A quasi-energy function for Pixton diffeomorphisms defined by generalized Mazur knots

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Abstract In this paper we give a lower estimate for the number of critical points of the Lyapunov function for Pixton diffeomorphisms (i.e. Morse-Smale diffeomorphisms in dimension 3 whose chain recurrent set consists of four points: one source, one saddle and two sinks). Ch. Bonatti and V. Grines proved that the class of topological equivalence of such diffeomorphism f is completely defined by the equivalency class of the Hopf knot L_f that is the knot in the generating class of the fundamental group of the manifold $\mathbb{S}^2 \times \mathbb{S}^1$. They also proved that there are infinitely many such classes and that any Hopf knot can be realized by a Pixton diffeomorphism. D. Pixton proved that diffeomorphisms defined by the standard Hopf knot $L_0 = \{s\} \times \mathbb{S}^1$ have an energy function (Lyapunov function) whose set of critical points coincide with the chain recurrent set whereas the set of critical points of any Lyapunov function for Pixton diffeomorphism with nontrivial (i.e. non equivalent to the standard) Hopf knot is strictly larger than the chain recurrent set of the diffeomorphism. The Lyapunov function for Pixton diffeomorphism with minimal number of critical points is called the quasi-energy function. In this paper we construct a quasi-energy function for Pixton diffeomorphisms defined by a generalized Mazur knot.

 $\mathbf{Keywords}$ Hopf knot \cdot Mazur knot \cdot Pixton diffeomorphism \cdot quasi-energy function

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1 Introduction and the main results

Let M^n be a smooth closed *n*-manifold with a metric *d* and let $f : M^n \to M^n$ be a diffeomorphism. For two given points $x, y \in M^n$ a sequence of points $x = x_0, \ldots, x_m = y$ is called an ε -chain of length $m \in \mathbb{N}$ connecting x to y if $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \leq i \leq m$ (Fig. 1).

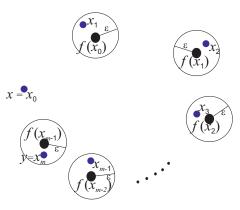


Fig. 1 An ε -chain of length $m \in \mathbb{N}$

A point $x \in M^n$ is called *chain recurrent* for the diffeomorphism f if for every $\varepsilon > 0$ there is an ε -chain of length m connecting x to itself for some m (m depends on $\varepsilon > 0$). The *chain recurrent set*, denoted by \mathcal{R}_f , is the set of all chain recurrent points of f. Define the equivalence on \mathcal{R}_f by the rule: $x \sim y$ if for every $\varepsilon > 0$ there is are ε -chains connecting x to y and y to x. This equivalence relation defines equivalence classes called *chain components*.

If the chain recurrent set of a diffeomorphism f is finite then it consists of periodic points. A periodic point $p \in \mathcal{R}_f$ of period m_p is said to be *hyperbolic* if absolute values of all the eigenvalues of the Jacobian matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ are not equal to 1. If absolute values of all these eigenvalues are greater (less) than 1 then p is called a *sink* (a *source*). Sinks and sources are called *knots*. If a periodic point is not a knot then it is called a *saddle*.

Let p be a hyperbolic periodic point of a diffeomorphism f whose chain recurrent set is finite. The Morse index of p, denoted by λ_p , is the number of eigenvalues of Jacobian matrix whose absolute values are greater than 1. The stable manifold $W_p^s = \{x \in M^n : \lim_{k \to +\infty} d(f^{km_p}(x), p) = 0\}$ and the unstable manifold $W_p^u = \{x \in M^n : \lim_{k \to +\infty} d(f^{-km_p}(x), p) = 0\}$ of p are smooth manifolds diffeomorphic to \mathbb{R}^{λ_p} and $\mathbb{R}^{n-\lambda_p}$, respectively. Stable and unstable manifolds are called invariant manifolds. A connected component of the set $W_p^u \setminus p(W_p^s \setminus p)$ is called a unstable (stable) separatrice of p.

A diffeomorphism $f: M^n \to M^n$ is called a *Morse-Smale* diffeomorphism if

- 1. its chain recurrent set \mathcal{R}_f consists of finite number of hyperbolic points;
- 2. for any two points $p, q \in \mathcal{R}_f$ the manifolds W_p^s, W_q^u intersect transversally.

C Conley in [3] gave the following definition: a Lyapunov function for a Morse-Smale diffeomorphism $f : M^n \to M^n$ is a continuous function $\varphi : M^n \to \mathbb{R}$ satisfying

$$-\varphi(f(x)) < \varphi(x) \text{ if } x \notin R_f; -\varphi(f(x)) = \varphi(x) \text{ if } x \in R_f.$$

Notice that every Morse-Smale diffeomorphism f has a Morse-Lyapunov function ¹, i.e. a Lyapunov function $\varphi : M^n \to \mathbb{R}$ which is a Morse function such that each periodic point $p \in \mathcal{R}_f$ is its non-degenerate critical point of index λ_p with Morse coordinates $(V_p, \phi_p : y \in V_p \mapsto (x_1(y), \ldots, x_n(y)) \in \mathbb{R}^n$ and

$$\phi_p^{-1}(Ox_1\dots x_{\lambda_p}) \subset W_p^u, \ \phi_p^{-1}(Ox_{\lambda_p+1}\dots x_n) \subset W_p^s.$$
(*)

If the function φ has no critical points outside \mathcal{R}_f then following [15] we call it the *energy function* for the Morse-Smale diffeomorphism f.

