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On families of constrictions in model of overdamped Josephson junction and Painlevé 3 equation*

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Abstract

The tunnelling effect predicted by Josephson (Nobel Prize, 1973) concerns the Josephson junction: two superconductors separated by a narrow dielectric. It states existence of a supercurrent through it and equations governing it. The *overdamped Josephson junction* is modelled by a family of differential equations on two-torus depending on three parameters: B (abscissa), A (ordinate), ω (frequency). We study its *rotation number* $\rho(B, A;\ \omega)$ as a function of (B, A) with fixed ω. The *phase-lock areas* are the level sets $L_r := \{\rho = r\}$ with non-empty interiors; they exist for $r \in \mathbb{Z}$ (Buchstaber, Karpov, Tertychnyi). Each L_r is an infinite chain of domains going vertically to infinity and separated by points. Those separating points for which $A \neq 0$ are called *constrictions*. We show that: (1) *all the constrictions in L_r lie on the axis $\{B = \omega r\}$*; (2) *each constriction* is *positive*: this means that some its punctured neighbourhood on the axis $\{B = \omega r\}$ lies in $\mathrm{Int}(L_r)$. These results confirm experiments by physicists (1970ths) and two mathematical conjectures. We first prove deformability of each constriction to another one, with arbitrarily small ω and the same $\ell := \frac{B}{\omega}$, using equivalent description of model by linear systems of differential equations on $\overline{\mathbb{C}}$ (Buchstaber, Karpov, Tertychnyi) and studying their isomonodromic deformations described by Painlevé 3 equations. Then

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non-existence of *ghost* constrictions (i.e., constrictions either with $\rho \neq \ell = \frac{B}{\omega}$, or of non-positive type) with a given ℓ for small ω is proved by slow-fast methods.

Keywords: Josephson junction, Painleve equation, isomonodromic deformation, slow–fast system, rotation number, phase-lock areas, dynamical system

Mathematics Subject Classification numbers: 34M03, 34A26, 34E15.

(Some figures may appear in colour only in the online journal)

Contents

1. Introduction

1.1. Model of Josephson junction: a brief survey and main results

The *Josephson effect* is a tunnelling effect in superconductivity predicted theoretically by Josephson in 1962 [41] (Nobel Prize in physics, 1973) and confirmed experimentally by Anderson and Rowell in 1963 [1]. It concerns the so-called *Josephson junction:* a system of two superconductors separated by a very narrow dielectric fibre. The Josephson effect is the existence of a supercurrent crossing the junction (provided that the dielectric fibre is narrow enough), described by equations discovered by Josephson[4].

The model of the so-called *overdamped Josephson junction*, see [45, 53, 58, 63], [6, p 306], [46, pp 337–40], [47, p 193], [49, p 88] is described by the family of nonlinear differential equations

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t, \quad \omega > 0,\ B \geqslant 0. \tag{1.1}$$

Here ϕ is the difference of phases (arguments) of the complex-valued wave functions describing the quantum mechanic states of the two superconductors. Its derivative is equal to the voltage up to known constant factor.

Equation (1.1) also arise in several models in physics, mechanics and geometry, e.g., in planimeters, see [26, 27]. Here ω is a fixed constant, and (B, A) are the parameters. The variable and parameter changes

$$\tau := \omega t, \qquad \theta := \phi + \frac{\pi}{2}, \qquad \ell := \frac{B}{\omega}, \qquad \mu := \frac{A}{2\omega}, \tag{1.2}$$

transform (1.1) to a non-autonomous ordinary differential equation on the two-torus $\mathbb{T}^2 = S^1 \times S^1$ with coordinates $(\theta, \tau) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$\frac{d\theta}{d\tau} = \frac{\cos\theta}{\omega} + \ell + 2\mu\cos\tau. \tag{1.3}$$

The graphs of its solutions are the orbits of the vector field

$$\begin{cases} \dot\theta = \dfrac{\cos\theta}{\omega} + \ell + 2\mu\cos\tau \\ \dot\tau = 1 \end{cases} \tag{1.4}$$

on $\mathbb{T}^2$. The *rotation number* of its flow, see [2, p 104], is a function $\rho(B, A)$ of parameters[5]:

$$\rho(B, A; \omega) = \lim_{k\to+\infty} \frac{\theta(2\pi k)}{2\pi k}.$$

Here $\theta(\tau)$ is a general $\mathbb{R}$-valued solution of the first equation in (1.4) whose parameter is the initial condition for $\tau = 0$. Recall that the rotation number is independent on the choice of the

[4] Analytic deformability of each constriction to constrictions of the same type, ρ, ℓ and arbitrarily small ω is a joint result of the authors. Non-existence of ghost constrictions with a given ℓ for every ω small enough is a result of the second author (Glutsyuk) Theorem 1.13.

[5] There is a misprint, missing 2π in the denominator, in analogous formulae in previous papers of the second author (Glutsyuk) with co-authors: [28, formula (2.2)], [13, the formula after (1.16)].

initial condition, see [2, p 104]. The parameter B is called *abscissa,* and A is called the *ordinate.* Recall the following well-known definition.

Definition 1.1 (cf [28, definition 1.1]). The rth *phase-lock area* is the level set

$$L_r = L_r(\omega) = \{\rho(B, A) = r\} \subset \mathbb{R}^2,$$

provided that it has a non-empty interior.

Remark 1.2 **phase-lock areas and Arnold tongues**. H.Poincaré introduced the rotation number of a circle diffeomorphism. The rotation number of the flow of the field (1.4) on $\mathbb{T}^2$ equals (modulo $\mathbb{Z}$) the rotation number of the circle diffeomorphism given by its time 2π flow mapping restricted to the cross-section $S^1_\theta \times \{0\}$. In Arnold family of circle diffeomorphisms $x \mapsto x + b + a \sin x$, $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the behaviour of its phase-lock areas for small a demonstrates the tongues effect discovered by Arnold [2, p 110]. That is why the phase-lock areas became 'Arnold tongues', see [28, definition 1.1].

Recall that the rotation number has physics meaning of the mean voltage over a long time interval up to known constant factor.

Relation of phase-lock effect in model (1.1) to dynamical systems on torus was discovered in [15]. In physics papers earlier than [15] the phase-lock area effect dealt with convergence of differences $\phi(t + (n + 1)T) - \phi(t + nT)$, where $\phi(t)$ is a solution of (1.1) and $T = \frac{2\pi}{\omega}$. This effect was defined there as a phenomenon of convergence of the above differences to $2\pi k$, $k \in \mathbb{Z}$, on an open subset of the parameter space. It was observed in [15] that their convergence is equivalent to the statement that the rotation number of the corresponding dynamical system (1.4) is equal to k.

Some figures of the phase-lock areas of family (1.4) are presented in physics books [46, p 339, figure 11.4], [47, p 193, figure 11.4], [49, p 88, figure 5.2]. See also figures of the phase-lock areas below.

The phase-lock areas of family (1.4) were studied by Buchstaber, Karpov, Tertychnyi *et al*, see [12–22, 28–30, 34, 36, 43, 44, 65, 66] and references therein. The following statements are known results proved mathematically:

(a) Phase-lock areas exist only for integer rotation number values (*quantization effect* observed and proved in [17], later also proved in [34, 36]).

(b) The boundary of the rth phase-lock area consists of two analytic curves, which are the graphs of two functions $B = G_{r,\alpha}(A)$, $\alpha = 0, \pi$, (see [18]; this fact was later explained by Klimenko via symmetry, see [44]).

(c) The latter functions have Bessel asymptotics

$$\begin{cases} G_{r,0}(s) = r\omega - J_r\left(-\dfrac{s}{\omega}\right) + O\left(\dfrac{\ln|s|}{s}\right) \\ G_{r,\pi}(s) = r\omega + J_r\left(-\dfrac{s}{\omega}\right) + O\left(\dfrac{\ln|s|}{s}\right) \end{cases}, \quad \text{as } s \to \infty \tag{1.5}$$

(observed and proved on physics level in [59], see also [46, p 338], [6, section 11.1], [16]; proved mathematically in [44]).

(d) Each phase-lock area is a garland of infinitely many bounded domains going to infinity in the vertical direction. In this chain each two subsequent domains are separated by one point. This was proved in [44] using the above statement (c). Those separation points that

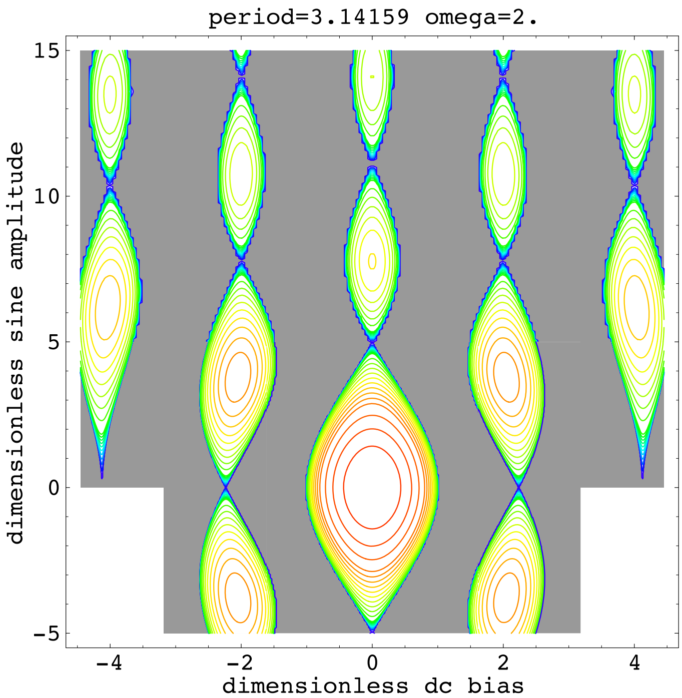

Figure 1. Phase-lock areas and their constrictions for $\omega = 2$. The abscissa is B, the ordinate is A. Reproduced from [13], with permission from Springer Nature.

lie on the horizontal B-axis, namely $A = 0$, were calculated explicitly, and we call them the *growth points*, see [18, corollary 3]. The other separation points, which lie outside the horizontal B-axis, are called the *constrictions*.

(e) For every $r \in \mathbb{Z}$ the rth phase-lock area is symmetric to the $-r$th one with respect to the vertical A-axis.

(f) Every phase-lock area is symmetric with respect to the horizontal B-axis. See figures 1–3 below.

Definition 1.3. For every $r \in \mathbb{Z}$ and $\omega > 0$ we consider the vertical line

$$\Lambda_r = \{B = \omega r\} \subset \mathbb{R}^2_{(B,A)}$$

and we will call it the *axis* of the phase-lock area L_r.

The figures for the phase-lock areas obtained experimentally are given in the physics books on Josephson effect, see [47, p 193, figure 11.4], [49, p 88, figure 5.2], [46, p 339, figure 11.4] (which refers to physics paper [48]). They had shown that in each phase-lock area L_r all the constrictions should lie on the same vertical line. No mathematical proof was presented there. Numerical illustrations which one can find in the paper [17] by Buchstaber, Karpov, Tertychnyi have shown the same effect and that the line containing the constrictions of the area L_r should coincide with its axis Λ_r. This constriction alignment phenomena was stated as an experimental fact and conjecture in [28, experimental fact A].

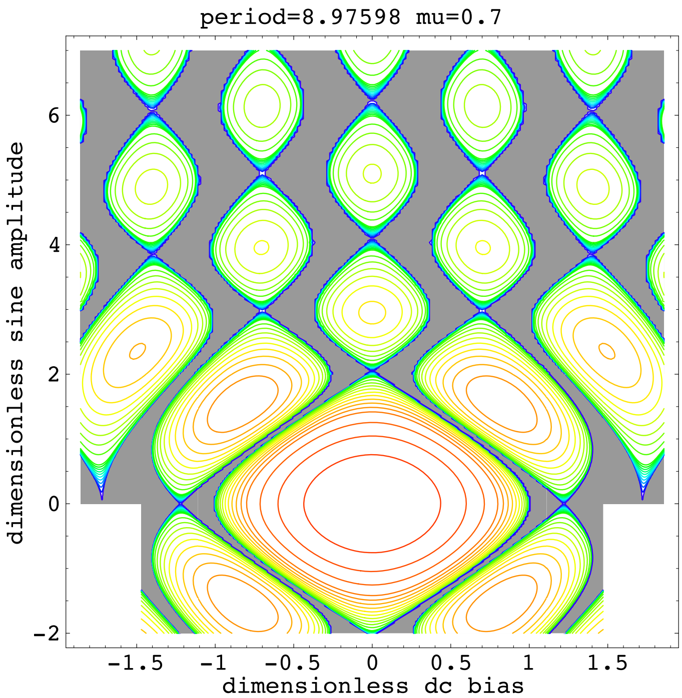

Figure 2. Phase-lock areas and their constrictions for $\omega = 0.7$. Reproduced from [13], with permission from Springer Nature.

The main results of the paper are the two following theorems. The first theorem confirms the above constriction alignment phenomena. The second one confirms another, positivity property of constrictions that can be also seen in the figures from physics books mentioned in the above paragraph.

Theorem 1.4. *For every $r \in \mathbb{Z}$ and every $\omega > 0$ all the constrictions of the phase-lock area L_r lie in its axis Λ_r.*

Remark 1.5. It was proved in [28, theorem 1.2] that for every $r \in \mathbb{Z}$ the constrictions in L_r have abscissas $B = \ell\omega$, $\ell \in \mathbb{Z}$, $\ell \equiv r (\mathrm{mod}\, 2)$, $\ell \in [0, r]$. For further results and discussion of the constriction alignment conjecture see [13, section 5] and [28–30].

Definition 1.6 [29, p 329]. A constriction (B_0, A_0) is said to be *positive,* if the corresponding germ of interior of phase-lock area contains the germ of punctured vertical line interval: that is, if there exists a punctured neighbourhood $U = U(A_0) \subset \mathbb{R}$ such that the punctured interval $B_0 \times (U \setminus \{A_0\}) \subset B_0 \times \mathbb{R}$ lies entirely in the interior of the corresponding phase-lock area. A constriction is called *negative,* if the above punctured interval can be chosen to lie in the complement to the union of the phase-lock areas. Otherwise it is called *neutral*. See figure 4.

Theorem 1.7. [6]*All the constrictions are positive.*

[6] The main results of the paper (theorems 1.4 and 1.7) with a sketch of proof were announced in [7].

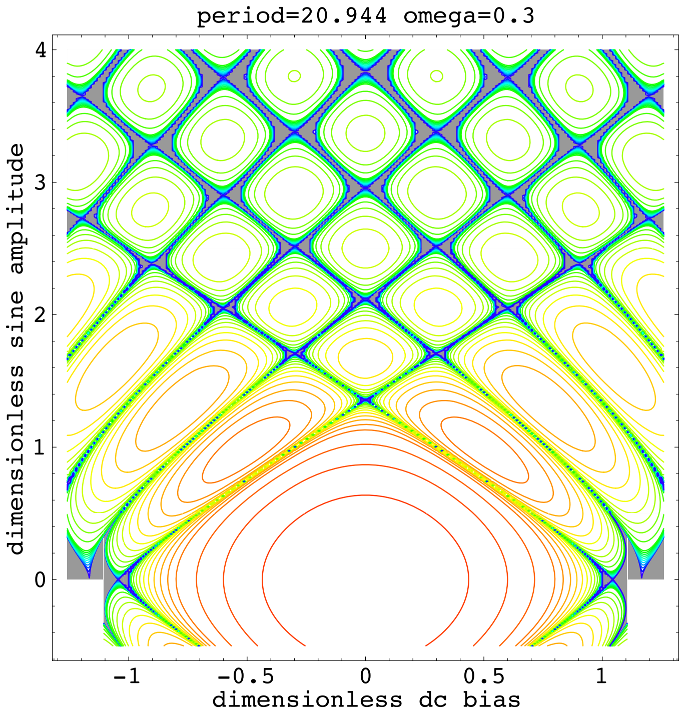

Figure 3. Phase-lock areas and their constrictions for $\omega = 0.3$. Reproduced from [13], with permission from Springer Nature.

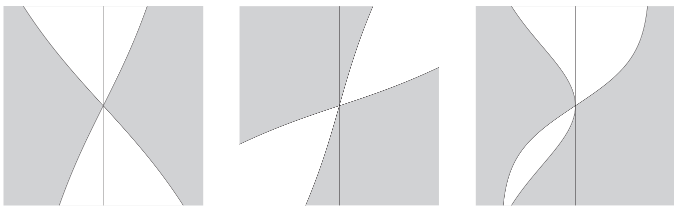

Figure 4. Positive, negative and neutral constrictions (figure made by Tertychnyi, included to the paper with his permission). It is proved that negative and neutral constrictions do not exist.

