Ba,$$

i.e. $\mathbf{QK} + X \vdash A$ implies $L \vdash \Box A$, where $L := \mathbf{QK} + \Box X + Ba$.

Note that theorems of $\mathbf{QK} + X$ are derivable from the axioms of $\mathbf{QK}$ and substitution instances of X by applying (MP), $\Box$-introduction, and $\forall$-introduction (Proposition 2.1). So we can consider derivations of this kind and argue by induction on the derivation of A. The proof is similar to Proposition 3.2.

We consider only two cases.

If A is a substitution instance of the form SB, where $B \in X$, S is a predicate substitution, then $\Box A = S\Box B$, and $\Box B \in \Box X$.

If $A = \forall x B$ is obtained by $\forall$-introduction from B, then $L \vdash \Box B$ by IH, hence by $\forall$-introduction, $L \vdash \forall x \Box B$. By applying substitution to Ba and next (MP), we have $L \vdash \Box \forall x B (= \Box A)$. □

Next, consider extensions of $\mathbf{QT} = \mathbf{QK} + \Box p \to p$.

Lemma 3.9. *For any set of modal sentences X,*

$$\Box \bullet (\mathbf{QK} + X) \subseteq \mathbf{QK} + \Box X + \Box \forall ref,$$

where

$$\forall ref := \forall x (\Box P(x) \to P(x)).$$

Proof. Similar to the previous lemma. Let $L := \mathbf{QK} + \Box X + \Box \forall ref$, and let us show that $\mathbf{QK} + X \vdash A$ implies $L \vdash \Box A$. Again we can use Proposition 2.1 and argue by induction on the derivation of A using $\mathbf{QK}$, substitution instances of X, (MP), $\Box$- and $\forall$-introduction.

The only nontrivial case is when $A = \forall x B$ is obtained by $\forall$-introduction from B. Then $L \vdash \Box B$ by IH, hence

$$L \vdash \Box \forall x \Box B \qquad (*)$$

by $\forall$- and $\Box$-introduction. By substitution into $\Box \forall ref$, we have $L \vdash \Box \forall x (\Box B \to B)$, hence by $\mathbf{QK}$ we obtain

$$L \vdash \Box (\forall x \Box B \to \forall x B),$$

and next

$$L \vdash \Box \forall x \Box B \to \Box \forall x B. \qquad (**)$$

Now $L \vdash \Box A$ follows from $(*), (**)$ by (MP). □

Proposition 3.10. *If $\mathbf{QT} \subseteq \mathbf{QK} + X$, then $\Box \bullet (\mathbf{QK} + X) = \mathbf{QK} + \Box X + \Box \forall ref$. In particular, for a propositional logic $\mathbf{\Lambda} \supseteq \mathbf{T}$,*

$$\Box \bullet \mathbf{Q\Lambda} = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}) + \Box \forall ref.$$

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Proof. We have $\mathbf{QT} \vdash \forall ref$ by substitution and $\forall$-introduction, so $\mathbf{QK} + X \vdash \forall ref$, and thus $\Box \bullet (\mathbf{QK} + X) \vdash \Box \forall ref$. Obviously, $\Box X \subseteq \Box(\mathbf{QK} + X)$, hence

$$\mathbf{QK} + \Box X + \Box \forall ref \subseteq \Box \bullet (\mathbf{QK} + X).$$

The converse inclusion holds by Lemma 3.9. $\Box$

Lemma 3.11. *For a Kripke sheaf model M*

$$M \models \Box \bullet L \text{ iff } M \models \Box \overline{L}.$$

Proof. 'Only if' holds, since $\Box \overline{L} \subseteq \Box \bullet L$.

Let us prove 'if'. Suppose $M \models \Box \overline{L}$.

If $A \in L$, then $\overline{\forall} A \in \overline{L}$, so $M \models \Box \overline{\forall} A$. Since $\mathbf{QK} \vdash \Box \overline{\forall} A \to \overline{\forall} \Box A$ (by the converse Barcan formula, cf. [11], p. 245), it follows that $M \models \overline{\forall} \Box A$, i.e. $M \models \Box A$.

Thus $M \models \Box L$. Now we can argue similarly to Lemma 3.8. Since the set $\Box L$ is substitution closed, theorems of $\Box \bullet L = \mathbf{QK} + \Box L$ can be obtained from $\mathbf{QK} \cup \Box L$ by applying (MP), $\forall$-introduction and $\Box$-introduction. These rules preserve the truth in M, therefore $M \models \Box \bullet L$. $\Box$

We also have an analogue of Lemma 3.3:

Lemma 3.12. *For any modal sentence A, $\Box \bullet L \vdash \Box A$ implies $L \vdash A$.*

Proof. We use the canonical model almost in the same way as in the proof of Lemma 3.3.

Suppose $L \nvdash A$. Then we have $PM_L, \Gamma \not\models A$, $PM_L \models L$ for some L-place Γ. Let $M := PM_L \uparrow \Gamma$ be the corresponding rooted generated submodel; by Lemma 2.9(1), (3), we have $M, \Gamma \not\models A$, $M \models L$ as well.

Consider a model M^+ obtained by adding the root 0 below M, so that only Γ is accessible from 0 and the domain at 0 is the same as at Γ. The valuation of predicate letters at 0 does not matter.

Then M is a generated submodel of M^+, so by Lemma 2.9(1), $M \models L$ implies $M^+, u \models \overline{L}$ for any $u \in M$. Hence $M^+, 0 \models \Box \overline{L}$, and also $M^+, u \models \Box \overline{L}$ for $u \in M$. Thus $M^+ \models \Box \overline{L}$ implying $M^+ \models \Box \bullet L$ by Lemma 3.11.

$M, \Gamma \not\models A$ implies $M^+, \Gamma \not\models A$ by Lemma 2.9(1), and thus $M^+, 0 \not\models \Box A$.

Since M^+ is a model of $\Box \bullet L$ refuting $\Box A$, we obtain $\Box \bullet L \nvdash \Box A$. $\Box$

Similarly to Proposition 3.4(1) we have

Proposition 3.13. *Boxing embeds the poset of modal predicate logics in itself:*

$$L_1 \subseteq L_2 \text{ iff } \Box \bullet L_1 \subseteq \Box \bullet L_2.$$

Proof. For the proof of 'if', note that $A \in (L_1 - L_2)$ implies $\Box A \in \Box \bullet L_1$ by Definition 3.1, $\Box A \notin \Box \bullet L_2$, by Lemma 3.12. $\Box$

Lemma 3.14. *For any set of sentences X*

$$\Box \bullet \widehat{\mathbf{QK} + X} \subseteq \widehat{\mathbf{QK} + \Box X}.$$

Proof. We have to show that for any sentence A, $\mathbf{QK} + X \models_{\mathcal{K}} A$ implies $\mathbf{QK} + \Box X \models_{\mathcal{K}} \Box A$. So assuming $\mathbf{QK} + X \models_{\mathcal{K}} A$, $\mathbf{F} = (W, R, D) \models \Box X$, let us prove that $\mathbf{F} \models \Box A$.

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Consider the set $V := \{v \in W \mid R^{-1}(v) \neq \varnothing\}$. We have $\mathbf{F}, u \vDash \Box X$ for any u, so uRv implies $\mathbf{F}, v \vDash X$ (Lemma 2.4). Thus $\mathbf{F}, v \vDash X$ for any $v \in V$. Since V is stable, we have $\mathbf{F} \restriction V, v \vDash X$ (Lemma 2.9 (2)). Hence $\mathbf{F} \restriction V \vDash X$, so $\mathbf{F} \restriction V \vDash A$, due to the assumption $\mathbf{QK} + X \vDash_{\mathcal{K}} A$.

Now for any u, uRv implies $v \in V$, so $\mathbf{F} \restriction V, v \vDash A$. Then $\mathbf{F}, v \vDash A$ (Lemma 2.9 (2)), and thus $\mathbf{F}, u \vDash \Box A$ by Lemma 2.4.

Eventually $\mathbf{F} \vDash \Box A$. □

Proposition 3.15.

$$\mathbf{QK} + \Box X \subseteq \Box \bullet (\mathbf{QK} + X) \subseteq \Box \bullet \widehat{\mathbf{QK} + X} \subseteq C_{\mathcal{K}}(\mathbf{QK} + \Box X) = C_{\mathcal{K}}(\Box \bullet (\mathbf{QK} + X)).$$

In particular, for a modal propositional logic $\mathbf{\Lambda}$,

$$\mathbf{Q}(\Box \cdot \mathbf{\Lambda}) \subseteq \Box \bullet \mathbf{Q\Lambda} \subseteq \Box \bullet \widehat{\mathbf{Q\Lambda}} \subseteq \widehat{\mathbf{Q}(\Box \cdot \mathbf{\Lambda})} = \widehat{\Box \bullet \mathbf{Q\Lambda}}.$$

Proof. The first inclusion holds by Lemma 3.7(2), the third one by Lemma 3.14. The second inclusion is obvious.

The inclusion $C_{\mathcal{K}}(\mathbf{QK} + \Box X) \subseteq C_{\mathcal{K}}(\Box \bullet (\mathbf{QK} + X))$ is also obvious. So, since $C_{\mathcal{K}}(\mathbf{QK} + \Box X)$ is a Kripke complete extension of $\Box \bullet (\mathbf{QK} + X)$, it should coincide with $C_{\mathcal{K}}(\Box \bullet (\mathbf{QK} + X))$. □

Corollary 3.16. *Boxing preserves Kripke incompleteness.*

Proof. Suppose L is incomplete, i.e. $L \subset \widehat{L}$. Then $\Box \bullet L \subset \Box \bullet \widehat{L}$ by Proposition 3.13; also $\Box \bullet \widehat{L} \subseteq \widehat{\Box \bullet L}$ by Proposition 3.15. Thus $\Box \bullet L \subset \widehat{\Box \bullet L}$. □

As we will see later on, it often happens that a logic $\Box \bullet (\mathbf{QK} + X)$ is Kripke complete, while $\mathbf{QK} + \Box X$ is Kripke incomplete.

Analogues of Lemma 3.14, Proposition 3.15, and Corollary 3.16 hold for Kripke sheaves. The proofs are almost the same.

Lemma 3.17. $\Box \bullet C_{\mathcal{KE}}(\mathbf{QK} + X) \subseteq C_{\mathcal{KE}}(\mathbf{QK} + \Box X)$.

Proof. Assuming that $\mathbf{QK} + X \vDash_{\mathcal{KE}} A$ for a sentence A and $\Phi = (W, R, D, \rho) \vDash \Box X$ we show $\Phi \vDash \Box A$.

Again consider $V := \{v \in W \mid R^{-1}(v) \neq \varnothing\}$. Then by Lemma 2.4 and Lemma 2.9 (2) we obtain $\Phi \restriction V \vDash X$, so by assumption $\Phi \restriction V \vDash A$. Hence by the same lemmas it follows that $\Phi, u \vDash \Box A$ for any u. □

Proposition 3.18.

$$\Box \bullet C_{\mathcal{KE}}(\mathbf{QK} + X) \subseteq C_{\mathcal{KE}}(\mathbf{QK} + \Box X) = C_{\mathcal{KE}}(\Box \bullet (\mathbf{QK} + X)).$$

In particular, for a propositional logic $\mathbf{\Lambda}$

$$\Box \bullet C_{\mathcal{KE}}(\mathbf{Q\Lambda}) \subseteq C_{\mathcal{KE}}(\mathbf{Q}(\Box \cdot \mathbf{\Lambda})) = C_{\mathcal{KE}}(\Box \bullet \mathbf{Q\Lambda}).$$

Corollary 3.19. *Boxing preserves Kripke sheaf incompleteness.*

To formulate the main theorem on axiomatization of boxing we first recall the definition of shifts from [9].

Let $P_1, \dots, P_k$ be all predicate letters (besides equality) occurring in a formula A, assume that P_i is n_i-ary, and put

$$\mathbf{P} := P_1(\mathbf{x}_1), \dots, P_k(\mathbf{x}_k),$$

where each $\mathbf{x}_i$ is a list of different variables of length n_i not occurring in A. Next, let $m \geq 0$, and let P'_i be different $(m+n_i)$-ary predicate letters $(i = 1, \dots, k)$, $\mathbf{z} = z_1 \dots z_m$ a list of distinct new variables that do not occur in $\mathbf{x}_1, \dots, \mathbf{x}_k$ and A. Then we call P'_i the *m-shift* of P_i; an *m-shift* of the formula A (*by* $\mathbf{z}$) is $A^m(\mathbf{z}) := [\mathbf{P}'/\mathbf{P}]A$, where

$$\mathbf{P}' = P'_1(\mathbf{x}_1, \mathbf{z}), \dots, P'_k(\mathbf{x}_k, \mathbf{z}).$$

We also put $A^0(\mathbf{z}) := A$.

Sometimes we omit $\mathbf{z}$ and use the notation A^m rather than $A^m(\mathbf{z})$.

We need the following decomposition lemma ([9], Lemma 2.5.30):

Lemma 3.20. *Let A be a modal sentence. Then every substitution instance of A is congruent to a formula of the form $S_0 A^m$, where S_0 is a strict substitution, $m \geq 0$.*

Recall that *congruent* formulas are obtained by renaming bound variables and two congruent formulas are $\mathbf{QK}$-equivalent. *Strict* substitutions do not introduce new parameters (cf. [9], sections 2.3, 2.5).

Theorem 3.21. *For a set of modal sentences X*

$$\Box \bullet (\mathbf{QK} + X) = \mathbf{QK} + \{\Box \overline{\forall} A^m \mid A \in X,\ m \geq 0\}.$$

(In more detail, for each $A \in X$ and $m \geq 0$ we choose a list of new variables $\mathbf{z}_{A,m}$; then $\overline{\forall} A^m$ denotes $\forall \mathbf{z}_{A,m} A^m(\mathbf{z}_{A,m})$.)