The proof of existence of an energy Morse function for a Morse-Smale diffeomorphism of the circle is an easy exercise. D. Pixton [15] in 1977 proved that every Morse-Smale diffeomorphism of a surface has an energy function. There he also constructed an example of a Morse-Smale diffeomorphism on the 3-sphere which admits no energy function. The obstacle to existence of an energy function in his example was the *wild embedding* of the saddle separatrices in the ambient manifold (i.e. the closure of the separatrice is not a submanifold of the ambient space). From [11] it follows that there are Morse-Smale diffeomorphisms with no energy function on manifolds of any dimension n > 2. Therefore, following [7] for a Morse-Smale diffeomorphism f we call a Morse-Lyapunov function with the minimal number of critical points (denote it by ρ_f) a quasi-energy function. Notice that ρ_f is a topological invariant, i.e. if two diffeomorphisms $f, f' : M^n \to M^n$ are topologically conjugate (that is there is a diffeomorphism $h : M^n \to M^n$ such that $h \circ f = f' \circ h$) then $\rho_f = \rho_{f'}$.

In this paper we give a lower estimate of ρ_f for Pixton diffeomorphisms. The class of Pixton diffeomorphisms \mathcal{P} is defined in the following way. Every diffeomorphism $f \in \mathcal{P}$ is a Morse-Smale 3-diffeomorphism whose chain recurrent set consists of four points: one source, one saddle and two sinks (for details see section 2). Notice that Pixton's example is a diffeomorphism of this class. According to [2] the class of topological conjugacy of a diffeomorphism $f \in \mathcal{P}$ is completely defined by the equivalence class of the Hopf knot L_f , i.e. the knot in the generating class of the fundamental group of the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ (see Proposition 1). Moreover, any Hopf knot can be realized as a Pixton diffeomorphism.

Recall that a knot in $\mathbb{S}^2 \times \mathbb{S}^1$ is a smooth embedding $\gamma : \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ or the image of this embedding $L = \gamma(\mathbb{S}^1)$. Two knots L, L' are said to be *equivalent* if there is a homeomorphism $h : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ such that h(L) = L'. Two knots

¹ This function can be constructed, for example, by suspension. Consider the topological flow \hat{f}^t on the manifold $M^n \times \mathbb{R}$ defined by $\hat{f}^t(x) = x + t$. Define the diffeomorphism $g: M^n \times \mathbb{R} \to M^n \times \mathbb{R}$ by $g(x,\tau) = (f(x),\tau-1)$ and let $G = \{g^k, k \in \mathbb{Z}\}$ and $W = (M^n \times \mathbb{R})/G$. Denote by $p_W: M^n \times \mathbb{R} \to W$ the natural projection and denote by f^t the flow on W defined by $f^t(x) = p_W(\hat{f}^t(p_W^{-1}(x)))$. The flow f^t is called the suspension over f. By construction the chain recurrent set of f^t consists of the finite number of periodic orbits $\delta_i = p_W(\mathcal{O}_i \times \mathbb{R}), i \in \{1, \ldots, k_f\}$ and this means that the suspension f^t is a Morse-Smale flow. A Lyapunov function for these flows is constructed in [12]. Then the restriction of this function on M is the desired Lyapunov function for f.

 γ, γ' are smoothly homotopic if there exists a smooth map $\Gamma : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^2 \times \mathbb{S}^1$ such that $\Gamma(s, 0) = \gamma(s)$ and $\Gamma(s, 1) = \gamma'(s)$ for every $s \in \mathbb{S}^1$. If $\Gamma|_{\mathbb{S}^1 \times \{t\}}$ is an embedding for every $t \in [0, 1]$ then the knots are said to be *isotopic*. Any Hopf knot $L \subset \mathbb{S}^2 \times \mathbb{S}^1$ is smoothly homotopic to the standard Hopf knot $L_0 = \{s\} \times \mathbb{S}^1$ (see, for example, [9]) but generally it is neither isotopic nor equivalent to it. B. Mazur [10] constructed the Hopf knot L_M which we call the Mazur land which is non-conjucted to the standard Land Kart $\Gamma(s, 0)$ is the standard hopf knot L_M which we call the Mazur knot and which is non-equivalent and non-isotopic to L_0 (see Fig. 2). It follows from the results of [1] that there exists a countable family of pairwise

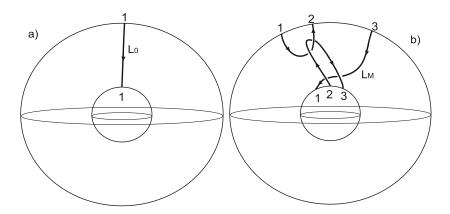


Fig. 2 Two non-isotopic and non equivalent Hopf knots L_0 and L_M : a) the standard Hopf knot L_0 ; b) the Mazur knot L_M

non-equivalent Hopf knots L_n , $n \in \mathbb{N}$ which are generalized Mazur knots (Fig. 3).

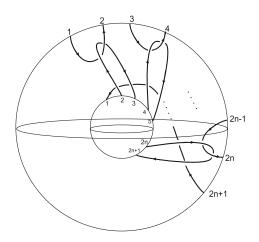


Fig. 3 A generalized Mazur knot L_n

According to [6] a Pixton diffeomorphism f admits an energy Morse function if and only if the knot L_f is trivial (i.e. equivalent to the standard one). If the knot L_f is not trivial then the number ρ_f of the critical points of a quasi-energy Morse function of f is evidently even and

 $\rho_f \ge 6.$

The main result of this paper is the proof of Theorem 1.

Theorem 1 Let f be a Pixton diffeomorphism $(f \in \mathcal{P})$ and let $L_n, n \in \mathbb{N}$ be its knot. Then the number ρ_f of critical points of a quasi-energy function of f is calculated by^2

$$\rho_f = 4 + 2n.$$

2 Construction of Pixton diffeomorphisms

In dynamics a wild Artin-Fox arc was for the first time introduced by D. Pixton in [15] where he constructed a Morse-Smale diffeomorphism on the 3-sphere with the unique saddle whose invariant manifolds form an Artin-Fox arc. We give the modern construction of these diffeomorphisms following Ch. Bonatti and V. Grines [2] where Pixton diffeomorphisms were also classified (see also [8], [11]).