Remark 1.8. It was shown in [29, theorem 1.8] that each constriction is either positive, or negative: there are no neutral constrictions. Positivity of constrictions was stated there as [29, conjecture 1.13]. It was also shown in [29] that theorem 1.7 would imply theorem 1.4.

Definition 1.9. A *ghost constriction* is a constriction $(B, A;\ \omega)$ in model of Josephson junction for which either $\ell := \frac{B}{\omega}$ is different from the corresponding rotation number $\rho(B, A;\ \omega)$, or the constriction is not of positive type. (Note that $\ell \in \mathbb{Z}$ for each constriction, see remark 1.5.)

Theorems 1.4 and 1.7 taken together are equivalent to the following theorem.

Theorem 1.10. *There are no ghost constrictions in the model of overdamped Josephson junction.*

Proof of theorem 1.10 is sketched in the next subsection, where the plan of the paper is presented. It is based on the following characterization of constrictions.

Proposition 1.11 [28, *proposition 2.2*]. *Consider the period* 2π *flow map* $h^{2\pi}$ *of system* (1.4) *acting on the transversal coordinate* θ*-circle* $\{\tau = 0\}$. *A point* $(B, A;\ \omega)$ *is a constriction, if and only if* $\omega, A \neq 0$ *and* $h^{2\pi} = \mathrm{Id}$.

Some applications of theorems 1.7 and 1.4 and open problems will be discussed in section 6.

1.2. Method of proof of theorem 1.10. Plan of the paper

We prove theorem 1.10 in two steps given by the two following theorems. To state them, let us introduce the following notation. We set $\eta := \omega^{-1}$. For every fixed $\ell \in \mathbb{Z}$ we consider the set

$$\mathrm{Constr}_\ell = \{(\mu, \eta) \in \mathbb{R}^2_+ | (B, A; \omega) = (\ell\eta^{-1}, 2\mu\eta^{-1}; \eta^{-1})\ \text{is a constriction}\}.$$

Theorem 1.12. *For every* $\ell \in \mathbb{Z}$ *the subset* $\mathrm{Constr}_\ell \subset \mathbb{R}^2_+$ *is a regular one-dimensional analytic submanifold in* $\mathbb{R}^2_+$. *The restriction of the coordinate* η *to each its connected component is unbounded from above* (*i.e.,* ω *is unbounded from below*). *The rotation number and the type of constriction* (*positive or negative, see remark* 1.8) *are constant on each component.*

Theorem 1.13. *For every* $\ell \in \mathbb{Z}$ *there are no ghost constrictions in the axis* $\Lambda_\ell := \{B = \omega\ell\}$ *whenever* $\omega > 0$ *is small enough* (*dependently on* ℓ).

Theorem 1.13 will be proved in section 5 by methods of the theory of slow–fast families of dynamical systems. Theorem 1.10 immediately follows from theorems 1.12 and 1.13, see the subsection 5.6.

The proof of theorem 1.12 is sketched below. It is based on the following equivalent description of model of Josephson junction by a family of two-dimensional linear systems of differential equations on the Riemann sphere, see [15, 17, 20, 26, 34, 36], [12, subsection 3.2]. The variable change

$$z = e^{i\tau} = e^{i\omega t}, \qquad \Phi = e^{i\theta} = ie^{i\phi}$$

transforms equation (1.3) on the function $\theta(\tau)$ to the Riccati equation

$$\frac{d\Phi}{dz} = z^{-2}\Big((\ell z + \mu(z^2+1))\Phi + \frac{z}{2\omega}(\Phi^2+1)\Big). \tag{1.6}$$

Recall that $\ell = \frac{B}{\omega}$, $\mu = \frac{A}{2\omega}$, see (1.2). Equation (1.6) is the projectivization of the two-dimensional linear system

$$Y' = \left(\frac{\mathrm{diag}(-\mu, 0)}{z^2} + \frac{\mathcal{B}}{z} + \mathrm{diag}(-\mu, 0)\right)Y, \qquad \mathcal{B} = \begin{pmatrix} -\ell & -\dfrac{1}{2\omega} \\ \dfrac{1}{2\omega} & 0 \end{pmatrix}, \tag{1.7}$$

in the following sense: a function $\Phi(z)$ is a solution of (1.6), if and only if $\Phi(z) = \frac{v}{u}(z)$, where the vector function $Y(z) = (u(z), v(z))$ is a solution of system (1.7). For $\mu > 0$ system (1.7) has two irregular nonresonant singular points at 0 and at ∞. Its *monodromy operator* acts on the space $\mathbb{C}^2$ of germs of its solutions at a given point $z_0 \in \mathbb{C}^*$ by analytic extension along a counterclockwise circuit around zero.

Remark 1.14. The variable change $E(z) := \mathrm{e}^{\mu z} v(z)$ transforms the family of systems (1.7) to the following family of *special double confluent Heun equations,* see [19–23, 66]:

$$z^2 E'' + ((\ell+1)z + \mu(1-z^2))E' + (\lambda - \mu(\ell+1)z)E = 0, \quad \lambda := \frac{1}{4\omega^2} - \mu^2. \tag{1.8}$$

We will also deal with the so-called *conjugate* Heun equation obtained from (1.8) by change of sign at ℓ:

$$z^2 E'' + ((-\ell+1)z + \mu(1-z^2))E' + (\lambda + \mu(\ell-1)z)E = 0. \tag{1.9}$$

Using this relation to well-known class of Heun equations a series of results on phase-lock area portrait of model of Josephson junction and related problems were obtained in [12, 13, 19–23, 66]. See also a brief survey in the next subsection.

Recall that an *isomonodromic family* of linear systems is a family in which the collection of residue matrices of formal normal forms at singular points, Stokes matrices and transition matrices between canonical solution bases at different singular points remain constant (up to appropriate conjugacies).

It is known that $(B, A;\ \omega)$ is a constriction, if and only if $A, \omega \neq 0$ and system (1.7) has trivial monodromy; then $\ell = \frac{B}{\omega} \in \mathbb{Z}$. See [28, proposition 3.2, lemma 3.3] and proposition 4.1 in section 4.

We denote by Jos the three-dimensional family of systems (1.7), which will be referred to, as *systems of Josephson type.* For the proof of theorem 1.12 we study their isomonodromic deformations in the four-dimensional space $\mathbf{J}^N(\mathbb{R}_+)$ of linear systems of the so-called *normalized* $\mathbb{R}_+$*-Jimbo type*

$$Y' = \left(-\tau\frac{K}{z^2} + \frac{R}{z} + \tau N\right)Y, \quad \tau \in \mathbb{R}_+,\ K, R, N \text{ are real } 2\times 2 - \text{matrices}, \tag{1.10}$$

$$N = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} -\ell & -R_{21} \\ R_{21} & 0 \end{pmatrix}, \qquad K = -GNG^{-1}, \quad R_{21} > 0,\ \ell \in \mathbb{R}, \tag{1.11}$$

where $G \in \mathrm{SL}_2(\mathbb{R})$ is a matrix such that

$$G^{-1}RG = \begin{pmatrix} -\ell & * \\ * & 0 \end{pmatrix}; \tag{1.12}$$

here the matrix elements * may be arbitrary.

Step 1. We study the real one-dimensional analytic foliation of the space $\mathbf{J}^N(\mathbb{R}_+)$ by isomonodromic families of linear systems. These isomonodromic families are given by differential equation (3.17). They are obtained (by gauge transformations and rescaling of the variable z) from well-known Jimbo isomonodromic deformations [38], which are given by real solutions of Painlevé 3 equation (P3). Namely, the function $w(\tau) = -\frac{R_{12}(\tau)}{\tau K_{12}(\tau)} = \frac{R_{21}(\tau)}{\tau K_{12}(\tau)}$ should satisfy the P3 equation

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + w^3 - 2\ell\frac{w^2}{\tau} - \frac{1}{w} + (2\ell - 2)\frac{1}{\tau} \tag{1.13}$$

along the isomonodromic leaves. We show that the hypersurface Jos $\subset \mathbf{J}^N(\mathbb{R}_+)$ corresponds to poles of order 1 with residue 1 of solutions of (1.13). This implies that Jos is *transversal* to the isomonodromic foliation. This is the key lemma in the proof. The only role of the Painlevé 3 equation in the proof is the above transversality statement. Relation to Painlevé 3 equation led us to a series of new open problems presented in section 6.

Step 2. We consider the subset $\Sigma \subset \mathbf{J}^N(\mathbb{R}_+)$ of systems (1.10) with trivial monodromy. We show that $\ell \in \mathbb{Z}$ for these systems, and their germs at 0 and at ∞ are analytically equivalent to their diagonal formal normal forms. Using this fact, we show that Σ is a real two-dimensional analytic submanifold in $\mathbf{J}^N(\mathbb{R}_+)$ with the following properties:

(a) Σ is a union of leaves of the isomonodromic foliation;

(b) (The key theorem in the proof) there exists a *submersive* projection $\mathcal{R} : \Sigma \to \mathbb{R}_x$ given by an analytic invariant $\mathcal{R}$ of linear systems, the so-called *transition cross-ratio*, that is constant along the leaves.

Statement (a) follows from definition. Statement (b) is proved by showing that (x, τ), $x = \mathcal{R}$, form local analytic coordinates on Σ. Fix an $\ell \in \mathbb{Z}$, and let $\Sigma_\ell \subset \Sigma$ denote the subset of systems with the given value of ℓ. For every pair (x_0, τ_0) corresponding to a system from Σ_ℓ realization of any pair (x, τ) close to (x_0, τ_0) by a system from Σ_ℓ can be viewed as a solution of the Riemann–Hilbert type problem. It is proved via holomorphic vector bundle argument, as in famous works by Bolibruch on the Riemann–Hilbert problem and related topics: see [9–11] and references therein. We glue a holomorphic vector bundle with connection on $\overline{\mathbb{C}}$ realizing given (x, τ) from the two trivial bundles: one over the disk $D_2 \subset \mathbb{C}$, and the other one on the complement of the closed disk $\overline{D}_{\frac{1}{2}} \subset \overline{\mathbb{C}}$. The connections on the latter trivial bundles are given by the diagonal normal forms prescribed by ℓ and τ. The gluing matrix, which is holomorphic on the annulus $D_2 \setminus \overline{D}_{\frac{1}{2}}$, depends analytically on (x, τ). The bundle thus obtained is trivial for (x_0, τ_0) (by definition). It remains trivial for all (x, τ) close enough to (x_0, τ_0). This follows from the classical theorem stating that *a holomorphic vector bundle close to a trivial one is also trivial* [11, appendix 3, lemma 1, theorem 2], [56, theorem 2.3], [31]. The connection on the trivial bundle thus obtained is given by a meromorphic system with order two poles at 0 and at ∞ and the same normal forms. Its gauge equivalence to a normalized $\mathbb{R}_+$-Jimbo type system (1.10), (1.11) is proved by a symmetry argument.

The submanifold Jos is transversal to Σ, by the result of step 1 and statement (a). Therefore, the intersection Jos $\cap\, \Sigma_\ell$ is a real one-dimensional submanifold in $\mathbf{J}^N(\mathbb{R}_+)$. It is transversal to the isomonodromic foliation of Σ_ℓ (step 1), and hence, is locally diffeomorphically projected to an open subset in $\mathbb{R}$ by the mapping $\mathcal{R}$. The above intersection is identified with Constr$_\ell$. This implies that Constr$_\ell$ is a one-dimensional submanifold; each its connected component is analytically parametrized by an interval $I = (a, b)$ of values of the parameter $x = \mathcal{R}$ and hence, is non-compact.

Step 3. We show that the coordinate $\eta = \omega^{-1}$ is unbounded from above on each component $\mathcal{C}$ in Constr$_\ell$. Assuming the contrary, i.e., that η is bounded on $\mathcal{C}$, we have that for every $c \in \{a, b\}$ at least one of the functions $\mu^{\pm 1}$, η^{-1} (depending on the choice of c) should be unbounded, as $x \to c$. Boundedness of μ is proved by using Klimenko–Romaskevich Bessel asymptotics of boundaries of the phase-lock areas [44]. For $c \neq 0$ we prove boundedness of the functions μ^{-1}, η^{-1}, as $x \to c$, by studying accumulation points of the set Constr_ℓ in the union of coordinate axes $\{\eta = 0\} \cup \{\mu = 0\}$.

Afterwards, to finish the proof of theorem 1.12, it remains to show that the rotation number and type of constriction are constant on each connected component in Constr_ℓ. We deduce constance of type from the fact that no constriction can be a limit of the so-called *generalized simple intersections:* those points of intersections $\Lambda_\ell \cap \partial L_r$, $r \equiv \ell (\mathrm{mod}\, 2)$, that are not constrictions and do not lie in the abscissa axis. This, in its turn, is implied by the two following facts:

- The generalized simple intersections correspond to Heun equations (1.9) having a polynomial solution [13, theorem 1.15]; this remains valid for their limits with $A \neq 0$;
- No constriction can correspond to a Heun equation (1.9) with polynomial solution [12, theorems 3.3, 3.10].

1.3. Historical remarks

Model (1.1) of overdamped Josephson junction was studied by Buchstaber, Karpov, Tertychnyi and other mathematicians and physicists, see [12–23, 28–30, 34, 36, 43–45, 65, 66] and references therein. Hereby we present a brief survey of results that were not mentioned in the introduction. Recall that the rotation number quantization effect for a family of dynamical systems on $\mathbb{T}^2$ containing (1.4) was discovered in [17]. Bizyaev, Borisov, and Mamaev noticed that a big family of dynamical systems on torus in which the rotation number quantization effect realizes was introduced by Hess (1890). It appears that in classical mechanics such systems were studied in problems on rigid body movement with fixed point in works by Hess, Nekrassov, Lyapunov, Mlodzejewski, Zhukovsky and others. See [8, 51, 54, 55, 67] and references therein. Nekrasov observed in [55] that the above-mentioned big family of systems considered by Hess can be equivalently described by a Riccati equation (or by a linear second order differential equation).

Transversal regularity of the fibration by level sets $\rho(B, A) = \mathrm{const} \notin \mathbb{Z}$ with fixed ω on the complement to the union of the phase-lock areas was proved in [13, proposition 5.3]. Conjectures on alignment and positivity of constrictions (now theorems 1.4 and 1.7 respectively) were stated respectively in [28, 29] and studied respectively in [28, 29], where some partial results were obtained. Theorem 1.4 for $\omega \geqslant 1$ was proved in [28]. For further survey on these conjectures see [13, 28, 29] and references therein. A conjecture saying that the semiaxis $\Lambda_\ell^+ := \Lambda_\ell \cap \{A > 0\}$ intersects the corresponding phase-lock area L_ℓ by a ray explicitly constructed in [29] was stated in [29, conjecture 1.14]. It was shown in [29, theorem 1.12] that the ray in question indeed lies in L_ℓ. An equivalent description of model (1.1) in terms of a family of special double confluent Heun equations (1.8) was found by Tertychnyi in [66] and further studied in a series of joint papers by Buchstaber and Tertychnyi [19–23]. They have shown that the constrictions are exactly those parameter values $(B, A;\ \omega)$ for which the corresponding double confluent Heun equation (1.8) has an entire solution: holomorphic on $\mathbb{C}$ [20]. Using this observation they stated a conjecture describing ordinates of the constrictions lying in a given axis Λ_ℓ as zeros of a known analytic function constructed via an infinite matrix product [20]. This conjecture was studied in [20, 21] and reduced to the conjecture stating that if the Heun equation (1.8) has an entire solution, then the conjugate Heun equation (1.9) cannot have polynomial solution. Both conjectures were proved in [12]. New automorphisms of solution space of Heun equations (1.8) were discovered and studied in [22, 23].

In [19] Buchstaber and Tertychnyi described those $(B, A;\ \omega)$, for which conjugate Heun equation (1.9) has a polynomial solution. Namely, for a given $\ell = \frac{B}{\omega} \in \mathbb{N}$ their set is a remarkable algebraic curve, the so-called spectral curve (studied in [19, 30]): zero locus of determinant of appropriate three-diagonal matrix with entries being linear non-homogeneous functions in the coefficients of equation (1.9). The fact that those points $(B, A;\ \omega)$ for which (1.9) has a polynomial solution are exactly the generalized simple intersections is a result of papers [13, 19], stated and proved in [13].