Proof. The inclusion $(\supseteq)$ follows easily, since $\mathbf{QK} + X$ is closed under m-shifts, $\forall$- and $\Box$-introduction.

To prove $(\subseteq)$ we have to show that $\mathbf{QK} + X \vdash B$ implies $\mathbf{QK} + Y \vdash \Box B$, where

$$Y := \{\Box \overline{\forall} A^m \mid A \in X,\ m \geq 0\}.$$

By Proposition 2.1, if $\mathbf{QK} + X \vdash B$, then $\Box^\infty \overline{Sub}(X) \vdash_{\mathbf{QK}} B$, so by Deduction Theorem,

$$\mathbf{QK} \vdash \bigwedge Z \to B$$

for some finite $Z \subseteq \Box^\infty \overline{Sub}(X)$. Then

$$\mathbf{QK} \vdash \Box(\bigwedge Z) \to \Box B,$$

or equivalently,

$$\mathbf{QK} \vdash (\bigwedge \Box Z) \to \Box B.$$

So it remains to prove that $\mathbf{QK} + Y \vdash \Box C$ for any $C \in \Box^\infty \overline{Sub}(X)$, or even (due to $\Box$-introduction) for any $C \in \overline{Sub}(X)$. Thus we can present C as $\overline{\forall} S A$ for some $A \in X$ and substitution S.

By Lemma 3.20 we have[8] $SA \doteq S_0 A^m$; then $\Box\overline{\forall} SA \doteq \Box\overline{\forall} S_0 A^m$. S_0 commutes with $\Box$ and also with $\overline{\forall}$ (since it is strict), so we obtain

$$\Box\overline{\forall} SA \doteq S_0 \Box\overline{\forall} A^m.$$

By definition, $\overline{\forall} A^m \in Y$, hence $\mathbf{QK} + Y \vdash S_0 \Box\overline{\forall} A^m$ by substitution and $\Box$-introduction, and eventually $\mathbf{QK} + Y \vdash \Box\overline{\forall} SA\,(= \Box C)$ as required. □

Corollary 3.22. *Boxing is a complete embedding of the upper semilattice of modal predicate logics in itself.*

Proof. Similar to Proposition 3.4 (2). Consider logics $L_i = \mathbf{QK} + X_i$ for $i \in I$; their join is

$$\sum_{i \in I} L_i = \mathbf{QK} + \bigcup_{i \in I} X_i.$$

By Theorem 3.21

$$\Box \bullet L_i = \mathbf{QK} + \{\Box\overline{\forall} A^m \mid A \in X_i,\ m \geq 0\},$$

hence

$$\sum_{i \in I} \Box \bullet L_i = \mathbf{QK} + \bigcup_{i \in I} \{\Box\overline{\forall} A^m \mid A \in X_i,\ m \geq 0\} = \mathbf{QK} + \{\Box\overline{\forall} A^m \mid A \in \bigcup_{i \in I} X_i,\ m \geq 0\},$$

which is $\Box \bullet \sum_{i \in I} L_i$ by Theorem 3.21 again. □

4. Transfer theorems for boxing

4.1. Transfer of canonicity

Theorem 4.1. *Let L be a modal predicate logic. If L is canonical, then $\Box \bullet L$ is canonical.*

Proof. Suppose L is canonical and consider the canonical frame $\mathbf{G} := PF_{\Box \bullet L}$. Let us show that $\mathbf{G} \vDash \Box \bullet L$, which is equivalent to $\mathbf{G} \vDash \Box \overline{L}$ (by Lemma 3.11).

We prove that $\mathbf{G}, \Gamma \vDash \Box \overline{L}$ for an arbitrary point Γ. We have $\Box \overline{L} \subseteq \overline{\Box \bullet L} \subseteq \Gamma$ by Lemma 2.13, so $\Gamma R_{\Box \bullet L} \Delta$ implies $\overline{L} \subseteq \Delta$, i.e. $\Delta \in PW_L$ (Lemma 2.19). Also, since L is canonical, we have $PF_L, \Delta \vDash \overline{L}$. Since $\Box \bullet L \subseteq L$, by Lemma 2.19, PF_L is a generated subframe of $\mathbf{G}$; then by Lemma 2.9(2), $\mathbf{G}, \Delta \vDash \overline{L}$. This holds for any $\Delta \in R_{\Box \bullet L}(\Gamma)$, hence $\mathbf{G}, \Gamma \vDash \Box \overline{L}$ (Lemma 2.4). □

In the propositional case we have a similar theorem:

Theorem 4.2. *Boxing preserves canonicity for propositional modal logics.*

The proof is by a straightforward modification of the previous one; use the following analogue of Lemma 2.19:

Lemma 4.3. *If $\mathbf{\Lambda} \subseteq \mathbf{\Lambda}'$, then the points of $M_{\mathbf{\Lambda}}$ containing $\mathbf{\Lambda}'$ are exactly the points of $M_{\mathbf{\Lambda}'}$.*

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[8] Recall that $\doteq$ denotes congruence ([9], section 2.3).

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4.2. Transfer of strong completeness

Theorem 4.4. *Boxing preserves strong $\mathcal{KE}$-completeness.*

Proof. Assume that L is strongly $\mathcal{KE}$-complete. Let us show that every $(\Box \bullet L)$-place Γ is satisfiable in a Kripke sheaf model over a $(\Box \bullet L)$-sheaf.

The proof of Theorem 4.1 shows that every $\Delta \in R_{\Box \bullet L}(\Gamma)$ is an L-place, so by strong $\mathcal{KE}$-completeness, Δ is satisfiable in a model M_Δ over some L-sheaf $\Phi_\Delta = (W_\Delta, R_\Delta, D_\Delta, \rho_\Delta)$. Note that $D_\Gamma \subseteq D_\Delta$.

By Generation Lemma 2.9 we may assume that M_Δ is rooted with root 0_Δ, and Δ is satisfiable at $M_\Delta, 0_\Delta$. So there exists an interpretation δ_Δ such that $M_\Delta, 0_\Delta \vDash \delta_\Delta \cdot \Delta$ $(:= \{\delta_\Delta \cdot A \mid A \in \Delta\})$.

Next, we add the root Γ to the disjoint sum $\bigsqcup_{\Gamma R_{\Box \cdot L} \Delta} M_\Delta$. To simplify notation, we identify M_Δ with its image in this disjoint sum.

Consider a Kripke sheaf $\Phi := (W, R, D^\star, \rho)$ such that

(i) $R(\Gamma) := \{0_\Delta \mid \Gamma R_{\Box \bullet L} \Delta\}$, $D^\star_\Gamma := D_\Gamma$,
(ii) $R(\Sigma) := R_\Delta(\Sigma)$ for $\Sigma \in W_\Delta$,
(iii) $\rho_{\Gamma 0_\Delta} := \delta_\Delta \restriction D_\Gamma$ (the restriction of δ_Δ to D_Γ) for $\Delta \in R_{\Box \bullet L}(\Gamma)$,
(iv) $\rho_{uv} := (\rho_\Delta)_{uv}$ for $u, v \in W_\Delta$,
(v) $\rho_{\Gamma v} := \rho_{0_\Delta v}\, \rho_{\Gamma 0_\Delta}$ for $v \in W_\Delta$,
(vi) $\rho_{uu} := id_{D^\star_u}$.

We define a model M over Φ by putting

$$M, \Gamma \vDash A \text{ iff } A \in \Gamma$$

for any atomic D_Γ-sentence A.

It is clear that ρ really defines a transition function, i.e. wR^*uR^*v implies

$$\rho_{wv} = \rho_{uv}\rho_{wu}. \quad (\circ)$$

In fact, this holds for $w \neq \Gamma$, since each ρ_Δ is a transition function. So suppose $w = \Gamma$.

If $u = w$, then $(\circ)$ is obvious.

If $u, v \in W_\Delta$, then by (v), $\rho_{\Gamma u} = \rho_{0_\Delta u}\, \rho_{\Gamma 0_\Delta}$, so

$$\rho_{uv}\rho_{\Gamma u} = \rho_{uv}\rho_{0_\Delta u}\, \rho_{\Gamma 0_\Delta} = \rho_{0_\Delta v}\, \rho_{\Gamma 0_\Delta} = \rho_{\Gamma v}$$

by (v) and since $(\circ)$ holds within W_Δ.

We claim that $\Phi \vDash \Box \bullet L$ and $M, \Gamma \vDash \Gamma$.

Indeed, for every $u \in W$, $u \neq \Gamma$ we have $\Phi, u \vDash \overline{L}$, since u is in some generated subsheaf Φ_Δ; so by Lemma 2.4, $\Phi, u \vDash \Box \overline{L}$. Similarly, since the points accessible from the root Γ are 0_Δ and (as noticed above) $\Phi, 0_\Delta \vDash \overline{L}$, it follows that $\Phi, \Gamma \vDash \Box \overline{L}$ as well. Thus $\Phi \vDash \Box \overline{L}$, which implies $\Phi \vDash \Box \cdot L$ (Lemma 3.11).

Now let us show by induction on the length of a D_Γ-sentence A that

$$M, \Gamma \vDash A \text{ iff } A \in \Gamma.$$

We can consider only the cases $A = \Box B$, $A = \exists x B$.

(i) Suppose $A = \Box B$. If $A \in \Gamma$, then by definition of the canonical model, $B \in \Delta$ for every $\Delta \in R_{\Box \bullet L}(\Gamma)$. So $M_\Delta, 0_\Delta \vDash \delta_\Delta \cdot B$ by the choice of M_Δ, and thus $M, 0_\Delta \vDash \delta_\Delta \cdot B$, since M_Δ is a generated submodel of

M. Note that[9] $B|0_\Delta = \rho_{\Gamma 0_\Delta} \cdot B = \delta_\Delta \cdot B$, since $\delta_\Delta \upharpoonright D_\Gamma = \rho_{\Gamma 0_\Delta}$ ((iii) in the definition of Φ) and B is a D_Γ-sentence. This implies $M, \Gamma \vDash \Box B$, since Γ sees exactly the points 0_Δ.

On the other hand, if $A \notin \Gamma$, then by properties of the canonical model, $\neg B \in \Delta$ for some $\Delta \in R_{\Box\cdot\mathbf{\Lambda}}(\Gamma)$. So $M_\Delta, 0_\Delta \vDash \delta_\Delta \cdot \neg B$, and thus $M, 0_\Delta \nvDash \delta_\Delta \cdot B$, since M_Δ is a generated submodel of M. Since $\Gamma R 0_\Delta$ and $\delta_\Delta \cdot B = B|0_\Delta$ (as noted above), this implies $M, \Gamma \nvDash \Box B$.

(ii) Suppose $A = \exists x B(x) \in \Gamma$. Then $B(c) \in \Gamma$ for some $c \in D_\Gamma$, since Γ is a $(\Box \bullet L)$-place. This implies $M, \Gamma \vDash B(c)$ by IH, and thus $M, \Gamma \vDash A$.

The other way round, if $M, \Gamma \vDash \exists x B(x)$, then $M, \Gamma \vDash B(c)$ for some $c \in D_\Gamma$. Hence $B(c) \in \Gamma$ by IH. Then $\Gamma \vdash_{\Box\bullet L} \exists x B(x)$, and so $\exists x B(x) \in \Gamma$, since Γ is a $(\Box \bullet L)$-place.

Therefore Γ is satisfiable in a $(\Box \bullet L)$-sheaf. □

Proposition 4.5. *For modal propositional logics, boxing preserves strong Kripke completeness.*

Proof. By slight changes in the proofs of Theorems 4.1, 4.2. Every $(\Box \cdot \mathbf{\Lambda})$-consistent set is contained in some $\Gamma \in W_{\Box\cdot\mathbf{\Lambda}}$. So it suffices to show that Γ is satisfied in some $(\Box\cdot\mathbf{\Lambda})$-frame.

Every $\Delta \in R_{\Box\cdot\mathbf{\Lambda}}(\Gamma)$ contains $\mathbf{\Lambda}$, so Lemma 4.3 implies that Δ is $\mathbf{\Lambda}$-consistent. Since $\mathbf{\Lambda}$ is strongly complete, Δ is satisfied in a model M_Δ over a $\mathbf{\Lambda}$-frame F_Δ; assume that M_Δ is rooted with root 0_Δ, and $M_\Delta, 0_\Delta \vDash \Delta$.

Consider the model $M = (W, R, \xi)$ obtained from the disjoint sum $\bigsqcup_{\Gamma R_{\Box\cdot\mathbf{\Lambda}}\Delta} M_\Delta$ by adding the root Γ, so that $R(\Gamma) = \{0_\Delta \mid \Gamma R_{\Box\cdot\mathbf{\Lambda}} \Delta\}$ and $M, \Gamma \vDash q$ iff $q \in \Gamma$ for every proposition letter q. Then we have $(W, R) \vDash \Box \cdot \mathbf{\Lambda}$ and $M, \Gamma \vDash \Gamma$.

By induction on the length of A it follows that $M, \Gamma \vDash A$ iff $A \in \Gamma$.

Therefore Γ is satisfiable in the $(\Box \cdot \mathbf{\Lambda})$-frame (W, R). □

Definition 4.6. Let $M = (\mathbf{F}, \xi)$ be a Kripke model over a predicate Kripke frame $\mathbf{F} = (W, R, D)$, and let $V \neq \varnothing$.