For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ denote $||\mathbf{x}|| = \sqrt{x_1^2 + \dots + x_n^2}$. Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be the diffeomorphism defined by $h(x_1, x_2, x_3) = \left(\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}\right)$. Define the map $p: \mathbb{R}^3 \setminus O \to \mathbb{S}^2 \times \mathbb{S}^1$ by

$$p(x_1, x_2, x_3) = \left(\frac{x_1}{||\mathbf{x}||}, \frac{x_2}{||\mathbf{x}||}, \log_2(||\mathbf{x}||) \pmod{1}\right).$$

Let $L \subset (\mathbb{S}^2 \times \mathbb{S}^1)$ be a Hopf knot and let U(L) be its tubular neighborhood. Then the set $\overline{L} = p^{-1}(L)$ is the *h*-invariant arc in \mathbb{R}^3 and $U(\overline{L}) = p^{-1}(U(L))$ is its *h*-invariant neighborhood diffeomorphic to $\mathbb{D}^2 \times \mathbb{R}^1$ (Fig. 4). Let $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2^2 + x_3^2 \leq 4\}$ and let $g^t : C \to C$ be the flow

defined by

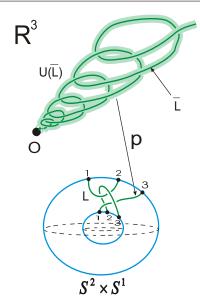
$$g^{t}(x_1, x_2, x_3) = (x_1 + t, x_2, x_3).$$

Then there is a diffeomorphism $\zeta: U(L) \to C$ that conjugates $h|_{U(L)}$ and $g = g^1|_C$. Define the flow ϕ^t on C by:

$$\begin{cases} \dot{x}_1 = \begin{cases} 1 - \frac{1}{9}(x_1^2 + x_2^2 + x_3^2 - 4)^2, & x_1^2 + x_2^2 + x_3^2 \leqslant 4\\ 1, & x_1^2 + x_2^2 + x_3^2 > 4 \end{cases} \\ \dot{x}_2 = \begin{cases} \frac{x_2}{2} \left(\sin\left(\frac{\pi}{2} \left(x_1^2 + x_2^2 + x_3^2 - 3\right)\right) - 1 \right), & 2 < x_1^2 + x_2^2 + x_3^2 \leqslant 4\\ -x_2, & x_1^2 + x_2^2 + x_3^2 \leqslant 2\\ 0, & x_1^2 + x_2^2 + x_3^2 > 4 \end{cases} \\ \dot{x}_3 = \begin{cases} \frac{x_3}{2} \left(\sin\left(\frac{\pi}{2} \left(x_1^2 + x_2^2 + x_3^2 - 3\right)\right) - 1 \right), & 2 < x_1^2 + x_2^2 + x_3^2 \leqslant 4\\ -x_3, & x_1^2 + x_2^2 + x_3^2 \leqslant 2\\ 0, & x_1^2 + x_2^2 + x_3^2 \leqslant 2 \end{cases} \\ 0, & x_1^2 + x_2^2 + x_3^2 \leqslant 4. \end{cases}$$

By construction the diffeomorphism $\phi = \phi^1$ has two fixed points: the saddle P(1,0,0) and the sink Q(-1,0,0) (Fig. 5), both being hyperbolic. One unstable

² For n = 1 Theorem 1 is proved in [7].



 ${\bf Fig. \ 4} \ {\rm Suspension \ of \ a \ Hopf \ knot}$

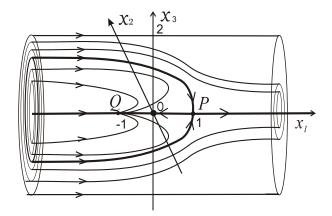


Fig. 5 Trajectories of the flow ϕ^t

separatrice of the saddle P coincides with the open interval $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| < 1, x_2 = x_3 = 0\}$ in the basin of the sink Q while the other coincides with the ray $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 1, x_2 = x_3 = 0\}$. Notice that ϕ coincides with the diffeomorphism $g = g^1$ outside the ball $\{(x_1, x_2, x_3) \in \mathbb{C} : x_1^2 + x_2^2 + x_3^2 \leq 4\}$.

Define the diffeomorphism $\bar{f}_L : \mathbb{R}^3 \to \mathbb{R}^3$ so that \bar{f}_L coincides with h outside U(L) and it coincides with $\zeta^{-1}\phi\zeta$ on U(L). Then \bar{f}_L has in U(L) two fixed points: the sink $\zeta^{-1}(Q)$ and the saddle $\zeta^{-1}(P)$, both being hyperbolic. The unstable separatrice of the saddle $\zeta^{-1}(P)$ lies in L (Fig. 6).

Now project the dynamics onto the 3-sphere. Denote by N(0, 0, 0, 1) the North Pole of the sphere $\mathbb{S}^3 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : ||\mathbf{x}|| = 1\}$. For every point $\mathbf{x} \in (\mathbb{S}^3 \setminus \{N\})$ there is the unique line passing through N and \mathbf{x} in \mathbb{R}^4 . This line intersects

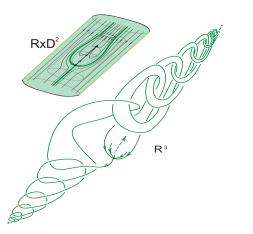


Fig. 6 The phase portrait of the diffeomorphism \bar{f}_L

 \mathbb{R}^3 in exactly one point $\vartheta_+(\mathbf{x})$ (Fig. 7). The point $\vartheta_+(\mathbf{x})$ is the stereographic projection of the point \mathbf{x} . One can easily check that

$$\vartheta_+(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1 - x_4}, \frac{x_2}{1 - x_4}, \frac{x_3}{1 - x_4}\right).$$

Thus, the stereographic projection $\vartheta_+: \mathbb{S}^3 \setminus \{N\} \to \mathbb{R}^3$ is a diffeomorphism.