There exists an antiquantization procedure that associates Painlevé equations to Heun equations; double confluent Heun equations correspond to Painlevé 3 equations. See [57, 61, 62] and references therein.

V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi, D.A.Filimonov, V.A.Kleptsyn, I.V.Schurov made numerical experiences that have shown that as $\omega \to 0$, the 'upper' part of the phase-lock area portrait converges to a kind of parquet in the renormalized coordinates $(\ell,\ \mu)$: the renormalized phase-lock areas tend to unions of pieces of parquet, and gaps between the phase-lock areas tend to zero. See figure 3 and the paper [43]. This is an open problem. In [43] Kleptsyn, Romaskevich, and Schurov proved some results on smallness of gaps and their rate of convergence to zero, as $\omega \to 0$, using methods of slow-fast systems.

A subfamily of family (1.3) of dynamical systems on two-torus was studied by Guckenheimer and Ilyashenko in [35] from the slow-fast system point of view. They obtained results on its limit cycles, as $\omega \to 0$.

An analogue of the rotation number integer quantization effect in braid groups was discovered by Malyutin [52].

2. Preliminaries: irregular singularities, normal forms, Stokes matrices and monodromy–Stokes data of linear systems

2.1. Normal forms, canonical solutions and Stokes matrices

All the results presented in this subsection are particular cases of classical results contained in [3–5, 37, 42, 60].

Recall that two germs of meromorphic linear systems of differential equations on a n-dimensional vector function $Y = Y(z)$ at a singular point (pole), say, 0 are *analytically equivalent,* if there exists a holomorphic $\mathrm{GL}_n(\mathbb{C})$-valued function $H(z)$ on a neighbourhood of 0 such that the Y-variable change $Y = H(z)\widetilde{Y}$ sends one system to the other one. Two systems are *formally* equivalent, if the above is true for a formal power series $\widehat{H}(z)$ with matrix coefficients that has an invertible free term.

Consider a two-dimensional linear system

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + O(1)\right)Y, \quad Y = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.1}$$

on a neighbourhood of 0; here the matrix K has distinct eigenvalues $\lambda_1 \neq \lambda_2$, and $O(1)$ is a holomorphic matrix-valued function on a neighbourhood of 0. Then we say that the singular point 0 of system (2.1) is *irregular non-resonant of Poincaré rank 1*. Then K is conjugate to $\widetilde{K} = \mathrm{diag}(\lambda_1, \lambda_2)$, $\widetilde{K} = \mathbf{H}^{-1}K\mathbf{H}$, $\mathbf{H} \in \mathrm{GL}_2(\mathbb{C})$, and one can achieve that $K = \widetilde{K}$ by applying the constant linear change (gauge transformation) $Y = \mathbf{H}\widehat{\mathbf{Y}}$. System (2.1) is formally equivalent to a unique *formal normal form*

$$\widetilde{Y}' = \left(\frac{\widetilde{K}}{z^2} + \frac{\widetilde{R}}{z}\right)\widetilde{Y}, \quad \widetilde{K} = \mathrm{diag}(\lambda_1, \lambda_2), \ \widetilde{R} = \mathrm{diag}(b_1, b_2), \tag{2.2}$$

$$\widetilde{R} \text{ is the diagonal part of the matrix } \mathbf{H}^{-1}R\mathbf{H}. \tag{2.3}$$

The matrix coefficient K in system (2.1) and the corresponding matrix $\widetilde{K}$ in (2.2) are called the *main term matrices*, and R, $\widetilde{R}$ the *residue matrices*.

Generically, the normalizing series $\widehat{H}(z)$ bringing (2.1) to (2.2) diverges. At the same time, there exists a covering of a punctured neighbourhood of zero by two sectors S_0 and S_1 with vertex at 0 in which there exist holomorphic matrix functions $H_j(z)$, $j = 0, 1$, that are C^∞ smooth on $\overline{S}_j \cap D_r$ for some $r > 0$, and such that the variable changes $Y = H_j(z)\widetilde{Y}$ transform (2.1) to (2.2). This sectorial normalization theorem holds for the so-called *good sectors* (or *Stokes sectors*.) Namely, consider the rays issued from 0 and forming the set $\{\mathrm{Re}\,\frac{\lambda_1-\lambda_2}{z} = 0\}$. They are called *imaginary dividing rays* (or *Stokes rays*). A sector S_j is *good,* if it contains one imaginary dividing ray and its closure does not contain the other one.

Let $W(z) = \mathrm{diag}(\widetilde{Y}_1(z), \widetilde{Y}_2(z))$ denote the canonical diagonal fundamental matrix solution of the formal normal form (2.2); here $\widetilde{Y}_\ell(z)$ are solutions of its one-dimensional equations. The matrices $X^j(z) := H_j(z)W(z)$ are fundamental matrix solutions of the initial equation (2.1) defining solution bases in S_j called the *canonical sectorial solution bases.* In their definition we choose the branches $W(z) = W^j(z)$ of the (*a priori* multivalued) matrix function $W(z)$ in S_j, $j = 0, 1$, so that $W^1(z)$ is obtained from $W^0(z)$ by counterclockwise analytic extension from S_0 to S_1. And in the same way we define yet another branch $W^2(z)$ of $W(z)$ in $S_2 := S_0$ that is obtained from $W^1(z)$ by counterclockwise analytic extension from S_1 to S_0. This yields another canonical matrix solution $X^2 := H_0(z)W^2(z)$ in S_0, which is obtained from $X^0(z)$ by multiplication from the right by the monodromy matrix $\exp(2\pi \mathrm{i}\widetilde{R})$ of the formal normal form (2.2). Let $S_{j,j+1}$ denote the connected component of intersection $S_{j+1} \cap S_j$, $j = 0, 1$, that is crossed when one moves from S_j to S_{j+1} counterclockwise, see figure 5. The transition matrices C_0, C_1 between thus defined canonical solution bases X^j,

$$X^1(z) = X^0(z)C_0 \quad \text{on} \ \ S_{0,1}; \qquad X^2(z) = X^1(z)C_1 \quad \text{on} \ \ S_{1,2} \tag{2.4}$$

are called the *Stokes matrices.*

Example 2.1. Let $A = \mathrm{diag}(\lambda_1, \lambda_2)$, and let $\lambda_2 - \lambda_1 \in \mathbb{R}$. Then the imaginary dividing rays are the positive and negative imaginary semiaxes. The good sectors S_0 and S_1 covering $\mathbb{C}^*$ satisfy the following conditions:

- The sector S_0 contains the positive imaginary semiaxis, and its closure does not contain the negative one;
- The sector S_1 satisfies the opposite condition. See figure 5.

Example 2.2. Let us numerate the sectors S_0, S_1 and the eigenvalues λ_1, λ_2 so that

$$\overline{S}_{0,1} \setminus \{0\} \subset \left\{\mathrm{Re}\left(\frac{\lambda_1 - \lambda_2}{z}\right) > 0\right\}, \qquad \overline{S}_{1,2} \setminus \{0\} \subset \left\{\mathrm{Re}\left(\frac{\lambda_1 - \lambda_2}{z}\right) < 0\right\}. \tag{2.5}$$

This holds, e.g., in the conditions of the above example, if $\lambda_2 - \lambda_1 > 0$. The canonical solutions of the formal normal form (2.2) are given by the solutions $c_k z^{b_k} \mathrm{e}^{-\frac{\lambda_k}{z}}$ of one-dimensional equations in (2.2). They are numerated by indices $k = 1, 2$ of the eigenvalues λ_k of the main term matrix K. The corresponding solutions of the initial system (2.1) in S_j, $j = 0, 1, 2$, i.e., the columns of the fundamental matrix $X^j(z)$, are also numerated by the same index k and will

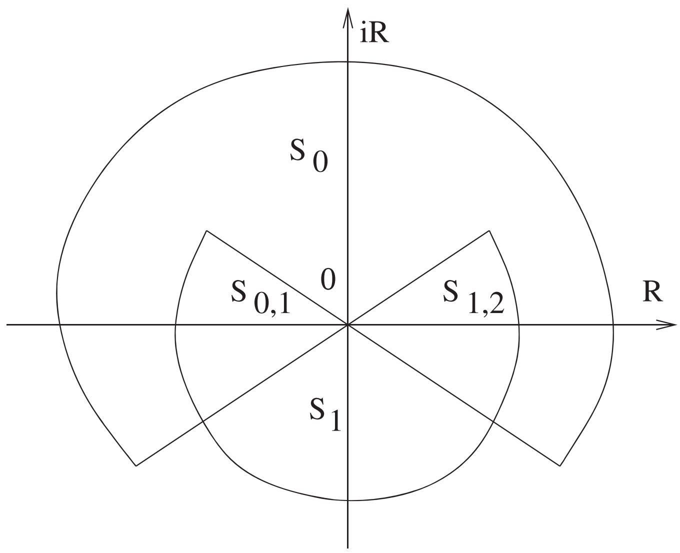

Figure 5. Good sectors in the case, when $\lambda_1 - \lambda_2 \in \mathbb{R}$.

be denoted by $f_{kj}(z)$. The norm $\|f_{1j}(z)\|$ is asymptotically dominated by $\|f_{2j}(z)\|$ in $S_{0,1}$, as $z \to 0$, and the converse asymptotic domination statement holds on $S_{1,2}$. This implies (and it is well-known) that $f_{10} \equiv f_{11}$ on $S_{0,1}$ and $f_{22} \equiv f_{21}$ on $S_{1,2}$. The Stokes matrices C_0 and C_1 are unipotent: C_0 is upper-triangular and C_1 is lower-triangular. If the numeration of either eigenvalues, or sectors (but not both) is opposite, or if the singular point under question is ∞, not zero (see remark 2.5 below), then the Stokes matrices are unipotent but of opposite triangular type.

Remark 2.3. The tautological projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 = \overline{\mathbb{C}}$ sends canonical sectorial basic solutions $f_{kj}(z)$ of system (2.1) to canonical sectorial solutions $q_{kj}(z)$ of its projectivization: the corresponding Riccati equation. These are the unique $\overline{\mathbb{C}}$-valued holomorphic solutions of the Riccati equation in the sector S_j that extend C^∞-smoothly to $\overline{S}_j \cap D_r$ for a sufficiently small $r > 0$. Their values at 0 are the projections of the eigenlines of the main term matrix K with eigenvalues λ_k.

Theorem 2.4 [3–5, 37, 42, 60]. *A germ of linear system at an irregular nonresonant singular point is analytically equivalent to its formal normal form, if and only if it has trivial Stokes matrices. Two germs of linear systems as above are analytically equivalent, if and only if their formal normal forms are the same and their Stokes matrix collections are equivalent in the following sense: they are simultaneously conjugated by one and the same diagonal matrix (independent on the choice of sector $S_{j,j+1}$).*

Recall that the *monodromy operator* of a germ of linear system at 0 acts on the space of germs of its solutions at a point $z_0 \neq 0$ sending a local solution to the result of its counterclockwise analytic extension along a circuit around the origin. Let the origin be an irregular nonresonant singular point of Poincaré rank 1, and let S_0, S_1 be the corresponding good sectors. Let M be the monodromy matrix written in the canonical sectorial basis of solutions in S_0. Let M_{norm} denote the diagonal monodromy matrix of the formal normal form in the canonical solution basis with diagonal fundamental matrix. We will call M_{norm} the *formal monodromy.*

Recall that $M_{\text{norm}} = \exp(2\pi i\widetilde{R})$. The matrix M is expressed in terms of the formal monodromy M_{norm} and the Stokes matrices C_0 and C_1 via the following well-known formula [37, p 35]:

$$M = M_{\text{norm}} C_1^{-1} C_0^{-1}. \tag{2.6}$$

Remark 2.5. We will also deal with the case, when the singular point under question is ∞, and the above statements hold in the local coordinate $\widetilde{z} := \frac{1}{z}$. In the coordinate z the corresponding equation and formal normal form take the form

$$Y' = \left(K + \frac{R}{z} + O\left(\frac{1}{z^2}\right)\right)Y, \qquad \widetilde{Y}' = \left(\widetilde{K} + \frac{\widetilde{R}}{z}\right)\widetilde{Y}.$$

The matrices $K, \widetilde{K}$ are called the *main term matrices,* and $R, \widetilde{R}$ the *residue matrices* of the corresponding systems at ∞. Let λ_1, λ_2 be the eigenvalues of the matrix K. An *imaginary dividing ray at infinity* is a ray issued from 0 and lying in the set $\{\operatorname{Re}(\lambda_1 - \lambda_2)z = 0\}$. This yields the definition of good sectors 'at infinity'. The sectorial normalization and analytic classification theorems and the definition of Stokes matrices at infinity are stated in the same way, as above; the sectors $S_0, S_{0,1}, S_1, S_{1,2}$ at infinity are also numerated counterclockwise. Formula (2.6) also holds at ∞.

2.2. Systems with two irregular singularities. Monodromy–Stokes data

Definition 2.6. By $\mathcal{H}^1_{0,\infty}$ we will denote the class of linear systems on the Riemann sphere having two singular points, at zero and at infinity, such that both of them are irregular nonresonant of Poincaré rank 1. Each system from the class $\mathcal{H}^1_{0,\infty}$ has the type

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N\right)Y, \quad K, R, N \in \operatorname{End}(\mathbb{C}^2), \tag{2.7}$$

where each one of the main term matrices K and N at zero and at ∞ has distinct eigenvalues.

Definition 2.7. Consider a linear system $\mathcal{L} \in \mathcal{H}^1_{0,\infty}$. Fix a point $z_0 \in \mathbb{C}^*$ and two pairs of good sectors (S_0^0, S_1^0), (S_0^∞, S_1^∞) for the main term matrices at 0 and ∞ respectively, see remark 2.5. Fix two paths α_p in $\mathbb{C}^*$ numerated by $p = 0, \infty$, going from the point z_0 to a point in S_0^p. Let f_{1p}, f_{2p} be a canonical sectorial solution basis for the system $\mathcal{L}$ at p in S_0^p. Consider the analytic extensions of the basic functions f_{kp} to the point z_0 along paths α_p^{-1}. Let $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ denote the tautological projection. Set $\Phi := \frac{Y_2}{Y_1}$,

$$q_{kp} := \pi(f_{kp}(z_0)) \in \mathbb{CP}^1 = \overline{\mathbb{C}}_\Phi. \tag{2.8}$$

Let M denote the monodromy operator of the system $\mathcal{L}$ acting on the local solution space at z_0 (identified with the space $\mathbb{C}^2$ of initial conditions at z_0) by analytic extension along counterclockwise circuit around zero. The tuple

$$(q, M) := (q_{10}, q_{20}, q_{1\infty}, q_{2\infty}; M) \tag{2.9}$$

taken up to the next equivalence is called the *monodromy–Stokes data* of the system $\mathcal{L}$. Namely, two tuples $(q, M), (q', M') \in (\mathbb{CP}^1)^4 \times \mathrm{GL}_2(\mathbb{C})$ are called *equivalent*[7], if there exists a linear

[7] Here is an equivalent group-action definition. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\overline{\mathbb{C}}^4 \times \mathrm{GL}_2(\mathbb{C})$ by action $h : q_{kp} \mapsto hq_{kp}$ on points in $\overline{\mathbb{C}}$ and conjugation $M \mapsto hMh^{-1}$ on matrices. The monodromy–Stokes data is the $\mathrm{PSL}_2(\mathbb{C})$-orbit of a collection (q, M) under this action.

operator $H \in \mathrm{GL}_2(\mathbb{C})$ whose projectivization sends q_{kp} to q'_{kp} and such that $H^{-1} \circ M' \circ H = M$. We will also deal with the *transition matrix* Q comparing the canonical bases at 0 and at ∞ at z_0: $(f_{1\infty}, f_{2\infty}) = (f_{10}, f_{20})Q$.

Remark 2.8. The monodromy–Stokes data of a system $\mathcal{L}$ depends only on the homotopy class of the pair of paths $(\alpha_0, \alpha_\infty)$ in the space of pairs of paths in $\mathbb{C}^*$ with a common (variable) starting point z_0 and with endpoints lying in given sectors S_0^0 and S_0^∞ respectively. Indeed, let a homotopy between two pairs of paths, $(\alpha_0, \alpha_\infty)$ with base point z_0 and $(\alpha'_0, \alpha'_\infty)$ with base point z'_0, move z_0 to z'_0 along a path β in $\mathbb{C}^*$. Let $X(z)$ be the germ of fundamental matrix of the system $\mathcal{L}$ at z_0 such that $X(z_0) = \mathrm{Id}$. Let $H = X(z'_0)$ denote the value at z'_0 of the analytic extension of the fundamental matrix function $X(z)$ along the path β. Then H transforms the monodromy–Stokes data corresponding to z_0 and the path pair $(\alpha_0, \alpha_\infty)$ to that corresponding to z'_0 and the path pair $(\beta^{-1}\alpha_0, \beta^{-1}\alpha_\infty)$, as at the end of the above definition.

