# A quasi-energy function for Pixton diffeomorphisms defined by generalized Mazur knots 

Timur Medvedev • Olga Pochinka

Received: date / Accepted: date


#### Abstract

In this paper we give a lower estimate for the number of critical points of the Lyapunov function for Pixton diffeomorphisms (i.e. Morse-Smale diffeomorphisms in dimension 3 whose chain recurrent set consists of four points: one source, one saddle and two sinks). Ch. Bonatti and V. Grines proved that the class of topological equivalence of such diffeomorphism $f$ is completely defined by the equivalency class of the Hopf knot $L_{f}$ that is the knot in the generating class of the fundamental group of the manifold $\mathbb{S}^{2} \times \mathbb{S}^{1}$. They also proved that there are infinitely many such classes and that any Hopf knot can be realized by a Pixton diffeomorphism. D. Pixton proved that diffeomorphisms defined by the standard Hopf knot $L_{0}=\{s\} \times \mathbb{S}^{1}$ have an energy function (Lyapunov function) whose set of critical points coincide with the chain recurrent set whereas the set of critical points of any Lyapunov function for Pixton diffeomorphism with nontrivial (i.e. non equivalent to the standard) Hopf knot is strictly larger than the chain recurrent set of the diffeomorphism. The Lyapunov function for Pixton diffeomorphism with minimal number of critical points is called the quasi-energy function. In this paper we construct a quasi-energy function for Pixton diffeomorphisms defined by a generalized Mazur knot.


Keywords Hopf knot • Mazur knot • Pixton diffeomorphism • quasi-energy function

Mathematics Subject Classification (2020) 37C15 • 37D15

[^0]
## 1 Introduction and the main results

Let $M^{n}$ be a smooth closed $n$-manifold with a metric $d$ and let $f: M^{n} \rightarrow$ $M^{n}$ be a diffeomorphism. For two given points $x, y \in M^{n}$ a sequence of points $x=x_{0}, \ldots, x_{m}=y$ is called an $\varepsilon$-chain of length $m \in \mathbb{N}$ connecting $x$ to $y$ if $d\left(f\left(x_{i-1}\right), x_{i}\right)<\varepsilon$ for $1 \leqslant i \leqslant m$ (Fig. 11).


Fig. 1 An $\varepsilon$-chain of length $m \in \mathbb{N}$

A point $x \in M^{n}$ is called chain recurrent for the diffeomorphism $f$ if for every $\varepsilon>0$ there is an $\varepsilon$-chain of length $m$ connecting $x$ to itself for some $m$ ( $m$ depends on $\varepsilon>0$ ). The chain recurrent set, denoted by $\mathcal{R}_{f}$, is the set of all chain recurrent points of $f$. Define the equivalence on $\mathcal{R}_{f}$ by the rule: $x \sim y$ if for every $\varepsilon>0$ there is are $\varepsilon$-chains connecting $x$ to $y$ and $y$ to $x$. This equivalence relation defines equivalence classes called chain components.

If the chain recurrent set of a diffeomorphism $f$ is finite then it consists of periodic points. A periodic point $p \in \mathcal{R}_{f}$ of period $m_{p}$ is said to be hyperbolic if absolute values of all the eigenvalues of the Jacobian matrix $\left.\left(\frac{\partial f^{m_{p}}}{\partial x}\right)\right|_{p}$ are not equal to 1 . If absolute values of all these eigenvalues are greater (less) than 1 then $p$ is called a $\operatorname{sink}$ (a source). Sinks and sources are called knots. If a periodic point is not a knot then it is called a saddle.

Let $p$ be a hyperbolic periodic point of a diffeomorphism $f$ whose chain recurrent set is finite. The Morse index of $p$, denoted by $\lambda_{p}$, is the number of eigenvalues of Jacobian matrix whose absolute values are greater than 1. The stable manifold $W_{p}^{s}=\left\{x \in M^{n}: \lim _{k \rightarrow+\infty} d\left(f^{k m_{p}}(x), p\right)=0\right\}$ and the unstable manifold $W_{p}^{u}=\left\{x \in M^{n}: \lim _{k \rightarrow+\infty} d\left(f^{-k m_{p}}(x), p\right)=0\right\}$ of $p$ are smooth manifolds diffeomorphic to $\mathbb{R}^{\lambda_{p}}$ and $\mathbb{R}^{n-\lambda_{p}}$, respectively. Stable and unstable manifolds are called invariant manifolds. A connected component of the set $W_{p}^{u} \backslash p\left(W_{p}^{s} \backslash p\right)$ is called a unstable (stable) separatrice of $p$.

A diffeomorphism $f: M^{n} \rightarrow M^{n}$ is called a Morse-Smale diffeomorphism if

1. its chain recurrent set $\mathcal{R}_{f}$ consists of finite number of hyperbolic points;
2. for any two points $p, q \in \mathcal{R}_{f}$ the manifolds $W_{p}^{s}, W_{q}^{u}$ intersect transversally.

C Conley in [3] gave the following definition: a Lyapunov function for a MorseSmale diffeomorphism $f: M^{n} \rightarrow M^{n}$ is a continuous function $\varphi: M^{n} \rightarrow \mathbb{R}$ satisfying

- $\varphi(f(x))<\varphi(x)$ if $x \notin R_{f}$;
$-\varphi(f(x))=\varphi(x)$ if $x \in R_{f}$.
Notice that every Morse-Smale diffeomorphism $f$ has a Morse-Lyapunov function ${ }^{1}$, i.e. a Lyapunov function $\varphi: M^{n} \rightarrow \mathbb{R}$ which is a Morse function such that each periodic point $p \in \mathcal{R}_{f}$ is its non-degenerate critical point of index $\lambda_{p}$ with Morse coordinates $\left(V_{p}, \phi_{p}: y \in V_{p} \mapsto\left(x_{1}(y), \ldots, x_{n}(y)\right) \in \mathbb{R}^{n}\right.$ and

$$
\begin{equation*}
\phi_{p}^{-1}\left(O x_{1} \ldots x_{\lambda_{p}}\right) \subset W_{p}^{u}, \phi_{p}^{-1}\left(O x_{\lambda_{p}+1} \ldots x_{n}\right) \subset W_{p}^{s} \tag{*}
\end{equation*}
$$

If the function $\varphi$ has no critical points outside $\mathcal{R}_{f}$ then following [15] we call it the energy function for the Morse-Smale diffeomorphism $f$.