The *inflation* of $\mathbf{F}$ by V is the frame $\mathbf{F} \odot V := (W, R, D')$, where $D'_u := D_u \times V$ for each $u \in W$.

The *inflation* of M by V is $M \odot V := (\mathbf{F} \odot V, \xi')$, where[10]

$$\xi'_u(P) := \{(d_1, \dots, d_n) \in (D'_u)^n \mid (pr_1(d_1), \dots, pr_1(d_n)) \in \xi_u(P)\}$$

for every n-ary predicate letter P.

Informally speaking, the inflation by V contains $|V|$ copies of each individual e behaving exactly as e. So we have

$$M \odot V, u \vDash P(d_1, \dots, d_n) \text{ iff } M, u \vDash P(pr_1(d_1), \dots, pr_1(d_n))$$

for n-ary P; in particular,

$$M \odot V, u \vDash P \text{ iff } M, u \vDash P$$

for 0-ary P.

Lemma 4.7. *Let M be a Kripke model over a predicate Kripke frame $\mathbf{F} = (W, R, D)$, $M \odot V = (W, R, D', \xi')$ its inflation. Then for any $u \in W$, for any D'_u-sentence $A(d_1, \dots, d_n)$ and $v \in V$*

[9] The notation $A|v$ was introduced in Definition 2.3.
[10] pr_1 denotes the first projection.

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$$M \odot V, u \vDash A(d_1, \dots, d_n) \text{ iff } M, u \vDash A(pr_1(d_1), \dots, pr_1(d_n)).$$

The proof is by induction on the length of A. In fact, this lemma is a particular case of Proposition 3.3.11 from [9]: here we have a morphism of Kripke models $(f_0, f_1) : M \odot V \longrightarrow M$, where $f_0 = id_W$, $f_{1u}(d) = pr_1(d)$, cf. [9], Definitions 3.3.1, 3.3.3.

Theorem 4.8. *If* $\mathbf{Q\Lambda}$ *is strongly Kripke complete, then* $\Box\bullet(\mathbf{Q\Lambda})$ *is strongly Kripke complete.*

Proof. We modify the proof of Theorem 4.4.

Let $L = \mathbf{Q\Lambda}$, and let Γ_0 be a $(\Box\bullet L)$-consistent theory. Recall that we assume that its set of constants is small, so by Lemma 2.13, Γ_0 is contained in some $(\Box\bullet L)$-place Γ. Let us show that Γ is satisfiable in some Kripke model over a $(\Box\bullet L)$-frame.

As in the proof of Theorem 4.4, we can see that every $\Delta \in R_{\Box\bullet L}(\Gamma)$ is an L-place. Since L is strongly complete, Δ is satisfiable in a model M_Δ over some L-frame $\mathbf{F}_\Delta$; let F_Δ be its propositional base. By Lemma 2.4(2) it follows that $F_\Delta \vDash \mathbf{\Lambda}$.

Note that $\Gamma R_{\Box\bullet L}\Delta$ implies $D_\Gamma \subseteq D_\Delta$.

We may assume that M_Δ is rooted with root 0_Δ, and Δ is satisfiable at $(M_\Delta, 0_\Delta)$. By Definition 2.5, there exists an interpretation δ_Δ such that

$$M_\Delta, 0_\Delta \vDash \delta_\Delta \cdot \Delta.$$

Now we would like to add the root Γ to the disjoint sum $\bigsqcup_{\Gamma R_{\Box\cdot L}\Delta} M_\Delta$. Unfortunately, this idea would not work directly, because the domain D_Γ may be condensed by the interpretations δ_Δ. To avoid this, we use inflation.

So for each $\Delta \in R_{\Box\cdot L}(\Gamma)$ we put $M'_\Delta := M_\Delta \odot D_\Gamma$; let D'_{0_Δ} be the root domain of M'_Δ. We call an interpretation δ'_Δ of Δ at $(M'_\Delta, 0_\Delta)$ *associated* with δ_Δ if $pr_1(\delta'_\Delta(c)) = \delta_\Delta(c)$ for any $c \in D_\Delta$. Then by Lemma 4.7, for any formula $B \in \mathcal{L}(\Delta)$

$$M'_\Delta, 0_\Delta \vDash \delta'_\Delta \cdot B \text{ iff } M_\Delta, 0_\Delta \vDash \delta_\Delta \cdot B,$$

and thus

$$M'_\Delta, 0_\Delta \vDash \delta'_\Delta \cdot \Delta. \qquad (*)$$

Now for every Δ consider a specific interpretation δ'_Δ associated with δ_Δ such that $\delta'_\Delta(c) = (\delta_\Delta(c), c)$ for every $c \in D_\Gamma$ and $\delta'_\Delta(c)$ is arbitrary for $c \in D_\Delta - D_\Gamma$ (with the only requirement $pr_1(\delta'_\Delta(c)) = \delta_\Delta(c)$). Then δ'_Δ is injective on D_Γ.

Next, we cross-identify some individuals in the domains D'_{0_Δ}.

Namely, let h be the map defined on the total domain of M'_Δ such that

$$h(a) := \begin{cases} c \text{ if } a = \delta'_\Delta(c),\ c \in D_\Gamma, \\ a \text{ otherwise.} \end{cases}$$

If D'_u is the domain at world u of M'_Δ, we define D''_u as the image set $h(D'_u)$; so h is a bijection[11] from D'_u onto D''_u. It is clear that $D_\Gamma \subseteq D''_{0_\Delta}$.

[11] For injectivity of h we need a technical assumption that D_Γ does not contain ordered pairs. This can be achieved by an appropriate choice of the basic set S^* (cf. subsection 2.6).

Then we define the Kripke model M''_Δ over the predicate frame $\mathbf{F}''_\Delta := (F_\Delta, D'')$ as follows:

$$M'_\Delta, u \vDash P(a_1, \dots, a_n) \text{ iff } M''_\Delta, u \vDash P(h(a_1), \dots, h(a_n))$$

for any world u, predicate letter P and $a_1, \dots, a_n \in D'_u$. So there is an isomorphism from M'_Δ onto M''_Δ sending every world to itself and every individual a to $h(a)$. Thus

$$M'_\Delta, u \vDash A(a_1, \dots, a_n) \text{ iff } M''_\Delta, u \vDash A(h(a_1), \dots, h(a_n)) \tag{$**$}$$

for any world u, formula $A(x_1, \dots, x_n)$ and tuple $(a_1, \dots, a_n) \in (D'_u)^n$. Next, we define an interpretation $\delta''_\Delta := h \cdot \delta'_\Delta$, so

$$\delta''_\Delta(c) = \begin{cases} c \text{ if } c \in D_\Gamma, \\ \delta'_\Delta(c) \text{ if } c \in D_\Delta - D_\Gamma. \end{cases}$$

Then by (**) for any formula B in $\mathcal{L}(\Delta)$ we have

$$M'_\Delta, 0_\Delta \vDash \delta'_\Delta \cdot B \text{ iff } M''_\Delta, 0_\Delta \vDash \delta''_\Delta \cdot B.$$

Therefore $(*)$ implies

$$M''_\Delta, 0_\Delta \vDash \delta''_\Delta \cdot \Delta. \tag{$***$}$$

Let $\mathbf{F}''_\Delta$ be the frame of M''_Δ. Its propositional base is the same F_Δ, so $\mathbf{F}''_\Delta \vDash \mathbf{\Lambda}$ by Lemma 2.4, which implies $\mathbf{F}''_\Delta \vDash L$.

Next, we construct the disjoint sum $M'' := \bigsqcup_{\Gamma R_{\Box \cdot L} \Delta} M''_\Delta$ and add the root Γ to it. This is possible, since $D_\Gamma \subseteq D''_{0_\Delta}$. Thus we obtain a Kripke model $M^\star$ over a frame $\mathbf{G} = (W, R, D^\star)$ such that

$$R(\Gamma) = \{0_\Delta \mid \Gamma R_{\Box \cdot L} \Delta\},\ D^\star_\Gamma = D_\Gamma,$$
$$M^\star, \Gamma \vDash A \text{ iff } A \in \Gamma$$

for any atomic D_Γ-sentence A.

We claim that $\mathbf{G} \vDash \Box \bullet L$ and $M^\star, \Gamma \vDash \Gamma$.

This is proved as in Theorem 4.4. By Lemma 3.11, it suffices to show that $\mathbf{G} \vDash \Box \overline{L}$. First note that for every $u \in W,\ u \neq \Gamma$ we have $\mathbf{G}, u \vDash \overline{L}$, since u is in some generated subframe $\mathbf{F}''_\Delta$ validating L; thus $\mathbf{G}, u \vDash \Box \overline{L}$. In particular, $\mathbf{G}, 0_\Delta \vDash \overline{L}$. Since $R(\Gamma)$ consists of the points 0_Δ, it follows that $\mathbf{G}, \Gamma \vDash \Box \overline{L}$ as well (by Lemma 2.4).

Next, let us show by induction on the length of a D_Γ-sentence A that

$$M^\star, \Gamma \vDash A \text{ iff } A \in \Gamma.$$

If A is atomic, this holds by definition.

We skip the cases $A = \bot$, $A = B \to C$.

Suppose $A = \Box B$. If $A \in \Gamma$, then by definition of the canonical model, $B \in \Delta$ for every $\Delta \in R_{\Box \bullet L}(\Gamma)$. So by (***), $M''_\Delta, 0_\Delta \vDash \delta''_\Delta \cdot B = B$ (since B is a D_Γ-sentence), and thus $M^\star, 0_\Delta \vDash B$, since M''_Δ is (isomorphic to) a generated submodel of $M^\star$. Therefore $M^\star, \Gamma \vDash \Box B$, since Γ sees only these 0_Δ.

On the other hand, if $A \notin \Gamma$, then by the properties of the canonical model, $\neg B \in \Delta$ for some $\Delta \in R_{\Box \bullet L}(\Gamma)$. So by (***) again, $M''_\Delta, 0_\Delta \vDash \neg B$, and thus $M^\star, 0_\Delta \nvDash B$, since M''_Δ is a generated submodel of $M^\star$. Since $\Gamma R 0_\Delta$, we have $M^\star, \Gamma \nvDash \Box B$.

Suppose $A = \exists x B(x) \in \Gamma$. Then $B(c) \in \Gamma$ for some $c \in D_\Gamma$, since Γ is a $(\Box \bullet L)$-place. Hence $M^\star, \Gamma \vDash B(c)$ by IH, which implies $M^\star, \Gamma \vDash A$.

The other way round, if $M^\star, \Gamma \vDash \exists x B(x)$, then $M^\star, \Gamma \vDash B(c)$ for some $c \in D_\Gamma$. Hence $B(c) \in \Gamma$ by IH. Then $\Gamma \vdash_{\Box \bullet L} \exists x B(x)$, and so $\exists x B(x) \in \Gamma$, since Γ is a $(\Box \bullet L)$-place.

Therefore Γ is satisfiable in a $(\Box \bullet L)$-frame. □

4.3. Transfer of Kripke completeness for propositional logics

Theorem 4.9. *For modal propositional logics, boxing preserves Kripke completeness.*

Proof. We modify the proof of Proposition 4.5. Assume that $\mathbf{\Lambda}$ is Kripke complete, let A be a $(\Box \cdot \mathbf{\Lambda})$-consistent formula, and consider the canonical model $M := M_{\Box \cdot \mathbf{\Lambda}}$. Then $M, \Gamma \vDash A$ for some Γ. For every subformula $\Box B$ of A refuted at M, Γ, there exists a point $\Delta_B \in R_{\Box \cdot \mathbf{\Lambda}}(\Gamma)$ such that $M, \Delta_B \nvDash B$. Δ_B is $\mathbf{\Lambda}$-consistent, since it contains $\mathbf{\Lambda}$.

Let Ψ be the set of all subformulas of A, and put

$$B^- := \neg B \wedge \bigwedge \{C \mid M, \Gamma \vDash \Box C,\ \Box C \in \Psi\}.$$

Then $M, \Delta_B \vDash B^-$, so B^- is $\mathbf{\Lambda}$-consistent.

Since $\mathbf{\Lambda}$ is complete, there exists a model N_B over a $\mathbf{\Lambda}$-frame with a root 0_B such that $N_B, 0_B \vDash B^-$. Consider the model $N = (W, R, \xi)$ obtained from the disjoint sum $\bigsqcup\limits_{\Box B \in (\Psi - \Gamma)} N_B$ by adding the root Γ, so that

$$R(\Gamma) = \{0_B \mid \Box B \in (\Psi - \Gamma)\}$$

and $N, \Gamma \vDash q$ iff $q \in \Gamma$ for every proposition letter q. We claim that $(W, R) \vDash \Box \cdot \mathbf{\Lambda}$ and $N, \Gamma \vDash A$.

The first claim is checked as in the proof of Proposition 4.5. For the second one, we show by induction that for any $E \in \Psi$

$$N, \Gamma \vDash E \text{ iff } E \in \Gamma.$$

Again the only nontrivial case is when E begins with $\Box$.

Suppose $E = \Box C \in \Gamma$. Then C occurs as a conjunct in each B^-, so $N_B, 0_B \vDash C$. Thus $N, 0_B \vDash C$, since N_B is a generated submodel of N. By the definition of $R(\Gamma)$, it follows that $N, \Gamma \vDash \Box C$.

Now suppose $E = \Box B \notin \Gamma$. Since $\neg B$ is a conjunct in B^-, we have $N_B, 0_B \vDash \neg B$, and thus $N, 0_B \vDash \neg B$, by Generation lemma. Therefore $N, \Gamma \nvDash \Box B$. □

4.4. Some examples

Let us recall examples of strongly complete logics of the form $\mathbf{Q\Lambda}$ for different $\mathbf{\Lambda}$.

