



There are No Structural Stable Axiom A 3-Diffeomorphisms with Dynamics “One-dimensional Surfaced Attractor-repeller”

O. Pochinka 

Abstract. In this paper, we study the structural stability of three-dimensional diffeomorphisms with source-sink dynamics. Here the role of source and sink is played by one-dimensional hyperbolic repeller and attractor. It is well known that in the case when the repeller and the attractor are solenoids (not embedded in the surface), the diffeomorphism is not structurally stable. The author proves that in the case when the attractor and the repeller are canonically embedded in a surface, the diffeomorphism is also not structurally stable.

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1. Introduction and Formulation of Results

The dynamics of any Ω -stable diffeomorphism f of a closed connected n -manifold¹ M^n can be represented as an attractor-repeller. In this situation, all points outside the attractor and the repeller are wandering and move from

¹All the manifolds considered in the paper are assumed to be orientable and all diffeomorphisms are assumed to preserve orientation.

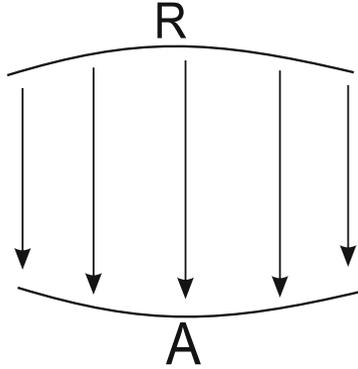


FIGURE 1. Attractor-repeller dynamic

the repeller to the attractor.² Such a representation is not unique in the general case and is a source of finding topological invariants of both regular and chaotic dynamical systems. If the attractor and repeller are basic sets, the wandering set is foliated by stable attractor manifolds and unstable repeller manifolds simultaneously. If the topological dimensions of the attractor and the repeller coincide, then the foliations also have the same dimension. The subject of many studies is the question on the existence and the structural stability of such a diffeomorphism.

Thus, we are dealing with the following situation (see Fig. 1).

- M^n is a smooth closed n -manifold
- $f : M^n \rightarrow M^n$ is a preserving orientation Ω -stable diffeomorphism whose non-wandering set consists of two basic sets, an attractor A and a repeller R of the same topological dimension $k \in \{0, \dots, n-1\}$.³

If $\mathbf{k} = \mathbf{0}$, then the attractor and the repeller are sink and source, respectively (see Fig. 2). In this case, all such diffeomorphisms are given on the n -sphere, are structurally stable, and form a unique class of topological conjugacy (see, for example, [11, Theorem 2.5]).

If $\mathbf{k} = \mathbf{1}$, $\mathbf{n} = \mathbf{2}$ the dynamics on the attractor and repeller are conjugated with the dynamics on non-trivial basic sets of the so-called DA (derived from an Anosov) or DPA (derived from a pseudo-Anosov) surface diffeomorphisms

²A set A is called an *attractor* of a diffeomorphism f if it has a compact neighborhood U_A such that $f(U_A) \subset \text{int } U_A$ and $A = \bigcap_{k \geq 0} f^k(U_A)$. U_A is called a *trapping neighborhood* of A , $\bigcup_{k \in \mathbb{Z}} f^k(U_A)$ is called a *basin* of the attractor A . A *repeller* is defined as the attractor for f^{-1} . By a *dimension* of the attractor (repeller) we mean its topological dimension.

³Since the topological dimension of the basic set does not exceed the dimension of the supporting manifold, k can take values from 0 to n . However, in the case of $k = n$, the basic set is unique, coincides with the ambient manifold and f is an Anosov diffeomorphism (see, for example, [11, Theorem 8.1]).

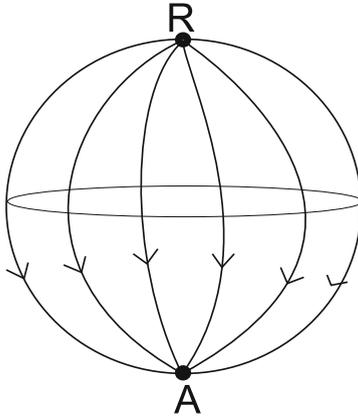


FIGURE 2. Sink-source diffeomorphism

(see, for example, [17]). Particular case is a surfaces diffeomorphism with one-dimensional attractor and repeller. Such dynamics is achieved, for example, by taking of the connected sum of two DA-models on 2-tori (see Fig. 3). R.C. Robinson and R.F. Williams [18] construct an open set of diffeomorphisms with such phase portraits no one of which is finitely stable. It follows from [7] that there are no structurally stable 2-diffeomorphisms whose non-wandering set is a disjoint union of one-dimensional hyperbolic attractor and repeller at all.

If $\mathbf{k} = 2$, $\mathbf{n} = 3$ then due to Brown result [4] the attractor (the repeller) is either expanding attractor (contracting repeller) or surface attractor (surface repeller). Recall that, due to [21], an attractor A of f is said to be *expanding* if the topological dimension of A is equal to the dimension of $W_x^u, x \in A$ (see Fig. 4). One says that R is a *contracting repeller* of f if it is an expanding attractor for f^{-1} . According to [9], a hyperbolic attractor of a diffeomorphism $f : M^3 \rightarrow M^3$ is called a *surface attractor* if it is contained in a compact surface (not necessarily connected and possible with boundary) Σ_A topologically embedded in M^3 such that $f(\Sigma_A) \subset \Sigma_A$. One says that R is a *surface repeller* of f if it is a surface attractor for f^{-1} .

It follows from results by V. Grines, V. Medvedev, E. Zhuzhoma [10], [15] that the dynamics is always not structural stable if either A is an expanding attractor or R is a contracting repeller. V. Grines, Yu. Levchenko, V. Medvedev, O. Pochinka [8] proved that dynamics where both the attractor and the repeller are surface there is only on mapping torus, it possible to be structural stable (see Fig. 5) and the authors obtained complete topological classification of such rough systems.

If $\mathbf{k} = 1$, $\mathbf{n} = 3$ then the attractor (the repeller) is automatically expanding (contracting) as it consists of the unstable (stable) manifolds of its

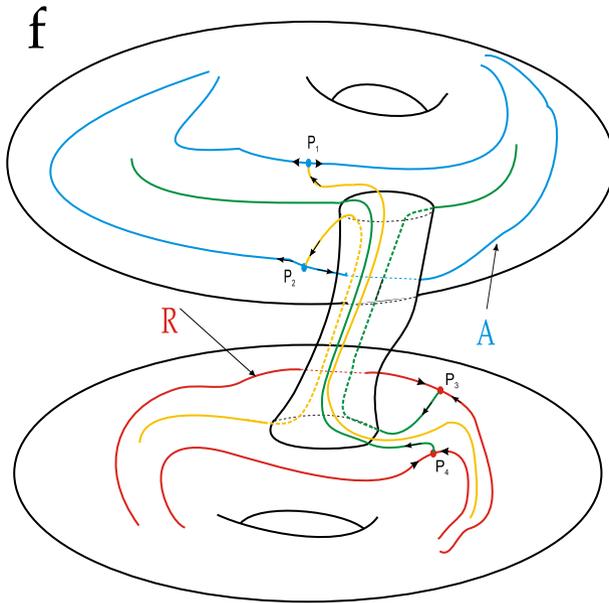


FIGURE 3. Robinson–Williams example

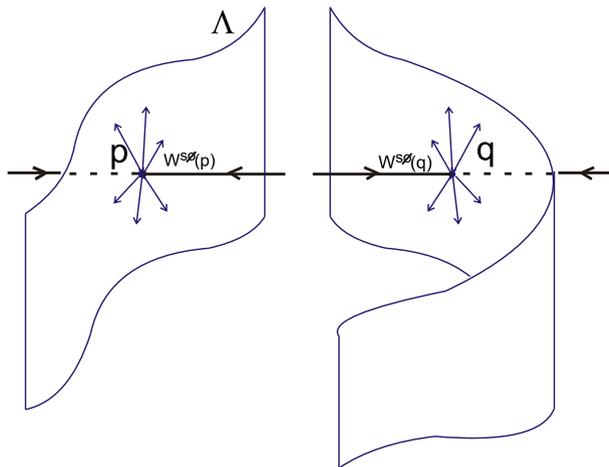


FIGURE 4. Expanding 2-dimensional attractor, obtained from Anosov diffeomorphism by the surgery operation