Proposition 2.9. *One has $q_{1p} \neq q_{2p}$ for every $p = 0, \infty$. The monodromy–Stokes data of a system $\mathcal{L} \in \mathcal{H}^1_{0,\infty}$ determines the collection of formal monodromies $M_{\mathrm{norm},p}$, the Stokes matrices C_{jp} at $p = 0, \infty$, $j = 0, 1$, and the transition matrix Q uniquely up to the following equivalence. Two collections $(M_{\mathrm{norm},p}, C_{jp}, Q)$ and $(M'_{\mathrm{norm},p}, C'_{jp}, Q')$ are* ***equivalent****, if $M_{\mathrm{norm},p} = M'_{\mathrm{norm},p}$ and there exists a pair of diagonal matrices D_0, D_∞ such that $C'_{jp} = D_p C_{jp} D_p^{-1}$ for all j, p, and $Q' = D_0 \circ Q \circ D_\infty^{-1}$.*

Proof. The inequality $q_{1p} \neq q_{2p}$ follows from linear independence of the basic functions f_{1p}, f_{2p}, which implies independence of their values at z_0. A given pair of distinct points $q_{1p}, q_{2p} \in \mathbb{CP}^1$ defines a basis (v_{1p}, v_{2p}) in $\mathbb{C}^2$ (whose vectors are projected to q_{jp}) uniquely up to multiplication of vectors by constants. Recall that in the basis (f_{1p}, f_{2p}) of the local solution space at z_0 with $f_{kp}(z_0) = v_{kp}$ the monodromy matrix is given by formula (2.6):

$$M = M_{\mathrm{norm},p} C_{1p}^{-1} C_{0p}^{-1}. \tag{2.10}$$

Here the Stokes matrices C_{0p}, C_{1p} are unipotent of opposite triangular types (determined by the main term matrix of the system $\mathcal{L}$ at p). Let they be, say, upper and lower triangular respectively with the corresponding triangular elements c_0 and c_1. Recall that the formal monodromy matrix $M_{\mathrm{norm},p}$ is diagonal, set $M_{\mathrm{norm},p} = \mathrm{diag}(m_{1p}, m_{2p})$; $m_{jp} \neq 0$. Then

$$m_{1p} = M_{11}, \qquad c_0 = -M_{12}M_{11}^{-1}, \qquad m_{2p}c_1 = -M_{21}, \tag{2.11}$$

$$m_{2p} = M_{22} - m_{2p}c_0c_1 = M_{22} - M_{12}M_{21}M_{11}^{-1}, \quad c_1 = -M_{21}m_{2p}^{-1}, \tag{2.12}$$

by (2.10). This yields expression for the formal monodromy $M_{\mathrm{norm},p}$ and the Stokes matrices in terms of M. All the latter matrices depend on choice of the basic functions f_{kp}, which are uniquely defined by q_{jp} up to multiplication by constant factors. These rescalings replace $M_{\mathrm{norm},p}$ and C_{jp} by their conjugates by a diagonal matrix D_p, and Q by $D_0QD_\infty^{-1}$. This does not change the diagonal matrix $M_{\mathrm{norm},p}$. The proposition is proved. □

Remark 2.10. Recall that two global linear systems on the Riemann sphere are globally analytically (gauge) equivalent, if and only if they are sent one to the other by constant linear change $Y \mapsto HY$, $H \in \mathrm{GL}_2(\mathbb{C})$ (i.e., constant gauge equivalent). For simplicity everywhere below whenever we work with global systems on $\overline{\mathbb{C}}$ we omit the word 'analytically' ('constant'), and '*gauge equivalence*' means 'constant gauge *equivalence*'.

Theorem 2.11. *Two systems $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{H}^1_{0,\infty}$ are gauge equivalent, if and only if they have the same formal normal forms at each singular point and the same monodromy–Stokes data. In this case each linear automorphism of the fibre $\{z = z_0\} \simeq \mathbb{C}^2$ sending the monodromy–Stokes data of one system to that of the other system extends to a gauge equivalence of systems. Here both monodromy–Stokes data correspond to the same sectors and path collections.*

Proof. The statement of the theorem holds if one replaces the monodromy–Stokes data by collection of Stokes matrices and the transition matrix up to equivalence from the above proposition, see [39, proposition 2.5, p 319]. The collection of Stokes and transition matrices (taken up to the latter equivalence) is uniquely determined by the monodromy–Stokes data, by the same proposition. Conversely, the monodromy–Stokes data can be restored from the formal monodromy and Stokes and transition matrices. Namely, the monodromy matrix M in the basis (f_{10}, f_{20}) is found from (2.10). Let us choose coordinates on $\mathbb{C}^2$ in which $f_{10}(z_0) = (1, 0)$, $f_{20}(z_0) = (0, 1)$. Then one has $q_{10} = (1 : 0)$, $q_{20} = (0 : 1)$, and $q_{1\infty}$, $q_{2\infty}$ are the projections of the columns of the transition matrix Q. Theorem 2.11 is proved. □

3. Isomonodromic deformations and Painlevé 3 equation

Here we introduce general Jimbo's isomonodromic deformations of linear systems in $\mathcal{H}^1_{0,\infty}$, which form a one-dimensional holomorphic foliation of the space $\mathcal{H}^1_{0,\infty}$ (subsection 3.2). Afterwards we study its restriction to the so-called Jimbo type systems, where isomonodromic deformations are described by solutions of Painlevé 3 equation (3.14) (subsection 3.3). In subsection 3.4 we consider the space of real Jimbo type systems (i.e., defined by real matrices) with $R_{21} > 0 > R_{12}$. We introduce the space $\mathbf{J}^N(\mathbb{R}_+)$ of their appropriate normalizations with $R_{21} = -R_{12} > 0$ by gauge transformations and variable rescalings: the so-called normalized $\mathbb{R}_+$-Jimbo type systems. Their space $\mathbf{J}^N(\mathbb{R}_+)$ contains the space Jos of systems (1.7) and is foliated by isomonodromic families obtained from Jimbo deformations by normalizations. We show that the family Jos is transversal to the isomonodromic foliation of $\mathbf{J}^N(\mathbb{R}_+)$, and it corresponds to poles of order 1 with residue 1 of solutions of Painlevé equations (3.14) (subsection 3.5). A background material on isomonodromic deformations is recalled in subsection 3.1.

3.1. Isomonodromic deformations: definition and Frobenius integrability sufficient condition

Let us give the following definition of isomonodromic family of linear systems in $\mathcal{H}^1_{0,\infty}$, which is equivalent to the classical definition, by proposition 2.9.

Definition 3.1. A family of systems in $\mathcal{H}^1_{0,\infty}$ is *isomonodromic,* if the residue matrices of formal normal forms at their singular points and the monodromy–Stokes data remain constant: independent on the parameter of the family.

Remark 3.2. If a family of systems in question is continuously parametrized by a connected parameter space, then constance of the monodromy–Stokes data automatically implies constance of the residue matrices $\widetilde{R}_p$ of the formal normal forms. Indeed, constance of formal monodromies $M_{\mathrm{norm},p}$ follows by proposition 2.9. The formula $M_{\mathrm{norm},p} = \exp(2\pi i\widetilde{R}_p)$ implies that the residue matrices $\widetilde{R}_p$ are uniquely determined by $M_{\mathrm{norm},p}$ up to addition of integer diagonal matrices. Hence, they are constant, by continuity and connectivity.

Theorem 3.3 [39], [25, *theorem 4.1*]. *A holomorphic family of linear systems in $\mathcal{H}^1_{0,\infty}$ depending on a parameter t from a simply connected domain $\mathcal{D} \subset \mathbb{C}$,*

$$Y' = \frac{\mathrm{d}Y}{\mathrm{d}z} = \left(\frac{K_2(t)}{z^2} + \frac{K_1(t)}{z} + K_0(t)\right)Y \tag{3.1}$$

is isomonodromic if there is a rational in z (with possible poles only at $z = 0, \infty$) and analytic in t matrix differential one-form $\Omega = \Omega(z,t)$ on $\overline{\mathbb{C}} \times \mathcal{D}$ such that

$$\Omega|_{\text{fixed } t} = \left(\frac{K_2(t)}{z^2} + \frac{K_1(t)}{z} + K_0(t)\right)\mathrm{d}z, \tag{3.2}$$

$$\mathrm{d}\Omega = \Omega \wedge \Omega. \tag{3.3}$$

Condition (3.3) means that Ω is integrable in the Frobenius sense. See, e.g., [11, proof of theorem 13.2].

An isomonodromic deformation with a scalar parameter is often defined by a system of PDEs [25]

$$\begin{cases} \dfrac{\partial Y}{\partial z} = U(z,t)Y \\ \dfrac{\partial Y}{\partial t} = V(z,t)Y, \end{cases} \tag{3.4}$$

where $U(z,t)$, $V(z,t)$ are rational in $z \in \overline{\mathbb{C}}$ and analytic in $t \in \mathcal{D}$. In that case, one can take $\Omega = U(z,t)\mathrm{d}z + V(z,t)\mathrm{d}t$. Then condition (3.2) of theorem 3.3 is satisfied if

$$U(z,t) = \frac{K_2(t)}{z^2} + \frac{K_1(t)}{z} + K_0(t).$$

Condition (3.3) is equivalent to the equation

$$[U,V] = UV - VU = \frac{\partial V}{\partial z} - \frac{\partial U}{\partial t}. \tag{3.5}$$

3.2. General Jimbo's isomonodromic deformation

In this section, we consider an isomonodromic deformation introduced by Jimbo in [38, p 1156, (3.11)] and describe its integrability condition (3.3). The deformation space will be a simply connected domain $\mathcal{D} \subset \mathbb{C}^*$ containing $\mathbb{R}_+$. Though the deformation in [38] was written in a seemingly special case, it works in the following general case. We are looking for isomonodromic families of systems $\mathcal{L}(t) \in \mathcal{H}^1_{0,\infty}$ given by system (3.4) of the following type:

$$\begin{cases} \dfrac{\partial Y}{\partial z} = \left(-\dfrac{\widetilde{K}(t)}{z^2} + \dfrac{R(t)}{z} + N(t)\right)Y := \mathcal{L}(t) \\ \dfrac{\partial Y}{\partial t} = \dfrac{1}{zt}\widetilde{K}(t)Y, \end{cases} \qquad t \in \mathcal{D}. \tag{3.6}$$

After the time variable change $t = \mathrm{e}^s$ (which cancels 't' in the latter denominator), the integrability condition (3.5) takes the form of a system of autonomous polynomial ordinary differential equations on matrix coefficients in $\widetilde{K}$, R, N (here and in what follows $[U, V] := UV - VU$):

$$\begin{cases} \widetilde{K}'_s = [R, \widetilde{K}] + \widetilde{K} \\ R'_s = [\widetilde{K}, N] \\ N'_s = 0. \end{cases} \tag{3.7}$$

In the initial time variable t and the new matrix variable $K := \frac{1}{t}\widetilde{K}$ system (3.7) takes the following simplified, though non-autonomous, form:

$$\begin{cases} tK' = [R, K] \\ R' = [K, N] \\ N' = 0. \end{cases} \tag{3.8}$$

Remark 3.4. Vector field (3.7) is a polynomial vector field on the space $\mathcal{H}^1_{0,\infty}$ identified with a connected open dense subset in the space $\mathbb{C}^{12}$ with coordinates being matrix coefficients. Its complex phase curves form a one-dimensional holomorphic foliation of the space $\mathcal{H}^1_{0,\infty}$ by isomonodromic families. The corresponding system (3.8) considered as a non-autonomous differential equation in the linear-system-valued function $\mathcal{L}(t) \in \mathcal{H}^1_{0,\infty}$ has the following first integrals:

- The matrix N;
- The conjugacy class of the matrix $K = \frac{1}{t}\widetilde{K}$;
- The residue matrices of the formal normal forms of $\mathcal{L}(t)$ at 0 and at ∞;
- The conjugacy class of the monodromy operator of the system $\mathcal{L}(t)$.

Invariance of residues and monodromy follows from theorem 3.3. Invariance of residues can be also deduced directly from (3.8) and (2.3).

Proposition 3.5. *Vector field (3.7) is equivariant under gauge transformations acting on $\mathcal{H}^1_{0,\infty}$. Its real flow preserves the space of systems in $\mathcal{H}^1_{0,\infty}$ defined by real matrices.*

The proposition follows immediately from expression (3.7).

3.3. Isomonodromic deformations of special Jimbo type systems

Definition 3.6. A *special Jimbo type* linear system is a system of type

$$Y' = \left(-t\frac{K}{z^2} + \frac{R}{z} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right) Y, \quad R = \begin{pmatrix} -\ell & * \\ * & 0 \end{pmatrix}, \tag{3.9}$$

such that there exists a matrix $G \in \mathrm{GL}_2(\mathbb{C})$ for which

$$K = G \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} G^{-1}, \quad G^{-1}RG = \begin{pmatrix} -\ell & * \\ * & 0 \end{pmatrix}. \tag{3.10}$$

In (3.9) and (3.10) the symbol * stands for an arbitrary unknown matrix element. Here all the matrices are complex.

Remark 3.7. The formal normal form at ∞ of a system (3.9) is

$$Y' = \left(\mathrm{diag}\left(-\frac{1}{2}, 0\right) + \frac{1}{z}\,\mathrm{diag}(-\ell, 0)\right)Y.$$

Condition (3.10) is equivalent to the statement saying that its formal normal form at 0 is

$$Y' = \left(\frac{t}{z^2}\,\mathrm{diag}\left(-\frac{1}{2}, 0\right) + \frac{1}{z}\,\mathrm{diag}(-\ell, 0)\right)Y,$$

by (2.3).

Proposition 3.8. *The space of Jimbo type systems (3.9), (3.10) corresponding to a given $\ell \in \mathbb{C}$ is invariant under the flow of field (3.7) and hence, is a union of its phase curves. The number ℓ is a first integral.*

The proposition follows from remarks 3.4 and 3.7.

We study Jimbo's isomonodromic families of systems (3.9) given by (3.8), which take the following form:

$$\begin{cases} tK' = -[K, R] \\ R' = \left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, K\right]. \end{cases} \tag{3.11}$$

We denote the upper right entries of $K(t)$ and $R(t)$ by $K_{12}(t)$ and $R_{12}(t)$ respectively.

Theorem 3.9 [38, *pp 1156–7*]. *Set*

$$y(t) = -\frac{R_{12}(t)}{K_{12}(t)}, \qquad \tau = \sqrt{t}, \quad w(\tau) = \frac{y(\tau^2)}{\tau}. \tag{3.12}$$

For every Jimbo's isomonodromic family (3.11) of Jimbo type systems (3.9) the corresponding function $w(\tau)$ satisfies the Painlevé 3 equation[8]

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + \alpha\frac{w^2}{\tau} + \beta\frac{1}{\tau} + \gamma w^3 + \delta\frac{1}{w}, \quad P_3(\alpha, \beta, \gamma, \delta)$$

whose parameters are expressed via the first integral ℓ in the following way

$$\alpha = -2\ell, \qquad \beta = 2\ell - 2, \qquad \gamma = 1, \qquad \delta = -1: \tag{3.13}$$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} - 2\ell\frac{w^2}{\tau} + (2\ell - 2)\frac{1}{\tau} + w^3 - \frac{1}{w}. \tag{3.14}$$

Remark 3.10. The deformation considered in Jimbo's paper [38, pp 1156–7] was of the type

[8] There is another frequently mentioned isomonodromic deformation that leads to the Painlevé 3 equation [25, 40].

$$\begin{cases} \dfrac{\partial Y}{\partial x} = \left(-\dfrac{\widetilde{t}A(\widetilde{t})}{x^2} + \dfrac{B(\widetilde{t})}{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) Y, \quad A(\widetilde{t}) = G(\widetilde{t}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G^{-1}(\widetilde{t}) \\ \dfrac{\partial Y}{\partial \widetilde{t}} = \dfrac{A(\widetilde{t})}{x} Y \end{cases} \tag{3.15}$$

with the (constant) residue matrices of formal normal forms at 0 and at ∞ being equal to $\frac{1}{2}\operatorname{diag}(\theta_0, -\theta_0)$ and $-\frac{1}{2}\operatorname{diag}(\theta_\infty, -\theta_\infty)$ respectively. Jimbo's family (3.15) with $\theta_\infty = -\theta_0 = \ell$ can be transformed to our family (3.9), (3.11) by multiplication of the vector function $Y(x)$ by the scalar monomial $x^{-\frac{\ell}{2}}$, variable rescaling $z = -2x$, and parameter rescaling $t = -4\widetilde{t}$. Our function $w(\tau)$ is obtained from analogous function $y(\sqrt{t})$ from [38, p 1157] by rescaling $w(\tau) = -\mathrm{i}y(-\frac{\mathrm{i}}{2}\tau)$, which transforms the Painlevé 3 equation satisfied by y (with parameters from [38, p 1157]) to (3.14).