1. One-way PTC logics.

These are logics axiomatized by formulas of the form $\Box p \to \Box^n p$ and closed propositional formulas. Then $\mathbf{Q\Lambda}$ is canonical ([9], Theorem 6.1.29).

2. Logics with confluence and density axioms.

Here we have several logics, for which $\mathbf{Q\Lambda}$ is strongly complete, but probably not canonical. The first example of this kind was **S4.2**, the logic of confluent (or even directed) **S4**-frames. Strong completeness of **QS4.2** was proved by G. Corsi and S. Ghilardi [5].[12] The proof can also be found in [9], section 6.6. Similar

[12] Their paper states only completeness, but actually proves strong completeness.

examples are constructed in [15]: these are extensions of **K4** by axioms of confluence, density, and 2-density in different combinations.

3. Logics with non-branching axioms.

This group of logics contains the axiom of non-branching, in reflexive or irreflexive versions, in combination with axioms of density or finite depth. Typical examples are the well-known logics **S4.3** and **K4.3**. The corresponding results on strong completeness of $\mathbf{Q\Lambda}$ are proved in G. Corsi's paper [4]. For the particular case of **QS4.3** cf. also [9], section 6.7.

5. **S5** and its extensions.

Kripke completeness for **QS5** was proved in the well-known S. Kripke's paper [12]. However, strong completeness (using a version of a canonical model with a constant domain) was first proved in [1]. The same method nicely works for all extensions of **S5**. For these extensions one can also apply the general Tanaka — Ono theorem (cf. [9], section 7.4) as these logics are tabular and their quantified extensions contain the Barcan formula.

5. Incompleteness

5.1. Some counterexamples

Now we will show that an analogue of Theorem 4.4 does not hold for Kripke semantics.

Consider the logics

$$\mathbf{Q\Lambda U_1} := \mathbf{Q\Lambda} + AU_1,$$

where $\mathbf{\Lambda}$ is a propositional modal logic, and

$$AU_1 := \exists x P(x) \to \forall x P(x)$$

is the axiom of singleton domains. So for a Kripke frame $\mathbf{F} = (W, R, D)$

$$\mathbf{F} \vDash AU_1 \text{ iff } \forall u \in W \, |D_u| = 1.$$

The axiom AU_1 allows us to eliminate all quantifiers. More exactly, we call a predicate formula *primitive* if it is quantifier-free and contains at most one individual variable x. Then we have

Lemma 5.1. *For any predicate formula A there exists a primitive formula A' such that $\mathbf{QK} + AU_1 \vdash A \leftrightarrow A'$.*

Proof. Let $L = \mathbf{QK} + AU_1$. First note that by AU_1, $L \vdash A \leftrightarrow \overline{\forall} A$ for any A.

Then the argument is by induction on the length of A.

If $A = P(x_1, \dots, x_n)$ is atomic, then A is L-equivalent to $A' = P(x, \dots, x)$, due to AU_1. In more detail, we have $L \vdash P(x, \dots, x) \to \exists x_1 \dots \exists x_n P(x_1, \dots, x_n)$ by classical logic and $L \vdash \exists x_1 \dots \exists x_n P(x_1, \dots, x_n) \to \overline{\forall} P(x_1, \dots, x_n)$ by AU_1. Also $L \vdash \overline{\forall} P(x_1, \dots, x_n) \to P(x, \dots, x)$ by classical logic.

If $A = \forall y B$, then $L \vdash A \leftrightarrow B$, so we can take $A' = B'$.

The remaining cases are trivial. □

Lemma 5.2. *Let $\mathbf{\Lambda}$ be a strongly complete modal propositional logic. Then $\mathbf{Q\Lambda U_1}$ is strongly Kripke complete.*

Proof. Let $L = \mathbf{Q\Lambda U_1}$. Note that L-frames are just propositional $\mathbf{\Lambda}$-frames with singleton domains.

Let Γ be an L-consistent theory. By Lemma 5.1 we can deal only with primitive formulas, so we may assume that formulas in Γ are of the form $A(c)$, where $A(x)$ is primitive, c is a constant. We may also assume that c is fixed, since we need only models with singleton domains.

For every formula $A(c)$ we construct a propositional formula A_π as follows.

Let $(P_i)_{i\in\omega}$ be an enumeration of all our predicate letters. We associate a proposition letter p_i with each P_i and put

$$P_i(c,\dots,c)_\pi := p_i,\ (A\to B)_\pi := (A_\pi \to B_\pi),\ \bot_\pi := \bot,\ (\Box A)_\pi := \Box A_\pi.$$

Then we have:

$$\mathbf{\Lambda} \vdash A_\pi \ \Rightarrow\ \vdash_L A(c). \tag{$*$}$$

In fact, $A(c)$ is obtained from A_π by substituting $P_i(c,\dots,c)$ for p_i.

Now put

$$\Gamma_\pi := \{A_\pi \mid A(c)\in\Gamma\}.$$

Then Γ_π is $\mathbf{\Lambda}$-consistent. Indeed, if we have $\mathbf{\Lambda}\vdash \neg\bigwedge_j (A_j)_\pi$ for some formulas $A_j(c)\in\Gamma$, then $\vdash_L \neg\bigwedge_j A_j(c)$ by (*), which implies the L-inconsistency of Γ.

By strong completeness of $\mathbf{\Lambda}$, Γ_π is satisfiable in a $\mathbf{\Lambda}$-frame F. So by adding a singleton domain to F we obtain an L-frame $\mathbf{F}$.

We claim that Γ is satisfiable in $\mathbf{F}$. Indeed, let $M,u\vDash\Gamma_\pi$ for a model M on F. Consider the model $\mathbf{M}$ on $\mathbf{F}$ such that for any v

$$\mathbf{M},v\vDash P_i(a,\dots,a) \text{ iff } M,v\vDash p_i,$$

where a is the unique individual in the domain. Then by induction we easily obtain that for any v

$$\mathbf{M},v\vDash A(a) \text{ iff } M,v\vDash A_\pi.$$

Hence $\mathbf{M},u\vDash\Gamma$. □

Lemma 5.3. *Let $\mathbf{\Lambda}$ be a modal propositional logic. Then* $\Box\bullet\mathbf{Q\Lambda U_1}\vDash_{\mathcal{K}} AU_1\vee\Box\bot$.

Proof. Let $L := \mathbf{Q\Lambda U_1}$. Every L-frame has singleton domains. In every $(\Box\bullet L)$-frame $\mathbf{F}=(W,R,D)$ for any u we have $\mathbf{F},u\vDash\Box AU_1$, so $|D_v|=1$ for any $v\in R(u)$. If $u\not\vDash\Box\bot$, there exists $v\in R(u)$. Since $D_u\subseteq D_v$, D_u is a singleton, thus $F,u\models AU_1$.

Hence $\mathbf{F}\vDash AU_1\vee\Box\bot$. □

Recall that $\mathbf{Triv} := \mathbf{K}+\Box p\leftrightarrow p$, $\mathbf{Ver} := \mathbf{K}+\Box\bot$.

Proposition 5.4. *Let $\mathbf{\Lambda}$ be a modal propositional logic. Then $\Box\bullet\mathbf{Q\Lambda U_1}$ is Kripke incomplete.*

Proof. Due to the previous lemma, it suffices to prove that

$$\Box\bullet\mathbf{Q\Lambda U_1}\nvdash AU_1\vee\Box\bot.$$

By Makinson's theorem ([2], Theorem 8.67) $\mathbf{\Lambda}\subseteq\mathbf{Triv}$ or $\mathbf{\Lambda}\subseteq\mathbf{Ver}$, so we can consider only the cases $\mathbf{\Lambda}=\mathbf{Triv}$ and $\mathbf{\Lambda}=\mathbf{Ver}$.

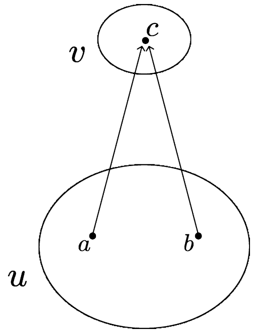

Fig. 1. Kripke sheaf Φ_0.

For **Triv** we take the Kripke sheaf Φ_0 with two worlds: irreflexive u and reflexive v, $D_u = \{a, b\}$, $D_v = \{c\}$. Fig. 1 shows the transition function from u to v.

We have $\Phi_0, v \vDash \mathbf{QTrivU_1}$, since v is reflexive with a singleton domain. Hence $\Phi_0 \vDash \Box \bullet \mathbf{QTrivU_1}$.

On the other hand, $|D_u| = 2$, $R(u) \neq \emptyset$, so $\Phi_0, u \nvDash AU_1 \vee \Box\bot$. Thus, $\Box\bullet\mathbf{QTrivU_1} \nvdash AU_1 \vee \Box\bot$ by Soundness theorem.

For $\mathbf{\Lambda} = \mathbf{Ver}$ replace Φ_0 with Φ_1, in which both u, v are irreflexive. □

Corollary 5.5. *Strong Kripke completeness of a modal predicate logic L does not imply Kripke completeness of $\Box\bullet L$: counterexamples are given by the logics $\mathbf{Q\Lambda U_1}$, where $\mathbf{\Lambda}$ is a strongly Kripke complete propositional logic.*[13]

5.2. *Some propositional logics*

Here are definitions of specific propositional logics; most of them are well known:

$$\begin{aligned}
&\mathbf{T} := \mathbf{K} + \Box p \to p,\ \mathbf{K4} := \mathbf{K} + \Box p \to \Box^2 p,\\
&\mathbf{S4} := \mathbf{T} + \Box p \to \Box^2 p,\ \mathbf{S5} := \mathbf{S4} + \Diamond\Box p \to p,\ \mathbf{Ver} := \mathbf{K} + \Box\bot,\\
&\mathbf{Triv} := \mathbf{K} + \Box p \leftrightarrow p,\ \mathbf{SL} := \mathbf{K} + \Box p \leftrightarrow \Diamond p,\\
&\mathbf{SL4} := \mathbf{SL} + \Box p \to \Box\Box p,\ \mathbf{SL4}_n := \mathbf{SL} + \Box^n p \to \Box^{n+1} p,\\
&\mathbf{K5} := \mathbf{K} + \Diamond\Box p \to \Box p,\\
&\mathbf{K45} := \mathbf{K5} + \Box p \to \Box^2 p,\\
&\mathbf{K4\Box S5} := \Box\cdot\mathbf{S5} + \Box p \to \Box^2 p.
\end{aligned}$$

All these logics are Sahlqvist, so they are elementary and canonical.

In this paper we are interested mainly in extensions of the logic $\Box\cdot\mathbf{T}$ including those of the form $\Box\cdot\mathbf{\Lambda}$ for $\mathbf{\Lambda} \supseteq \mathbf{T}$. By Makinson's theorem [2], $\mathbf{\Lambda} \subseteq \mathbf{Triv}$; thus $\Box\cdot\mathbf{T} \subseteq \Box\cdot\mathbf{\Lambda} \subseteq \Box\cdot\mathbf{Triv}$.

Fig. 2 shows inclusions between some extensions of $\Box\cdot\mathbf{T}$.

Recall that **K5** is determined by 'Euclidean frames', i.e., by those satisfying

$$\forall x \forall y \forall z (xRy \,\&\, xRz \to yRz).$$

So **K45** is determined by transitive Euclidean frames.

The paper [13] describes all extensions of **K5** and proves that **K5** is locally tabular.[14]

[13] These $\mathbf{\Lambda}$ were discussed in subsection 4.4.
[14] One can also show local tabularity for $\Box\cdot\mathbf{S5}$ (and so, for all its extensions); this fact is stated in [14].

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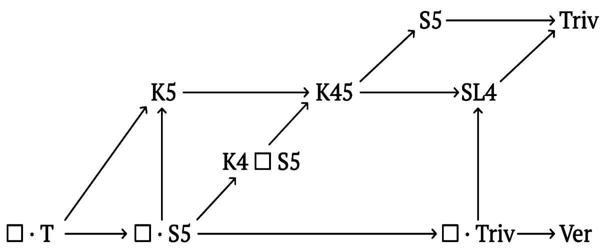

Fig. 2. Extensions of $\Box\cdot\mathbf{T}$.

The inclusion $\mathbf{K5} \supset \Box\cdot\mathbf{S5}$ follows from semantical characterizations of these logics — they are complete, and every Euclidean frame validates $\Box\cdot\mathbf{S5}$. In fact, in Euclidean frames xRy implies yRy, and the points accessible from y make a cluster.[15]

The logic $\mathbf{SL4}$ is determined by transitive functional frames and also by a single two-element frame $(\{0,1\},\{(0,1),(1,1)\})$. The inclusion $\mathbf{K45} \subset \mathbf{SL4}$ is easily checked, semantically or syntactically.

Similarly, the logic $\mathbf{SL4}_n$ is determined by the frame $(\{0,1,\dots,n\},R)$, where xRy iff $y = x+1 \vee x = y = n$. This logic will be used in section 8.

5.3. Kripke bundles

In the rest of this section we prove Kripke-incompleteness for a large family of predicate modal logics including $\mathbf{QK5}$. For this purpose we apply the Kripke bundle semantics. Let us briefly recall the corresponding definitions, see [9], chapter 5 for further details.

Definition 5.6. A *Kripke bundle* over a propositional frame $F = (W,R)$ is a triple $\mathbb{F} = (F,D,\rho)$, in which $D = (D_u)_{u\in W}$ is a family of (non-empty) disjoint domains and $\rho = (\rho_{uv})_{(u,v)\in R}$ is a family of inheritance relations $\rho_{uv} \subseteq D_u \times D_v$ such that $\rho_{uv}(a) \neq \varnothing$ whenever $uRv,\ a \in D_u$.