The proof of existence of an energy Morse function for a Morse-Smale diffeomorphism of the circle is an easy exercise. D. Pixton [15] in 1977 proved that every Morse-Smale diffeomorphism of a surface has an energy function. There he also constructed an example of a Morse-Smale diffeomorphism on the 3 -sphere which admits no energy function. The obstacle to existence of an energy function in his example was the wild embedding of the saddle separatrices in the ambient manifold (i.e. the closure of the separatrice is not a submanifold of the ambient space). From [11] it follows that there are Morse-Smale diffeomorphisms with no energy function on manifolds of any dimension $n>2$. Therefore, following 7 for a Morse-Smale diffeomorphism $f$ we call a Morse-Lyapunov function with the minimal number of critical points (denote it by $\rho_{f}$ ) a quasi-energy function. Notice that $\rho_{f}$ is a topological invariant, i.e. if two diffeomorphisms $f, f^{\prime}: M^{n} \rightarrow M^{n}$ are topologically conjugate (that is there is a diffeomorphism $h: M^{n} \rightarrow M^{n}$ such that $\left.h \circ f=f^{\prime} \circ h\right)$ then $\rho_{f}=\rho_{f^{\prime}}$.

In this paper we give a lower estimate of $\rho_{f}$ for Pixton diffeomorphisms. The class of Pixton diffeomorphisms $\mathcal{P}$ is defined in the following way. Every diffeomorphism $f \in \mathcal{P}$ is a Morse-Smale 3-diffeomorphism whose chain recurrent set consists of four points: one source, one saddle and two sinks (for details see section 2). Notice that Pixton's example is a diffeomorphism of this class. According to [2] the class of topological conjugacy of a diffeomorphism $f \in \mathcal{P}$ is completely defined by the equivalence class of the Hopf knot $L_{f}$, i.e. the knot in the generating class of the fundamental group of the manifold $\mathbb{S}^{2} \times \mathbb{S}^{1}$ (see Proposition 11). Moreover, any Hopf knot can be realized as a Pixton diffeomorphism.

Recall that a knot in $\mathbb{S}^{2} \times \mathbb{S}^{1}$ is a smooth embedding $\gamma: \mathbb{S}^{1} \rightarrow \mathbb{S}^{2} \times \mathbb{S}^{1}$ or the image of this embedding $L=\gamma\left(\mathbb{S}^{1}\right)$. Two knots $L, L^{\prime}$ are said to be equivalent if there is a homeomorphism $h: \mathbb{S}^{2} \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{2} \times \mathbb{S}^{1}$ such that $h(L)=L^{\prime}$. Two knots

[^1]$\gamma, \gamma^{\prime}$ are smoothly homotopic if there exists a smooth map $\Gamma: \mathbb{S}^{1} \times[0,1] \rightarrow \mathbb{S}^{2} \times \mathbb{S}^{1}$ such that $\Gamma(s, 0)=\gamma(s)$ and $\Gamma(s, 1)=\gamma^{\prime}(s)$ for every $s \in \mathbb{S}^{1}$. If $\left.\Gamma\right|_{\mathbb{S}^{1} \times\{t\}}$ is an embedding for every $t \in[0,1]$ then the knots are said to be isotopic.

Any Hopf knot $L \subset \mathbb{S}^{2} \times \mathbb{S}^{1}$ is smoothly homotopic to the standard Hopf knot $L_{0}=\{s\} \times \mathbb{S}^{1}$ (see, for example, [9]) but generally it is neither isotopic nor equivalent to it. B. Mazur [10 constructed the Hopf knot $L_{M}$ which we call the Mazur knot and which is non-equivalent and non-isotopic to $L_{0}$ (see Fig. 2). It follows from the results of [1] that there exists a countable family of pairwise


Fig. 2 Two non-isotopic and non equivalent Hopf knots $L_{0}$ and $L_{M}$ : a) the standard Hopf knot $L_{0} ;$ b) the Mazur $\operatorname{knot} L_{M}$
non-equivalent Hopf knots $L_{n}, n \in \mathbb{N}$ which are generalized Mazur knots (Fig. 3).


Fig. 3 A generalized Mazur knot $L_{n}$

According to [6] a Pixton diffeomorphism $f$ admits an energy Morse function if and only if the knot $L_{f}$ is trivial (i.e. equivalent to the standard one). If the
knot $L_{f}$ is not trivial then the number $\rho_{f}$ of the critical points of a quasi-energy Morse function of $f$ is evidently even and

$$
\rho_{f} \geqslant 6 .
$$

The main result of this paper is the proof of Theorem 1.
Theorem 1 Let $f$ be a Pixton diffeomorphism $(f \in \mathcal{P})$ and let $L_{n}, n \in \mathbb{N}$ be its knot. Then the number $\rho_{f}$ of critical points of a quasi-energy function of $f$ is calculated by ${ }^{2}$

$$
\rho_{f}=4+2 n .
$$

## 2 Construction of Pixton diffeomorphisms

In dynamics a wild Artin-Fox arc was for the first time introduced by D. Pixton in [15] where he constructed a Morse-Smale diffeomorphism on the 3-sphere with the unique saddle whose invariant manifolds form an Artin-Fox arc. We give the modern construction of these diffeomorphisms following Ch. Bonatti and V. Grines [2] where Pixton diffeomorphisms were also classified (see also [8], [11]).

For $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$ denote $\|\mathbf{x}\|=\sqrt{x_{1}^{2}+\cdots+x_{n}^{2}}$. Let $h: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ be the diffeomorphism defined by $h\left(x_{1}, x_{2}, x_{3}\right)=\left(\frac{x_{1}}{2}, \frac{x_{2}}{2}, \frac{x_{3}}{2}\right)$. Define the map $p: \mathbb{R}^{3} \backslash O \rightarrow \mathbb{S}^{2} \times \mathbb{S}^{1}$ by

$$
p\left(x_{1}, x_{2}, x_{3}\right)=\left(\frac{x_{1}}{\|\mathbf{x}\|}, \frac{x_{2}}{\|\mathbf{x}\|}, \log _{2}(\|\mathbf{x}\|) \quad(\bmod 1)\right)
$$

Let $L \subset\left(\mathbb{S}^{2} \times \mathbb{S}^{1}\right)$ be a Hopf knot and let $U(L)$ be its tubular neighborhood. Then the set $\bar{L}=p^{-1}(L)$ is the $h$-invariant arc in $\mathbb{R}^{3}$ and $U(\bar{L})=p^{-1}(U(L))$ is its $h$-invariant neighborhood diffeomorphic to $\mathbb{D}^{2} \times \mathbb{R}^{1}$ (Fig. 4).