3.4. Isomonodromic families of normalized $\mathbb{R}_+$-Jimbo systems

Definition 3.11. An $\mathbb{R}_+$-*Jimbo type system* is a system (3.9) given by real matrices K, R satisfying (3.10) with $R_{12} < 0 < R_{21}$ and $t > 0$. (The matrix G, whose inverse diagonalizes K, can be chosen real and unimodular.) The space of $\mathbb{R}_+$-Jimbo type systems will be denoted by $\mathbf{J}(\mathbb{R}_+)$. A linear system is of *normalized* $\mathbb{R}_+$-*Jimbo type*, if it has the form

$$Y'_\zeta = \left(-\tau\frac{K}{\zeta^2} + \frac{R}{\zeta} + \tau \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \right) Y, \quad R = \begin{pmatrix} -\ell & -R_{21} \\ R_{21} & 0 \end{pmatrix}, \quad R_{21} > 0, \tag{3.16}$$

where K and R satisfy (3.10). The space of normalized $\mathbb{R}_+$-Jimbo type systems will be denoted by $\mathbf{J}^N(\mathbb{R}_+)$.

Remark 3.12. The space $\mathbf{J}(\mathbb{R}_+)$ of $\mathbb{R}_+$-Jimbo type systems is a union of real isomonodromic families $\mathcal{L}(t)$, real phase curves of vector field (3.7) (proposition 3.5). Each $\mathbb{R}_+$-Jimbo type system can be transformed to a normalized one by composition of a unique diagonal gauge transformation $(Y_1, Y_2) \mapsto (Y_1, \lambda Y_2)$, $\lambda > 0$ and the variable change $z = \tau\zeta$, $\tau = \sqrt{t} > 0$.

Example 3.13. The space Jos of linear *systems of Josephson type*, i.e., family (1.7), is contained in $\mathbf{J}^N(\mathbb{R}_+)$. Its natural inclusion to $\mathbf{J}^N(\mathbb{R}_+)$ transforms a system (1.7) with parameters (μ, ℓ, ω) to a system (3.16) with $K = \operatorname{diag}(\frac{1}{2}, 0)$ and the parameters $\tau = 2\mu$, ℓ, $R_{21} = \frac{1}{2\omega}$.

Proposition 3.14 ***(rigidity)****. No two distinct systems in $\mathbf{J}^N(\mathbb{R}_+)$ are gauge equivalent.*

Proof. A gauge equivalence must be diagonal: it should keep the main term matrix at ∞ diagonal. It should also preserve the equality $R_{12} = -R_{21}$ and the inequality $R_{21} > 0$. Therefore, it is a constant multiple of identity, and leaves the system in question invariant. This proves the proposition. □

Lemma 3.15. *Let $\mathcal{H}^{1,0}_{0,\infty} \subset \mathcal{H}^1_{0,\infty}$ be the open subset consisting of systems with $R_{21}, R_{12} \neq 0$. The set $\mathbf{J}^N(\mathbb{R}_+)$ is a four-dimensional real-analytic submanifold in $\mathcal{H}^{1,0}_{0,\infty}$. It carries a real* *analytic foliation by isomonodromic families* (*which will be referred to, as* ***normalized real isomonodromic families***) *given by the differential equation*

$$\begin{cases} R'_\tau = 2\tau[K,N] + u[N,R] \\ K'_\tau = \dfrac{2}{\tau}[R,K] + u[N,K], \end{cases} \qquad u = \tau\frac{K_{21}-K_{12}}{R_{21}},\ N = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.17)$$

Along its solutions the function $w(\tau) = -\frac{R_{12}(\tau)}{\tau K_{12}(\tau)} = \frac{R_{21}(\tau)}{\tau K_{12}(\tau)}$ *satisfies Painlevé 3 equation* (3.14). *The above foliation will be denoted by* $\mathcal{F}$.

Proof. Let us show that the closed subset $\mathbf{J}^N(\mathbb{R}_+) \subset \mathcal{H}^{1,0}_{0,\infty}$ is a four-dimensional submanifold. For every matrix $K \in \mathrm{Mat}_2(\mathbb{C})$ with distinct eigenvalues and any their fixed order (λ_1, λ_2), the matrix G such that $G^{-1}KG = \mathrm{diag}(\lambda_1, \lambda_2)$ is uniquely defined up to multiplication from the right by a non-degenerate diagonal matrix. We will cover $\mathcal{H}^{1,0}_{0,\infty}$ by two open subsets $W_1, W_2 \subset \mathcal{H}^{1,0}_{0,\infty}$:

$$W_1 := \{G_{11} \neq 0\}; \qquad W_2 := \{G_{12} \neq 0\}.$$

Let us show that the intersection $\mathbf{J}^N(\mathbb{R}_+) \cap W_1$ is a four-dimensional submanifold in W_1. Then we prove the similar statement for the intersection $\mathbf{J}^N(\mathbb{R}_+) \cap W_2$. For every system in $\mathbf{J}^N(\mathbb{R}_+) \cap W_1$ the corresponding matrix G can be normalized as above in a unique way so that

$$\det G = 1, \quad G_{11} = 1; \qquad G_{22} = 1 + G_{12}G_{21}. \quad (3.18)$$

Hence, its matrix K is defined by two parameters G_{12} and G_{21}, and the correspondence $(G_{12}, G_{21}) \mapsto K$ is bijective. Let us write the second equation in (3.10) for a normalized system with $R_{12} = -R_{21}$. It says that the matrix

$$G^{-1}RG = \begin{pmatrix} * & * \\ G_{21}\ell + R_{21} & G_{21}R_{21} \end{pmatrix} \begin{pmatrix} * & G_{12} \\ * & 1 + G_{21}G_{12} \end{pmatrix}$$

has zero right-lower element. This is the equation

$$G_{21}G_{12}\ell + R_{21}G_{12} + G_{21}R_{21} + G_{21}^2G_{12}R_{21} = 0, \quad (3.19)$$

which is equivalent to the equation

$$G_{12} = -\frac{G_{21}R_{21}}{G_{21}\ell + R_{21}(1 + G_{21}^2)} \quad (3.20)$$

saying that G_{12} is a known rational function of three independent variables G_{21}, R_{21}, ℓ. The latter equivalence holds outside the exceptional set where the numerator and the denominator in (3.20) vanish simultaneously. Vanishing of the numerator is equivalent to vanishing of G_{21} (since $R_{21} \neq 0$, by assumption), and in this case the denominator equals $R_{21} \neq 0$. Thus, the exceptional set is empty. This implies that $W_1 \cap \mathbf{J}^N(\mathbb{R}_+)$ is a real four-dimensional analytic submanifold in W_1 (the fourth parameter is $\tau = \sqrt{t}$).

Let us now prove the above statement for $\mathbf{J}^N(\mathbb{R}_+) \cap W_2$. If $G_{12} \neq 0$, then we can normalize the matrix G in a unique way so that

$$\det G = 1, \quad G_{12} = 1; \qquad G_{21} = G_{11}G_{22} - 1. \quad (3.21)$$

Then the second equation in (3.10), which says that the matrix

$$G^{-1}RG = \begin{pmatrix} G_{22} & -1 \\ -G_{21} & G_{11} \end{pmatrix}\begin{pmatrix} -\ell & -R_{21} \\ R_{21} & 0 \end{pmatrix}\begin{pmatrix} G_{11} & 1 \\ G_{21} & G_{22} \end{pmatrix}$$

has zero right-lower element, is $(G_{11}G_{22}-1)(\ell+G_{22}R_{21})+G_{11}R_{21}=0$, which is equivalent to the equation

$$G_{11} = \frac{\ell + G_{22}R_{21}}{R_{21}(1+G_{22}^2)+\ell G_{22}}.$$

Now it suffices to show that the above numerator and denominator cannot vanish simultaneously, as in the previous discussion. Indeed, their vanishing means that $G_{22}R_{21}=-\ell$ and $R_{21}-\ell G_{22}+\ell G_{22}=R_{21}=0$, which is impossible. The first statement of the lemma is proved.

The space $\mathbf{J}(\mathbb{R}_+)$ of $\mathbb{R}_+$-Jimbo type systems is a manifold projected to the space $\mathbf{J}^N(\mathbb{R}_+)$ via the diagonal gauge normalizations from remark 3.12. The projection is an analytic bundle with fibre $\mathbb{R}_+$, by the same remark and since $\mathbf{J}^N(\mathbb{R}_+)$ is a submanifold. It sends isomonodromic families in $\mathbf{J}(\mathbb{R}_+)$ given by (3.11) to normalized isomonodromic families in $\mathbf{J}^N(\mathbb{R}_+)$. Let us find the differential equation describing them. Fix a $\tau_0>0$ and matrices K_0, R_0 defining a system in $\mathbf{J}^N(\mathbb{R}_+)$. Set $t_0=\tau_0^2$. Consider the $\mathbb{R}_+$-Jimbo type system (3.9) defined by the same matrices. Let $\widetilde{\mathcal{L}}(t)$ be its isomonodromic deformation given by equation (3.11), and let $\widetilde{K}(t)$, $\widetilde{R}(t)$ denote the corresponding matrices. Let $\mathcal{L}(t)$ denote its projection to $\mathbf{J}^N(\mathbb{R}_+)$, which is given by a gauge transformation family $(Y_1,Y_2)\mapsto(Y_1,\lambda(t)Y_2)$ and z-variable rescalings $z=\tau\zeta$: the matrices defining the systems $\mathcal{L}(t)$ are

$$K(t)=\Lambda(t)\widetilde{K}(t)\Lambda^{-1}(t), \qquad R(t)=\Lambda(t)\widetilde{R}(t)\Lambda^{-1}(t), \qquad \Lambda(t)=\mathrm{diag}(1,\lambda(t)),$$

$\lambda(t_0)=1$. Isomonodromicity equation (3.11) on $\widetilde{\mathcal{L}}(t)$ at $t=t_0$ yields

$$\begin{cases} t_0K'_t = -[K, R+t_0\nu\,\mathrm{diag}(0,1)] \\ R' = \left[\mathrm{diag}\left(\frac{1}{2},0\right),K\right]+\nu[\mathrm{diag}(0,1),R], \end{cases} \qquad \nu=(\ln\,\lambda(t_0))'=\lambda'(t_0). \tag{3.22}$$

In the second equation in (3.22) $R'_{21}=-R'_{12}$, since $R_{12}\equiv-R_{21}$. This yields

$$K_{12}-K_{21}+2\nu(R_{21}-R_{12})=0, \quad \nu=\frac{K_{21}-K_{12}}{4R_{21}}.$$

Substituting the above formula for ν to (3.22), replacing the matrix diag$(0,1)$ in the commutators by $\mathrm{diag}(-1,0)=\mathrm{diag}(0,1)-\mathrm{Id}$, and changing the variable t to $\tau=\sqrt{t}$ yields (3.17). The function $w(\tau)$ defined in (3.12) for the family $\widetilde{\mathcal{L}}(t)$ coincides with the analogous function defined for the family $\mathcal{L}(t)$, since $\frac{R_{12}}{K_{12}}=\frac{\lambda^{-1}\widetilde{R}_{12}}{\lambda^{-1}\widetilde{K}_{12}}=\frac{\widetilde{R}_{12}}{\widetilde{K}_{12}}$. It satisfies equation (3.14), by theorem 3.9. Lemma 3.15 is proved. □

3.5. Transversality property of Josephson type systems

Lemma 3.16. *Consider an arbitrary system $\mathcal{L}\in$ Jos. Let $w(\tau)$ be the germ of solution of Painlevé equation (3.14) defining its real isomonodromic deformation in the space $\mathbf{J}^N(\mathbb{R}_+)$ at* *the point τ_0 corresponding to the system $\mathcal{L}$. Then $w(\tau)$ has first order pole at τ_0 with residue 1. Conversely, every system in $\mathbf{J}^N(\mathbb{R}_+)$ corresponding to a first order pole $\tau_0 > 0$ of solution of equation (3.14) with residue 1 lies in* Jos.

Proof. It is well-known that non-zero singular points of solutions of equation (3.14) are poles of order 1 with residues ± 1 [33, p 158]. Let us check that systems in Jos correspond to poles with residue 1. Consider a system in $\mathbf{J}^N(\mathbb{R}_+)$ with $\tau = \tau_0$ and $K_{12} = 0$ (e.g., a system lying in Jos) and its isomonodromic deformation given by (3.17). The upper triangular term in the second matrix equation in (3.17) has the form

$$K'_{12} = \frac{2}{\tau} R_{12}(K_{22} - K_{11}) + O(K_{12}), \quad \text{as } \tau \to \tau_0; \quad K_{12}(\tau_0) = 0. \tag{3.23}$$

Therefore, $K_{12}(\tau) = \frac{2}{\tau_0} R_{12}(\tau_0)(K_{22}(\tau_0) - K_{11}(\tau_0))(\tau - \tau_0) + o(\tau - \tau_0)$,

$$w(\tau) = -\frac{R_{12}(\tau)}{\tau K_{12}(\tau)} = -\frac{1 + o(1)}{2(K_{22}(\tau_0) - K_{11}(\tau_0))(\tau - \tau_0) + o(\tau - \tau_0)}. \tag{3.24}$$

If the initial system corresponding to $\tau = \tau_0$ lies in Jos, then $K_{22}(\tau_0) - K_{11}(\tau_0) = -\frac{1}{2}$, hence $w(\tau) = \frac{1}{\tau - \tau_0}(1 + o(1))$, and w has simple pole with residue 1 at τ_0.

Conversely, let $w(\tau)$ have a simple pole with residue 1 at τ_0. Then $K_{12}(\tau_0) = 0$, and $K_{22}(\tau_0) - K_{11}(\tau_0) = -\frac{1}{2}$, by (3.24). Note that the trace of the matrix K is constant and equal to $\frac{1}{2}$. Hence, $K_{22}(\tau_0) = 0$, $K_{11}(\tau_0) = \frac{1}{2}$. Now to show that the system in question lies in Jos, it suffices to prove that $K_{21}(\tau_0) = 0$. Suppose the contrary: $K_{21}(\tau_0) \neq 0$. Then the matrix G, whose inverse conjugates $K(\tau_0)$ to $\mathrm{diag}(\frac{1}{2}, 0)$, is lower triangular with $G_{21} \neq 0$. We normalize it by constant factor to have $G_{11} = 1$. Equation (3.19) together with $G_{12} = 0$ yield $G_{21}R_{21}(\tau_0) = 0$, while $G_{21}, R_{21}(\tau_0) \neq 0$. The contradiction thus obtained proves that $K(\tau_0) = \mathrm{diag}(\frac{1}{2}, 0)$ and the system in question lies in Jos. Lemma 3.16 is proved. □

Lemma 3.17 ***(key lemma)****. The submanifold* Jos $\subset \mathbf{J}^N(\mathbb{R}_+)$ *is transversal to the isomonodromic foliation $\mathcal{F}$ from lemma 3.15.*

Proof. Way 1 of proof. In an isomonodromic family given by (3.17) the derivative K'_{12} is nonzero at τ corresponding to a system lying in Jos, see (3.23). Hence, this family is transversal to the hypersurface Jos.

Way 2 of proof. Points of the hypersurface Jos correspond to simple poles of solutions of equation (3.14) satisfied along leaves. This together with the fact that a simple pole of an analytic family of functions depends analytically on parameter implies the statement of lemma 3.17. □

4. Analytic families of constrictions. Proof of theorem 1.12

For every linear system $\mathcal{L}$ let $M(\mathcal{L})$ denote its monodromy operator.