Models on $\mathbb{F}$ are of the form $(\mathbb{F},\xi)$, where $\xi = (\xi_u)_{u\in W}$, $\xi_u(P) \subseteq D_u^m$ for each m-ary P.

For a Kripke bundle model M the forcing relation $M,u \vDash A$ between worlds $u \in F$ and D_u-sentences is defined recursively. In particular, $M,u \vDash \Box A(a_1,\dots,a_n)$ iff

$$\forall v \in R(u)\, \forall b_1 \in \rho_{uv}(a_1) \dots \forall b_n \in \rho_{uv}(a_n)\, M,v \vDash B(b_1,\dots,b_n).$$

Definition 5.7. A predicate formula A is *true* in a Kripke bundle model if its universal closure $\overline{\forall}A$ is true at every world of this model. A is *strongly valid* on a Kripke bundle $\mathbb{F}$ (notation: $\mathbb{F} \vDash^+ A$) all its substitution instances are true in every model over $\mathbb{F}$.

Proposition 5.8. *The set* $\{A \mid\ \mathbb{F} \vDash^+ A\}$ *is a modal predicate logic.*

Definition 5.9. For a Kripke bundle $\mathbb{F} = (F,D,\rho)$, put $D^0 := W$ and

$$D^n := \bigcup\{D_u^n \mid u \in W\}$$

for $n > 0$.

The relations R^n on D^n are defined as follows:

$$R^1 := \bigcup\{\rho_{uv} \mid uRv\},\ R^0 := R,$$

[15] An alternative syntactic proof is an exercise for the reader.

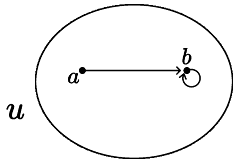

Fig. 3. Kripke bundle $\mathbb{F}$.

$$\mathbf{a} R^n \mathbf{b} \text{ iff } \forall j \ a_j R^1 b_j \ \& \ \forall j, k \, (a_j = a_k \Rightarrow b_j = b_k)$$

for $n > 1$.

Thus $\mathbb{F}$ corresponds to the family of propositional frames $F_n = (D^n, R^n)$, in which $F_0 = F$.

Proposition 5.10. *For a Kripke bundle* $\mathbb{F}$ *and a modal propositional formula* A*,*

$$\mathbb{F} \vDash^+ A \textit{ iff } \forall n \, F_n \vDash A.$$

For the proof cf. [9], Proposition 5.3.7.

Theorem 5.11. *Let* $\mathbf{\Lambda}$ *be a modal propositional logic between* $\Box \cdot \mathbf{T}$ *and* $\mathbf{SL4}$*. Then* $\mathbf{Q\Lambda}$ *is Kripke (and Kripke sheaf) incomplete. In particular, this holds for* $\mathbf{\Lambda} = \Box \cdot \mathbf{\Lambda}_1$*, where* $\mathbf{\Lambda}_1$ *is a consistent extension of* $\mathbf{T}$*.*

Proof. Here the crucial formula is $\Box \forall ref$. First we prove

Lemma 5.12. $\mathbf{QSL4} \nvdash \Box \forall ref$*.*

Proof. By Proposition 5.8 it suffices to construct a Kripke bundle $\mathbb{F} = (F, D, \rho)$ strongly validating $\mathbf{QSL4}$, but refuting $\Box \forall ref$.

Let $F = F_0$ be a reflexive singleton $(\{u\}, \{(u, u)\})$. Let $F_1 = (D, \rho)$, where $D = \{a, b\}$ and $\rho := \{(a, b), (b, b)\}$, see Fig. 3. Then $\mathbf{ML}(F_1) = \mathbf{SL4}$, and we obtain

$$D^n = (D^1)^n; \ \mathbf{d} R^n \mathbf{e} \text{ iff } \forall i \leq n \, e_i = b.$$

So every R^n is functional; every $\mathbf{d} \in D^n$ (for $n > 0$) has a unique successor $(\underbrace{b, b, \ldots, b}_{n})$. Hence $F_n \vDash \mathbf{SL4}$ for $n \geq 0$, and thus $\mathbb{F} \vDash^+ \mathbf{QSL4}$ by Proposition 5.10.

On the other hand, consider a model $M = (\mathbb{F}, \xi)$, where $\xi_u(P) := \{b\}$. We claim that $M, u \nvDash \Box \forall ref$.

Indeed, $M, u \vDash \Box P(a)$, since $M, u \vDash P(b)$ and b is the unique inheritor of a. At the same time $M, u \nvDash P(a)$. Thus $M, u \nvDash \Box P(a) \to P(a)$, and so

$$M, u \nvDash \forall x (\Box P(x) \to P(x)).$$

This implies $M, u \nvDash \Box \forall ref$, since u is reflexive. $\Box$

Now to prove Theorem 5.11 note that $\mathbf{Q\Lambda} \vDash_{\mathcal{KE}} \Box \forall ref$, since $\mathbf{Q}(\Box \cdot \mathbf{T}) \vDash_{\mathcal{KE}} \Box \forall ref$ (Propositions 3.10, 3.18); however $\mathbf{Q\Lambda} \nvdash \Box \forall ref$, since $\mathbf{QSL4} \nvdash \Box \forall ref$ (Lemma 5.12). $\Box$

Remark 5.13. It is well known (cf. [8]) that there is a continuum of modal logics between $\mathbf{T}$ and $\mathbf{Triv}$. By Proposition 3.4, their boxings are different; hence by Theorem 5.11 we obtain a continuum of incomplete logics of the form $\mathbf{Q\Lambda}$.

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Theorem 5.14. *For any modal propositional logic $\mathbf{\Lambda}$ between $\Box \cdot \mathbf{T}$ and* **SL4***:*

(1) *If $\mathbf{Q\Lambda} + \Box\forall ref$ is Kripke complete, then*

$$\widehat{\mathbf{Q\Lambda}} = \mathbf{Q\Lambda} + \Box\forall ref.$$

In particular, this holds whenever $\mathbf{\Lambda} = \Box \cdot \mathbf{\Lambda}_1$, $\mathbf{\Lambda}_1 \supseteq \mathbf{T}$ and $\mathbf{Q\Lambda}_1$ is strongly Kripke complete.

(2) *Similarly, if $\mathbf{Q\Lambda} + \Box\forall ref$ is Kripke sheaf complete, then*

$$C_{\mathcal{KE}}(\mathbf{Q\Lambda}) = \mathbf{Q\Lambda} + \Box\forall ref.$$

In particular, this holds for $\mathbf{\Lambda} = \Box \cdot \mathbf{\Lambda}_1$, where $\mathbf{\Lambda}_1 \supseteq \mathbf{T}$ and $\mathbf{Q\Lambda}_1$ is strongly Kripke sheaf complete.

Proof. (1) Since $\mathbf{Q\Lambda} \vDash_{\mathcal{K}} \Box\forall ref$, we have $\mathbf{Q\Lambda} + \Box\forall ref \subseteq \widehat{\mathbf{Q\Lambda}}$. But $\widehat{\mathbf{Q\Lambda}}$ is the least $\mathcal{K}$-complete extension of $\mathbf{Q\Lambda}$, so the inclusion becomes equality if $\mathbf{Q\Lambda} + \Box\forall ref$ is $\mathcal{K}$-complete.

Also note that strong Kripke completeness transfers from $\mathbf{Q\Lambda}_1$ to $\Box \bullet (\mathbf{Q\Lambda}_1)$, by Theorem 4.8, and $\Box \bullet (\mathbf{Q\Lambda}_1) = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}_1) + \Box\forall ref$, by Proposition 3.10.

(2) Similar to (1), now using $\mathbf{Q\Lambda} \vDash_{\mathcal{KE}} \Box\forall ref$ and Theorem 4.4 (for $\mathbf{\Lambda} = \Box \cdot \mathbf{\Lambda}_1$). □

6. Kripke completions of QK4□S5 and QK45

6.1. C-canonicity

Let us first recall the construction of canonical models with constant domains from [9], chapter 7.

We begin with the well-known characterization of the Barcan formula:

Lemma 6.1. *The Barcan formula*

$$\forall x \Box P(x) \to \Box \forall x P(x)$$

is valid on a rooted Kripke frame $\mathbf{F} = (W, R, D)$ iff $\mathbf{F}$ has a constant domain, i.e. all the domains D_u for $u \in W$ coincide.

Definition 6.2. For a modal logic L containing Ba let CW_L be the set of all L-places (from PW_L) with a fixed countable set of constants S_0. The *canonical frame* and the *canonical model* with a constant domain for L are the restrictions $CF_L := PF_L \upharpoonright CW_L$, $CM_L := PM_L \upharpoonright CW_L$.

For these models we have

Theorem 6.3. *For any $\Gamma \in CW_L$ and modal S_0-sentence A,*

$$CM_L, \Gamma \vDash A \text{ iff } A \in \Gamma.$$

Corollary 6.4. *For any modal predicate formula A,*

$$CM_L \vDash A \text{ iff } L \vdash A.$$

Definition 6.5. A modal predicate logic L containing Ba is called *C-canonical* if $CF_L \vDash L$.

Corollary 6.6. *Every C-canonical logic is strongly Kripke complete.*

Proof. Assume that L is C-canonical, Γ is an L-consistent theory. We may also assume that $D_\Gamma \subseteq S_0$. Then Γ can be extended to some $\Delta \in CW_L$. By Theorem 6.3, $CM_L, \Delta \vDash \Delta$. Thus Γ is satisfiable in the L-frame CF_L. $\square$

The definition of CW_L and CM_L depends on the basic set of constants S_0, but sometimes we need to indicate this set explicitly. Then we use the notation $CW_L(S_0)$, $CF_L(S_0)$, $CM_L(S_0)$. Obviously, all the models $CM_L(S_0)$ are isomorphic for countable S_0.

The well-known example of a C-canonical logic is **QS5**:

Theorem 6.7. **QS5** *is C-canonical.*

This is a particular case of Tanaka – Ono completeness theorem, cf. [9], Theorem 7.4.7. However, in this case C-canonicity follows readily: reflexivity and transitivity are inherited from the larger frame $PF_{\mathbf{QS5}}$; symmetry for $CF_{\mathbf{QS5}}$ is easily checked.

6.2. **QK4□S5**

Lemma 6.8. *Let L be a modal predicate logic containing* $\square \bullet$ **QS5***, and let $\Gamma R_L \Delta$ in the canonical model PM_L. Then*

(1) *Δ is a* **QS5***-place.*
(2) $\Delta R_L \Delta$.

Proof. (1) By Lemma 2.19, it suffices to prove that $\overline{\mathbf{QS5}} \subseteq \Delta$. Indeed, suppose $\mathbf{QS5} \vdash A$ for a sentence A; then by definition $\square \bullet \mathbf{QS5} \vdash \square A$, hence $L \vdash \square A$, and so $\square A \in \Gamma$ (Lemma 2.13). Thus $A \in \Delta$ by the definition of R_L.

(2) Since Δ is a **QS5**-place, we have $(\square A \to A) \in \Delta$ for $A \in \mathcal{L}(\Delta)$. So $\square A \in \Delta$ implies $A \in \Delta$, i.e. $\Delta R_L \Delta$. $\square$

Lemma 6.9. *Let L be a modal predicate logic containing* $\square \bullet$ **QS5***, and assume that Γ is an L-place with an infinite domain. Then in the canonical model PM_L*

$$\Gamma R_L \Delta \,\&\, \Delta R_L \Sigma \,\&\, D_\Delta = D_\Sigma \Rightarrow \Sigma R_L \Delta.$$

Proof. By the previous lemma, $\Delta, \Sigma \in PW_{\mathbf{QS5}}$. Since $D_\Delta = D_\Sigma$, we have $\Delta, \Sigma \in CW_{\mathbf{QS5}}(D_\Delta)$. Then the claim follows from the symmetry of $CF_{\mathbf{QS5}}$. $\square$

To prove the required completeness results we use selective submodels of canonical models. Note that their definition differs from the one given in [9]. It resembles the Tarski–Vaught test for elementary submodels in classical model theory.

Definition 6.10. Let $M = (W, R, D, \xi)$ be a Kripke model. A *weak submodel* of M is a Kripke model $M_1 = (U, R_1, D_1, \xi_1)$ such that[16]

$$U \subseteq W,\ R_1 \subseteq R,\ D_1 = D \restriction U,\ \xi_1 = \xi \restriction U.$$

The weak submodel M_1 is called *selective* if for any $u \in U$, for any D_u-sentence A

[16] I.e. $(\xi_1)_u = \xi_u$, $(D_1)_u = D_u$ for each $u \in U$, cf. Definition 2.8.

$$M, u \vDash \Diamond A \Rightarrow \exists v \in R_1(u)\, M, v \vDash A.$$

Lemma 6.11. *Let M, M_1 be the same as in the previous definition. Then M_1 is a reliable submodel of M: for any $u \in U$, for any D_u-sentence A*

$$M, u \vDash A \text{ iff } M_1, u \vDash A.$$

Proof. We may assume that A is constructed from atomic formulas using $\Diamond$, $\rightarrow$, $\bot$, $\exists$ and argue by induction on the length of A. The only nontrivial case is when $A = \Diamond B$.

Then, if $M, u \vDash \Diamond B$, there exists $v \in R_1(u)$ such that $M, v \vDash B$, since M_1 is selective. By IH, we have $M_1, v \vDash B$, hence $M_1, u \vDash \Diamond B$.