Let $C=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}: x_{2}^{2}+x_{3}^{2} \leqslant 4\right\}$ and let $g^{t}: C \rightarrow C$ be the flow defined by

$$
g^{t}\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}+t, x_{2}, x_{3}\right) .
$$

Then there is a diffeomorphism $\zeta: U(L) \rightarrow C$ that conjugates $\left.h\right|_{U(L)}$ and $g=\left.g^{1}\right|_{C}$. Define the flow $\phi^{t}$ on $C$ by:

$$
\left\{\begin{array}{l}
\dot{x}_{1}=\left\{\begin{array}{l}
1-\frac{1}{9}\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}-4\right)^{2}, \quad x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant 4 \\
1, \quad x_{1}^{2}+x_{2}^{2}+x_{3}^{2}>4
\end{array}\right. \\
\dot{x}_{2}=\left\{\begin{array}{l}
\frac{x_{2}}{2}\left(\sin \left(\frac{\pi}{2}\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}-3\right)\right)-1\right), \quad 2<x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant 4 \\
-x_{2}, \quad x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant 2 \\
0, \quad x_{1}^{2}+x_{2}^{2}+x_{3}^{2}>4
\end{array}\right. \\
\dot{x}_{3}=\left\{\begin{array}{l}
\frac{x_{3}}{2}\left(\sin \left(\frac{\pi}{2}\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}-3\right)\right)-1\right), \quad 2<x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant 4 \\
-x_{3}, \quad x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant 2 \\
0, \quad x_{1}^{2}+x_{2}^{2}+x_{3}^{2}>4 .
\end{array}\right.
\end{array}\right.
$$

By construction the diffeomorphism $\phi=\phi^{1}$ has two fixed points: the saddle $P(1,0,0)$ and the sink $Q(-1,0,0)$ (Fig. 5), both being hyperbolic. One unstable

[^2]

Fig. 4 Suspension of a Hopf knot


Fig. 5 Trajectories of the flow $\phi^{t}$
separatrice of the saddle $P$ coincides with the open interval $\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}:\left|x_{1}\right|<1, x_{2}=x_{3}=0\right\}$ in the basin of the $\operatorname{sink} Q$ while the other coincides with the ray $\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}: x_{1}>1, x_{2}=x_{3}=0\right\}$. Notice that $\phi$ coincides with the diffeomorphism $g=g^{1}$ outside the ball $\left\{\left(x_{1}, x_{2}, x_{3}\right) \in\right.$ $\left.C: x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant 4\right\}$.

Define the diffeomorphism $\bar{f}_{L}: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ so that $\bar{f}_{L}$ coincides with $h$ outside $U(L)$ and it coincides with $\zeta^{-1} \phi \zeta$ on $U(L)$. Then $\bar{f}_{L}$ has in $U(L)$ two fixed points: the sink $\zeta^{-1}(Q)$ and the saddle $\zeta^{-1}(P)$, both being hyperbolic. The unstable separatrice of the saddle $\zeta^{-1}(P)$ lies in $L$ (Fig. 6).

Now project the dynamics onto the 3 -sphere. Denote by $N(0,0,0,1)$ the North Pole of the sphere $\mathbb{S}^{3}=\left\{\mathbf{x}=\left(x_{1}, x_{2}, x_{3}, x_{4}\right):\|\mathbf{x}\|=1\right\}$. For every point $\mathbf{x} \in\left(\mathbb{S}^{3} \backslash\right.$ $\{N\})$ there is the unique line passing through $N$ and $\mathbf{x}$ in $\mathbb{R}^{4}$. This line intersects


Fig. 6 The phase portrait of the diffeomorphism $\bar{f}_{L}$
$\mathbb{R}^{3}$ in exactly one point $\vartheta_{+}(\mathbf{x})$ (Fig. 7). The point $\vartheta_{+}(\mathbf{x})$ is the stereographic projection of the point $\mathbf{x}$. One can easily check that

$$
\vartheta_{+}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\left(\frac{x_{1}}{1-x_{4}}, \frac{x_{2}}{1-x_{4}}, \frac{x_{3}}{1-x_{4}}\right) .
$$

Thus, the stereographic projection $\vartheta_{+}: \mathbb{S}^{3} \backslash\{N\} \rightarrow \mathbb{R}^{3}$ is a diffeomorphism.


Fig. 7 The stereographic projection.

By construction $\bar{f}_{L}$ coincides with $h$ in some neighborhood of the point $O$ and in some neighborhood of the infinity. Therefore, it induces on $\mathbb{S}^{3}$ the Morse-Smale

$$
f_{L}(\mathbf{x})=\left\{\begin{array}{l}
\vartheta_{+}^{-1}\left(\bar{f}_{L}\left(\vartheta_{+}(\mathbf{x})\right)\right), \mathbf{x} \neq N \\
N, \mathbf{x}=N
\end{array}\right.
$$

It follows directly from the construction that the non-wandering set of $f_{L}$ consists of exactly four fixed hyperbolic points: two sinks $\omega=\vartheta_{+}^{-1}\left(\zeta^{-1}(Q)\right)$, $S$, one saddle $\sigma=\vartheta_{+}^{-1}\left(\zeta^{-1}(P)\right)$ and one source $N$. We say the constructed diffeomorphism to be model and it is of Pixton class.

Proposition 1 ([2])

- Any diffeomorphism $f \in \mathcal{P}$ is topologically conjugate to some model diffeomorphism $f_{L}$.
- Two model diffeomorphisms $f_{L}, f_{L^{\prime}}$ are topologically conjugate if and only if their knots $L, L^{\prime}$ are equivalent.


## 3 Genus of Hopf knot

In this section we introduce the notion of genus for a Hopf knot and use it to estimate the number of critical points of the quasi-energy function of the Pixton diffeomorphism defined by this knot.

Let $L$ be a Hopf knot and let $\bar{L}=p^{-1}(L)$ be its cover in $\mathbb{R}^{3} \backslash O$. We say a closed orientable surface $\Sigma \subset \mathbb{S}^{2} \times \mathbb{S}^{1}$ to be a secant surface of the knot $L$ if it intersects $L$ in a unique point and there is an $h$-compressible 3 -manifold $Q_{\Sigma} \subset \mathbb{R}^{3}$ (that is $\left.h\left(Q_{\Sigma}\right) \subset \operatorname{int} Q_{\Sigma}\right)$ with the boundary $\bar{\Sigma}$ such that $\Sigma=p(\bar{\Sigma})$ and the intersection $\bar{L} \cap \bar{\Sigma}$ is the unique point $\bar{y}$. The minimally possible genus $g_{L}$ of the secant surface is called the genus of the knot $L$. The secant surface of $L$ of genus $g_{L}$ is said to be minimal.

Lemma 1 If $\Sigma$ is a minimal secant surface of the knot $L$ then the surface $\bar{\Sigma} \backslash \bar{y}$ is non-compressible in $\mathbb{R}^{3} \backslash(O \cup \bar{L})$, i.e. any simple closed curve $c \subset$ int $(\bar{\Sigma} \backslash \bar{y})$ is contractible on $\bar{\Sigma} \backslash \bar{y}$ if it bounds a smoothly embedded 2-disk $D \subset \operatorname{int}\left(\mathbb{R}^{3} \backslash(O \cup \bar{L})\right)$ such that $D \cap(\bar{\Sigma} \backslash \bar{y})=\partial D=c$.