points, that was proved by R. Plykin [16]. R. Williams [21] shows that the dynamics on such a basic set is conjugate to the shift on the reverse limit of a branched 1-manifold with respect to an expanding map. A construction

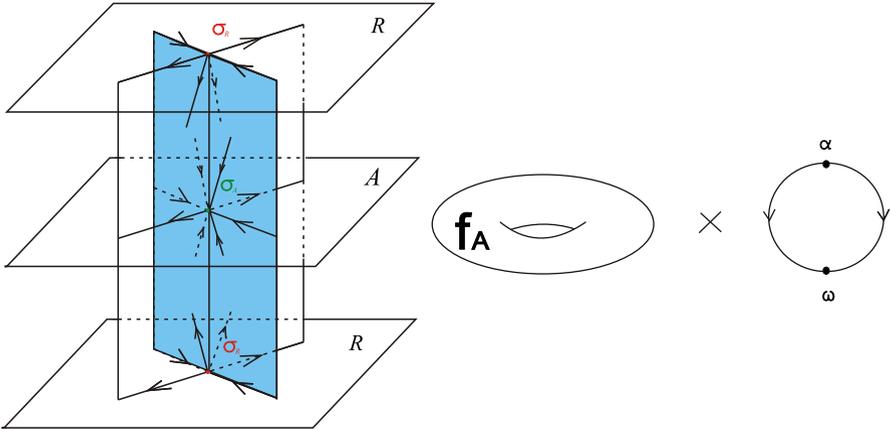


FIGURE 5. Structural stable 3-diffeomorphism with 2-dimensional surfaced attractor and repeller

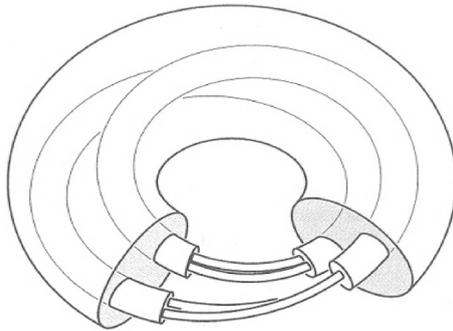


FIGURE 6. Smale's solenoid

of 3-diffeomorphisms with one-dimensional attractor-repeller dynamics firstly was suggested by J. Gibbons [5]. He construct many models on 3-sphere with Smale's solenoid basic sets (see Fig. 6) and proves that all examples are not structurally stable. B. Jiang, Y. Ni and S. Wang [13] proved that a 3-manifold M^3 admits a diffeomorphism f whose non-wandering set consists of Smale's solenoid attractors and repellers if and only if M^3 is a lens space $L(p, q)$ with $p \neq 0$. They also shown that such f are not structural stable.

All generalizations of Smale's solenoid as the intersections of nested handlebodies are not surface. Moreover, all known examples of diffeomorphisms with the generalized solenoids as the attractor and the repeller are not structurally stable.

A natural way to get a surface one-dimensional attractor for a 3-diffeomorphism f is to take an attractor A of some 2-diffeomorphism and

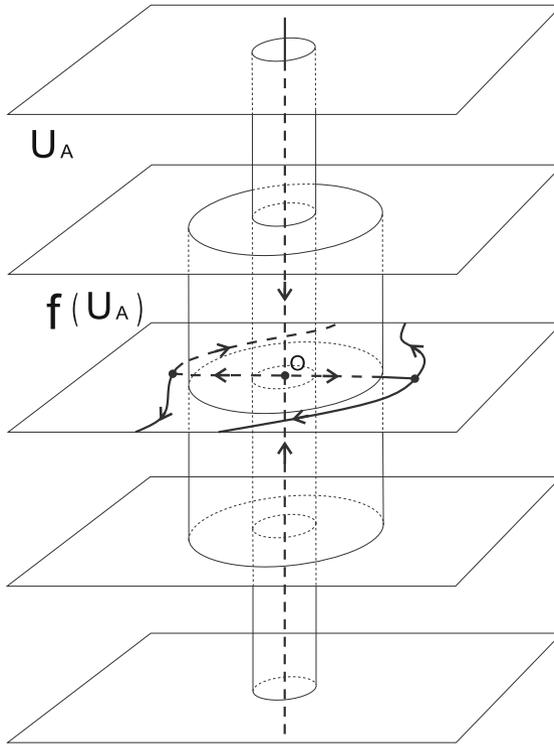


FIGURE 7. Trapping neighborhood of a canonically embedded surface attractor A

multiply its trapping neighbourhood by a contraction in transversal direction (see Fig. 7). According to [1] such attractor A is called *canonically embedded surface attractor*. One says that R is a *canonically embedded surface repeller* of f if it is a surface attractor for f^{-1} .

Infinitely many pairwise Ω -non-conjugated diffeomorphisms with such attractors and repellers were constructed in [1]. Moreover, there a conjecture was formulated that all such diffeomorphisms are not structurally stable. The main result of this paper is the proof of the conjecture.

Theorem 1.1. *There are no structurally stable 3-diffeomorphisms whose non-wandering set is a disjoint union of one-dimensional hyperbolic canonically embedded surface attractor and repeller.*

Notice, that in [3], [19] structurally stable 3-diffeomorphisms with one-dimensional attractor-repeller dynamics were constructed, but the constructed basic sets were not canonically embedded in surfaces in that examples.

2. One Dimensional Basic Sets for Diffeomorphisms of Surfaces

Let M^2 be a closed surface and $\psi : M^2 \rightarrow M^2$ be an Ω -stable diffeomorphism. In this section we describe important properties of one-dimensional basic sets for diffeomorphisms of surfaces following by [16], [6] (see also [11]).

For simplicity everywhere below we assume that the basic set is a connected attractor A . Then

- $A = \bigcup_{x \in A} W_x^u$;
- at least one of the connected components of the set $W_x^s \setminus \{x\}, x \in A$ contains a dense set in A ;
- there is a finite number of points $x \in A$ for which one of the connected components W_x^{s-} of the set $W_x^s \setminus \{x\}$ does not intersect A , denote by W_x^{s+} other connected component. Such points are called *s-boundary*, their set P_A is not empty and consists of periodic points;
- the set $W_A^s \setminus A$ consists of a finite number path-connected components.