The other way round, if $M_1, u \vDash \Diamond B$, there is $v \in R_1(u)$ such that $M_1, v \vDash B$. We have $M, v \vDash B$, by IH, and $v \in R(u)$, since $R_1 \subseteq R$. Thus $M, u \vDash \Diamond B$. $\square$

Theorem 6.12. *The logic $\mathbf{QK4\Box S5} + \Box\forall ref$ is strongly Kripke complete. Thus $\widehat{\mathbf{QK4\Box S5}} = \mathbf{QK4\Box S5} + \Box\forall ref$.*

Proof. We denote this logic by L. Then $L \supseteq \mathbf{Q}(\Box \cdot \mathbf{S5}) + \Box\forall ref = \Box \bullet \mathbf{QS5}$.

Since $L \supseteq \mathbf{QK4}$, a standard argument shows that the canonical relation R_L is transitive.

Given an L-consistent theory Γ_0, we may assume that its set of constants is small (by an appropriate choice of the set S^* in the canonical model, cf. subsection 2.6) and extend Γ_0 to an L-place Γ (Lemma 2.15) So it is sufficient to satisfy Γ in a Kripke model over an L-frame.

<u>CASE 1</u> $\Box\bot \in \Gamma$. Then Γ is an endpoint in the canonical model PM_L, and $PM_L, \Gamma \vDash \Gamma$ by the Canonical model theorem. By Lemma 2.9(1) we have $PM_L{\uparrow}\Gamma, \Gamma \vDash \Gamma$.

Since Γ is an endpoint, the cone $\mathbf{F} := PF_L{\uparrow}\Gamma$ has a single world Γ, which is irreflexive. Then obviously $\mathbf{F} \vDash \Box B$ for any formula B; hence $\mathbf{F} \vDash \Box\forall ref$, $\mathbf{F} \vDash \mathbf{K4\Box S5}$, and thus $\mathbf{F}$ is an L-frame.

<u>CASE 2</u> $\Gamma R_L \Gamma$.

Then by Lemma 6.8, Γ is a $\mathbf{QS5}$-place. So by Theorem 6.7 and Corollary 6.6, Γ is satisfiable in the $\mathbf{QS5}$-frame $CF_{\mathbf{QS5}}$. This is an L-frame, since $L \subseteq \mathbf{QS5}$.

<u>CASE 3</u> $\Diamond\top \in \Gamma$, but Γ is R_L-irreflexive. Then $R_L(\Gamma) \neq \emptyset$.

Consider the set $U := \{\Gamma\} \cup R_L(\Gamma)$ with the relation

$$\Delta R \Sigma \text{ iff } \Delta R_L \Sigma \,\&\, (\Delta = \Gamma \vee D_\Delta = D_\Sigma).$$

Let M be the restriction of PM_L to (U, R). We claim that M is selective.

Indeed, suppose $\Diamond C \in \Delta$, $\Delta \in U$.

(a) $\Delta \neq \Gamma$.

Then Δ is a $\mathbf{QS5}$-place, by Lemma 6.8. So $CM_{\mathbf{QS5}}(D_\Delta), \Delta \vDash \Diamond C$ (Theorem 6.3); hence in the C-canonical model $CM_{\mathbf{QS5}}(D_\Delta)$ there exists Θ such that $\Delta R_{\mathbf{QS5}} \Theta$, $C \in \Theta$. Certainly, Δ, Θ are points in the larger canonical model $PM_{\mathbf{QS5}}$, which is a generated submodel of PM_L (Lemma 2.19). Thus Θ is an L-place and $\Delta R_L \Theta$.

We also have $\Gamma R_L \Theta$ by transitivity. Therefore $\Theta \in U$ and $PM_L, \Theta \vDash C$.

(b) $\Delta = \Gamma$. In the canonical model there exists Σ such that $\Gamma R_L \Sigma$ and $C \in \Sigma$. This Σ is in U, and $\Delta = \Gamma R \Sigma$.

Besides selectivity, we need to show that $PF_L \restriction (U, R) \vDash L$, i.e., that R is transitive, and R is an equivalence on all points but Γ. In fact, reflexivity for these points follows from Lemma 6.8, and symmetry from Lemma 6.9.

The transitivity of R_L is provided by $\mathbf{K4}$, so R is transitive on $U - \{\Gamma\}$. Also $\Gamma R \Delta R \Sigma$ implies $\Gamma R_L \Delta R_L \Sigma$, and thus $\Gamma R_L \Sigma$, i.e. $\Gamma R \Sigma$. $\square$

6.3. QK45

Theorem 6.13. *The logic* $\mathbf{QK45} + \Box\forall ref$ *is strongly Kripke complete. Thus* $\widehat{\mathbf{QK45}} = \mathbf{QK45} + \Box\forall ref$.

Proof. It goes along the same lines as in the previous theorem. Let $L = \mathbf{QK45} + \Box\forall ref$ and let Γ be an L-place (with an infifnite domain). Since $\mathbf{K45} \supseteq \Box \cdot \mathbf{S5}$, we see again that $L \supseteq \Box \bullet \mathbf{QS5}$.

Now we have the same three cases as in the previous theorem. In cases 1, 2 the argument does not change.

CASE 3 Suppose $\Diamond\top \in \Gamma$, but Γ is R_L-irreflexive. Then there exists an L-place Δ such that $\Gamma R_L \Delta$.

Let $S := D_\Delta$, and put $M := PM_L \upharpoonright U$, where

$$U := \{\Gamma\} \cup \{\Sigma \in R_L(\Delta) \mid D_\Sigma = S\}.$$

By Lemma 6.8, Δ is reflexive, so $\Delta \in U$.

Let us show that M is selective. Let $B = \Diamond C \in \Sigma \in U$ and consider two cases.

(a) $\Sigma \in R_L(\Delta)$, $D_\Sigma = S$.

As in the previous theorem, case 3(a), we notice that Σ is a $\mathbf{QS5}$-place, by Lemma 6.8. Hence we obtain $\Theta \in R_{\mathbf{QS5}}(\Sigma)$ such that $C \in \Theta$. By Lemma 2.19 it follows that Θ is an L-place and $\Sigma R_L \Theta$. By transitivity, $\Delta R_L \Theta$ and also $D_\Theta = S$. Thus $\Theta \in U$ and $PM_L, \Theta \vDash C$.

(b) $\Sigma = \Gamma$.

We claim that there exists $\Theta \in R_L(\Delta) \cap R_L(\Gamma)$ such that $D_\Theta = S$ and $C \in \Theta$.

Since the relation R_L is transitive, this claim follows easily from the case (a). Indeed, $\vdash_L \Diamond C \to \Box\Diamond C$, so $\Box\Diamond C \in \Gamma$, and since $\Gamma R_L \Delta$, we have $\Diamond C \in \Delta$. By (a) we obtain $\Theta \in R_L(\Delta)$ with $D_\Theta = S$ and $M, C \vDash \Theta$; also $\Gamma R_L \Theta$ by transitivity.

It remains to show that the frame of M (we again denote it by $\mathbf{F}$) validates L, i.e. it is transitive, Euclidean and validates $\Box\forall ref$.

The transitivity is already known.

For the Euclideanness note that Γ is R_L-related to all other points by transitivity, and let us show that all these points are R_L-related.

Indeed, suppose $\Delta R_L \Sigma$, $\Delta R_L \Sigma'$, $D_\Sigma = D_{\Sigma'} = S$. Since Δ is reflexive, $\Sigma R_L \Delta$ by Lemma 6.9. Hence $\Sigma R_L \Sigma'$ by transitivity.

The formula $\Box\forall ref$ is valid on $\mathbf{F}$, since every its point but Γ is reflexive and Γ is not accessible (from any point): $\Gamma R_L \Sigma R_L \Gamma$ would imply $\Gamma R_L \Gamma$ by transitivity. □

Remark. The equality $\widehat{\mathbf{Q\Lambda}} = \mathbf{Q\Lambda} + \Box\forall ref$ also holds for $\mathbf{\Lambda} = \mathbf{SL4}$. We leave the proof as an exercise for the reader.

7. Kripke completion of QK5

In this section we also prove completeness using canonical models, but the argument becomes more involved.

We begin with a useful lemma on adding witnesses for logics containing the Barcan formula. Implicitly it is contained in the proof of Lemma 7.1.2 from [9].

Lemma 7.1. *Let L be a predicate logic containing Ba, Σ an L-consistent theory such that $\Sigma \vdash_L \Diamond B$ for some sentence B in the language of Σ. Let $\exists x A(x)$ also be a sentence in the language of Σ and let c be a new constant not occurring in Σ, B and $A(x)$. Then the theory*

$$\Sigma' := \Sigma \cup \{\Diamond(B \wedge (\exists x A(x) \to A(c)))\}$$

is L-consistent.

Proof. Supposing the inconsistency of Σ' we obtain

$$\Sigma \vdash_L \Box(B \to \exists x A(x) \wedge \neg A(c)),$$

and thus

$$\Sigma \vdash_L \forall y \Box(B \to \exists x A(x) \wedge \neg A(y))$$

by lemma on new constants ([9], Lemma 2.7.11). Hence by applying Ba we have

$$\Sigma \vdash_L \Box \forall y(B \to \exists x A(x) \wedge \neg A(y)).$$

Next, by classical logic, we can move the quantifier inside:

$$\Sigma \vdash_L \Box(B \to \exists x A(x) \wedge \forall y \neg A(y)),$$

which implies

$$\Sigma \vdash_L \Box \neg B.$$

Together with the assumption $\Sigma \vdash_L \Diamond B$ this implies the inconsistency of Σ.

Therefore Σ' is consistent. □

Theorem 7.2. *The logic* $\mathbf{QK5} + \Box\forall ref$ *is strongly Kripke complete. Thus* $\widehat{\mathbf{QK5}} = \mathbf{QK5} + \Box\forall ref$.

Proof. The proof follows the same lines as in Theorem 6.13, with an essential difference in case 3.

Let L be our logic, and consider the canonical model PM_L. Since $\mathbf{K5} \supseteq \Box \cdot \mathbf{S5}$, we have

$$L \supseteq \mathbf{Q}(\Box \cdot \mathbf{S5}) + \Box\forall ref = \Box \bullet \mathbf{QS5}$$

(Proposition 3.10), so Lemmas 6.8, 6.9 hold for PM_L.

Let Γ be an L-place and consider three cases as in Theorems 6.12, 6.13.

In cases 1 and 2 the argument is preserved. However, in case 3 (Γ is irreflexive, $\Diamond\top \in \Gamma$), we cannot rely on the transitivity of R_L. To see the difficulties, consider a formula $\Diamond C \in \Gamma$. In the proof of Theorem 6.13, case 3(b), we constructed $\Theta \in R_L(\Gamma) \cap R_L(\Delta)$ such that $D_\Theta = D_\Delta$ and $C \in \Theta$. To obtain Θ we noticed that $\Diamond C \in \Delta$ and then used the transitivity of R_L. But now R_L is not transitive, so we cannot claim that $\Theta \in R_L(\Gamma)$.

To modify the proof properly, we do not start from a certain $\Delta \in R_L(\Gamma)$, but construct successors of Γ gradually. The whole procedure goes in three stages.

For a theory Σ denote

$$\Diamond\Sigma := \{\Diamond A \mid A \in \Sigma\}, \ \Box^-\Sigma := \{A \mid \Box A \in \Sigma\}.$$

Stage 1 Consider all formulas in Γ beginning with $\Diamond$: $\Diamond C_1, \ \Diamond C_2, \ldots$ We first construct **QS5**-consistent theories with the Henkin property $\Delta_1, \ \Delta_2, \ldots$ (all in the same language) containing $\Box^-\Gamma$ and such that $C_i \in \Delta_i$. This is done by induction.

Base We start with the theories $\Delta_i^0 := [\{C_i\} \cup \Box^-\Gamma]$, where $[\ldots]$ denotes closure under $\vdash_{\mathbf{QS5}}$ (i.e. **QS5**-derivability, cf. subsection 2.2).

(1) Let us show their consistency.[17] It is sufficient to show that $\{C_i\} \cup \Box^-\Gamma$ are consistent. So suppose $\Box^-\Gamma \vdash \neg C_i$. Since $\Box^-\Gamma$ is closed under conjunction, by Deduction theorem it follows that $\vdash G \to \neg C_i$ for some $G \in \Box^-\Gamma$. Recall that by Lemma 3.9, $\mathbf{QS5} \vdash A$ implies $\mathbf{Q}(\Box \cdot \mathbf{S5}) + \Box\forall ref \vdash \Box A$ and thus $L \vdash \Box A$. So we obtain $\vdash_L \Box(G \to \neg C_i)$, and thus $\vdash_L \Box G \to \neg\Diamond C_i$ (by $\mathbf{K}$). Since $\Box G \in \Gamma$ and Γ is an L-place, it follows that $\neg\Diamond C_i \in \Gamma$, which contradicts $\Diamond C_i \in \Gamma$. Therefore Δ_i^0 is consistent.

(2) We claim that $\Diamond\Delta_i^0 \subseteq \Delta_1^0$ for any i.

Indeed, let $A \in \Delta_i^0$; then $C_i, B_1, \dots, B_n \vdash A$ for some $B_1, \dots, B_n \in \Box^-\Gamma$. Since $\Box^-\Gamma$ is closed under conjunction, there is $B \in \Box^-\Gamma$ such that $C_i, B \vdash A$, so by Deduction theorem $\vdash C_i \wedge B \to A$. Hence $\vdash \Diamond(C_i \wedge B) \to \Diamond A$.

However, $\Diamond C_i, \Box B \in \Gamma$, and note that $\vdash \Diamond C_i \wedge \Box B \to \Diamond(C_i \wedge B)$ (by $\mathbf{K}$). It follows that $\Diamond A \in \Gamma$, so $\Box\Diamond A \in \Gamma$ (by $\mathbf{K5}$); hence $\Diamond A \in \Box^-\Gamma \subseteq \Delta_1^0$. Thus (2) holds.