Proof Let $\Sigma$ be a minimal secant surface of $L$ and let $\bar{y}$ be the unique point of the intersection $\bar{L} \cap \bar{\Sigma}$. Assume the opposite: the surface $\bar{\Sigma} \backslash \bar{y}$ is compressible in $\mathbb{R}^{3} \backslash(O \cup \bar{L})$. Then there is a non-contractible simple closed curve $c \subset$ int $(\bar{\Sigma} \backslash \bar{y})$ and there is the smoothly embedded 2-disk $D \subset \operatorname{int}\left(\mathbb{R}^{3} \backslash(O \cup \bar{L})\right)$ such that $D \cap(\bar{\Sigma} \backslash \bar{y})=\partial D=c$ (see, for example, [14]). Then we have two possibilities:

$$
\begin{align*}
& (\text { int } D) \cap\left(\bigcup_{k \in \mathbb{Z}} h^{k}(\bar{\Sigma})\right)=\emptyset,  \tag{1}\\
& (\text { int } D) \cap\left(\bigcup_{k \in \mathbb{Z}} h^{k}(\bar{\Sigma})\right) \neq \emptyset \tag{2}
\end{align*}
$$

In case (1) two subcases are possible: (1a) $D \subset Q_{\Sigma}$, (1b) $D \subset\left(\mathbb{R}^{3} \backslash \operatorname{int} Q_{\Sigma}\right)$. For case 1a) let $N(D) \subset Q_{\Sigma}$ be a tubular neighborhood of the disk $D$. Then exactly one connected component of the set $Q_{\Sigma} \backslash \operatorname{int} N(D)$ intersects $\bar{L}$. According to (1) this component is $h$-compressible and its boundary intersects $\bar{L}$ at a unique point. The projection of this boundary into $\mathbb{S}^{2} \times \mathbb{S}^{1}$ is, therefore, the secant surface of $L$ of genus less than $g_{L}$. This contradicts the fact that the surface $\Sigma$ is minimal. In case 1b) let $N(D) \subset\left(\mathbb{R}^{3} \backslash \operatorname{int} Q_{\Sigma}\right)$ be a tubular neighborhood of $D$. Then due to (1) the set $Q_{\Sigma} \cup N(D)$ is $h$-compressible and its boundary intersects $\bar{L}$ at a unique point. The projection of this boundary into $\mathbb{S}^{2} \times \mathbb{S}^{1}$ is, therefore, the secant surface of $L$ of genus less than $g_{L}$ and we have the same contradiction.

In case (2) without loss of generality assume the intersection int $D \cap\left(\bigcup_{k \in \mathbb{Z}} h^{k}(\bar{\Sigma})\right)$ to be transversal and denote it by $\Gamma$. Let $\gamma$ be a curve from $\Gamma$. We say the curve $\gamma$ to be innermost if it is the boundary of the disk $D_{\gamma} \subset D$ such that int $D_{\gamma}$ contains no curves of $\Gamma$. Consider this innermost curve $\gamma \subset f^{k}(\Sigma)$. There are two subcases: a) $\gamma$ is essential on $f^{k}(\Sigma)$ and b) $\gamma$ is contractible on $f^{k}(\Sigma)$. In case a) the arguments of the case (1) apply for the body $f^{k}\left(Q_{\Sigma}\right)$ and the disk $D_{\gamma}$ and we get the contradiction to the minimality of the surface $\Sigma$. In case b) denote by $d_{\gamma} \subset f^{k}(\Sigma)$ the 2 -disk bounded by $\gamma$ and denote by $B_{\gamma} \subset\left(\mathbb{R}^{3} \backslash O\right)$ the 3 -ball bounded by the 2 -sphere $D_{\gamma} \cup d_{\gamma}$. Consider: b1) $B_{\gamma} \subset f^{k}\left(Q_{\Sigma}\right)$ and b2) $B_{\gamma} \subset\left(\mathbb{R}^{3} \backslash \operatorname{int} f^{k}\left(Q_{\Sigma}\right)\right)$. For b1) let $N\left(B_{\gamma}\right) \subset f^{k}\left(Q_{\Sigma}\right)$ be a tubular neighborhood of $B_{\gamma}$. Then the set $Q_{\Sigma} \backslash$ int $N\left(B_{\gamma}\right)$ is $h$-compressible because the curve $\gamma$ lies in its interior and the boundary of $Q_{\Sigma} \backslash i n t N\left(B_{\gamma}\right)$ intersects $\bar{L}$ at a unique point. The projection of this boundary into $\mathbb{S}^{2} \times \mathbb{S}^{1}$ is, therefore, the secant surface of the knot $L$ of genus $g_{L}$ for which the number of connected components of the set $\Gamma$ is less. We get the same result for b2) for the set $Q_{\Sigma} \cup N\left(B_{\gamma}\right)$. Thus, iterating the process we come either to the case a) or to the case (1) and get a contradiction.

Lemma 2 For any diffeomorphism $f \in \mathcal{P}$ the following estimation holds

$$
\begin{equation*}
\rho_{f} \geqslant 4+2 g_{L_{f}} . \tag{3}
\end{equation*}
$$

Proof Since Proposition 1 is true and since the number $\rho_{f}$ of the critical points of a quasi-energy function of $f \in \mathcal{P}$ is invariant, from now on we consider model Pixton diffeomorphisms $f_{L}$ with the Hopf knot $L$. Denote by $\ell$ the non-stable separatrice of the saddle $\sigma$ lying in the basin of the sink $S$. Let

$$
p_{S}: W_{S}^{s} \backslash S \rightarrow \mathbb{S}^{2} \times \mathbb{S}^{1}
$$

be the natural projection sending a point $w \in\left(W_{S}^{s} \backslash S\right)$ to the point $p\left(f^{k_{w}}(w)\right), f^{k_{w}}(w) \in$ $V_{S}$. Since the diffeomorphism $f_{L}$ coincides with the homothety $h$ in some neighborhood $V_{S}$ of $S$, the natural projection $p_{S}$ is well defined and $p_{S}(\ell)=L$ by construction.

Consider an arbitrary Morse-Lyapunov function $\varphi: \mathbb{S}^{3} \rightarrow \mathbb{R}$ of the diffeomorphism $f_{L}$. To be definite let $\varphi(S)=0, \varphi(\sigma)=1$ and $\varphi(N)=3$. From the definition of the Morse-Lyapunov function it follows that $\left.\varphi\right|_{\ell}$ monotonically decreases in some neighborhood of the saddle $\sigma$. Therefore, there is $\varepsilon_{1} \in(0,1)$ such that the interval $\left(1-\varepsilon_{1}, 1\right)$ contains no critical values of $\varphi$ and the connected component $\bar{\Sigma}_{1}$ of the level set $\varphi^{-1}\left(1-\varepsilon_{1}\right)$ intersects the separatrice $\ell$ at the unique point. Denote this point by $w_{1}$.