In the proof of theorem 1.12 we use the following proposition.

Proposition 4.1. *A point $(B, A;\ \omega)$ is a constriction, if and only if $A, \omega \neq 0$ and the corresponding system (1.7) has trivial monodromy.*

Proof. Proposition 3.2 from [28] states that a point is a constriction, if and only if (1.7) has projectively trivial monodromy: the monodromy matrix is a scalar multiple of identity. Another criterion given by [28, lemma 3.3] states that a point is a constriction, if and only if $\ell \in \mathbb{Z}$ and the germ of linear system (1.7) at the origin is analytically equivalent to its formal normal form. In this case system (1.7) and its formal normal form have the same monodromy matrices in appropriate bases. The monodromy of the normal form is given by the diagonal matrix $\mathrm{diag}(\mathrm{e}^{-2\pi\mathrm{i}\ell}, 1)$, which is identity if $\ell \in \mathbb{Z}$. Proposition 4.1 is proved. □

Corollary 4.2. *The systems* (1.7) *corresponding to constrictions lie in the set*

$$\Sigma := \{\mathcal{L} \in \mathbf{J}^N(\mathbb{R}_+) | M(\mathcal{L}) = \mathrm{Id}\}.$$

For every system $\mathcal{L} \in \mathbf{J}^N(\mathbb{R}_+)$ let us choose good sectors S_0 and S_1 that contain the upper (respectively, lower) half-plane punctured at 0, see figure 5. Consider its monodromy–Stokes data $(q_{10}, q_{20}, q_{1\infty}, q_{2\infty}; M)$ defined by the base point $z_0 = 1 \in S_{1,2} \subset S_0 \cap S_1$ and trivial paths $\alpha_0, \alpha_\infty \equiv 1$. Set

$$\mathcal{R}(\mathcal{L}) := \frac{(q_{10} - q_{1\infty})(q_{20} - q_{2\infty})}{(q_{10} - q_{2\infty})(q_{20} - q_{1\infty})} \in \overline{\mathbb{C}}. \tag{4.1}$$

We will call $\mathcal{R}(\mathcal{L})$ the *transition cross-ratio* of the system $\mathcal{L}$. It depends only on the monodromy–Stokes data and not on choice of its representative.

For the proof of theorem 1.12 we first prove the following theorem and lemma in subsections 4.1 and 4.2 respectively.

Theorem 4.3 ***(key theorem)***. *The subset $\Sigma \subset \mathbf{J}^N(\mathbb{R}_+)$ is a two-dimensional analytic submanifold, a union of leaves of the real isomonodromic foliation $\mathcal{F}$ from lemma 3.15. One has $\ell \in \mathbb{Z}$ for every system in Σ. The function $\mathcal{R}$ is constant on leaves of $\mathcal{F}$ in $\mathbf{J}^N(\mathbb{R}_+)$. The restriction $\mathcal{R}|_\Sigma$ is real-valued; it is an analytic submersion $\Sigma \to \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$. The map $(\mathcal{R}, \tau) : \Sigma \to \mathbb{RP}^1 \times \mathbb{R}_+$ is a local diffeomorphism.*

For every $\ell \in \mathbb{Z}$ by $\Sigma_\ell \subset \Sigma$ we denote the subset of systems with given ℓ.

Lemma 4.4. *For every $\ell \in \mathbb{Z}$ the subset $\mathrm{Constr}_\ell \subset (\mathbb{R}^2_+)_{(\mu,\eta)}$, $\eta = \omega^{-1}$, is a real-analytic one-dimensional submanifold identified with the intersection $\mathrm{Jos} \cap \Sigma_\ell$. The restriction of the function $\mathcal{R}$ to the latter intersection yields a mapping $\mathrm{Constr}_\ell \to \mathbb{R} \setminus \{0, 1\}$ that is a local analytic diffeomorphism.*

Afterwards in subsection 4.3 we prove the following more precise version of the first two statements of theorem 1.12.

Theorem 4.5.

(*a*) *For every connected component $\mathcal{C}$ of the submanifold Constr_ℓ the mapping $\mathcal{R} : \mathcal{C} \to \mathbb{R}$ is a diffeomorphism onto an interval $I = (a, b)$.*

(*b*) *Let $C := \mathcal{R}^{-1} : I \to \mathcal{C}$ denote the inverse function. For every $c \in \{a, b\} \setminus \{0\}$ there exists a sequence $x_n \in I$, $x_n \to c$, as $n \to \infty$, such that $\eta_n = \eta(C(x_n)) \to \infty$, i.e., $\omega(C(x_n)) \to 0$.*

In subsection 4.4 we prove constance of the rotation number and type of constriction on each connected component in Constr_ℓ and finish the proof of theorem 1.12.

4.1. Systems with trivial monodromy. Proof of theorem 4.3

In the proof of theorem 4.3 we use a series of propositions.

Proposition 4.6. *Every system $\mathcal{L} \in \mathcal{H}^1_{0,\infty}$ with trivial monodromy (e.g., every system in Σ) has trivial Stokes matrices and trivial formal monodromies at both singular points 0, ∞. In particular, the residue matrices of its formal normal forms have integer elements. If $\mathcal{L} \in \Sigma$, then one has $\ell \in \mathbb{Z}$.*

Proof. The proof repeats arguments from [28, proof of lemma 3.3]. Triviality of the Stokes matrices follows from formulae (2.11) and (2.12). Then $M = M_{\mathrm{norm}} = \mathrm{Id}$, by (2.6). Hence, $\ell \in \mathbb{Z}$, if $\mathcal{L} \in \Sigma$. □

Proposition 4.7. *Let in a system $\mathcal{L} \in \mathcal{H}^1_{0,\infty}$, see (2.7), the matrices K, R, N be real, and let each one of the matrices K, N have distinct real eigenvalues. Let the Stokes matrices of the system $\mathcal{L}$ at 0 and at ∞ be trivial. Then the transition cross-ratio $\mathcal{R}(\mathcal{L})$ is either real, or infinite.*

Proof. Let $f_{1j,p}, f_{2j,p}$ denote the canonical sectorial solution basis of the system $\mathcal{L}$ at point $p = 0, \infty$ in the sector S_j, $j = 0, 1$, see figure 5. The complex conjugation $\widehat{\sigma} : (Y_1, Y_2; z) \mapsto (\overline{Y}_1, \overline{Y}_2; \bar{z})$ leaves $\mathcal{L}$ invariant and sends graphs of its solutions to graphs of solutions. Its projectivization $\sigma : (\Phi, z) \mapsto (\overline{\Phi}, \bar{z})$, $\Phi := \frac{Y_2}{Y_1}$, permutes the sectors S_0, S_1 and graphs of the projectivized solutions

$$g_{k0,p} := \pi \circ f_{k0,p}, \qquad g_{k1,p} := \pi \circ f_{k1,p}.$$

Here $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 = \overline{\mathbb{C}}_\Phi$ is the tautological projection. This follows from uniqueness of projectivized sectorial basic solutions (remark 2.3). Triviality of Stokes matrices implies that $g_{k0,p} = g_{k1,p}$ is a global holomorphic $\overline{\mathbb{C}}$-valued function on $\mathbb{C}^*$. In particular, for every $z \in \mathbb{R}$ one has $g_{k0,p}(z) = g_{k1,p}(z)$; hence, $(g_{k0,p}(z), z)$ is a fixed point of the involution σ and $g_{k0,p}(z) \in \mathbb{R} \cup \{\infty\}$. Finally, $q_{kp} = g_{k0,p}(1) \in \mathbb{R} \cup \{\infty\}$ for every $k = 1, 2$ and $p = 0, \infty$, and thus, $\mathcal{R}(\mathcal{L}) \in \mathbb{R} \cup \{\infty\}$. Proposition 4.7 is proved. □

Proposition 4.8. *For every system $\mathcal{L} \in \Sigma$ the corresponding collection of points q_{kp}, $k = 1, 2$, $p = 0, \infty$ consists of at least three distinct points. One has $q_{1p} \neq q_{2p}$ for every $p = 0, \infty$.*

Proof. One has $q_{kp} = g_{k0,p}(1) = \pi \circ f_{k0,p}(1)$, where $f_{10,p}$, $f_{20,p}$ form the canonical basis of solutions of the system in S_0. Their linear independence implies linear independence of their values at $z = 1$, and hence, the inequality $q_{1p} \neq q_{2p}$. Let us now prove that among the points q_{kp} there are at least three distinct ones. To do this, we use the fact that $g_{k,p}(z) := g_{k0,p}(z) = g_{k1,p}(z)$ are two meromorphic functions on $\mathbb{C}^* \cup \{p\}$, $p = 0, \infty$. Meromorphicity on $\mathbb{C}^*$ follows from proposition 4.6 and the proof of proposition 4.7. Meromorphicity at p follows from remark 2.3. Suppose the contrary: there are only two distinct points among q_{kp}. Then $g_{1,0} \equiv g_{k_1,\infty}$, $g_{2,0} \equiv g_{k_2,\infty}$, where (k_1, k_2) is some permutation of $(1, 2)$. Therefore, $g_{1,p}$ and $g_{2,p}$ are meromorphic on $\overline{\mathbb{C}}$, by the above discussion. Their graphs are disjoint, since so are graphs of their restrictions to $\mathbb{C}^*$ (being phase curves of the Riccati foliation on $\mathbb{CP}^1 \times \overline{\mathbb{C}}$ defined by $\mathcal{L}$), and their values at each point $p \in \{0, \infty\}$ are distinct and equal to the projections of the eigenlines of the main term matrix at p (remark 2.3). The main term matrix at infinity being diagonal, one has $g_{1,\infty}(\infty) = [1:0]$, $g_{2,\infty}(\infty) = [0:1]$. But graphs of two meromorphic functions on $\overline{\mathbb{C}}$ with values in $\mathbb{CP}^1 = \overline{\mathbb{C}}$ may be disjoint only if the functions are constant. Indeed, $\mathrm{H}_2(\overline{\mathbb{C}} \times \overline{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ (Künneth formula), and the intersection form on the latter homology group is given by the formula $\langle (m_1, n_1), (m_2, n_2) \rangle = m_1 n_2 + m_2 n_1$. See the corresponding background material in [32, chapter 0, section 4]. The homology class of graph of a rational function F of degree n is $(1, n)$; $n > 0$, if $F \not\equiv \mathrm{const}$. Therefore, if $F \not\equiv \mathrm{const}$, then the intersection index of its graph with the graph of any rational function is positive. Hence, $g_{1,\infty} \equiv [1:0]$, $g_{2,\infty} \equiv [0:1]$, and the constant functions $\Phi(z) \equiv 0$, $\Phi(z) \equiv \infty$ are solutions of the Riccati equation corresponding to $\mathcal{L}$. This implies that the matrices of the system $\mathcal{L}$ are diagonal, which is obviously impossible for a system from $\mathbf{J}^N(\mathbb{R}_+)$. The contradiction thus obtained proves the proposition. □

Proposition 4.9. *For every collection $q_0 = (q_{10}, q_{20}, q_{1\infty}, q_{2\infty}) \in \overline{\mathbb{C}}^4$ that has at least three distinct points there exists a neighbourhood $\mathcal{V} = \mathcal{V}(q_0) \subset \overline{\mathbb{C}}^4$ such that two collections in $\mathcal{V}$ lie in the same $\mathrm{PSL}_2(\mathbb{C})$-orbit, if and only if they have the same cross-ratio.*

Proof. Fix a neighbourhood $\mathcal{V}$ such that three distinct points in q_0 remain distinct in each collection from $\mathcal{V}$. Let us normalize them by the $\mathrm{PSL}_2(\mathbb{C})$ action in such a way that these points be $0, 1, \infty$: such normalization is unique. Then the fourth point is uniquely determined by the cross-ratio. □

Proof of theorem 4.3. A system $\mathcal{L} \in \mathbf{J}^N(\mathbb{R}_+)$ is uniquely defined by the formal invariants ℓ, τ and the monodromy–Stokes data (theorem 2.11 and proposition 3.14). Let now $M(\mathcal{L}) = \mathrm{Id}$. Then the latter data are reduced to the $\mathrm{PSL}_2(\mathbb{C})$-orbit of the collection $(q_{10}, q_{20}, q_{1\infty}, q_{2\infty})$. The latter collection consists of at least three distinct points (proposition 4.8). Therefore, *each system $\mathcal{L} \in \Sigma_\ell$ has a* neighbourhood $\mathcal{W} = \mathcal{W}(\mathcal{L}) \subset \Sigma_\ell$ *such that two systems in $\mathcal{W}$ have the same monodromy–Stokes data, if and only if the corresponding cross-ratios $\mathcal{R}$ are equal.* This follows from proposition 4.9 and the above discussion. One has $(\mathcal{R}, \tau)(\mathcal{L}) \in \mathbb{RP}^1 \times \mathbb{R}$, by propositions 4.6 and 4.7. This together with the above statement on unique local determination by $\mathcal{R}$ imply that the mapping $\Pi : (\mathcal{R}, \tau) : \Sigma \to \mathbb{RP}^1 \times \mathbb{R}$ is locally injective.

Proposition 4.10. *For every $\ell \in \mathbb{Z}$ and $\mathcal{L}_0 \in \Sigma_\ell$, set $T_0 = (\mathcal{R}_0, \tau_0) := (\mathcal{R}, \tau)(\mathcal{L}_0)$, there exist neighbourhoods $V_1 = V_1(\mathcal{L}_0) \subset \mathbf{J}^N(\mathbb{R}_+)$, $V_2 = V_2(T_0) \subset \mathbb{RP}^1 \times \mathbb{R}$ and an analytic inverse $g = (\mathcal{R}, \tau)^{-1} : V_2 \to V_1$ with $g(V_2) = V_1 \cap \Sigma_\ell$.*

Proof. We have to realize each $T = (\mathcal{R}, \tau)$ close to T_0 by a linear system from Σ. To this end, we first realize T by an abstract two-dimensional holomorphic vector bundle over $\overline{\mathbb{C}}$ with connection. Namely, we take two linear systems defined by the given formal normal forms at 0 and ∞ respectively:

$$\mathcal{H}_0 : Y' = \left(\frac{1}{z^2}\operatorname{diag}\left(-\frac{\tau}{2}, 0\right) + \frac{1}{z}\operatorname{diag}(-\ell, 0)\right)Y; \tag{4.2}$$

$$\mathcal{H}_\infty : Y' = \left(\operatorname{diag}\left(-\frac{\tau}{2}, 0\right) + \frac{1}{z}\operatorname{diag}(-\ell, 0)\right)Y. \tag{4.3}$$

We consider the following trivial bundles with connections over discs covering $\overline{\mathbb{C}}$: the bundle $F_0 := \mathbb{C}^2_{Y^0} \times D_2$ equipped with the system $\mathcal{H}_0$; the bundle $F_\infty := \mathbb{C}^2_{Y^\infty} \times (\overline{\mathbb{C}} \setminus \overline{D}_{\frac{1}{2}})$ equipped with the system $\mathcal{H}_\infty$. The bundle realizing T is obtained by the following gluing F_0 and F_∞ over the annulus $\mathcal{A} := D_2 \setminus \overline{D}_{\frac{1}{2}}$. Let $v_1 = (1, 0)$, $v_2 = (0, 1)$ denote the standard basis in $\mathbb{C}^2$. For every $\mathcal{R}$ close enough to $\mathcal{R}_0$ fix a linear isomorphism $\mathbf{L}_1 : \mathbb{C}^2 \to \mathbb{C}^2$ such that the tautological projection to $\mathbb{CP}^1 = \overline{\mathbb{C}}$ of the collection of vectors $\mathbf{L}_1 v_1$, $\mathbf{L}_1 v_2$, v_1, v_2 has the given cross-ratio $\mathcal{R}$ and $\mathbf{L}_1$ depends analytically on $\mathcal{R}$. Let $W^0(z) = \operatorname{diag}(\mathrm{e}^{\frac{\tau}{2}(\frac{1}{z}-1)} z^{-\ell}, 1)$, $W^\infty(z) = \operatorname{diag}(\mathrm{e}^{-\frac{\tau}{2}(z-1)} z^{-\ell}, 1)$ be the standard fundamental matrix solutions of systems $\mathcal{H}_0$, $\mathcal{H}_\infty$ normalized to be equal to the identity at $z = 1$. Set

$$\mathbf{L}_z = \mathbf{L}_{z,\mathcal{R},\tau} = W^\infty(z)\mathbf{L}_1(W^0(z))^{-1}. \tag{4.4}$$

Let $E = E(\mathcal{R}, \tau)$ denote the disjoint union $F_0 \sqcup F_\infty$ pasted by the following identification: for every $z \in \mathcal{A}$ the point $(Y^0, z) \in F_0$ is equivalent to $(Y^\infty, z) \in F_\infty$, if $Y^\infty = \mathbf{L}_z Y^0$. The space E inherits a structure of holomorphic vector bundle over $\overline{\mathbb{C}}$ with a well-defined meromorphic connection induced by the formal normal forms $\mathcal{H}_0$, $\mathcal{H}_\infty$ in the charts F_0 and F_∞ (which paste together by $\mathbf{L}_z$ to the same connection over $\mathcal{A}$). This connection has two Poincaré rank

1 irregular nonresonant singular points at 0 and ∞ where it is analytically equivalent to $\mathcal{H}_0$ and $\mathcal{H}_\infty$. Note that the monodromy–Stokes data and the transition cross-ratio are well-defined for bundles with connections as well, provided that the singularities at 0 and at ∞ are irregular nonresonant of Poincaré rank 1. The transition cross-ratio of the bundle $E(\mathcal{R}, \tau)$ coincides with $\mathcal{R}$, by construction.