A *bunch* b of the attractor A is the union of the maximal number r_b of the unstable manifolds $W_{p_1}^u, \dots, W_{p_{r_b}}^u$ of the s -boundary points p_1, \dots, p_{r_b} of the set A whose separatrices $W_{p_1}^{s-}, \dots, W_{p_{r_b}}^{s-}$ belong to the same path-connected components of $W_A^s \setminus A$. The number r_b is called a *degree of the bunch* (see Fig. 8). Let B_A be the set of all bunches of the attractor A .

For attractor A let m_A denote the number of its bunches and let r_A denote the sum of the degrees of these bunches. Then

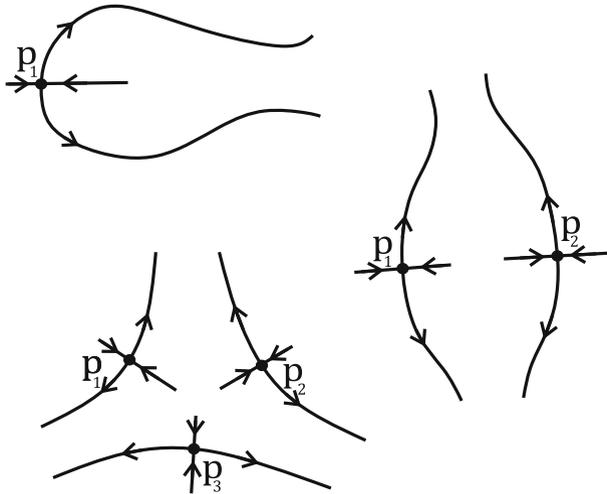


FIGURE 8. Bunches of degrees 1, 2 and 3

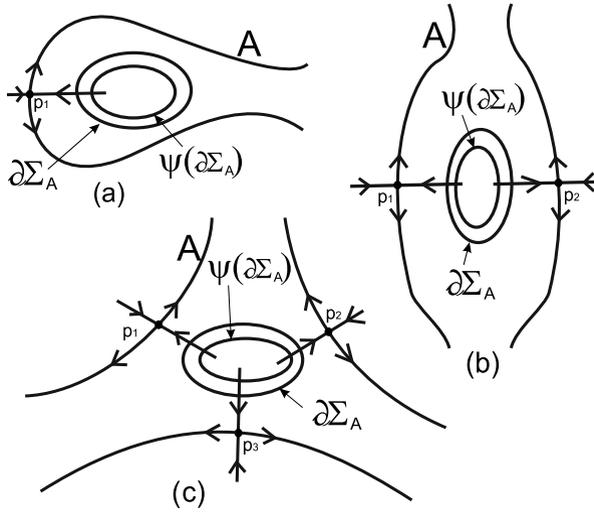


FIGURE 9. Trapping neighborhoods for 1-dimensional attractors

- there is a trapping neighborhood $\Sigma_A \subset M^2$ ($\psi(\Sigma_A) \subset \text{int} \Sigma_A$) of the attractor A such that Σ_A is a compact orientable surface of genus $g_A = 1 + \frac{r_A}{4} - \frac{m_A}{2}$, it has m_A boundary components and it has negative Euler characteristic (see Fig. 9).

3. Point of View from Lobachevsky Plane

In this section, we describe the dynamics on a one-dimensional surface basic set from the point of view of its lifting to the Lobachevsky plane.

3.1. Properties of Automorphisms of Deck Transformation Groups

Firstly we describe important properties of automorphisms of deck transformation groups for universal cover of surfaces following by [12] (see also [11]).

We consider the Poincaré disk model of the hyperbolic plane as the unit open 2-disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane with the hyperbolic metric d . The boundary of the disc \mathbb{U} is called the *absolute of the hyperbolic plane* denoted by \mathbb{E} ($\mathbb{E} = \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$).

Let us describe a characteristic of so called *hyperbolic isometry* $g : \mathbb{U} \rightarrow \mathbb{U}$.

- g uniquely extends to the absolute \mathbb{E} and has exactly two fixed points in \mathbb{E} and it has no fixed points in \mathbb{U} ;
- there is the unique geodesic l_g which is invariant under g . This geodesic is called the *axis of the hyperbolic isometry* g . The axis joins the points P_g and Q_g on the absolute. The restriction of g to its axis is a shift, i.e. $d(x, g(x)) = d_g$ for every point $x \in l_g$ (see Fig. 10);

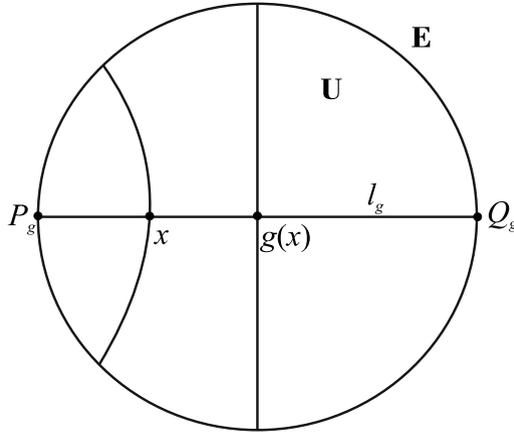


FIGURE 10. Action of a hyperbolic isometry on the Poincaré disk

- two distinct hyperbolic isometries g_1, g_2 have a common fixed point if and only if there is a hyperbolic isometry γ and there are integers k_1, k_2 such that $g_1 = \gamma^{k_1}$ and $g_2 = \gamma^{k_2}$. Therefore, two hyperbolic isometries g_1, g_2 either have no common fixed points or both fixed points of g_1 are the fixed points of g_2 as well.

We say the fixed points of each hyperbolic isometry to be *rational* and call *irrational* any other point of the absolute.

Now let Σ be an orientable surface with boundary (possibly empty) of negative Euler characteristic. If the boundary is empty then we put $S = \Sigma$, in the opposite case we glue two copies of Σ along the boundary components we get a surface S without boundary. The curves along which we glue are essential and, therefore, they can be assumed to be geodesics. Then

- there is a subgroup \mathbb{G}_Σ of the hyperbolic isometries of \mathbb{U} , which is isomorphic to the fundamental group of the manifold Σ and a connected set $\mathbb{U}_\Sigma \subset \mathbb{U}$, which universally covers Σ . That is $\Sigma = \mathbb{U}_\Sigma / \mathbb{G}_\Sigma$ and $p_\Sigma : \mathbb{U}_\Sigma \rightarrow \Sigma$ is a universal cover.
- the set \mathbb{Q}_Σ of the rational points of the isometries from \mathbb{G}_Σ is countable and dense on $\mathbb{E}_\Sigma = \partial\mathbb{U}_\Sigma \cap \mathbb{E}$;
- \mathbb{E}_Σ is the Cantor perfect set on the absolute and, therefore, it can be expressed as $\mathbb{E} \setminus \bigcup_{k \in \mathbb{N}} (P_k, Q_k)$ where (P_k, Q_k) are the adjacent intervals of the Cantor set \mathbb{E}_Σ . Every geodesic $l_k \subset \mathbb{U}$ with the boundary points P_k, Q_k belongs to \mathbb{U}_Σ , is the axis of some non-identity element $g \in \mathbb{G}_\Sigma$ and $p_\Sigma(l_k)$ is a connected component of $\partial\Sigma$ (see Fig. 11).