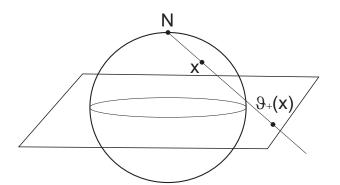


Fig. 7 The stereographic projection.

By construction \bar{f}_L coincides with h in some neighborhood of the point O and in some neighborhood of the infinity. Therefore, it induces on S^3 the Morse-Smale

$$f_L(\mathbf{x}) = \begin{cases} \vartheta_+^{-1}(\bar{f}_L(\vartheta_+(\mathbf{x}))), \ \mathbf{x} \neq N; \\ N, \ \mathbf{x} = N \end{cases}$$

It follows directly from the construction that the non-wandering set of f_L consists of exactly four fixed hyperbolic points: two sinks $\omega = \vartheta_+^{-1}(\zeta^{-1}(Q))$, S, one saddle $\sigma = \vartheta_+^{-1}(\zeta^{-1}(P))$ and one source N. We say the constructed diffeomorphism to be *model* and it is of Pixton class.

Proposition 1 ([2])

- Any diffeomorphism $f \in \mathcal{P}$ is topologically conjugate to some model diffeomorphism f_L .
- Two model diffeomorphisms $f_L, f_{L'}$ are topologically conjugate if and only if their knots L, L' are equivalent.

3 Genus of Hopf knot

In this section we introduce the notion of genus for a Hopf knot and use it to estimate the number of critical points of the quasi-energy function of the Pixton diffeomorphism defined by this knot.

Let L be a Hopf knot and let $\overline{L} = p^{-1}(L)$ be its cover in $\mathbb{R}^3 \setminus O$. We say a closed orientable surface $\Sigma \subset \mathbb{S}^2 \times \mathbb{S}^1$ to be a secant surface of the knot L if it intersects L in a unique point and there is an *h*-compressible 3-manifold $Q_{\Sigma} \subset \mathbb{R}^3$ (that is $h(Q_{\Sigma}) \subset int Q_{\Sigma}$) with the boundary $\overline{\Sigma}$ such that $\Sigma = p(\overline{\Sigma})$ and the intersection $\overline{L} \cap \overline{\Sigma}$ is the unique point \overline{y} . The minimally possible genus g_L of the secant surface is called the genus of the knot L. The secant surface of L of genus g_L is said to be minimal.

Lemma 1 If Σ is a minimal secant surface of the knot L then the surface $\overline{\Sigma} \setminus \overline{y}$ is non-compressible in $\mathbb{R}^3 \setminus (O \cup \overline{L})$, i.e. any simple closed curve $c \subset int (\overline{\Sigma} \setminus \overline{y})$ is contractible on $\overline{\Sigma} \setminus \overline{y}$ if it bounds a smoothly embedded 2-disk $D \subset int (\mathbb{R}^3 \setminus (O \cup \overline{L}))$ such that $D \cap (\overline{\Sigma} \setminus \overline{y}) = \partial D = c$.

Proof Let Σ be a minimal secant surface of L and let \bar{y} be the unique point of the intersection $\bar{L} \cap \bar{\Sigma}$. Assume the opposite: the surface $\bar{\Sigma} \setminus \bar{y}$ is compressible in $\mathbb{R}^3 \setminus (O \cup \bar{L})$. Then there is a non-contractible simple closed curve $c \subset int \ (\bar{\Sigma} \setminus \bar{y})$ and there is the smoothly embedded 2-disk $D \subset int \ (\mathbb{R}^3 \setminus (O \cup \bar{L}))$ such that $D \cap (\bar{\Sigma} \setminus \bar{y}) = \partial D = c$ (see, for example, [14]). Then we have two possibilities:

$$(int D) \cap \left(\bigcup_{k \in \mathbb{Z}} h^k(\bar{\Sigma})\right) = \emptyset,$$
 (1)

$$(int D) \cap \left(\bigcup_{k \in \mathbb{Z}} h^k(\bar{\Sigma})\right) \neq \emptyset.$$
 (2)

In case (1) two subcases are possible: (1a) $D \subset Q_{\Sigma}$, (1b) $D \subset (\mathbb{R}^3 \setminus int Q_{\Sigma})$. For case 1a) let $N(D) \subset Q_{\Sigma}$ be a tubular neighborhood of the disk D. Then exactly one connected component of the set $Q_{\Sigma} \setminus int N(D)$ intersects \overline{L} . According to (1) this component is *h*-compressible and its boundary intersects \overline{L} at a unique point. The projection of this boundary into $\mathbb{S}^2 \times \mathbb{S}^1$ is, therefore, the secant surface of L of genus less than g_L . This contradicts the fact that the surface Σ is minimal. In case 1b) let $N(D) \subset (\mathbb{R}^3 \setminus int Q_{\Sigma})$ be a tubular neighborhood of D. Then due to (1) the set $Q_{\Sigma} \cup N(D)$ is *h*-compressible and its boundary intersects \overline{L} at a unique point. The projection of this boundary into $\mathbb{S}^2 \times \mathbb{S}^1$ is, therefore, the secant surface of L of genus less than g_L and we have the same contradiction.