Step The further construction is ruled by a fixed enumeration of all possible pairs $(k, \exists x A(x))$, where $k > 0$ and $\exists x A(x)$ is a sentence (with x arbitrary) in the language of Γ with extra constants from a certain countable set S.

Suppose we have a collection of consistent $\wedge$-closed theories Δ_i^n, $i = 1, 2, \dots$ such that $\Diamond\Delta_i^n \subseteq \Delta_1^n$ for any i and infinitely many constants from S do not appear in these theories.[18] Let us construct theories Δ_i^{n+1} with the same properties.

Consider the first new pair $(k, \exists x A(x))$ and assume that $k \neq 1$. Then we create a witness for $\exists x A(x)$ in Δ_k^{n+1}: we choose a new constant $c \in S$ that does not occur in $\Delta_1^n \cup \Delta_k^n \cup \{A(x)\}$ and put

$$\Delta_k^{n+1} := [\Delta_k^n \cup \{\exists x A(x) \to A(c)\}],$$
$$\Delta_1^{n+1} := [\Delta_1^n \cup \Diamond\Delta_k^{n+1}],\ \Delta_i^{n+1} := \Delta_i^n \text{ for } i \neq 1, k.$$

It is clear that the theories Δ_i^{n+1} are $\wedge$-closed. The consistency of Δ_k^{n+1} is checked as in classical logic; for Δ_i^{n+1} with $i \neq 1, k$ it holds by IH.

(3) Let us prove that Δ_1^{n+1} is consistent.

Suppose the contrary. Then

$$\Delta_1^n, \Diamond A_1, \dots, \Diamond A_r \vdash \bot$$

for some $A_1, \dots, A_r \in \Delta_k^{n+1}$. Since $\Diamond(A_1 \wedge \ldots \wedge A_r) \vdash \Diamond A_1, \ldots \Diamond A_r$, we also have

$$\Delta_1^n, \Diamond(A_1 \wedge \ldots \wedge A_r) \vdash \bot.$$

Since Δ_k^n is $\wedge$-closed, we can join the A_i from this set together, so we obtain

$$\Delta_1^n, \Diamond(B \wedge (\exists x A(x) \to A(c))) \vdash \bot$$

for some $B \in \Delta_k^n$. However, by IH, $\Diamond\Delta_k^n \subseteq \Delta_1^n$, so $\Diamond B \in \Delta_1^n$, while

$$\Delta_1^n \cup \{\Diamond(B \wedge (\exists x A(x) \to A(c)))\}$$

is inconsistent. This contradicts Lemma 7.1.

Therefore Δ_1^{n+1} is consistent, and we have $\Diamond\Delta_i^{n+1} \subseteq \Delta_1^{n+1}$ for $i \neq 1$ by construction and IH.

The inclusion $\Diamond\Delta_1^{n+1} \subseteq \Delta_1^{n+1}$ also holds for $i = 1$, since $\vdash A \to \Diamond A$ for any A (remember that we argue in $\mathbf{QS5}$) and Δ_1^{n+1} is closed under derivability.

[17] Henceforth in this proof, ‘consistency’ means ‘**QS5**-consistency’, $\vdash$ means $\vdash_{\mathbf{QS5}}$.
[18] In [9] theories with infinitely many unused constants are called ‘small’.

Recall that the above argument refers to the pair $(k, \exists x A(x))$ with $k \neq 1$. If the first new pair is $(1, \exists x A(x))$, we choose a new constant c and put

$$\Delta_1^{n+1} := [\Delta_1^n \cup \{\exists x A(x) \to A(c)\}],$$
$$\Delta_i^{n+1} := \Delta_i^n \quad \text{for } i \neq 1.$$

Then the consistency of Δ_1^{n+1} is proved by a standard classical argument; the inclusion $\Diamond\Delta_1^{n+1} \subseteq \Delta_1^{n+1}$ is explained as above.

Finally we put

$$\Delta_i := \bigcup_n \Delta_i^n.$$

By the construction these theories are also $\wedge$-closed, consistent, satisfy the condition

$$\Diamond\Delta_i \subseteq \Delta_1 \tag{5}$$

and enjoy the Henkin property. Indeed, if $\exists x A(x) \in \mathcal{L}(\Delta_k)$ and $(k, \exists x A(x))$ is the n-th pair in our fixed enumeration, then by construction, $\exists x A(x)$ gets a witness in Δ_k^{n+1}.

Stage 2 Completing the theories Δ_i.

We will now construct **QS5**-places $\overline{\Delta}_i$ containing Δ_i.

By Lindenbaum lemma there is a complete (i.e. maximal consistent) theory $\overline{\Delta}_1 \supseteq \Delta_1$. We also claim that for any $i > 1$ the theory $\Delta_i \cup \Box^-\overline{\Delta}_1$ is consistent.

Indeed, suppose not. Since both Δ_i, $\Box^-\overline{\Delta}_1$ are $\wedge$-closed, there exist $A \in \Delta_i$, $B \in \Box^-\overline{\Delta}_1$ such that $\vdash B \to \neg A$, so $\vdash \Box B \to \Box\neg A$, and thus $\Box\neg A \in \overline{\Delta}_1$ by completeness of $\overline{\Delta}_1$. But $\Diamond A \in \Delta_1$ by (5), which contradicts the consistency of $\overline{\Delta}_1$.

Then we can also construct complete theories in the same language

$$\overline{\Delta}_i \supseteq \Delta_i \cup \Box^-\overline{\Delta}_1$$

for all $i > 1$.

Therefore due to Henkin property, we have **QS5**-places $\overline{\Delta}_1, \overline{\Delta}_2, \ldots$, and by construction $\Gamma R_L \overline{\Delta}_i$ for all i, $\overline{\Delta}_1 R_L \overline{\Delta}_i$ for all $i > 1$. All the theories Δ_i have the same set of constants S. So $\overline{\Delta}_i \in CW_{\mathbf{QS5}}(S)$. Since the relation R_L on $CW_{\mathbf{QS5}}(S)$ is an equivalence (Theorem 6.7), we obtain

$$\overline{\Delta}_i R_L \overline{\Delta}_j \text{ for all } i, j. \tag{6}$$

Stage 3 Extending the model.

This is done as in the proof of Theorem 6.13. Put

$$M := PM_L \restriction (\{\Gamma\} \cup \{\Sigma \in R_L(\overline{\Delta}_1) \mid D_\Sigma = S\}).$$

Since all the points except Γ are R_L-related, and Γ is not accessible from any other point (otherwise it would be reflexive by Lemma 6.8), the frame of M is Euclidean. The validity of $\Box\forall ref$ is checked as in the proof of Theorem 6.13.

It remains to prove selectivity. This is done again as in Theorem 6.13, case 3, with the only difference in the subcase (b). Namely, if $B = \Diamond C \in \Gamma$, C is some C_i (see Stage 1), so by construction $C \in \overline{\Delta}_i$. □

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8. Remarks on iterated boxing

Definition 8.1. For a modal predicate logic L we define $\Box^n \bullet L$ by recursion:

$$\Box^0 \bullet L := L,\ \Box^{n+1} \bullet L := \Box \bullet (\Box^n \bullet L).$$

Similarly for a modal propositional logic $\mathbf{\Lambda}$ we define $\Box^n \cdot \mathbf{\Lambda}$.

Some results on boxing are rather easily transferred to iterated boxing.

From Proposition 3.13 we obtain

Proposition 8.2. *Iterated boxing embeds the poset of modal predicate logics in itself:*

$$L_1 \subseteq L_2 \ \textit{iff}\ \Box^n \bullet L_1 \subseteq \Box^n \bullet L_2.$$

Theorem 8.3. *Iterated boxing preserves canonicity and strong $\mathcal{KE}$-completeness for modal predicate logics.*

Proof. Follows from Theorems 4.1, 4.4. □

In the next lemma $\mathbf{F} = (F, D)$ is a predicate Kripke frame with a propositional base $F = (W, R)$. The set $V := \{v \in W \mid R^{-1}(v) \neq \varnothing\}$ is stable in F, so we have generated subframes $\mathbf{F}^- := \mathbf{F} \restriction V$, $F^- := F \restriction V$ (cf. Definition 2.8).

$\mathbf{\Lambda}$ denotes an arbitrary modal propositional logic, L an arbitrary modal predicate logic.

Lemma 8.4.

(1) *For any modal sentence A, $\mathbf{F} \vDash \Box A$ iff $\mathbf{F}^- \vDash A$.*
(2) *For any modal propositional formula A, $F \vDash \Box A$ iff $F^- \vDash A$.*
(3)

$$\mathbf{F} \vDash \Box \bullet L \ \textit{iff}\ \mathbf{F}^- \vDash L.$$

(4)

$$F \vDash \Box \cdot \mathbf{\Lambda} \ \textit{iff}\ F^- \vDash \mathbf{\Lambda}.$$

(5)

$$\mathbf{F} \vDash \Box^n \cdot \mathbf{\Lambda} \ \textit{iff}\ \mathbf{F} \vDash \Box^n \bullet \mathbf{Q\Lambda}.$$

(6)

$$\Box^n \bullet \widehat{\mathbf{Q\Lambda}} \subseteq C_{\mathcal{K}}(\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda})).$$

(7)

$$\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda}) \subseteq \Box^n \bullet \mathbf{Q\Lambda} \subseteq \Box^n \bullet \widehat{\mathbf{Q\Lambda}} \subseteq C_{\mathcal{K}}(\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda})) = \widehat{\Box^n \bullet \mathbf{Q\Lambda}}.$$

Proof. (1) Assume $\mathbf{F}^- \vDash A$. Then $\mathbf{F}, v \vDash A$ for any $v \in V$, by Lemma 2.9 (2). Since $R(u) \subseteq V$ for any $u \in W$, it follows that $\mathbf{F}, u \vDash \Box A$ (Lemma 2.4 (1)). Thus $\mathbf{F} \vDash \Box A$.

The other way round, assume $\mathbf{F} \vDash \Box A$. Since every $v \in V$ is in some $R(u)$, we have $\mathbf{F}, v \vDash A$ (Lemma 2.4 (1)), and thus $\mathbf{F}^-, v \vDash A$ (Lemma 2.9 (2)). Hence $\mathbf{F}^- \vDash A$.

(2) The same argument as in (1) can be applied to the propositional frames F and F^-.

(3) We have $\mathbf{F} \vDash \Box \bullet L \Leftrightarrow \mathbf{F} \vDash \Box \overline{L}$ by Lemma 3.11, and $\mathbf{F} \vDash \Box \overline{L} \Leftrightarrow \mathbf{F}^- \vDash \overline{L}$ by (1). Obviously, $\mathbf{F}^- \vDash \overline{L} \Leftrightarrow \mathbf{F}^- \vDash L$.

(4) We have $F \vDash \Box \cdot \mathbf{\Lambda} = \mathbf{QK} + \Box\mathbf{\Lambda} \Leftrightarrow F \vDash \Box\mathbf{\Lambda}$ by soundness, and $F \vDash \Box\mathbf{\Lambda} \Leftrightarrow F^- \vDash \mathbf{\Lambda}$ by (2).

(5) By induction on n. Denote $L_n := \Box^n \bullet \mathbf{Q\Lambda}$, $\mathbf{\Lambda}_n := \Box^n \cdot \mathbf{\Lambda}$. The case $n = 0$ is obvious, by soundness. For the induction step, suppose the equivalence holds for n. Note that $L_{n+1} = \Box \bullet L_n$, $\mathbf{\Lambda}_{n+1} = \Box \cdot \mathbf{\Lambda}_n$. Then by (3), IH and (4)

$$\mathbf{F} \vDash L_{n+1} \text{ iff } \mathbf{F}^- \vDash L_n \text{ iff } F^- \vDash \mathbf{\Lambda}_n \text{ iff } F \vDash \mathbf{\Lambda}_{n+1}.$$

Finally, $F \vDash \mathbf{\Lambda}_{n+1} \Leftrightarrow \mathbf{F} \vDash \mathbf{\Lambda}_{n+1}$ by Lemma 2.4 (2).

(6) Also by induction. The case $n = 0$ is trivial.

Assuming $\Box^n \bullet \widehat{\mathbf{Q\Lambda}} \subseteq C_{\mathcal{K}}(\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda}))$ we obtain

$$\Box^{n+1} \bullet \widehat{\mathbf{Q\Lambda}} \subseteq \Box \bullet C_{\mathcal{K}}(\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda})) \subseteq C_{\mathcal{K}}(\mathbf{Q}(\Box^{n+1} \cdot \mathbf{\Lambda}))$$

by monotonicity of boxing and Lemma 3.14 (or Proposition 3.15).

(7) For the first inclusion it is sufficient to show that $\Box^n \cdot \mathbf{\Lambda} \subseteq \Box^n \bullet \mathbf{Q\Lambda}$. The latter follows easily by induction: $\mathbf{\Lambda} \subseteq \mathbf{Q\Lambda}$ is obvious, and $\Box^n \cdot \mathbf{\Lambda} \subseteq \Box^n \bullet \mathbf{Q\Lambda}$ implies $\Box(\Box^n \cdot \mathbf{\Lambda}) \subseteq \Box(\Box^n \bullet \mathbf{Q\Lambda})$, and hence $\Box \cdot (\Box^n \cdot \mathbf{\Lambda}) \subseteq \Box \bullet (\Box^n \bullet \mathbf{Q\Lambda})$.

The second inclusion follows from $\mathbf{Q\Lambda} \subseteq \widehat{\mathbf{Q\Lambda}}$ by monotonicity of boxing.