Every automorphism $\tau : \mathbb{G}_\Sigma \rightarrow \mathbb{G}_\Sigma$ induces on the set \mathbb{Q}_Σ the map which sends the fixed points of an element g onto the fixed points of the element $\tau(g)$. This map uniquely extends to the homeomorphism $\tau^e : \mathbb{E}_\Sigma \rightarrow \mathbb{E}_\Sigma$. Also

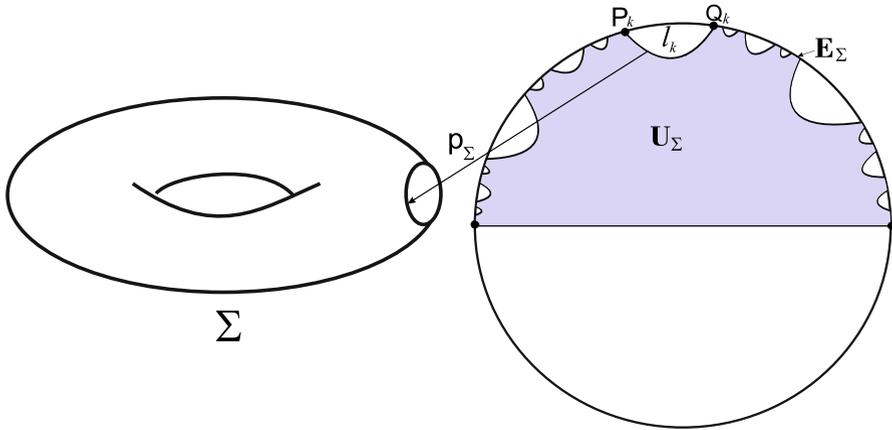


FIGURE 11. A universal cover for Σ

every automorphism $\gamma : \mathbb{G}_\Sigma \rightarrow \mathbb{G}_\Sigma$ induces the *internal automorphism* $A_\gamma : \mathbb{G}_\Sigma \rightarrow \mathbb{G}_\Sigma$ by the formula

$$A_\gamma(g) = \gamma g \gamma^{-1}, \quad g \in \mathbb{G}_\Sigma.$$

Let $\mathcal{A}_\tau = \{A_\gamma \tau^k, \gamma \in \mathbb{G}_\Sigma, k \in \mathbb{Z}\}$. An automorphism τ of the group \mathbb{G}_Σ is said to be *hyperbolic* if $a(g) \neq g$ for every non-identity automorphism $a \in \mathcal{A}_\tau$ and for every non-identity element g except those that $p_\Sigma(l_g) \subset \partial\Sigma$.

Let $h : \Sigma \rightarrow \Sigma$ be a homeomorphism. Since \mathbb{U}_Σ is the universal covering space, it implies that there is a homeomorphism (not unique) $\bar{h} : \mathbb{U}_\Sigma \rightarrow \mathbb{U}_\Sigma$ which is a lift of a map h (i.e. the homeomorphism \bar{h} satisfies $p_\Sigma \bar{h} = h p_\Sigma$). Let $h(x) = y$ and $\bar{x} \in p_\Sigma^{-1}(x), \bar{y} \in p_\Sigma^{-1}(y), \bar{h}(\bar{x}) = \bar{y}$ and $g \in \mathbb{G}_\Sigma$. Since $p_\Sigma(g(\bar{x})) = x$ we have $p_\Sigma(\bar{h}(g(\bar{x}))) = y$. Hence, $\bar{h}(g(\bar{x})) = g'(\bar{y})$ for some element $g' \in \mathbb{G}_\Sigma$ and, therefore, $\bar{h}(g(\bar{x})) = g'(\bar{h}(\bar{x}))$. Thus, the lift \bar{h} induces the automorphism $\tau_{\bar{h}}$ of the group \mathbb{G}_Σ which assigns the element g' to an element g by $\tau_{\bar{h}}(g) = \bar{h} g \bar{h}^{-1}$. Every lift $\bar{h} : \mathbb{U}_\Sigma \rightarrow \mathbb{U}_\Sigma$ of a homeomorphism $h : \Sigma \rightarrow \Sigma$ possesses to following properties:

- \bar{h} uniquely extends onto \mathbb{E}_Σ by the homeomorphism $\bar{h}^* : \mathbb{E} \rightarrow \mathbb{E}$ and $\bar{h}^* = \tau_{\bar{h}}^e$;
- if $h_1, h : \Sigma \rightarrow \Sigma$ are homotopic homeomorphisms then there is a lift $\bar{h}_1 : \mathbb{U}_\Sigma \rightarrow \mathbb{U}_\Sigma$ of h_1 such that $\bar{h}_1^* = \bar{h}^*$.

Homeomorphisms $h_1, h_2 : \Sigma \rightarrow \Sigma$ are called *π_1 -conjugate* if there are their lifts $\bar{h}_1, \bar{h}_2 : \mathbb{U}_\Sigma \rightarrow \mathbb{U}_\Sigma$ and an automorphism $\tau : \mathbb{G}_\Sigma \rightarrow \mathbb{G}_\Sigma$ such that $\tau \tau_{\bar{h}_1} = \tau_{\bar{h}_2} \tau$.

3.2. Lifting of One-Dimensional Attractor to the Lobachevsky Plane

In this section we describe important properties of the lifting to the Lobachevsky plane of one-dimensional basic sets for diffeomorphisms of surfaces following by [16], [6] (see also [11]).

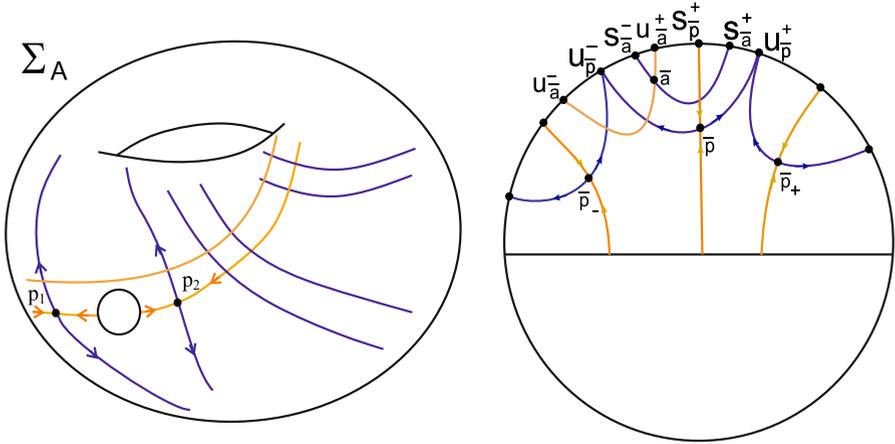


FIGURE 12. Asymptotic behaviour of invariant manifolds on the universal cover, here $p = p_1, p_+ = p_- = p_2$

Let A be a hyperbolic one-dimensional attractor of an Ω -stable diffeomorphism $f : M^2 \rightarrow M^2$ and Σ_A be its trapping neighborhood (see Sect. 2). Let $p_{\Sigma_A} : \mathbb{U}_{\Sigma_A} \rightarrow \Sigma_A$ be a universal covering and let \mathbb{G}_{Σ_A} be the group of its covering transformations. According to the Sect. 3.1 every lifting $\tilde{\psi}_A$ induces an automorphism $\tau_{\tilde{\psi}_A}$ of the group \mathbb{G}_{Σ_A} . Then

- the automorphism $\tau_{\tilde{\psi}_A}$ is hyperbolic.