In case (2) without loss of generality assume the intersection $int D \cap (\bigcup_{k \in \mathbb{Z}} h^k(\bar{\Sigma}))$

to be transversal and denote it by Γ . Let γ be a curve from Γ . We say the curve γ to be *innermost* if it is the boundary of the disk $D_{\gamma} \subset D$ such that $int D_{\gamma}$ contains no curves of Γ . Consider this innermost curve $\gamma \subset f^{k}(\Sigma)$. There are two subcases: a) γ is essential on $f^{k}(\Sigma)$ and b) γ is contractible on $f^{k}(\Sigma)$. In case a) the arguments of the case (1) apply for the body $f^{k}(Q_{\Sigma})$ and the disk D_{γ} and we get the contradiction to the minimality of the surface Σ . In case b) denote by $d_{\gamma} \subset f^{k}(\Sigma)$ the 2-disk bounded by γ and denote by $B_{\gamma} \subset (\mathbb{R}^{3} \setminus O)$ the 3-ball bounded by the 2-sphere $D_{\gamma} \cup d_{\gamma}$. Consider: b1) $B_{\gamma} \subset f^{k}(Q_{\Sigma})$ and b2) $B_{\gamma} \subset (\mathbb{R}^{3} \setminus int f^{k}(Q_{\Sigma}))$. For b1) let $N(B_{\gamma}) \subset f^{k}(Q_{\Sigma})$ be a tubular neighborhood of B_{γ} . Then the set $Q_{\Sigma} \setminus int N(B_{\gamma})$ is *k*-compressible because the curve γ lies in its interior and the boundary of $Q_{\Sigma} \setminus int N(B_{\gamma})$ intersects \overline{L} at a unique point. The projection of this boundary into $\mathbb{S}^{2} \times \mathbb{S}^{1}$ is, therefore, the secant surface of the knot L of genus g_{L} for which the number of connected components of the set Γ is less. We get the same result for b2) for the set $Q_{\Sigma} \cup N(B_{\gamma})$. Thus, iterating the process we come either to the case a) or to the case (1) and get a contradiction.

Lemma 2 For any diffeomorphism $f \in \mathcal{P}$ the following estimation holds

$$\rho_f \geqslant 4 + 2g_{L_f}.\tag{3}$$

Proof Since Proposition 1 is true and since the number ρ_f of the critical points of a quasi-energy function of $f \in \mathcal{P}$ is invariant, from now on we consider model Pixton diffeomorphisms f_L with the Hopf knot L. Denote by ℓ the non-stable separatrice of the saddle σ lying in the basin of the sink S. Let

$$p_S: W^s_S \setminus S \to \mathbb{S}^2 \times \mathbb{S}^1$$

be the natural projection sending a point $w \in (W_S^s \setminus S)$ to the point $p(f^{k_w}(w)), f^{k_w}(w) \in V_S$. Since the diffeomorphism f_L coincides with the homothety h in some neighborhood V_S of S, the natural projection p_S is well defined and $p_S(\ell) = L$ by construction.

Consider an arbitrary Morse-Lyapunov function $\varphi : \mathbb{S}^3 \to \mathbb{R}$ of the diffeomorphism f_L . To be definite let $\varphi(S) = 0$, $\varphi(\sigma) = 1$ and $\varphi(N) = 3$. From the definition of the Morse-Lyapunov function it follows that $\varphi|_{\ell}$ monotonically decreases in some neighborhood of the saddle σ . Therefore, there is $\varepsilon_1 \in (0, 1)$ such that the interval $(1 - \varepsilon_1, 1)$ contains no critical values of φ and the connected component $\overline{\Sigma}_1$ of the level set $\varphi^{-1}(1 - \varepsilon_1)$ intersects the separatrice ℓ at the unique point. Denote this point by w_1 .

Let \bar{Q}_1 be the connected component of the set $\varphi^{-1}([0, 1 - \varepsilon_1])$ which contains the segment $[w_1, S]$ of the closure of the separatrice ℓ . Since φ decreases along the trajectories of f, the values of φ on W^s_{σ} are greater than 1. Therefore, the manifold \bar{Q}_1 lies in the manifold W^s_S diffeomorphic to \mathbb{R}^3 . Let the function $\varphi|_{\bar{Q}_1}$ have $k_q, q \in \{0, \ldots, 3\}$ critical points of index q. Due to [5, Theorem 6.1] on the manifold \bar{Q}_1 there exists a self-indexing Morse function ψ (the value of the function in a critical point equals the index of this point) which has k_q critical points of index q and which is constant on $\partial \bar{Q}_1$. Thus, the manifold \bar{Q}_1 is the surface \tilde{Q}_1 of genus $g_1 = 1 + k_1 - k_0$ with attached handles of indexes 2 and 3. Then the genus of any surface of the set $\partial \bar{Q}_1$ cannot be greater than g_1 .

On the other hand, the number of critical points of $\varphi|_{\bar{Q}_1}$ is not less than k_0+k_1 . If $k_0 \ge 1$ and $g_1 = 1 + k_1 - k_0$ then one gets $k_0 + k_1 = g_1 + 2k_0 - 1 \ge g_1 + 1$. Thus, $\varphi|_{\bar{Q}_1}$ has at least $g_1 + 1$ critical points.

Denote by $\overline{\Sigma}_1$ the connected component of $\partial \overline{Q}_1$ which intersects the separatrice ℓ . Then the surface $\overline{\Sigma}_1$ divides the manifold $W_S^s \cong \mathbb{R}^3$ into two parts, one of which Q_1 being an *h*-compressible body. This means that $\Sigma_1 = p_S(\overline{\Sigma}_1)$ is the secant surface of L and, therefore,

 $g_1 \geqslant g_{\scriptscriptstyle L}.$

Analogously, there is $\varepsilon_2 \in (0, 1)$ for which the interval $(1, 1 + \varepsilon_2)$ contains no critical points of φ and the connected component \bar{Q}_2 of the level set $\varphi^{-1}([0, 1+\varepsilon_2)]$ contains $cl(W_{\sigma}^u)$ in its interior while the intersection \bar{Q}_2 with W_{σ}^s is the unique 2-disk. Due to construction the function $\varphi|_{\bar{Q}_2}$ has at least $g_1 + 3$ critical points and genus of the connected components of ∂Q_2 is less or equals g_1 . Denote by $\bar{\Sigma}_2$ the connected component of $\partial \bar{Q}_2$ which intersects W_{σ}^s and denote by g_2 its genus. The surface $\bar{\Sigma}_2$ divides the manifold $W_N^u \cong \mathbb{R}^3$ into two parts, one of which Q_2 being an h^{-1} -compressible body. Arguing as above one comes to conclusion that the number of critical points of $\varphi|_{Q_2}$ is at least $g_2 + 1$. Therefore, the total number of critical points of φ is greater or equal to

$$g_1 + 3 + g_2 + 1 \ge 4 + 2g_1 \ge 4 + 2g_{L_f}$$

4 The generalized Mazur knot L_n

In this section we show that the genus g_{L_n} of a generalized Mazur knot equals n. At first we give a detailed description of construction of L_n .