The third inclusion is (6).

The last equality follows from (5). Indeed, $A \in C_{\mathcal{K}}(\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda}))$ if and only if $\mathbf{F} \vDash A$ for any predicate frame $\mathbf{F}$ validating $\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda})$, or $\Box^n \cdot \mathbf{\Lambda}$ (by soundness), or $\Box^n \cdot \mathbf{Q\Lambda}$ (by (5)). □

Now we prove an analogue of Theorem 4.8:

Theorem 8.5. *If $\mathbf{Q\Lambda}$ is strongly Kripke complete, then $\Box^n \bullet (\mathbf{Q\Lambda})$ is strongly Kripke complete.*

Proof. By induction. Denote again $L_n := \Box^n \bullet (\mathbf{Q\Lambda})$, $\mathbf{\Lambda}_n := \Box^n \cdot \mathbf{\Lambda}$.

The base is trivial. For the induction step, suppose L_n is strongly Kripke complete. Let Γ be an L_{n+1}-place. The further argument follows the proof of Theorem 4.8, where we replace L with L_n and $\mathbf{\Lambda}$ with $\mathbf{\Lambda}_n$.

So for every $\Delta \in R_{L_{n+1}}(\Gamma)$ we construct a model M_Δ over a frame $\mathbf{F}_\Delta$ with root 0_Δ and an interpretation δ_Δ such that $M_\Delta, 0_\Delta \vDash \delta_\Delta \cdot \Delta$ and $\mathbf{F}_\Delta \vDash L_n$. The latter is equivalent to $\mathbf{F}_\Delta \vDash \mathbf{\Lambda}_n$, by Lemma 8.4 (5).

Next, in two steps we construct the models M''_Δ over frames $\mathbf{F}''_\Delta$ such that $M''_\Delta, 0_\Delta \vDash \Delta$. Since the propositional bases of frames $\mathbf{F}_\Delta$ and $\mathbf{F}''_\Delta$ are the same, it follows that $\mathbf{F}''_\Delta \vDash \mathbf{\Lambda}_n$ and $\mathbf{F}''_\Delta \vDash L_n$ (Lemma 8.4 (5)).

Finally, from the models M''_Δ we construct the model M^* with root Γ over a frame $\mathbf{G}$ and show by the same argument as in Theorem 4.8 that $M^*, \Gamma \vDash \Gamma$. Since $\mathbf{G}^-$ is a disjoint union of frames $\mathbf{F}''_\Delta$, it follows that $\mathbf{G}^- \vDash L_n$, and thus by Lemma 8.4 (3), $\mathbf{G} \vDash \Box \bullet L_n = L_{n+1}$.

Therefore L_{n+1} is strongly Kripke complete. □

The previous theorem can be slightly generalized.

Theorem 8.6. *Assume that a logic $\widehat{\mathbf{Q\Lambda}}$ is strongly Kripke complete. Then*

(1) *$\Box^n \bullet \widehat{\mathbf{Q\Lambda}}$ is strongly Kripke complete.*
(2) $\Box^n \bullet \widehat{\mathbf{Q\Lambda}} = \widehat{\Box^n \bullet \mathbf{Q\Lambda}} = C_{\mathcal{K}}(\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda}))$.

Proof. We can almost repeat the proof of Theorem 8.5 and argue by induction. Let $L'_n := \Box^n \bullet \widehat{\mathbf{Q\Lambda}}$, $\mathbf{\Lambda}_n := \Box^n \cdot \mathbf{\Lambda}$.

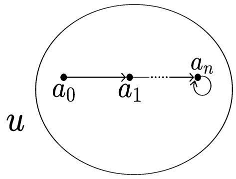

Fig. 4. Kripke bundle $\mathbb{F}'$.

If $n = 0$, there is nothing to prove. For the step, suppose L'_n is strongly Kripke complete and satisfies (2), and consider an L'_{n+1}-place Γ. Every $\Delta \in R_{L'_{n+1}}(\Gamma)$ is an L'_n-place and so satisfiable at the root of some model M_Δ over an L'_n-frame $\mathbf{F}_\Delta$ with a propositional base F_Δ.

Then (2) implies $\mathbf{F}_\Delta \vDash \widehat{\Box^n \bullet \mathbf{Q\Lambda}}$, which is obviously equivalent to $\mathbf{F}_\Delta \vDash \Box^n \bullet \mathbf{Q\Lambda}$, or to $\mathbf{F}_\Delta \vDash \mathbf{\Lambda}_n$ (Lemma 8.4 (5)), or to $F_\Delta \vDash \mathbf{\Lambda}_n$ (Lemma 2.4(2)).

By the same construction as in Theorems 4.8, 8.5 we obtain models M''_Δ satisfying Δ over frames $\mathbf{F}''_\Delta$ with the same propositional bases F_Δ. Thus $\mathbf{F}''_\Delta \vDash \mathbf{\Lambda}_n$, hence $\mathbf{F}''_\Delta \vDash \Box^n \bullet \mathbf{Q\Lambda}$ by Lemma 8.4 (5) and $\mathbf{F}''_\Delta \vDash L'_n$ by (2).

Next, as in Theorem 8.5, we obtain a model M^* with the root Γ over a frame $\mathbf{G}$ such that $M^*, \Gamma \vDash \Gamma$. By Lemma 8.4 (3) we also have $\mathbf{G} \vDash \Box \bullet L'_n = L'_{n+1}$, since $\mathbf{G}^- \vDash L'_n$ as a disjoint sum of frames $\mathbf{F}''_\Delta$. Therefore L'_{n+1} is strongly Kripke complete.

By Lemma 8.4 (7) we have

$$\Box^{n+1} \bullet \mathbf{Q\Lambda} \subseteq \Box^{n+1} \bullet \widehat{\mathbf{Q\Lambda}} \subseteq \widehat{\Box^{n+1} \bullet \mathbf{Q\Lambda}} = C_{\mathcal{K}}(\mathbf{Q}(\Box^n \cdot \mathbf{\Lambda})).$$

The logic $L'_{n+1} = \Box^{n+1} \bullet \widehat{\mathbf{Q\Lambda}}$ is Kripke complete by (1), so from these inclusions it follows that $\Box^{n+1} \bullet \widehat{\mathbf{Q\Lambda}} = \widehat{\Box^{n+1} \cdot \mathbf{Q\Lambda}}$, thus (2) holds for $n + 1$. $\Box$

Theorem 8.7. *Let $\mathbf{\Lambda}$ be a modal propositional logic between $\Box^n \cdot \mathbf{T}$ and $\mathbf{SL4}_n$ (for $n > 0$). Then*

$$\mathbf{Q\Lambda} \subset \mathbf{Q\Lambda} + \Box^n \forall ref \subseteq \widehat{\mathbf{Q\Lambda}}.$$

Thus $\mathbf{Q\Lambda}$ is Kripke incomplete.

Proof. By an easy generalization of Theorem 5.11.

First note that $\mathbf{QT} \vdash \forall ref$ obviously implies $\Box^n \bullet \mathbf{QT} \vdash \Box^n \forall ref$ by induction. Hence by Lemma 8.4 (6), $\Box^n \forall ref \in \widehat{\mathbf{Q}(\Box^n \cdot \mathbf{T})} \subseteq \widehat{\mathbf{Q\Lambda}}$.

We also have

Lemma 8.8. $\mathbf{QSL4}_n \nvdash \Box^n \forall ref$.

Proof. We construct a Kripke bundle $\mathbb{F}' = (F, D')$ strongly validating $\mathbf{QSL4}_n$ and refuting $\Box^n \forall ref$.

The propositional base F is again a reflexive singleton $(\{u\}, \{(u, u)\})$. The frame (D', ρ') determines $\mathbf{SL4}_n$. Viz.,

$$D' = \{a_0, \ldots, a_n\},\ a_i \rho' a_j \text{ iff } j = i + 1 \vee i = j = n,$$

see Fig. 4.

Then D'^m consists of m-tuples of individuals with the relation

$$\mathbf{d} \rho'^m \mathbf{e} \text{ iff } \forall i \leq m\, d_i \rho' e_i.$$

(sub does not matter here, since ρ' is functional.) So ρ'^m is functional, and $(\underbrace{a_n, \dots, a_n}_{m})$ is the successor of itself. The nth and the $(n+1)$th iterations of ρ'^m coincide. Hence $F'_m \vDash \mathbf{SL4}_n$, and thus $\mathbb{F}' \vDash^+ \mathbf{QSL4}_n$ by Proposition 5.10.

On the other hand, consider a model $M = (\mathbb{F}, \xi)$, where $\xi_u(P) := \{a_1\}$. Then $M, u \nvDash \Box^n \forall ref$, since $M, u \vDash \Box P(a_0) \wedge \neg P(a_0)$ implies $M, u \vDash \Diamond^n \exists x(\Box P(x) \wedge \neg P(x))$. □

Eventually we obtain $\mathbf{Q\Lambda} \vDash_{\mathcal{K}} \Box^n \forall ref$, $\mathbf{Q\Lambda} \nvdash \Box^n \forall ref$. □

Remark 8.9. A similar assertion holds for Kripke sheaf semantics. We leave the corresponding proofs to the reader.

9. Conclusion

This paper makes a little next step in systematic study of completeness for modal predicate logics of the form $\mathbf{Q\Lambda}$, and large *terra incognita* is lying ahead. Let us outline some topics for further development.

1. Axiomatization of completions.

For logics considered in this paper Kripke completions are obtained by adding a single axiom $\Box\forall ref$ (or $\Box^n\forall ref$). How far do these results extend? In particular, is it true that $\mathbf{Q\Lambda} + \Box\forall ref$ is Kripke complete for any $\mathbf{\Lambda}$ between $\Box \cdot \mathbf{S5}$ and $\mathbf{SL4}$?

2. Finite axiomatizability of boxing.

The general Theorems 3.21, **??** suggest infinite axiomatizations for boxing and iterated boxing. However, the logics $\Box \bullet \mathbf{Q\Lambda}$ for $\mathbf{\Lambda}$ containing $\mathbf{T}$ are finitely axiomatizable (Proposition 3.10). It seems we are lucky here. What happens for other propositional logics? We conjecture that in many cases $\Box \bullet \mathbf{Q\Lambda}$ should not be finitely axiomatizable, in particular, for $\mathbf{\Lambda} = \mathbf{K5}$ and $\mathbf{\Lambda} = \mathbf{SL4}$. Finite axiomatizability of iterated boxing is also an open problem. We hope to return to this topic in later publications.

3. Boxing vs Δ-operation.

A certain analogue of boxing is Suzuki's Δ-operation for superintuitionistic logics [18]. The definitions and properties of these two operations are very similar. Is it always the case, i.e. do general theorems on boxing for modal predicate logics transfer to Δ-operation and vice versa?

In particular, from sections 4, 5 we know that boxing preserves strong Kripke sheaf completeness, but does not preserve strong Kripke completeness. However, for Δ-operation there is a better result ([9], Proposition 6.9.9): it preserves Kripke completeness for *intermediate* predicate logics. A modal analogue of this result might be the following: boxing preserves Kripke completeness for modal predicate logics included in $\mathbf{QTriv}$. Is this assertion true?

4. Correlation between Kripke completeness and strong Kripke completeness.

For intermediate predicate logics these two properties are non-equivalent. It is very likely that similar counterexamples can be constructed for modal predicate logics. However, we do not know if Kripke completeness implies strong Kripke completeness for logics of the form $\mathbf{Q\Lambda}$ (modal or superintuitionistic).

5. Correlation between Kripke completeness and Kripke sheaf completeness.

There are many examples of Kripke sheaf complete, but Kripke incomplete predicate logics, both in the modal and the intuitionistic fields. On the other hand, Suzuki showed that Kripke and Kripke sheaf completeness are equivalent for logics of the form $\mathbf{Q\Lambda} + CD$, where $\mathbf{\Lambda}$ is an intermediate propositional logic, CD is the axiom of constant domains [17]. Apparently this result extends to modal logics of the form $\mathbf{Q\Lambda} + Ba$, where Ba is the Barcan formula. But for the logics $\mathbf{Q\Lambda}$ the problem remains open. Moreover, the following weaker problem is open: does there exist a propositional logic $\mathbf{\Lambda}$ such that $C_{\mathcal{KE}}(\mathbf{Q\Lambda}) \neq \widehat{\mathbf{Q\Lambda}}$?

6. Completeness of $\mathbf{Q\Lambda}$ in other semantics.

When syntax and semantics mismatch, one can try to change semantics appropriately. There is a sequence of generalizations leading from Kripke to simplicial semantics, cf. [9], [16].[19] For Kripke incomplete logics considered in this paper simplicial semantics can be helpful: viz., for d-persistent (in particular, Sahlqvist) propositional logics $\mathbf{\Lambda}$, both $\mathbf{Q\Lambda}$ and $\mathbf{Q\Lambda} + Ba$ are simplicially complete [16]. But the general case makes a problem: e.g. for **QGL** and **QGrz** we do not know any semantical characterization.

7. Reflexive boxing.

$\Box \cdot \mathbf{\Lambda}$-frames for are obtained from $\mathbf{\Lambda}$-frames by adding an irreflexive root below. So boxing destroys reflexivity. To stay within reflexive frames, we should add a reflexive root or a reflexive cluster. Both options lead to some operations on modal logics similar to Suzuki's Δ. These operations deserve a special study. Note that iterated reflexive boxing of **QS5** gives predicate modal logics of finite depth. In particular, the logic $\widehat{\mathbf{QS4.4}}$ axiomatized by M. Cresswell [7] is a singleton reflexive boxing of **QS5**.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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