Let $\bar{A} = p_{\Sigma_A}^{-1}(A)$. If $a \in A$ then let $\bar{a} \in \bar{A}$ denote the point in the preimage $p_{\Sigma_A}^{-1}(a)$. Let $\delta \in \{u, s\}$ and $\nu \in \{+, -\}$. Denote by w_a^δ the curve on \mathbb{U}_{Σ_A} such that $p_{\Sigma_A}(w_a^\delta) = W_a^\delta$. If $t \in \mathbb{R}$ is a parameter on the curve W_a^δ such that $W_a^\delta(0) = a$ then $w_a^\delta(t)$ is the point on w_a^δ such that $p_{\Sigma_A}(w_a^\delta(t)) = W_a^\delta(t)$ and $w_a^{\delta+}, w_a^{\delta-}$ are the connected components of the curve $w_a^\delta \setminus \bar{a}$ for $t > 0, t < 0$ respectively.

Let $a \in A$. We say that a curve $w_a^{\delta\nu}$ has the asymptotic direction δ_a^ν for $t \rightarrow \nu\infty$ if the set $cl(w_a^{\delta\nu}) \setminus w_a^{\delta\nu}$ consists of the point \bar{a} and the point δ_a^ν which belongs to \mathbb{E}_{Σ_A} . For the attractor A the following asymptotic properties take place (see Fig. 12):

- if for a point $a \in A$ the component $W_a^{\delta\nu}$ is densely situated in A then $w_a^{\delta\nu}$ has an irrational asymptotic direction δ_a^ν for $t \rightarrow \nu\infty$;
- the curve w_a^u has two distinct boundary points (asymptotic directions) u_a^+, u_a^- ;
- the curves w_a^s for some $\bar{a} \in (\bar{A} \setminus w_{P_A}^{s+})$ has two distinct boundary points s_a^+, s_a^- ;
- for every point $\bar{a} \in \bar{A}$ the intersection $cl(w_a^u) \cap cl(w_a^s)$ consists of a unique point \bar{a} ;

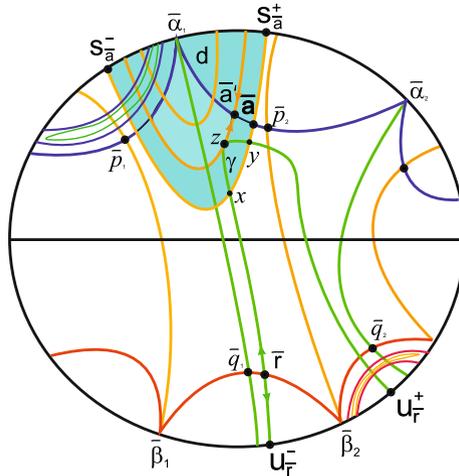


FIGURE 13. The lifting of the Robinson–Williams example on the Lobachevsky plane

- if $w_a^s \cap w_b^s = \emptyset$ for some points $\bar{a}, \bar{b} \in \bar{A}$ then $cl(w_a^s) \cap cl(w_b^s) = \emptyset$;
- if $w_a^u \cap w_b^u = \emptyset$ for some points $\bar{a} \in \bar{A}, \bar{b} \in (\bar{A} \setminus w_{\bar{P}_A}^u)$ then $cl(w_a^u) \cap cl(w_b^u) = \emptyset$;
- for every point $\bar{p} \in \bar{P}_A$ there are two distinct points $\bar{p}_+, \bar{p}_- \in \bar{P}_A$ such that $w_{\bar{p}_+}^u \cap w_{\bar{p}}^u = \emptyset, w_{\bar{p}_-}^u \cap w_{\bar{p}}^u = \emptyset, cl(w_{\bar{p}_+}^u) \cap cl(w_{\bar{p}}^u) = u_{\bar{p}_+}^+, cl(w_{\bar{p}_-}^u) \cap cl(w_{\bar{p}}^u) = u_{\bar{p}_-}^+$ and the points p, p_-, p_+ are the s -boundary points of the same bunch of the attractor A ;
- if $w_a^u \cap w_b^s = \emptyset$ for some points $\bar{a}, \bar{b} \in \bar{A}$ then $cl(w_a^u) \cap cl(w_b^s) = \emptyset$.

At the end of this section we illustrate why the Robinson–Williams example is not structural stable. A phase portrait of the lifting of the Robinson–Williams example on the Lobachevsky plane is given on Fig. 13. It follows from the description above that there are point $\bar{a} \in \bar{A}, \bar{r} \in \bar{R}$ such that $w_a^s \cap w_r^u \neq \emptyset$ and w_a^s, w_r^u have asymptotic directions $s_a^-, s_a^+, u_r^-, u_r^+$ which bound disjoint arcs on the absolute. Moreover, the closure of the leaf w_a^s divides \mathbb{U} into two connected components, exactly one of them is a 2-disk (denote it d) which does not contain the points u_r^-, u_r^+ .

If we assume that the leaves w_a^s and w_r^u are transversally intersected, then there is a connected component γ of the intersection $w_r^u \cap d$ which has exactly two intersection points with w_a^s , say x, y . By Rolle’s theorem there is a point $z \in \gamma$ where w_r^u has a contact with a leaf $w_{a'}^s$, for a point $\bar{a}' \in \bar{A}$, that contradict to the assumption.

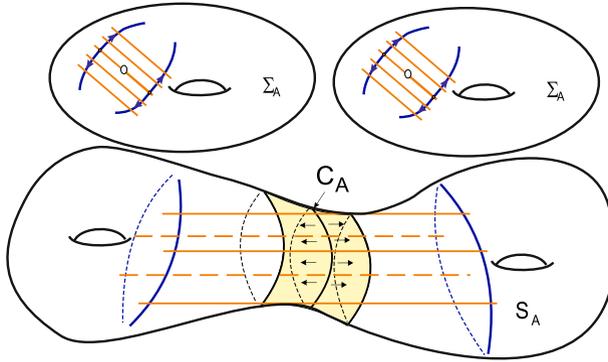


FIGURE 14. Phase portrait of the diffeomorphism Ψ_A

4. “One-dimensional canonically embedded surface attractor-repeller” Dynamics for 3-Diffeomorphism

Let $f : M^3 \rightarrow M^3$ be an Ω -stable diffeomorphism whose non-wandering set consists of a one-dimensional canonically embedded surface attractor A and one-dimensional canonically embedded surface repeller R . Then $V_f = M^3 \setminus (A \cup R)$ is the wandering set of f which coincides with $W_A^s \setminus A$ and $W_R^u \setminus R$ simultaneously. As our aim to refute the structural stability of f , everywhere below we will suppose that all boundary points of the attractor A are fixed (in the opposite case we can consider an appropriate degree of f).

4.1. The Topology of the Wandering Set

By definition of canonically embedded attractor and due to results in Sect. 2, A has a trapping neighborhood U_A of the form $\Sigma_A \times [-1, 1]$, where $\Sigma_A = \Sigma_A \times \{0\}$ is a trapping neighborhood of attractor A as an attractor of a 2-diffeomorphism ψ_A and diffeomorphism $f_A = f|_{U_A} : U_A \rightarrow f(U_A)$ has a form $f_A(w, z) = (\psi_A(w), z/2) : \Sigma_A \times [-1, 1] \rightarrow f(\Sigma_A) \times [-1/2, 1/2]$ (see Fig. 7).