Let $\bar{Q}_{1}$ be the connected component of the set $\varphi^{-1}\left(\left[0,1-\varepsilon_{1}\right]\right)$ which contains the segment $\left[w_{1}, S\right]$ of the closure of the separatrice $\ell$. Since $\varphi$ decreases along the trajectories of $f$, the values of $\varphi$ on $W_{\sigma}^{s}$ are greater than 1 . Therefore, the manifold $\bar{Q}_{1}$ lies in the manifold $W_{S}^{s}$ diffeomorphic to $\mathbb{R}^{3}$. Let the function $\left.\varphi\right|_{\bar{Q}_{1}}$ have $k_{q}, q \in\{0, \ldots, 3\}$ critical points of index $q$. Due to [5, Theorem 6.1] on the manifold $\bar{Q}_{1}$ there exists a self-indexing Morse function $\psi$ (the value of the function in a critical point equals the index of this point) which has $k_{q}$ critical points of index $q$ and which is constant on $\partial \bar{Q}_{1}$. Thus, the manifold $\bar{Q}_{1}$ is the surface $\tilde{Q}_{1}$ of
genus $g_{1}=1+k_{1}-k_{0}$ with attached handles of indexes 2 and 3 . Then the genus of any surface of the set $\partial \bar{Q}_{1}$ cannot be greater than $g_{1}$.

On the other hand, the number of critical points of $\left.\varphi\right|_{\bar{Q}_{1}}$ is not less than $k_{0}+k_{1}$. If $k_{0} \geqslant 1$ and $g_{1}=1+k_{1}-k_{0}$ then one gets $k_{0}+k_{1}=g_{1}+2 k_{0}-1 \geqslant g_{1}+1$. Thus, $\left.\varphi\right|_{\bar{Q}_{1}}$ has at least $g_{1}+1$ critical points.

Denote by $\bar{\Sigma}_{1}$ the connected component of $\partial \bar{Q}_{1}$ which intersects the separatrice $\ell$. Then the surface $\bar{\Sigma}_{1}$ divides the manifold $W_{S}^{s} \cong \mathbb{R}^{3}$ into two parts, one of which $Q_{1}$ being an $h$-compressible body. This means that $\Sigma_{1}=p_{S}\left(\bar{\Sigma}_{1}\right)$ is the secant surface of $L$ and, therefore,

$$
g_{1} \geqslant g_{L}
$$

Analogously, there is $\varepsilon_{2} \in(0,1)$ for which the interval $\left(1,1+\varepsilon_{2}\right)$ contains no critical points of $\varphi$ and the connected component $\bar{Q}_{2}$ of the level set $\varphi^{-1}\left(\left[0,1+\varepsilon_{2}\right)\right]$ contains $\operatorname{cl}\left(W_{\sigma}^{u}\right)$ in its interior while the intersection $\bar{Q}_{2}$ with $W_{\sigma}^{s}$ is the unique 2-disk. Due to construction the function $\left.\varphi\right|_{\bar{Q}_{2}}$ has at least $g_{1}+3$ critical points and genus of the connected components of $\partial Q_{2}$ is less or equals $g_{1}$. Denote by $\bar{\Sigma}_{2}$ the connected component of $\partial \bar{Q}_{2}$ which intersects $W_{\sigma}^{s}$ and denote by $g_{2}$ its genus. The surface $\bar{\Sigma}_{2}$ divides the manifold $W_{N}^{u} \cong \mathbb{R}^{3}$ into two parts, one of which $Q_{2}$ being an $h^{-1}$-compressible body. Arguing as above one comes to conclusion that the number of critical points of $\left.\varphi\right|_{Q_{2}}$ is at least $g_{2}+1$. Therefore, the total number of critical points of $\varphi$ is greater or equal to

$$
g_{1}+3+g_{2}+1 \geqslant 4+2 g_{1} \geqslant 4+2 g_{L_{f}}
$$

## 4 The generalized Mazur $\operatorname{knot} L_{n}$

In this section we show that the genus $g_{L_{n}}$ of a generalized Mazur knot equals $n$. At first we give a detailed description of construction of $L_{n}$.

### 4.1 Construction of the generalized Mazur knot $L_{n}$

Recall that $h: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ is the homothety defined by

$$
h\left(x_{1}, x_{2}, x_{3}\right)=\left(\frac{x_{1}}{2}, \frac{x_{2}}{2}, \frac{x_{3}}{2}\right)
$$

and $p: \mathbb{R}^{3} \backslash O \rightarrow \mathbb{S}^{2} \times \mathbb{S}^{1}$ is the natural projection defined by

$$
p\left(x_{1}, x_{2}, x_{3}\right)=\left(\frac{x_{1}}{\|\mathbf{x}\|}, \frac{x_{2}}{\|\mathbf{x}\|}, \log _{2}(\|\mathbf{x}\|) \quad(\bmod 1)\right)
$$

Consider the annulus

$$
K=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}: \frac{1}{4} \leq x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leq 1\right\}
$$

bounded by the spheres

$$
\mathbb{S}^{2}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}: x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=1\right\}, h\left(\mathbb{S}^{2}\right)
$$

Pick on the circle

$$
\mathbb{S}^{1}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}: x_{1}^{2}+x_{2}^{2}=1, x_{3}=0\right\}
$$

pairwise distinct points $\alpha_{1}, \ldots, \alpha_{2 n+1}$ numbered in counter-clockwise order (Fig. 8). Let $a_{i}, i \in\{1, \ldots, 2 n\}$ be the arc of the circle $\mathbb{S}^{1}$ bounded by $\alpha_{i}, \alpha_{i+1}$ whose interior contains no points of $\left\{\alpha_{1}, \ldots, \alpha_{2 n+1}\right\}$. Let $B, A_{i} \subset \operatorname{int} K, i \in\{1, \ldots, 2 n\}$ be pairwise disjoint smooth arcs such that:

1. the boundary points of $B$ are $\alpha_{2 n+1}, h\left(\alpha_{1}\right)$; the boundary points of $A_{2 j-1}$ are $\alpha_{2 j-1}, \alpha_{2 j}$ and the boundary points of $A_{2 j}$ are $h\left(\alpha_{2 j}\right), h\left(\alpha_{2 j+1}\right)$ for $j \in$ $\{1, \ldots, n\}$;
2. the closed curves $c_{2 j-1}=\operatorname{cl}\left(a_{2 j-1} \cup A_{2 j-1}\right), c_{2 j}=c l\left(h\left(a_{2 j}\right) \cup A_{2 j}\right)$ bound the 2-disks $d_{2 j-1}, d_{2 j}$, the transversal intersection of these disks being the arc $l_{j}$ with the boundary points $b_{2 j-1}=d_{2 j-1} \cap A_{2 j}$ and $b_{2 j}=d_{2 j} \cap A_{2 j-1}$;
3. the $\operatorname{arc} c l\left(h\left(A_{1}\right) \cup A_{2} \cup \cdots \cup h\left(A_{2 n-1}\right) \cup A_{2 n} \cup B\right)$ is smooth.