4.1 Construction of the generalized Mazur knot L_n

Recall that $h: \mathbb{R}^3 \to \mathbb{R}^3$ is the homothety defined by

$$h(x_1, x_2, x_3) = \left(\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}\right)$$

and $p:\mathbb{R}^3\setminus O\to \mathbb{S}^2\times \mathbb{S}^1$ is the natural projection defined by

$$p(x_1, x_2, x_3) = \left(\frac{x_1}{||\mathbf{x}||}, \frac{x_2}{||\mathbf{x}||}, \log_2(||\mathbf{x}||) \pmod{1}\right).$$

Consider the annulus

$$K = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{1}{4} \le x_1^2 + x_2^2 + x_3^2 \le 1 \right\}$$

bounded by the spheres

$$\mathbb{S}^{2} = \left\{ (x_{1}, x_{2}, x_{3}) \in \mathbb{R}^{3} : x_{1}^{2} + x_{2}^{2} + x_{3}^{2} = 1 \right\}, h(\mathbb{S}^{2})$$

Pick on the circle

$$\mathbb{S}^{1} = \left\{ (x_{1}, x_{2}, x_{3}) \in \mathbb{R}^{3} : x_{1}^{2} + x_{2}^{2} = 1, x_{3} = 0 \right\}$$

pairwise distinct points $\alpha_1, \ldots, \alpha_{2n+1}$ numbered in counter-clockwise order (Fig. 8). Let $a_i, i \in \{1, \ldots, 2n\}$ be the arc of the circle \mathbb{S}^1 bounded by α_i, α_{i+1} whose interior contains no points of $\{\alpha_1, \ldots, \alpha_{2n+1}\}$. Let $B, A_i \subset int K, i \in \{1, \ldots, 2n\}$ be pairwise disjoint smooth arcs such that:

- 1. the boundary points of B are α_{2n+1} , $h(\alpha_1)$; the boundary points of A_{2j-1} are α_{2j-1} , α_{2j} and the boundary points of A_{2j} are $h(\alpha_{2j})$, $h(\alpha_{2j+1})$ for $j \in \{1, \ldots, n\}$;
- 2. the closed curves $c_{2j-1} = cl(a_{2j-1} \cup A_{2j-1}), c_{2j} = cl(h(a_{2j}) \cup A_{2j})$ bound the 2-disks d_{2j-1}, d_{2j} , the transversal intersection of these disks being the arc l_j with the boundary points $b_{2j-1} = d_{2j-1} \cap A_{2j}$ and $b_{2j} = d_{2j} \cap A_{2j-1}$;
- 3. the arc $cl(h(A_1) \cup A_2 \cup \cdots \cup h(A_{2n-1}) \cup A_{2n} \cup B)$ is smooth.

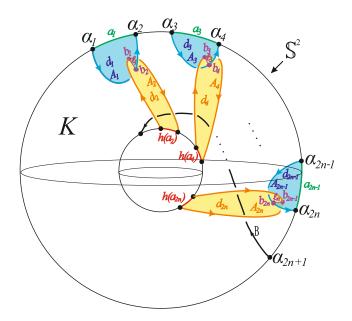


Fig. 8 Construction of the knot L_n

Let

$$\bar{L}_n = \bigcup_{k \in \mathbb{Z}} h^k (B \cup A_1 \cup \dots \cup A_{2n}), L_n = p(\bar{L}_n).$$

4.2 The genus of the knot L_n

Lemma 3 The genus g_{L_n} of the knot L_n equals n.

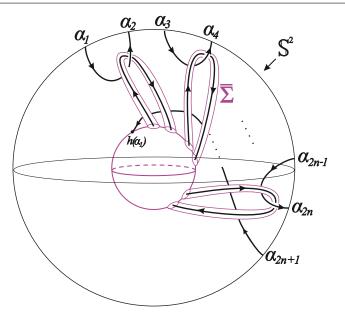


Fig. 9 A secant surface of L_n of genus n

Proof Since there is a secant surface of L_n of genus n, we have $g_{L_n} \leq n$ (Fig. 9). Now we show that $g_{L_n} \geq n$. To that end we prove that for L_n there exists a minimal secant surface Σ such that $\overline{\Sigma} \subset K$ and $\overline{L_n} \cap \overline{\Sigma} = h(\alpha_1)$.

Indeed, let Σ_0 be some minimal secant surface of L_n . Then there exists the connected component $\bar{\Sigma}_0$ of $p^{-1}(\Sigma_0)$ such that it intersects the curve \bar{L}_n at the point \bar{y}_0 situated on \bar{L}_n between $\alpha_1, h(\alpha_1)$ and that bounds the *h*-compressible body Q_{Σ_0} . Without loss of generality let $\bar{y}_0 = h(\alpha_1)$ (otherwise the desired surface is constructed by removing the tubular neighborhood of the arc $[\bar{y}_0, h(\alpha_1)] \subset \bar{L}_n$ from Q_{Σ_0}).