Recall that Σ_A is a compact orientable surface of genus g_A with m_A boundary components and negative Euler characteristic. Let us glue two copies of Σ_A along the boundary components we get a closed surface S_A (see Fig. 14) of a positive genus $\rho_A = 2g_A + m_A - 1 > 1$. Denote by C_A the coinciding copies of $\partial\Sigma_A$ in S_A .

Lemma 4.1. *Diffeomorphism $f|_{V_f}$ is smoothly conjugate with a diffeomorphism $\chi_A : S_A \times \mathbb{R} \rightarrow S_A \times \mathbb{R}$ given by the formula*

$$\chi_A(s, r) = (\Psi_A(s), r + 1),$$

where $\Psi_A : S_A \rightarrow S_A$ is a reducible diffeomorphism which coincides with ψ_A on each copy of Σ_A outside a neighborhood of C_A and is an extension from C_A in the neighborhood, being identical on C_A (see Fig. 14).

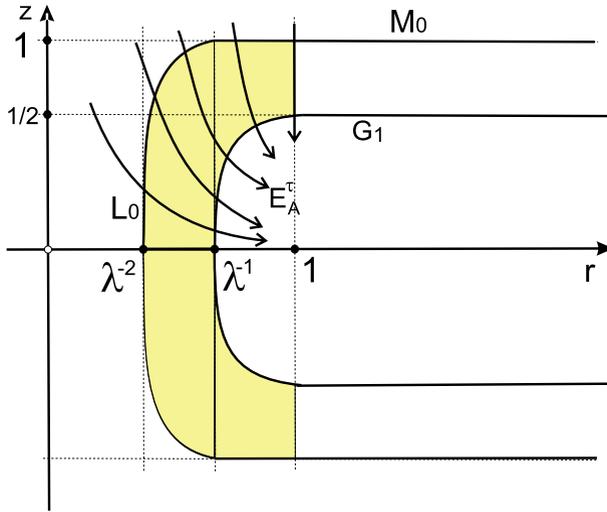


FIGURE 15. Construction of the domain K_A

Proof. Let us compactify the surface Σ_A by discs D_A to get a closed surface $\tilde{\Sigma}_A$ and a diffeomorphism $\tilde{\psi}_A : \tilde{\Sigma}_A \rightarrow \tilde{\Sigma}_A$ which is ψ_A on Σ_A and every disc in D_A is a part of the basin of a linear source point in it. Let $\tilde{f}_A : \tilde{\Sigma}_A \times \mathbb{R} \rightarrow \tilde{\Sigma}_A \times \mathbb{R}$ be a diffeomorphism given by the formula

$$\tilde{f}_A(w, z) = (\tilde{\psi}_A(w), z/2).$$

Let (x, y) be local coordinates in a neighborhood the discs D_A such that for $r = \sqrt{x^2 + y^2}$ and $\lambda > 1$, $D_A = \{(x, y) : r \leq \lambda^{-4}\}$, $\partial\Sigma_A = \{(x, y) : r = 1\}$ and the diffeomorphism $\tilde{\psi}_A$ has a form

$$\tilde{\psi}_A(x, y) = (\lambda x, \lambda y)$$

for $\lambda^{-2} \leq r \leq 1$. Let (see Fig. 15)

$$L_0 = \left\{ (r, z) : r = \lambda^{-1} - (\lambda^{-1} - \lambda^{-2})\sqrt{1 - z^2}, \lambda^{-2} \leq r \leq \lambda^{-1} \right\}.$$

Let G_0 coincides with L_0 for $\lambda^{-2} \leq r \leq \lambda^{-1}$ and coincides with $M_0 = \tilde{\Sigma}_A \times \{-1, 1\}$ outside the set $\{(r, z) : r < \lambda^{-1}\}$ (see Fig. 15). By the construction $G_0 \cong S_A$ and we will identify their.

By the construction the diffeomorphism \tilde{f}_A on the set $\{(r, z) : \lambda^{-2} \leq r \leq 1\}$ is a one-time shift of the flow $\xi_A^\tau(w, z) = (\lambda^\tau w, 2^{-\tau} z)$ and L_0 is transversal to trajectories of this flow. Then $G_0 \cap \tilde{f}_A(G_0) = \emptyset$ and G_0 with $G_1 = \tilde{f}_A(G_0)$ bounds a compact subset K_A which is a fundamental domain of the restriction of the diffeomorphism f to V_f .

Let us construct a diffeomorphism $\eta_A : K_A \rightarrow S_A \times [0, 1]$. For this aim let us define the flow $\zeta_A^\tau(w, z) = (w, 4^{-\tau} z)$ on the set $\{(r, z) : r \geq 1\}$. Finitely,

let us E_A^τ be a confluence of the flows ξ_A^τ and ζ_A^τ on K_A (see Fig. 15) whose trajectories transversally intersects G_0 and G_1 at exactly one point. Then for every point $s \in G_0$ it is correctly defined a time $\tau(s)$ such that $E^{\tau(s)}(s) \in G_1$. Thus the desired diffeomorphism η_A assigns for a point $(w, s) = E^{t\tau(s)}(s)$, $s \in S_A$, $t \in [0, 1]$ from K_A the point $(s, t) \in S_A \times [0, 1]$.

Define a diffeomorphism $\Psi_A : S_A \rightarrow S_A$ by the formula

$$\eta_A f \eta_A^{-1}(s, 0) = (\Psi_A(s), 1).$$

By the construction Ψ_A satisfies all requirements of the lemma.

Using similar notation with index R for the repeller of the diffeomorphism f we conclude that:

- Σ_R is a compact orientable surface of genus g_R with m_R boundary components and negative Euler characteristic
- R has a trapping neighborhood U_R of the form $\Sigma_R \times [-1, 1]$, where $\Sigma_R = \Sigma_R \times \{0\}$ is a trapping neighborhood of repeller R as a repeller of a 2-diffeomorphism ψ_R and diffeomorphism $f_R^{-1} = f^{-1}|_{U_R} : U_R \rightarrow f^{-1}(U_R)$ has a form $f_R^{-1}(w, z) = (\psi_R(w), z/2) : \Sigma_R \times [-1, 1] \rightarrow f^{-1}(\Sigma_R) \times [-1/2, 1/2]$;
- S_R is a closed surface of a positive genus $\rho_R = 2g_R + m_R - 1 > 1$ formed by gluing two copies of Σ_R along the boundary components, C_R is the coinciding copies of $\partial\Sigma_R$ in S_R ;
- diffeomorphism $f|_{V_f}$ is smoothly conjugate with a diffeomorphism $\chi_R : S_R \times \mathbb{R} \rightarrow S_R \times \mathbb{R}$ given by the formula

$$\chi_R(s, r) = (\Psi_R(s), r + 1),$$

where $\Psi_R : S_R \rightarrow S_R$ coincides with ψ_R on each copy of Σ_R outside a neighborhood of $\partial\Sigma_R$ and is a contraction to $\partial\Sigma_R$ in the neighborhood, being identical on $\partial\Sigma_R$.