Let now $\widehat{V}_2$ be a small ball centred at $T_0 = (\mathcal{R}_0, \tau_0)$ in the complex product $\overline{\mathbb{C}}_{\mathcal{R}} \times \mathbb{C}_\tau$ (in its local chart centred at T_0). Set $\widehat{E} := \sqcup_{(\mathcal{R},\tau)\in\widehat{V}_2} E(\mathcal{R}, \tau)$. This is a holomorphic vector bundle over the product $\overline{\mathbb{C}} \times \widehat{V}_2$.

Claim 1. *The bundle $\widehat{E}$ is trivial, if the ball $\widehat{V}_2$ is small enough.*

Proof. The bundle $E(T_0)$ is trivial, since it has the same monodromy–Stokes data and formal normal forms, as the system $\mathcal{L}_0$ (which is a connection on trivial bundle), and by theorem 2.11 (which remains valid for bundles with connections). It is glued from two trivial bundles over the domains D_2 and $\overline{\mathbb{C}} \setminus \overline{D}_{\frac{1}{2}}$ by the transition matrix function $\mathbf{L}_{z,T_0}$. Triviality implies that there exist (and unique) $\mathrm{GL}_2(\mathbb{C})$-valued matrix functions $U_0(z)$ and $U_\infty(z)$ holomorphic on D_2 and $\overline{\mathbb{C}} \setminus \overline{D}_{\frac{1}{2}}$ respectively such that $U_0(z) = U_\infty(z)\mathbf{L}_{z,T_0}$ on $\mathcal{A}$ and $U_\infty(\infty) = \mathrm{Id}$. They are holomorphic on bigger domains $D_3 \Supset \overline{D}_2, \overline{\mathbb{C}} \setminus \overline{D}_{\frac{1}{3}} \Supset \overline{\mathbb{C}} \setminus D_{\frac{1}{2}}$, by the above statement applied to the latter bigger domains and holomorphicity of the transition matrix function $\mathbf{L}_{z,T_0}$ on $\mathbb{C}^*$. Consider the following new trivializations of the trivial bundles $\mathbb{C}^2_{Y^0} \times (D_2 \times \widehat{V}_2)$ and $\mathbb{C}^2_{Y^\infty} \times ((\overline{\mathbb{C}} \setminus \overline{D}_{\frac{1}{2}}) \times \widehat{V}_2)$:

$$\widetilde{Y}^0 := U_0(z)Y^0, \qquad \widetilde{Y}^\infty := U_\infty(z)Y^\infty.$$

In the new coordinates $\widetilde{Y}^0$ and $\widetilde{Y}^\infty$ the fibre identifications gluing $\widehat{E}$ of the above trivial bundles over points $(z, T) \in \mathcal{A} \times \widehat{V}_2$ become the following: a point $(\widetilde{Y}^0, z, T)$ is identified with $(\widetilde{Y}^\infty, z, T)$, if $M(z, T)\widetilde{Y}^0 = \widetilde{Y}^\infty$, where

$$M(z, T) = U_\infty(z)\mathbf{L}_{z,T}U_0^{-1}(z).$$

Therefore, $\widehat{E}$ can be viewed as the bundle glued from two trivial bundles on $D_2 \times \widehat{V}_2$ and $(\overline{\mathbb{C}} \setminus \overline{D}_{\frac{1}{2}}) \times \widehat{V}_2$ by the transition matrix function $M(z, T)$ holomorphic on $\overline{\mathcal{A}} \times \widehat{V}_2$. One has $M(z, T_0) = \mathrm{Id}$, by construction. Choosing $\widehat{V}_2$ small enough, one can make $M(z, T)$ continuous on $\overline{\mathcal{A} \times \widehat{V}_2}$ and make the C^0-norm $\|M(z, T) - \mathrm{Id}\|$ on $\overline{\mathcal{A} \times \widehat{V}_2}$ arbitrarily small. Therefore, the bundle $\widehat{E}$ glued by $M(z, T)$ is 'close to trivial', and hence, is trivial, whenever $\widehat{V}_2$ is small enough, by [11, appendix 3, lemma 1]. (Formally speaking, this lemma should be applied after rescaling the coordinates in the chart containing $\widehat{V}_2$ in the parameter space to make $\widehat{V}_2$ the unit ball.) The claim is proved. □

Let $V_2 \subset \widehat{V}_2$ be the subset of real points of the complex ball $\widehat{V}_2$, which is a real planar disk. The claim implies that the family $E(T)|_{T\in V_2}$ yields a family of connections on the trivial bundle $\mathbb{C}^2 \times \overline{\mathbb{C}}$ depending analytically on the parameter $T \in V_2$. They should be linear systems in $\mathcal{H}^1_{0,\infty}$, since the singularities at 0 and ∞ are irregular non-resonant of Poincaré rank 1. This yields an analytic map $g : V_2 \to V_1$ from a neighbourhood $V_2 = V_2(T_0) \subset \mathbb{RP}^1 \times \mathbb{R}$ to a domain $V_1 \subset \mathcal{H}^1_{0,\infty}$ such that for every $(\mathcal{R}, \tau) \in V_2$ the system $g(\mathcal{R}, \tau)$ has trivial monodromy, transition cross-ratio equal to $\mathcal{R}$, and is analytically equivalent to formal normal forms (4.2), (4.3) near 0 and ∞ respectively. Without loss of generality we consider that $g(T_0) = \mathcal{L}_0$, applying a gauge transformation independent on $(\mathcal{R}, \tau)$. For every system in $g(V_2)$ the corresponding points $q_{kp} \in \overline{\mathbb{C}}_\Phi$ from the monodromy–Stokes data given by the base point $z_0 = 1$ and trivial paths $\alpha_0 \equiv \alpha_\infty \equiv 1$ lie on the same circle, since their cross-ratio $\mathcal{R}$ lies in $\mathbb{R} \cup \{\infty\}$. The latter circle is unique, since there are at least three distinct points q_{kp}: this is true for $T = T_0$ (proposition 4.8) and remains valid for all $T \in V_2$, provided that $\widehat{V}_2$ is chosen small enough. We normalize the systems in $g(V_2)$ so that the latter circle is the real line, applying an analytic family of gauge transformations depending on $(\mathcal{R}, \tau)$.

Claim 2. The systems in $g(V_2)$ are defined by real matrices.

Proof. The transformation $\widehat{\sigma} : (Y_1, Y_2; z) \mapsto (\overline{Y}_1, \overline{Y}_2; \bar{z})$ applied to systems in $g(V_2)$ preserves formal normal forms and monodromy–Stokes data, by construction and the above normalization. Therefore, it sends each system in $g(V_2)$ to a system gauge equivalent to it, and the collections of points q_{kp} in the fibre $\overline{\mathbb{C}} \times \{1\}$ are the same for both systems. Their gauge equivalence restricted to the fibre $\mathbb{C}^2 \times \{1\}$ should fix the lines corresponding to q_{kp}. Hence, it is identity up to scalar factor, since the number of distinct points q_{kp} is at least three. Therefore, the systems in question coincide. Thus, $\widehat{\sigma}$ fixes each system in $g(V_2)$, which means that its matrices are real. □

The main term matrix N at ∞ of each system in $g(V_2)$ is real, and its eigenvalues are $-\frac{\tau}{2}$, 0. It is close to $\mathrm{diag}(-\frac{\tau}{2}, 0)$, if V_2 is small enough. Therefore, it is conjugated to the diagonal matrix $\mathrm{diag}(-\frac{\tau}{2}, 0)$ by a real matrix H close to the identity. The matrix H is unique up to left multiplication by a real diagonal matrix. It can be chosen in a unique way so that the gauge transformation $Y = H^{-1}\widetilde{Y}$ makes $R_{21} = -R_{12} > 0$. This yields a family of gauge transformations sending systems in $g(V_2)$ to systems lying in $\mathbf{J}^N(\mathbb{R}_+)$, and hence, in Σ_ℓ (triviality of monodromy). From now on, the mapping $V_2 \to \mathbf{J}^N(\mathbb{R}_+)$ thus constructed will be denoted by g. By construction, its image lies in Σ_ℓ, and for every $(\mathcal{R}', \tau') \in V_2$ the transition cross-ratio $\mathcal{R}$ and the formal invariant τ of the system $g(\mathcal{R}', \tau')$ are respectively $\mathcal{R}'$ and τ'. Conversely, every system $\mathcal{L} \in \Sigma_\ell$ close enough to $\mathcal{L}_0$ has invariants $(\mathcal{R}, \tau)$ lying in V_2, and hence $\mathcal{L} = g(\mathcal{R}, \tau)$, by construction, theorem 2.11 and proposition 3.14. This proves proposition 4.10. □

The mapping g is an immersion, since the projection $\mathcal{L} \mapsto (\mathcal{R}, \tau)(\mathcal{L})$ is real-analytic and $(\mathcal{R}, \tau) \circ g = \mathrm{Id}$. This together with proposition 4.10 implies that Σ_ℓ is a two-dimensional submanifold, and $(\mathcal{R}, \tau) : \Sigma_\ell \to \mathbb{RP}^1 \times \mathbb{R}$ is a local diffeomorphism. Hence, the projection $\mathcal{R} : \Sigma_\ell \to \mathbb{RP}^1$ (which is constant along isomonodromic leaves) is a submersion. Theorem 4.3 is proved. □

4.2. The manifold of constrictions. Proof of lemma 4.4

The space of systems (1.7) with given ℓ is identified with $(\mathbb{R}_+)^2_{\mu,\eta}$, $\eta = \omega^{-1}$. They are represented as systems in $\mathrm{Jos} \subset \mathbf{J}^N(\mathbb{R}_+)$ with parameters $\tau = 2\mu$, ℓ, $R_{21} = \frac{\eta}{2}$. The constriction subset $\mathrm{Constr}_\ell \subset (\mathbb{R}_+)^2_{\mu,\eta}$ is thus identified with the intersection $\mathrm{Jos} \cap \Sigma_\ell$, by proposition 4.1. The latter intersection is transversal, since Σ_ℓ is a union of leaves of the isomonodromic foliation $\mathcal{F}$ and Jos is transversal to $\mathcal{F}$ (lemma 3.17). Therefore, Constr_ℓ is a one-dimensional submanifold transversal to the isomonodromic foliation on Σ_ℓ. Hence, $\mathcal{R} : \mathrm{Constr}_\ell \to \mathbb{RP}^1$ is a local analytic diffeomorphism (submersivity of the projection $\mathcal{R} : \Sigma_\ell \to \mathbb{RP}^1$, see theorem 4.3). It remains to show that $\mathcal{R} \neq 0, 1, \infty$ on Constr_ℓ.

Proposition 4.11. *For every constriction $(B, A;\ \omega)$ the collection of points q_{kp} from the monodromy–Stokes data of the corresponding linear system (1.7) consists of four distinct points. Or equivalently, $\mathcal{R} \neq 0, 1, \infty$.*

Proof. One has $q_{1p} \neq q_{2p}$. Hence, the only *a priori* possible coincidences are the following.

Case (1): $q_{k0} = q_{k\infty}$ for some k. Then the same equality holds for the other k, by symmetry $(\Phi, z) \mapsto (\Phi^{-1}, z^{-1})$ of the corresponding Riccati equation (1.6). Thus, the collection of points q_{kp} consists of two distinct points. This contradicts to proposition 4.8.

Case (2): $q_{k0} = q_{(3-k)\infty}$ for some k. This means that the transition matrix between the canonical solution base of system (1.7) at 0 and the canonical base at ∞ taken in inverse order is a triangular matrix. But this contradicts to [29, theorem 2.10, statement (2.19)].

Finally none of cases (1) and (2) is possible. Proposition 4.11 is proved. □

Lemma 4.4 follows from proposition 4.11 and the discussion before it.

4.3. Asymptotics and unboundedness. Proof of theorem 4.5

The subset $\mathrm{Constr}_\ell \subset (\mathbb{R}^2_+)_{\mu,\eta}$ is a submanifold that admits a locally diffeomorphic projection $\mathcal{R}$ to $\mathbb{R} \setminus \{0, 1\}$ (lemma 4.4). This implies that it has no compact components, since no compact component can admit a locally diffeomorphic mapping to $\mathbb{R}$. Therefore, each its component $\mathcal{C}$ is diffeomorphic to some interval $I = (a, b)$ with coordinate $x := \mathcal{R}$. This implies the first statement of theorem 4.5. To prove its second statement, the existence of a sequence $x_n \to c$ with $\eta(C(x_n)) \to \infty$ for $c \in \{a, b\}\setminus\{0\}$, $C = \mathcal{R}^{-1}$, we will

- Use the following Klimenko–Romaskevich Bessel asymptotic result [44] to show that boundedness of η implies boundedness of μ;
- Prove that $(\mu, \eta)(x_n)$ cannot converge to $(0, 0)$, by using solution of variational equation to (1.3) and studying local parametrization of the analytic subset in $\mathbb{R}^2$ containing Constr_ℓ;
- Show that if $\eta(x_n) \to 0$, then $c = \lim x_n = 0$.

Let us recall that the boundary of the phase-lock area L_r consists of two curves $\partial L_{r,0}$, $\partial L_{r,\pi}$, corresponding to those parameter values, for which the Poincaré map of the corresponding dynamical system (1.4) acting on the circle $\{\tau = 0\}$ has fixed points 0 and π respectively. These are graphs

$$\partial L_{r,\alpha} = \{B = G_{r,\alpha}(A)\}, \quad G_{r,\alpha} \text{ are analytic functions on } \mathbb{R}; \ \alpha = 0, \pi.$$

Theorem 4.12 [44, *theorem* 2]. *There exist positive constants ξ_1, ξ_2, K_1, K_2, K_3 such that the following statement holds. Let $r \in \mathbb{Z}$, A, $\omega > 0$ be such that*

$$|r\omega| + 1 \leqslant \xi_1\sqrt{A\omega}, \quad A \geqslant \xi_2\omega. \tag{4.5}$$

Let J_r denote the rth Bessel function. Then

$$\left|\frac{1}{\omega}G_{r,0}(A) - r + \frac{1}{\omega}J_r\left(-\frac{A}{\omega}\right)\right| \leqslant \frac{1}{A}\left(K_1 + \frac{K_2}{\omega^3} + K_3\,\ln\left(\frac{A}{\omega}\right)\right), \tag{4.6}$$

$$\left|\frac{1}{\omega}G_{r,\pi}(A) - r - \frac{1}{\omega}J_r\left(-\frac{A}{\omega}\right)\right| \leqslant \frac{1}{A}\left(K_1 + \frac{K_2}{\omega^3} + K_3\,\ln\left(\frac{A}{\omega}\right)\right). \tag{4.7}$$

Proposition 4.13. *Fix an $\ell \in \mathbb{Z}$. For every $\eta_0 > 0$ the intersection*

$$\mathrm{Constr}_{\ell,\eta_0} := \mathrm{Constr}_\ell \cap \{0 < \eta < \eta_0\} \subset \mathbb{R}_+ \times (0, \eta_0)$$

is a one-dimensional analytic submanifold with infinitely many connected components, and each component is bounded.

Proof. Let $u_1 < u_2 < \ldots$ denote the sequence of points of local maxima of the modulus $|J_\ell(-u)|$, which tends to plus infinity.