Denote by $k_+, k_- \ge 0$ the maximal integers for which $f^k(\bar{\Sigma}_0) \cap \bar{\Sigma}_0 \ne \emptyset$, $f^{-k}(\bar{\Sigma}_0) \cap \bar{\Sigma}_0 \ne \emptyset$, $k \ge 0$, respectively. If $k_+ = k_- = 0$ then $\bar{\Sigma}_0$ is the desired surface. Otherwise we show the way to decrease by 1 the number $k_+ > 0$ (for k_- the arguments are the same) using isotopy of the secant surface.

Notice that $\bar{\Sigma}_0 \cap f^{k_+}(c_{2j-1}) = \emptyset$, $j \in \{1, \ldots, n\}$. Without loss of generality let the intersection $\Gamma = \bigcup_{j=1}^n f^{k_+}(d_{2j-1}) \cap \bar{\Sigma}_0$ be transversal. Let γ be a curve from

 Γ . Then γ bounds the unique disk $D_{\gamma} \subset f^{k_+}(d_{2j-1})$. There are two possibilities: 1) $b_{2j-1} \notin D_{\gamma}$, 2) $b_{2j-1} \in D_{\gamma}$. In case 1) we say the curve γ to be *innermost* if it bounds the disk $D_{\gamma} \subset f^{k_+}(d_{2j-1})$ such that $int D_{\gamma}$ contains no curves of Γ . Consider this innermost curve γ . Due to Lemma 1 the surface $\overline{\Sigma}_0 \setminus \overline{y}_0$ is noncompressible in $\mathbb{R}^3 \setminus (O \cup \overline{L}_n)$ and, therefore, there exists the disk $d_{\gamma} \subset (\overline{\Sigma}_0 \setminus \overline{y}_0)$ bounded by γ . Denote by $B_{\gamma} \subset (\mathbb{R}^3 \setminus (O \cup \overline{L}_n))$ the 3-ball bounded by the 2-sphere $D_{\gamma} \cup d_{\gamma}$. Consider two subcases: 1a) $B_{\gamma} \subset Q_{\Sigma_0}$ and 1b) $B_{\gamma} \subset (\mathbb{R}^3 \setminus int Q_{\Sigma_0})$.

In case 1a) let $N(B_{\gamma}) \subset Q_{\Sigma_0}$ be a tubular neighborhood of the ball B_{γ} . Then the set $Q_{\Sigma} \setminus int N(B_{\gamma})$ is *h*-compressible because the curve γ lies in its interior and its boundary intersects \overline{L}_n at a unique point. The projection of this boundary to $\mathbb{S}^2 \times \mathbb{S}^1$ is, therefore, a secant surface of L_n of the same genus as Σ_0 . For it the number of the connected components of the set Γ is less. One gets the same result in case 1b) for the set $Q_{\Sigma_0} \cup N(B_{\gamma})$.

If we continue this process then we get the secant surface of L_n of the same genus as Σ_0 and for which the set Γ contains no curves of type 1). Denote the resulting surface again by Σ_0 . Now the set Γ consists only of the curves γ bounding the disk $D_{\gamma} \subset b_{2j-1}$ which contains the point b_{2j-1} . Since $(b_{2j-1} \sqcup c_{2j-1}) \subset (\mathbb{R}^3 \setminus Q_{\Sigma_0})$, the number of these curves on the disk d_{2j-1} is even. Since the surface $\overline{\Sigma}_0 \setminus \overline{y}_0$ is non-compressible in $\mathbb{R}^3 \setminus (O \cup \overline{L}_n)$, all these curves are pairwise homotopic on $\overline{\Sigma}_0 \setminus \overline{y}_0$ and, therefore, they lie in the annulus $\kappa \subset (\overline{\Sigma}_0 \setminus \overline{y}_0)$ bounded by the pair of these curves γ_1, γ_2 . Denote by $\tilde{\kappa} \subset d_{2j-1}$ the annulus bounded by the same curves on the disk d_{2j-1} . Let $\tilde{\Sigma}_0 = \overline{\Sigma}_0 \setminus \kappa \cup \tilde{\kappa}$. Due to construction the surface $\tilde{\Sigma}_0$ is of the same genus as the surface $\overline{\Sigma}_0$ and it bounds an *h*-compressible body. Having removed a tubular neighborhood of the annulus $\tilde{\kappa}$ from this body we get a *h*-compressible body whose boundary does not intersect the disk d_{2j-1} and whose projection to $\mathbb{S}^2 \times \mathbb{S}^1$ is the secant surface of the knot L_n of the same genus as Σ_0 .

If we continue this process then we get a secant surface of L_n of the same genus as Σ_0 and for which the set Γ is not empty. Denote this surface again by Σ_0 . Without loss of generality let the intersections of the surface $\bar{\Sigma}_0$ with the spheres $f^k(\mathbb{S}^2)$ be transversal. Denote by \mathcal{F} the set of the connected components of the intersection $f^{k_+}(K) \cap \bar{\Sigma}_0$. Now we show the way to reduce by 1 the number of the components in \mathcal{F} using isotopy of the secant surface.

Denote by Q the set obtained by removal from the annulus $f^{k_+}(K)$ of the tubular neighborhoods of the disks d_{2j-1} as well as the tubular neighborhoods of the curves $A_{2j}, j \in \{1, \ldots, n\}$. Then Q is homeomorphic to the direct product of the 2-sphere with 2n + 1 deleted points and the segment. Since $Q \cap \overline{\Sigma}_0 = f^{k_+}(K) \cap \overline{\Sigma}_0$ and since $\overline{\Sigma}_0 \setminus \overline{y}_0$ is non-compressible in $\mathbb{R}^3 \setminus (O \cup \overline{L}_n)$, each connected component of $F \in \mathcal{F}$ is non-compressible in Q. Due to [16, Corollary 3.2] there exists a surface $\tilde{F} \subset f^{k_+-1}(\mathbb{S}^2)$ diffeomorphic to F for which $\partial F = \partial \tilde{F}$ and the surface $F \cup \tilde{F}$ bounds in Q the body Δ diffeomorphic to the direct product $F \times [0, 1]$. Then we replace the part F of $\overline{\Sigma}_0$ with \tilde{F} . If we continue the process we get the desired secant surface $\Sigma \subset K$.