Let us introduce orbit spaces $\hat{V}_f = V_f/f$, $\hat{V}_A = (S_A \times \mathbb{R})/\chi_A$, $\hat{V}_R = (S_R \times \mathbb{R})/\chi_R$ and the natural projections $p_f : V_f \rightarrow \hat{V}_f$, $p_A : S_A \times \mathbb{R} \rightarrow \hat{V}_A$, $p_R : S_R \times \mathbb{R} \rightarrow \hat{V}_R$. Also let $T_A = C_A \times \mathbb{R}$, $\hat{T}_A = p_A(T_A)$, $T_R = C_R \times \mathbb{R}$, $\hat{T}_R = p_R(T_R)$.

Lemma 4.2. *For every Ω -stable diffeomorphism $f : M^3 \rightarrow M^3$ with one-dimensional canonically embedded surface attractor-repeller the following is true:*

1. $\rho_A = \rho_R$;
2. \hat{T}_A (\hat{T}_R) is a collection of m_A (m_R) incompressible tori which form characteristic submanifold in \hat{V}_A (\hat{V}_R), that is \hat{V}_A (\hat{V}_R) should be cut along these tori to yield pieces that each have geometric structures (the JSJ-decomposition);
3. $m_A = m_R$, $g_A = g_R$.

- Proof.* 1. It follows from Lemma 4.1 that V_f is diffeomorphic to $S_A \times \mathbb{R}$ and $S_R \times \mathbb{R}$ simultaneously, then $S_A \times \mathbb{R}$ and $S_R \times \mathbb{R}$ are diffeomorphic. Hence, the fundamental group $\pi_1(S_A \times \mathbb{R})$ and $\pi_1(S_R \times \mathbb{R})$ are isomorphic, that implies $\rho_A = \rho_R$.
2. By the construction Ψ_A (Ψ_R) is a diffeomorphism reducible by curves C_A (C_R). As Ψ_A (Ψ_R) is identical on C_A (C_R) then \hat{T}_A (\hat{T}_R) are m_A (m_R) pairwise disjoint tori. It follows from the Nilsen-Thurston theory [20] that they are incompressible and form characteristic submanifold in \hat{V}_A (\hat{V}_R), that is \hat{V}_A (\hat{V}_R) should be cut along these tori to yield pieces that each have geometric structures (the JSJ-decomposition).
3. It follows from Lemma 4.1 that \hat{V}_A and \hat{V}_R are diffeomorphic. As characteristic submanifold is unique up to isotopy (see, for example [2]) then $m_A = m_R$.

Thus, without loss of generality, we can assume that $\Sigma_A = \Sigma_R = \Sigma$, $S_A = S_R = S$ and $C_A = C_R = C$. Let $V = S \times \mathbb{R}$. It follows from Lemma 4.1 that there are diffeomorphisms $\eta_A, \eta_R : V_f \rightarrow V$ such that

$$\eta_A f = \chi_A \eta_A, \eta_R f = \chi_R \eta_R.$$

Then the diffeomorphism $\eta = \eta_A \eta_R^{-1} : V \rightarrow V$ possesses the property

$$\eta \chi_R = \chi_A \eta.$$

Thus the set V is foliated by foliations $F^s = \{\eta_A(W_a^s), a \in A\}$ and $F^u = \{\eta_R(W_r^u), r \in R\}$. To prove Theorem 1.1 we have to show that the foliations F^s and $\eta(F^u)$ are not transversal. We are going to do it using the universal covering space for V .

4.2. Lifting Wandering Dynamics to the Universal Covering Space

Let \mathbb{G}_S be a group of the hyperbolic isometrics of the Lobachevsky plane \mathbb{U} , which is isomorphic to the fundamental group of the surface S (see Sect. 3.1) and $S = \mathbb{U}/\mathbb{G}_S$. Then the group $\mathbb{G}_V = \{g \times id : g \in \mathbb{G}_S\}$ is a group of isometrics of $\mathbb{U} \times \mathbb{R}$, which is isomorphic to the fundamental group of V and $V = (\mathbb{U} \times \mathbb{R})/\mathbb{G}_V$. Denote by $q : \mathbb{U} \times \mathbb{R} \rightarrow V$ the natural projection and by \tilde{A} a connected component of the set $q^{-1}(A)$ for a connected subset $A \subset V$. Let $H : V \rightarrow V$ be a homeomorphism and $\tilde{H} : \mathbb{U} \times \mathbb{R} \rightarrow \mathbb{U} \times \mathbb{R}$ be its a lift. Then \tilde{H} induces the automorphism $\tau_{\tilde{H}}$ of the group \mathbb{G}_S which assigns the element $g' = \tau_{\tilde{H}}(g)$ to an element g by

$$(g' \times id) \circ \tilde{H} = \tilde{H} \circ (g \times id).$$

By the construction every lifts $\tilde{\chi}_A, \tilde{\chi}_R : \mathbb{U} \times \mathbb{R} \rightarrow \mathbb{U} \times \mathbb{R}$ of the diffeomorphisms $\chi_A, \chi_R : V \rightarrow V$ have forms $\tilde{\chi}_A(\tilde{s}, r) = (\tilde{\Psi}_A(\tilde{s}), r + 1), \tilde{\chi}_R(\tilde{s}, r) = (\tilde{\Psi}_R(\tilde{s}), r + 1), (\tilde{s}, r) \in \mathbb{U} \times \mathbb{R}$, where $\tilde{\Psi}_A, \tilde{\Psi}_R : \mathbb{U} \rightarrow \mathbb{U}$ are lifts of $\Psi_A, \Psi_R : S \rightarrow S$. Herewith, $\tau_{\tilde{\chi}_A} = \tau_{\tilde{\Psi}_A}, \tau_{\tilde{\chi}_R} = \tau_{\tilde{\Psi}_R}$.

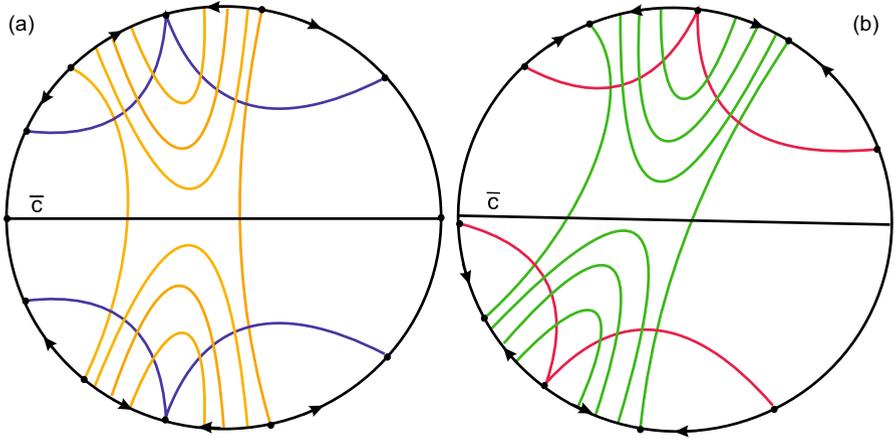


FIGURE 16. Dynamics of the maps: **a** $\bar{\Psi}_A$ on \mathbb{U} and $\tau_{\bar{\Psi}_A}^e$ on \mathbb{E} ; **b** $\bar{\Psi}_R$ on \mathbb{U} and $\tau_{\bar{\Psi}_R}^e$ on \mathbb{E}