Fig. 8 Construction of the knot $L_{n}$

Let

$$
\bar{L}_{n}=\bigcup_{k \in \mathbb{Z}} h^{k}\left(B \cup A_{1} \cup \cdots \cup A_{2 n}\right), L_{n}=p\left(\bar{L}_{n}\right) .
$$

4.2 The genus of the knot $L_{n}$

Lemma 3 The genus $g_{L_{n}}$ of the knot $L_{n}$ equals $n$.


Fig. 9 A secant surface of $L_{n}$ of genus $n$

Proof Since there is a secant surface of $L_{n}$ of genus $n$, we have $g_{L_{n}} \leqslant n$ (Fig. 9). Now we show that $g_{L_{n}} \geqslant n$. To that end we prove that for $L_{n}$ there exists a minimal secant surface $\Sigma$ such that $\bar{\Sigma} \subset K$ and $\bar{L}_{n} \cap \bar{\Sigma}=h\left(\alpha_{1}\right)$.

Indeed, let $\Sigma_{0}$ be some minimal secant surface of $L_{n}$. Then there exists the connected component $\bar{\Sigma}_{0}$ of $p^{-1}\left(\Sigma_{0}\right)$ such that it intersects the curve $\bar{L}_{n}$ at the point $\bar{y}_{0}$ situated on $\bar{L}_{n}$ between $\alpha_{1}, h\left(\alpha_{1}\right)$ and that bounds the $h$-compressible body $Q_{\Sigma_{0}}$. Without loss of generality let $\bar{y}_{0}=h\left(\alpha_{1}\right)$ (otherwise the desired surface is constructed by removing the tubular neighborhood of the $\operatorname{arc}\left[\bar{y}_{0}, h\left(\alpha_{1}\right)\right] \subset \bar{L}_{n}$ from $Q_{\Sigma_{0}}$ ).

Denote by $k_{+}, k_{-} \geqslant 0$ the maximal integers for which $f^{k}\left(\bar{\Sigma}_{0}\right) \cap \bar{\Sigma}_{0} \neq \emptyset, f^{-k}\left(\bar{\Sigma}_{0}\right) \cap$ $\bar{\Sigma}_{0} \neq \emptyset, k \geqslant 0$, respectively. If $k_{+}=k_{-}=0$ then $\bar{\Sigma}_{0}$ is the desired surface. Otherwise we show the way to decrease by 1 the number $k_{+}>0$ (for $k_{-}$the arguments are the same) using isotopy of the secant surface.

Notice that $\bar{\Sigma}_{0} \cap f^{k_{+}}\left(c_{2 j-1}\right)=\emptyset, j \in\{1, \ldots, n\}$. Without loss of generality let the intersection $\Gamma=\bigcup_{j=1}^{n} f^{k_{+}}\left(d_{2 j-1}\right) \cap \bar{\Sigma}_{0}$ be transversal. Let $\gamma$ be a curve from $\Gamma$. Then $\gamma$ bounds the unique disk $D_{\gamma} \subset f^{k+}\left(d_{2 j-1}\right)$. There are two possibilities: 1) $\left.b_{2 j-1} \notin D_{\gamma}, 2\right) b_{2 j-1} \in D_{\gamma}$. In case 1) we say the curve $\gamma$ to be innermost if it bounds the disk $D_{\gamma} \subset f^{k_{+}}\left(d_{2 j-1}\right)$ such that int $D_{\gamma}$ contains no curves of $\Gamma$. Consider this innermost curve $\gamma$. Due to Lemma 1 the surface $\bar{\Sigma}_{0} \backslash \bar{y}_{0}$ is noncompressible in $\mathbb{R}^{3} \backslash\left(O \cup \bar{L}_{n}\right)$ and, therefore, there exists the disk $d_{\gamma} \subset\left(\bar{\Sigma}_{0} \backslash \bar{y}_{0}\right)$ bounded by $\gamma$. Denote by $B_{\gamma} \subset\left(\mathbb{R}^{3} \backslash\left(O \cup \bar{L}_{n}\right)\right)$ the 3-ball bounded by the 2-sphere $D_{\gamma} \cup d_{\gamma}$. Consider two subcases: 1a) $B_{\gamma} \subset Q_{\Sigma_{0}}$ and 1b) $B_{\gamma} \subset\left(\mathbb{R}^{3} \backslash \operatorname{int} Q_{\Sigma_{0}}\right)$.

In case 1a) let $N\left(B_{\gamma}\right) \subset Q_{\Sigma_{0}}$ be a tubular neighborhood of the ball $B_{\gamma}$. Then the set $Q_{\Sigma} \backslash \operatorname{int} N\left(B_{\gamma}\right)$ is $h$-compressible because the curve $\gamma$ lies in its interior and its boundary intersects $\bar{L}_{n}$ at a unique point. The projection of this boundary
to $\mathbb{S}^{2} \times \mathbb{S}^{1}$ is, therefore, a secant surface of $L_{n}$ of the same genus as $\Sigma_{0}$. For it the number of the connected components of the set $\Gamma$ is less. One gets the same result in case 1b) for the set $Q_{\Sigma_{0}} \cup N\left(B_{\gamma}\right)$.

If we continue this process then we get the secant surface of $L_{n}$ of the same genus as $\Sigma_{0}$ and for which the set $\Gamma$ contains no curves of type 1). Denote the resulting surface again by $\Sigma_{0}$. Now the set $\Gamma$ consists only of the curves $\gamma$ bounding the disk $D_{\gamma} \subset b_{2 j-1}$ which contains the point $b_{2 j-1}$. Since $\left(b_{2 j-1} \sqcup c_{2 j-1}\right) \subset$ $\left(\mathbb{R}^{3} \backslash Q_{\Sigma_{0}}\right.$ ), the number of these curves on the disk $d_{2 j-1}$ is even. Since the surface $\bar{\Sigma}_{0} \backslash \bar{y}_{0}$ is non-compressible in $\mathbb{R}^{3} \backslash\left(O \cup \bar{L}_{n}\right)$, all these curves are pairwise homotopic on $\bar{\Sigma}_{0} \backslash \bar{y}_{0}$ and, therefore, they lie in the annulus $\kappa \subset\left(\bar{\Sigma}_{0} \backslash \bar{y}_{0}\right)$ bounded by the pair of these curves $\gamma_{1}, \gamma_{2}$. Denote by $\tilde{\kappa} \subset d_{2 j-1}$ the annulus bounded by the same curves on the disk $d_{2 j-1}$. Let $\tilde{\Sigma}_{0}=\bar{\Sigma}_{0} \backslash \kappa \cup \tilde{\kappa}$. Due to construction the surface $\tilde{\Sigma}_{0}$ is of the same genus as the surface $\bar{\Sigma}_{0}$ and it bounds an $h$-compressible body. Having removed a tubular neighborhood of the annulus $\tilde{\kappa}$ from this body we get a $h$-compressible body whose boundary does not intersect the disk $d_{2 j-1}$ and whose projection to $\mathbb{S}^{2} \times \mathbb{S}^{1}$ is the secant surface of the knot $L_{n}$ of the same genus as $\Sigma_{0}$.