Notice (see, for instance, [4, Exercise 2.8.1]) that the fundamental group $\pi_1(K \setminus \overline{L}_n)$ has 2n generators $\gamma_1, \ldots, \gamma_{2n}$, each of which $\gamma_i, i \in \{1, \ldots, 2n\}$ being the generator of the punctured disk $d_i \setminus b_i$ (Fig. 10). Since $b_{2j-1} \in int Q_{\Sigma}$ and $c_{2j-1} \cap Q_{\Sigma} = \emptyset$, there exists the connected component of \tilde{d}_{2j-1} of the intersection $d_{2j-1} \cap Q_{\Sigma}$ which contains the point b_{2j-1} . This component is the 2-disk bounded by the curve $\tilde{\gamma}_{2j-1} \subset (\bar{\Sigma} \setminus h(\alpha_1))$ with holes and the curves $\gamma_{2j-1}, \tilde{\gamma}_{2j-1}$ are homotopic on the punctured disk $d_{2j-1} \setminus b_{2j-1}$. In the same way one finds the curves $\tilde{\gamma}_{2j} \subset (\bar{\Sigma} \setminus h(\alpha_1))$ homotopic to the curves γ_{2j} on the punctured disk $d_{2j} \setminus b_{2j}$ (Fig. 10). Due to Lemma 1 the surface $\bar{\Sigma} \setminus h(\alpha_1)$ is non-compressible in $K \setminus \bar{L}_n$. Then the curves $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{2n}$ are pairwise non-homotopic to the generators on the surface $\bar{\Sigma} \setminus h(\alpha_1)$.

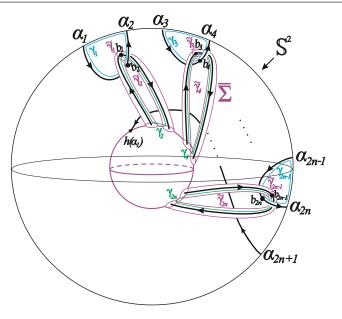


Fig. 10 Generators of the group $\pi_1(K \setminus \overline{L}_n)$

5 Construction of a quasi-energy function for a Pixton diffeomorphism with the Hopf knot ${\cal L}_n$

Let f be a Pixton diffeomorphism constructed for a generalized Mazur knot L_n . Then its non-wandering set Ω_f consists of four points: two sinks ω , S, a source Nand a saddle σ . Then $W_{\sigma}^u \setminus \sigma$ consists of two separatrices ℓ_{ω} , ℓ_S respective closures of which contain the sinks ω , S, the separatrice ℓ_{ω} being tame while ℓ_S being wild. Let $\bar{\Sigma}$ be the surface of genus n bounding the handle-body Q_{Σ} of the same genus. Now we construct for f a Morse-Lyapunov function with 6 + 2n critical points.

Our construction of a quasi-energy function is analogous to the construction of an energy function in [7].

- 1. Choose an energy function $\varphi_p : U_p \to \mathbb{R}$ in the neighborhood of each fixed point p of f so that $\varphi_p(p) = \dim W_p^u$. Let B_ω , B_S be the 3-balls which are the level sets of respective functions φ_ω , φ_S and such that $B_S \subset int Q_\Sigma$. Choose a tubular neighborhood T_σ of the arc $W_\sigma^u \setminus (B_\omega \cup Q_\Sigma)$ so that the handle-body $B_\omega \cup Q_\Sigma \cup T_\sigma$ of genus n is f-compressible and its intersection with W_σ^s is the 2-disk. Denote by P^+ the smoothing of this body by addition of a small exterior collar.
- 2. Due to [7, Section 4.3] there exists an energy function $\varphi : P^+ \setminus int Q_{\Sigma}$ whose value on ∂P^+ is 4/3, whose value on $\bar{\Sigma}$ is 2/3 and which has exactly two critical points ω , σ of respective Morse indexes 0, 1. The disks d_1, \ldots, d_{2n-1} cut the handle-body Q_{Σ} making the 3-ball. Denote by B_{Σ} the smoothing of this ball by removal of the interior collar. The results of the classic Morse theory (see, for example, [13]) allow to extend the function φ to the set $Q_{\Sigma} \setminus int B_{\Sigma}$ in such way that it has n critical points of Morse index 1, one point lying on each disk d_1, \ldots, d_{2n-1} , while the value of φ on ∂B_{Σ} is 1/3. Due to [7, Lemma

4.2] the function φ can be extended to the ball B_{Σ} by an energy function with the unique critical point S of Morse index 0. Since $f(Q_{\Sigma}) \subset int B_{\Sigma}$, the constructed function decreases along the trajectories of the diffeomorphism f.

3. It follows from the definition of the knot L_n that $P^- = \mathbb{S}^3 \setminus int P^+$ is the handle-body of genus n. Moreover, the disks d_2, \ldots, d_{2n} cut P^- making the 3-ball. Denote by B^- smoothing of this ball by removal of the interior collar. The results of the classic Morse theory (see, for example, [13]) allow extension of the function φ to the set $P^- \setminus int B^-$ in such way that it has n critical points of Morse index 2, one point lying on each disk d_2, \ldots, d_{2n} , while the value of φ on ∂B^- is 5/3. According to [7, Lemma 4.2] the function φ can be extended to the ball B^- by an energy function with unique critical point Nof Morse index 3. Since $f(B^-) \subset int P^-$, the constructed function decreases along the trajectories of the diffeomorphism f and, therefore, it is the desired quasi-energy function.

Conflict of interest

The authors declare that they have no conflict of interest.

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