Lemma 4.3. *The diffeomorphisms Ψ_A and Ψ_R are π_1 -conjugate.⁴*

Proof. Recall that $\eta\chi_R = \chi_A\eta$. Let $\tilde{\chi}_A : V \rightarrow V$ be a lift of χ_A and $\tilde{\eta} : V \rightarrow V$ be a lift of η . Then $\tilde{\chi}_R = \tilde{\eta}^{-1}\tilde{\chi}_A\tilde{\eta} : V \rightarrow V$ be a lift of χ_R . Thus $\tau_{\tilde{\eta}}\tau_{\tilde{\chi}_R} = \tau_{\tilde{\chi}_A}\tau_{\tilde{\eta}}$ and, hence,

$$\tau_{\tilde{\eta}}\tau_{\bar{\Psi}_R} = \tau_{\bar{\Psi}_A}\tau_{\tilde{\eta}}.$$

Let us fixed a curve c in C , a lift $\bar{\Psi}_A : \mathbb{U} \rightarrow \mathbb{U}$ such that $\bar{\Psi}_A(\bar{s}) = \bar{s}$ for every point $\bar{s} \in \bar{c}$ (see Fig. 16a)) and the lift $\tilde{\chi}_A(\bar{s}, r) = (\bar{\Psi}_A(\bar{s}), r + 1)$. Due to Lemma 4.3, without loss of generality we can assume that $\tau_{\tilde{\eta}} = id$ in the equality⁵ $\tau_{\tilde{\eta}}\tau_{\tilde{\chi}_R} = \tau_{\tilde{\chi}_A}\tau_{\tilde{\eta}}$. Then the diffeomorphism $\bar{\Psi}_R$ in the lift $\tilde{\chi}_R(\bar{s}, r) = (\bar{\Psi}_R(\bar{s}), r + 1)$ has a dynamics represented on Fig. 16.

5. There Are No Structural Stable 3-Diffeomorphisms with Dynamic “one-dimensional canonically embedded surface attractor-repeller”

In this section we prove Theorem 1.1.

⁴Notice that this fact follows independently from the other side. The manifold \hat{V}_A (\hat{V}_R) is the mapping torus $S_A \times [0, 1]/\sim$ ($S_R \times [0, 1]/\sim$) with the monodromy Ψ_A (Ψ_R), that is $(s, 1) \sim (\Psi_A(s), 0)$ ($(s, 1) \sim (\Psi_R(s), 0)$). As the mapping tori \hat{V}_A (\hat{V}_R) are homeomorphic then the monodromies Ψ_A (Ψ_R) are π_1 -conjugate (see, for example, [14]).

⁵For this aim it is enough to consider a diffeomorphism $g\Psi_Rg^{-1}$ instead Ψ_R where $g : S \rightarrow S$ has a lift $\bar{g} : \mathbb{U} \rightarrow \mathbb{U}$ such that $\tau_{\bar{g}} = \tau_{\tilde{\eta}}$.

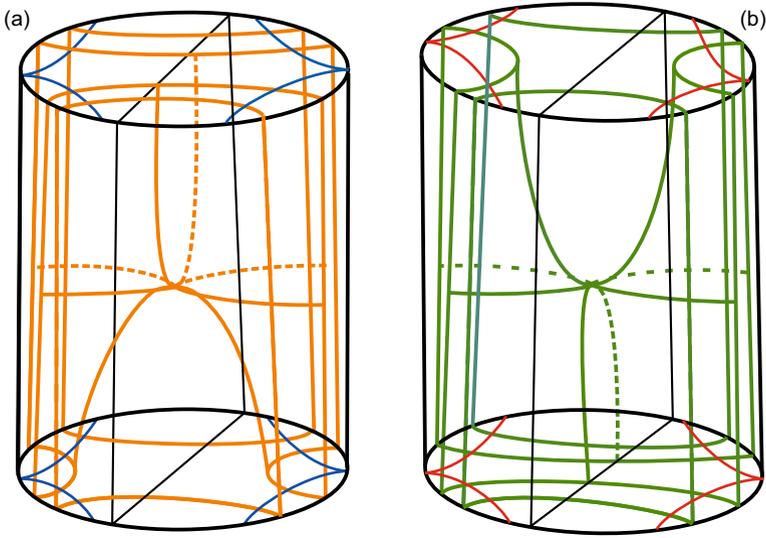


FIGURE 17. Leaves of the foliations on V : a) \tilde{F}^s ; b) \tilde{F}^u

Proof. Let $\tilde{F}^s = q^{-1}(F^s)$, $\tilde{F}^u = q^{-1}(F^u)$ (see Fig. 17). Due to results from the previous section it is enough to show that the foliations \tilde{F}^s , $\tilde{\eta}(\tilde{F}^u)$ are not transversal. Suppose the contrary.

As $\tau_{\tilde{\eta}} = id$ then $\tilde{\eta}$ preserves asymptotic behaviour of leaves of the foliation \tilde{F}^u and, hence, mutual configuration of the leaves \tilde{w}^s , \tilde{w}^u of the foliations \tilde{F}^s , $\tilde{\eta}(\tilde{F}^u)$ has a form represented on Fig. 18. Moreover, if we consider a compactification of $\mathbb{U} \times \mathbb{R}$ as $cl(\mathbb{U}) \times [0, 1]$ then we will see that the closure of every leaf \tilde{w}^s (\tilde{w}^u), except the leaves of the boundary points, has two intersection points s_1, s_2 (curves) with $cl(\mathbb{U}) \times \{0\}$ and two intersection curves (points u_1, u_2) with $cl(\mathbb{U}) \times \{1\}$ (see Fig. 18). Moreover, one of the curve joints the points u_1, u_2 (s_1, s_2). Then there is a connected component γ_u (γ_s) of the intersection $\tilde{w}^u \cap \tilde{w}^s$ joining the points u_1, u_2 (s_1, s_2).

On the other side the leaf \tilde{w}^s is homeomorphic to 2-disc and completely foliated by leaves of the foliation \tilde{F}^s . By Rolle's theorem there is a point $z \in \tilde{w}^s$ where a leaf of the foliation \tilde{F}^s has a contact with \tilde{w}^s , that contradict to the assumption.

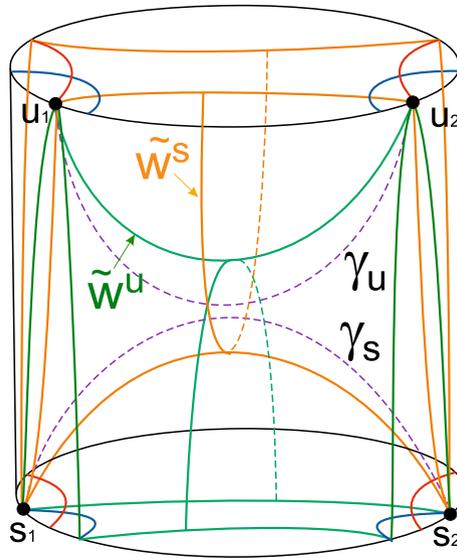


FIGURE 18. Intersection of the Leaves of the Leaves \tilde{w}^s, \tilde{w}^u

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Data Availability Data sharing is not applicable to this article as no new data were created or analysed in this study.

Declarations

Conflict of interest I will disclose all relationships that could be viewed as potential conflicts of interest.

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O. Pochinka
DSA Laboratory
HSE University
Moscow
Russia
e-mail: olga-pochinka@yandex.ru

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