If we continue this process then we get a secant surface of $L_{n}$ of the same genus as $\Sigma_{0}$ and for which the set $\Gamma$ is not empty. Denote this surface again by $\Sigma_{0}$. Without loss of generality let the intersections of the surface $\bar{\Sigma}_{0}$ with the spheres $f^{k}\left(\mathbb{S}^{2}\right)$ be transversal. Denote by $\mathcal{F}$ the set of the connected components of the intersection $f^{k_{+}}(K) \cap \bar{\Sigma}_{0}$. Now we show the way to reduce by 1 the number of the components in $\mathcal{F}$ using isotopy of the secant surface.

Denote by $Q$ the set obtained by removal from the annulus $f^{k_{+}}(K)$ of the tubular neighborhoods of the disks $d_{2 j-1}$ as well as the tubular neighborhoods of the curves $A_{2 j}, j \in\{1, \ldots, n\}$. Then $Q$ is homeomorphic to the direct product of the 2 -sphere with $2 n+1$ deleted points and the segment. Since $Q \cap \bar{\Sigma}_{0}=$ $f^{k_{+}}(K) \cap \bar{\Sigma}_{0}$ and since $\bar{\Sigma}_{0} \backslash \bar{y}_{0}$ is non-compressible in $\mathbb{R}^{3} \backslash\left(O \cup \bar{L}_{n}\right)$, each connected component of $F \in \mathcal{F}$ is non-compressible in $Q$. Due to [16, Corollary 3.2] there exists a surface $\tilde{F} \subset f^{k_{+}-1}\left(\mathbb{S}^{2}\right)$ diffeomorphic to $F$ for which $\partial F=\partial \tilde{F}$ and the surface $F \cup \tilde{F}$ bounds in $Q$ the body $\Delta$ diffeomorphic to the direct product $F \times[0,1]$. Then we replace the part $F$ of $\bar{\Sigma}_{0}$ with $\tilde{F}$. If we continue the process we get the desired secant surface $\Sigma \subset K$.

Notice (see, for instance, [4, Exercise 2.8.1]) that the fundamental group $\pi_{1}(K \backslash$ $\bar{L}_{n}$ ) has $2 n$ generators $\gamma_{1}, \ldots, \gamma_{2 n}$, each of which $\gamma_{i}, i \in\{1, \ldots, 2 n\}$ being the generator of the punctured disk $d_{i} \backslash b_{i}$ (Fig. 10). Since $b_{2 j-1} \in \operatorname{int} Q_{\Sigma}$ and $c_{2 j-1} \cap$ $Q_{\Sigma}=\emptyset$, there exists the connected component of $\tilde{d}_{2 j-1}$ of the intersection $d_{2 j-1} \cap$ $Q_{\Sigma}$ which contains the point $b_{2 j-1}$. This component is the 2 -disk bounded by the curve $\tilde{\gamma}_{2 j-1} \subset\left(\bar{\Sigma} \backslash h\left(\alpha_{1}\right)\right)$ with holes and the curves $\gamma_{2 j-1}, \tilde{\gamma}_{2 j-1}$ are homotopic on the punctured disk $d_{2 j-1} \backslash b_{2 j-1}$. In the same way one finds the curves $\tilde{\gamma}_{2 j} \subset$ ( $\bar{\Sigma} \backslash h\left(\alpha_{1}\right)$ ) homotopic to the curves $\gamma_{2 j}$ on the punctured disk $d_{2 j} \backslash b_{2 j}$ (Fig. 10). Due to Lemma 1 the surface $\bar{\Sigma} \backslash h\left(\alpha_{1}\right)$ is non-compressible in $K \backslash \bar{L}_{n}$. Then the curves $\tilde{\gamma}_{1}, \ldots, \tilde{\gamma}_{2 n}$ are pairwise non-homotopic to the generators on the surface $\bar{\Sigma} \backslash h\left(\alpha_{1}\right)$. Therefore, the genus of the surface $\bar{\Sigma}$ cannot be less than $n$.


Fig. 10 Generators of the group $\pi_{1}\left(K \backslash \bar{L}_{n}\right)$

## 5 Construction of a quasi-energy function for a Pixton diffeomorphism with the Hopf knot $L_{n}$

Let $f$ be a Pixton diffeomorphism constructed for a generalized Mazur knot $L_{n}$. Then its non-wandering set $\Omega_{f}$ consists of four points: two sinks $\omega, S$, a source $N$ and a saddle $\sigma$. Then $W_{\sigma}^{u} \backslash \sigma$ consists of two separatrices $\ell_{\omega}, \ell_{S}$ respective closures of which contain the sinks $\omega, S$, the separatrice $\ell_{\omega}$ being tame while $\ell_{S}$ being wild. Let $\bar{\Sigma}$ be the surface of genus $n$ bounding the handle-body $Q_{\Sigma}$ of the same genus. Now we construct for $f$ a Morse-Lyapunov function with $6+2 n$ critical points.

Our construction of a quasi-energy function is analogous to the construction of an energy function in (7].

1. Choose an energy function $\varphi_{p}: U_{p} \rightarrow \mathbb{R}$ in the neighborhood of each fixed point $p$ of $f$ so that $\varphi_{p}(p)=\operatorname{dim} W_{p}^{u}$. Let $B_{\omega}, B_{S}$ be the 3-balls which are the level sets of respective functions $\varphi_{\omega}, \varphi_{S}$ and such that $B_{S} \subset \operatorname{int} Q_{\Sigma}$. Choose a tubular neighborhood $T_{\sigma}$ of the arc $W_{\sigma}^{u} \backslash\left(B_{\omega} \cup Q_{\Sigma}\right)$ so that the handle-body $B_{\omega} \cup Q_{\Sigma} \cup T_{\sigma}$ of genus $n$ is $f$-compressible and its intersection with $W_{\sigma}^{s}$ is the 2-disk. Denote by $P^{+}$the smoothing of this body by addition of a small exterior collar.
2. Due to [7] Section 4.3] there exists an energy function $\varphi: P^{+} \backslash \operatorname{int} Q_{\Sigma}$ whose value on $\partial P^{+}$is $4 / 3$, whose value on $\bar{\Sigma}$ is $2 / 3$ and which has exactly two critical points $\omega, \sigma$ of respective Morse indexes 0,1 . The disks $d_{1}, \ldots, d_{2 n-1}$ cut the handle-body $Q_{\Sigma}$ making the 3 -ball. Denote by $B_{\Sigma}$ the smoothing of this ball by removal of the interior collar. The results of the classic Morse theory (see, for example, [13]) allow to extend the function $\varphi$ to the set $Q_{\Sigma} \backslash$ int $B_{\Sigma}$ in such way that it has $n$ critical points of Morse index 1, one point lying on each disk $d_{1}, \ldots, d_{2 n-1}$, while the value of $\varphi$ on $\partial B_{\Sigma}$ is $1 / 3$. Due to [7] Lemma
4.2] the function $\varphi$ can be extended to the ball $B_{\Sigma}$ by an energy function with the unique critical point $S$ of Morse index 0 . Since $f\left(Q_{\Sigma}\right) \subset$ int $B_{\Sigma}$, the constructed function decreases along the trajectories of the diffeomorphism $f$.
3. It follows from the definition of the knot $L_{n}$ that $P^{-}=\mathbb{S}^{3} \backslash$ int $P^{+}$is the handle-body of genus $n$. Moreover, the disks $d_{2}, \ldots, d_{2 n}$ cut $P^{-}$making the 3 -ball. Denote by $B^{-}$smoothing of this ball by removal of the interior collar. The results of the classic Morse theory (see, for example, [13]) allow extension of the function $\varphi$ to the set $P^{-} \backslash$ int $B^{-}$in such way that it has $n$ critical points of Morse index 2, one point lying on each disk $d_{2}, \ldots, d_{2 n}$, while the value of $\varphi$ on $\partial B^{-}$is $5 / 3$. According to [7, Lemma 4.2] the function $\varphi$ can be extended to the ball $B^{-}$by an energy function with unique critical point $N$ of Morse index 3. Since $f\left(B^{-}\right) \subset$ int $P^{-}$, the constructed function decreases along the trajectories of the diffeomorphism $f$ and, therefore, it is the desired quasi-energy function.

## Conflict of interest

The authors declare that they have no conflict of interest.

## References

1. Akhmetiev, P., Medvedev, T., Pochinka, O.: On the number of the classes of topological conjugacy of Pixton diffeomorphisms. Qualitative Theory of Dynamical Systems 20(3), 1-15 (2021)
2. Bonatti, C., Grines, V.: Knots as topological invariants for gradient-like diffeomorphisms of the sphere $S^{3}$. Journal of Dynamical and Control Systems 6(4), 579-602 (2000)
3. Conley, C.: Isolated invariant sets and the morse index. American Mathematical Society, CBMS, Providence, RI 38 (1978)
4. Daverman, R.J., Venema, G.: Embeddings in manifolds, vol. 106. American Mathematical Soc. (2009)
5. Fomenko, A.: Differential Geometry and Topology: Additional Chapters. Moscow University Press (1983)
6. Grines, V., Laudenbach, F., Pochinka, O.: The energy function for gradient-like diffeomorphisms on 3-manifolds. Doklady Mathematics 78(2), 702-704 (2008)
7. Grines, V.Z., Laudenbach, F., Pochinka, O.V.: Quasi-energy function for diffeomorphisms with wild separatrices. Mathematical Notes 86(1), 163-170 (2009)
8. Grines, V.Z., Medvedev, T.V., Pochinka, O.V.: Dynamical Systems on 2- and 3-Manifolds, Developments in Mathematics, vol. 46. Springer International Publishing (2016). DOI 10.1007/978-3-319-44847-3
9. Kirk, P., Livingston, C.: Knot invariants in 3-manifolds and essential tori. Pacific Journal of Mathematics $\mathbf{1 9 7}(1), 73-96$ (2001)
10. Mazur, B.: A note on some contractible 4-manifolds. Annals of Mathematics 79(1), 221228 (1961)
11. Medvedev, T.V., Pochinka, O.V.: The wild Fox-Artin arc in invariant sets of dynamical systems. Dynamical Systems 33(4), 660-666 (2018). DOI 10.1080/14689367.2017.1421903. URL https://doi.org/10.1080/14689367.2017.1421903
12. Meyer, K.R.: Energy functions for morse smale systems. American Journal of Mathematics $\mathbf{9 0}(4), 1031-1040$ (1968). URL http://www.jstor.org/stable/2373287
13. Milnor, J.: Morse theory.(am-51), volume 51. In: Morse Theory.(AM-51), Volume 51. Princeton university press (2016)
14. Neumann, W.D.: Notes on geometry and 3-manifolds. Citeseer (1996)
15. Pixton, D.: Wild unstable manifolds. Topology 16, 167-172 (1977). DOI 10.1016/ 0040-9383(77)90014-3
16. Waldhausen, F.: On irreducible 3-manifolds which are sufficiently large. Annals of Mathematics pp. 56-88 (1968)

[^0]:    * The research was done with the support of Russian National Foundation (project 21-1100010) except construction of the quasi-energy function which was supported by International Laboratory of Dynamical Systems and Applications of National Research University Higher School of Economics, grant of Government of Russian Federation 075-15-2022-1101.
    T. Medvedev

    Laboratory of Algorithms and Technologies for Network Analysis; HSE University
    136 Rodionova Street, Niznhy Novgorod, Russia
    E-mail: mtv2001@mail
    O. Pochinka

    International Laboratory of Dynamical Systems and Applications; HSE University, 25/12 Bolshaya Pecherckaya Street, Niznhy Novgorod, Russia

[^1]:    ${ }^{1}$ This function can be constructed, for example, by suspension. Consider the topological flow $\hat{f}^{t}$ on the manifold $M^{n} \times \mathbb{R}$ defined by $\hat{f}^{t}(x)=x+t$. Define the diffeomorphism $g$ : $M^{n} \times \mathbb{R} \rightarrow M^{n} \times \mathbb{R}$ by $g(x, \tau)=(f(x), \tau-1)$ and let $G=\left\{g^{k}, k \in \mathbb{Z}\right\}$ and $W=\left(M^{n} \times \mathbb{R}\right) / G$. Denote by $p_{W}: M^{n} \times \mathbb{R} \rightarrow W$ the natural projection and denote by $f^{t}$ the flow on $W$ defined by $f^{t}(x)=p_{W}\left(\hat{f}^{t}\left(p_{W}^{-1}(x)\right)\right)$. The flow $f^{t}$ is called the suspension over $f$. By construction the chain recurrent set of $f^{t}$ consists of the finite number of periodic orbits $\delta_{i}=p_{W}\left(\mathcal{O}_{i} \times \mathbb{R}\right), i \in$ $\left\{1, \ldots, k_{f}\right\}$ and this means that the suspension $f^{t}$ is a Morse-Smale flow. A Lyapunov function for these flows is constructed in 12. Then the restriction of this function on $M$ is the desired Lyapunov function for $f$.

[^2]:    ${ }^{2}$ For $n=1$ Theorem 1 is proved in 7.