Claim. *Fix an* $\eta_0 > 0$ *and an* $\ell \in \mathbb{Z}$. *For every* $k \in \mathbb{N}$ *large enough (dependently on* η_0 *and* ℓ*) the interval* $\widehat{I}_k := \{\mu = \frac{u_k}{2}\} \times (0, \eta_0)$ *does not intersect the constriction set* Constr_ℓ.

Proof. In the coordinates (μ, η) inequalities (4.5)–(4.7) can be rewritten for $r = \ell$ respectively as

$$|\frac{\ell}{\eta}| + 1 \leqslant \frac{\xi_1}{\eta}\sqrt{2\mu}, \quad \mu \geqslant \frac{\xi_2}{2}, \tag{4.8}$$

$$\left|\eta G_{\ell,0}\left(\frac{2\mu}{\eta}\right) - \ell + \eta J_\ell(-2\mu)\right| \leqslant \frac{\eta}{2\mu}\big(K_1 + K_2\eta^3 + K_3 \ln(2\mu)\big), \tag{4.9}$$

$$\left|\eta G_{\ell,\pi}\left(\frac{2\mu}{\eta}\right) - \ell - \eta J_\ell(-2\mu)\right| \leqslant \frac{\eta}{2\mu}\big(K_1 + K_2\eta^3 + K_3 \ln(2\mu)\big). \tag{4.10}$$

For every k large enough the value $\mu = \frac{u_k}{2}$ satisfies inequality (4.8) for all $\eta \in (0, \eta_0)$. Substituting $\mu = \frac{u_k}{2}$ to the right-hand side in (4.9) transforms it to a sequence of functions of $\eta \in (0, \eta_0)$ with uniform asymptotics $\eta(O(\frac{1}{u_k}) + O(\frac{\ln u_k}{u_k}))$, as $k \to \infty$. The values $|J_\ell(-u_k)|$ are known to behave asymptotically as $\frac{1}{\sqrt{u_k}}$ (up to a known constant factor). Therefore, they dominate the right-hand sides in (4.9) and (4.10). This together with (4.9), (4.10) implies that for every k large enough, set $A_k = \frac{u_k}{\eta}$,

$$G_{\ell,0}(A_k) = \ell\omega - J_\ell(-u_k)(1 + o(1)), \qquad G_{\ell,\pi}(A_k) = \ell\omega + J_\ell(-u_k)(1 + o(1)),$$

as $k \to \infty$, uniformly in $\omega > \omega_0 = \eta_0^{-1}$. This implies that $\ell\omega$ lies between $G_{\ell,0}(A_k)$ and $G_{\ell,\pi}(A_k)$ for large k. Therefore, for every k large enough and every $\omega > \omega_0$ the point $(\ell\omega, A_k;\ \omega)$ lies in the interior of the phase-lock area L_ℓ, and hence, is not a constriction. This proves the claim. □

For every point $q \in \mathrm{Constr}_\ell$ and every k large enough dependently on q the connected component of the point q in Constr_ℓ is separated from infinity by the segment $\widehat{I}_k$ from the above claim. This proves boundedness of connected components. Infiniteness of number of connected components follows from their boundedness and the fact that for every given $\ell \in \mathbb{Z}$ and $\omega > 0$ the vertical line $\Lambda_\ell = \{B = \omega\ell\}$ contains an infinite sequence of constrictions with A-ordinates converging to $+\infty$; the latter fact follows from [44, the discussion after definition 2] and [28, theorem 1.2]. This finishes the proof of proposition 4.13. □

Lemma 4.14. *For every* $\ell \in \mathbb{Z}$ *the subset* $\mathrm{Constr}_\ell \subset \mathbb{R}^2_+$ *does not accumulate to zero. That is, there exists no sequence of constrictions* $(B_k, A_k;\ \omega_k)$ *with* $B_k = \ell\omega_k$ *where* $\omega_k \to +\infty$ *and* $\mu_k := \frac{A_k}{2\omega_k} \to 0$, *as* $k \to \infty$.

For the proof of lemma 4.14 (given below) let us recall that the first equation in system (1.4) describing model of Josephson junction takes the following form in the new parameters μ and η:

$$\dot{\theta} := \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \eta \cos\theta + \ell + 2\mu\cos\tau, \quad \eta = \omega^{-1},\ \mu = \frac{A}{2\omega}. \tag{4.11}$$

The constrictions correspond to those values of $(\mu, \eta) \in \mathbb{R}^2_+$ for which the time 2π flow map

$$h = h^{2\pi} = h_{\mu,\eta}$$

of equation (4.11) acting on the θ-circle $\{\tau = 0\}$ is identity (proposition 1.11). For the proof of the lemma it suffices to show that $(0, 0)$ is an isolated point in the analytic subset $\{h_{\mu,\eta} = \mathrm{Id}\} \subset \mathbb{R}^2_{\mu,\eta}$. This is done by using the following formulae for a solution $\theta(\tau)$ of (4.11) and its derivatives in parameters for $\eta = 0$.

Proposition 4.15. *Let $\theta(\tau, \theta_0;\ \mu, \eta)$ denote the solution of equation (4.11) with initial condition $\theta(0) = \theta_0$. One has the following formulae for the solution and its partial derivatives in the parameters (μ, η):*

$$\theta(\tau, \theta_0; \mu, 0) = \theta_0 + \ell\tau + 2\mu \sin \tau, \qquad h_{\mu,0} = \mathrm{Id}, \tag{4.12}$$

$$\theta(\tau, \theta_0) := \theta(\tau, \theta_0, 0, 0) = \theta_0 + \ell\tau, \tag{4.13}$$

$$\theta'_\eta = \frac{\partial\theta}{\partial\eta} = \frac{1}{\ell}(\sin(\theta_0 + \ell\tau) - \sin\theta_0) \quad \text{at the locus } \{\mu = \eta = 0\}, \tag{4.14}$$

$$\theta'_\mu = 2 \sin \tau, \theta^{(k)}_{\mu\ldots\mu} = \frac{\partial^k\theta}{\partial\mu^k} = 0 \quad \text{for } k \geqslant 2 \text{ at the locus } \{\eta = 0\}. \tag{4.15}$$

The following two formulae hold at the locus $\{\mu = \eta = 0\}$:

$$\frac{\partial^2\theta}{\partial\eta^2} = -\frac{\tau}{\ell} + \frac{1}{2\ell^2}(\sin 2(\theta_0 + \ell\tau) - \sin 2\theta_0) - \frac{2}{\ell^2} \sin \theta_0(\cos(\theta_0 + \ell\tau) - \cos \theta_0); \tag{4.16}$$

$$\frac{\partial^{k+1}\dot\theta}{\partial\eta\partial\mu^k} = 2^k s_k(\theta_0 + \ell\tau)\sin^k \tau, \quad \text{where } s_k(y) = \begin{cases} (-1)^{\frac{k}{2}} \cos y \text{ for even } k \\ (-1)^{\frac{k+1}{2}} \sin y \text{ for odd } k. \end{cases} \tag{4.17}$$

Here 'dot' is the derivative in τ.

Proof. Formulae (4.12) and (4.13) are obvious. The equation in variations for the derivative θ'_η is

$$\dot\theta'_\eta = \cos(\theta_0 + \ell\tau + 2\mu \sin \tau) + O(\eta), \quad \text{as } \eta \to 0. \tag{4.18}$$

The derivative θ'_η is a solution of (4.18) vanishing at $\tau = 0$. Therefore, for $\mu = \eta = 0$ it is given by (4.14). Formulae (4.15) follow immediately by differentiating (4.12) in μ. Formula (4.17) follows by differentiating (4.18) in μ and taking the value thus obtained at $\mu = \eta = 0$. It remains to prove (4.16). Differentiating equation (4.11) in η twice at $\eta = \mu = 0$ and substituting (4.14) yields the following differential equation for the derivative $\theta''_{\eta\eta} = \frac{\partial^2\theta}{\partial\eta^2}$:

$$\dot\theta''_{\eta\eta} = -2 \sin \theta\theta'_\eta = -\frac{2}{\ell} \sin(\theta_0 + \ell\tau)(\sin(\theta_0 + \ell\tau) - \sin \theta_0).$$

Taking the primitive in τ of the right-hand side that vanishes at $\tau = 0$ yields (4.16). The proposition is proved. □

Proposition 4.16. *Let $\ell \in \mathbb{N}$. The Taylor expansion in (μ, η) of the time 2π flow map $h_{\mu,\eta}(\theta_0)$ takes the form*

$$h_{\mu,\eta}(\theta_0) = \theta_0 - \frac{\pi}{\ell}\eta^2 + g(\theta_0)\eta\mu^\ell + o(\eta^2) + o(\eta\mu^\ell), \quad \text{as } \mu, \eta \to 0, \tag{4.19}$$

where $g(\theta_0)$ is a non-constant function of θ_0 that is equal to either $\sin\theta_0$, or $\cos\theta_0$ up to non-zero constant factor.

Proof. The Taylor coefficient of the difference $h_{\mu,\eta}(\theta_0) - \theta_0$ at $\mu^k\eta^m$ at the locus $\mu = \eta = 0$ equals $\frac{1}{k!m!}\frac{\partial^{k+m}\theta}{\partial\mu^k\partial\eta^m}(\tau, \theta_0; 0, 0)$ where $\tau = 2\pi$. The latter derivative at $\tau = 2\pi$ vanishes for $(k, m) = (0, 1), (n, 0)$, by (4.14), (4.15); it equals $-\frac{2\pi}{\ell}$ for $(k, m) = (0, 2)$, by (4.16).

Claim. The above $(k, 1)$th derivative is 2π-periodic in τ, if $1 \leqslant k \leqslant \ell - 1$. If $k = \ell$, it is equal to $g(\theta_0)\tau$ plus a 2π-periodic function; here $g(\theta_0)$ has the same type, as in proposition 4.16.

Proof. The $(k, 1)$th derivative equals the primitive of the right-hand side in (4.17). The latter right-hand side is a linear combination of values of sin (or cos) of $\theta_0 + r\tau$, $r \in \mathbb{Z}$, $\ell - k \leqslant r \leqslant \ell + k$. Moreover, the coefficient at the 'lower term', the sin (cos) of $\theta_0 + (\ell - k)\tau$, is non-zero, by elementary trigonometry. Therefore, the primitive of the latter right-hand side in (4.17) is a linear combination of cos (sin) of the above arguments, except for a possible term with $r = 0$, which is $\tau\cos\theta_0$ ($\tau\sin\theta_0$) up to constant factor. For $k \leqslant \ell - 1$ the latter term does not arise. For $k = \ell$ it arises with a non-zero constant factor, by the above discussion. The claim is proved. □

One has $\frac{\partial^{k+m}\theta}{\partial\mu^k\partial\eta^m}(0, \theta_0; 0, 0) = 0$, by definition. This together with the above claim and discussion implies the statements of proposition 4.16. □

Proof of lemma 4.14. Suppose the contrary: the set Constr_ℓ accumulates to zero. Recall that it lies in the ambient analytic subset in $\mathbb{R} \times \mathbb{R}$ defined by the equation $h_{\mu,\eta} = \mathrm{Id}$. (The Poincaré map $h_{\mu,\eta}$ is Möbius, being the restriction to $S^1 = \{|\Phi| = 1\}$ of the monodromy map of Riccati equation (1.6); the equation $h_{\mu,\eta} = \mathrm{Id}$ is written in the Lie group $\mathrm{Aut}(D_1) \simeq \mathrm{PSL}_2(\mathbb{R})$.) Therefore, the latter analytic subset contains an irreducible germ of analytic curve Γ at 0 with $\eta|_\Gamma$, $\mu|_\Gamma \not\equiv 0$, since η, $\mu \neq 0$ on Constr_ℓ. Hence, Γ can be considered as a graph of (may be singular) analytic function $\mu = c\eta^\alpha(1 + o(1))$, $\alpha > 0$, $c \neq 0$. Substituting the latter expression for μ to the Taylor formula (4.19) yields

$$h_{\mu,\eta}(\theta_0) = \theta_0 - \frac{\pi}{\ell}\eta^2 + c^\ell g(\theta_0)\eta^{1+\ell\alpha} + o(\eta^2) + o(\eta^{1+\ell\alpha}). \tag{4.20}$$

The right-hand side in (4.20) should be identically equal to θ_0, since $h_{\mu,\eta} = \mathrm{Id}$ for $(\mu, \eta) \in \Gamma$. This together with (4.20) implies that its second and third terms should cancel out: $1 + \ell\alpha = 2$ and $g(\theta_0) \equiv c^{-\ell}\frac{\pi}{\ell}$. But we know that $g(\theta_0) \not\equiv \mathrm{const}$. The contradiction thus obtained proves lemma 4.14. □

Proof of the second statement of theorem 4.5. Suppose the contrary: as $x \in I$ tends to a non-zero endpoint $c \in \{a, b\}$ of the interval I, the function $\eta = \eta(C(x))$ is bounded from above. But then $\mu(C(x))$ is also bounded from above, by proposition 4.13. The component $\mathcal{C}$ being a non-compact submanifold in $\mathbb{R}^2_+$, it should go to 'infinity' (to the boundary), as $x \to c$. Therefore, there exists a sequence $x_k \to c$ such that $C(x_k) \to C^* \in \{\mu\eta = 0 | \mu, \eta \geqslant 0\}$ (boundedness of μ and η). One has $C^* \neq (0, 0)$, by lemma 4.14. Let show that two other possible cases treated below are impossible.

Case (1): $C^* = (0, \eta)$, $\eta > 0$. Then the equation (4.11) corresponding to $C(x_k) = (\mu_k, \eta_k)$ have identity Poincaré map and limit to the equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \eta\cos\theta + \ell, \tag{4.21}$$

which should also have identity Poincaré map. In the case, when $\ell = 0$, this is obviously impossible, since the dynamical system on $\mathbb{T}^2$ given by (4.21) is hyperbolic with an attracting periodic orbit $\theta \equiv \frac{\pi}{2}$. In the case, when $\ell \in \mathbb{N}$, the rotation number of the above system is an integer non-negative number $\rho < \ell$. This follows from the fact that the ℓ-th phase-lock area L_ℓ intersects the B-axis $\{A = 0\} = \{\mu = 0\}$ at the so-called growth point with known abscissa $B(\ell, \omega) = \sqrt{\ell^2\omega^2 + 1}$, $\omega = \eta^{-1}$, see [18, corollary 3], while $C(x_k)$ correspond to constrictions with the abscissas $\ell\omega_k < B(\ell, \omega_k)$. Therefore, the points $C(x_k) \in \mathrm{Constr}_\ell$ also correspond to the same rotation number $\rho < \ell$, whenever k is large enough (continuity of the rotation number function and its integer-valuedness on the points $(B_k, A_k;\ \omega_k)$ corresponding to $C(x_k)$). Thus, the points $C(x_k)$ correspond to constrictions lying on the axis $\Lambda_\ell = \{B = \ell\omega\}$ with non-negative rotation number $\rho < \ell$. But all the constrictions lying in Λ_ℓ should correspond to rotation numbers no less than ℓ, by [28, theorem 1.2]. The contradiction thus obtained shows that the case under consideration is impossible.

Case (2): $C^* = (\mu, 0)$, $\mu > 0$. Then the linear system (1.7) corresponding to C^* is diagonal, and hence, has zero cross-ratio $\mathcal{R} = x$. Hence, the cross-ratios x_k corresponding to $C(x_k)$ tend to zero. But their limit c is non-zero, by assumption. The contradiction thus obtained shows that case (2) is also impossible and finishes the proof of theorem 4.5. □

4.4. Constance of rotation number and type. Proof of theorem 1.12

Without loss of generality we can and will restrict ourselves to the case, when $\ell \in \mathbb{Z}_{\geqslant 0}$, due to symmetry.

All the statements of theorem 1.12 except for the last one follow immediately from theorem 4.5. Let us prove its last statement: constance of rotation number and type. Fix a connected component $\mathcal{C}$ of the manifold Constr_ℓ. Constance of the rotation number function on $\mathcal{C}$ follows from its continuity and integer-valuedness. Constance of the constriction type is obvious for $\ell = 0$: the A-axis lies in L_0, hence, all its constrictions are positive. Thus, everywhere below we consider that $\ell \in \mathbb{N}$ (symmetry). To prove constance of type, we use the following proposition. To state it, let us recall that for every $\omega > 0$ a *generalized simple intersection* is a point $(B, A;\ \omega)$ with $\ell = \frac{B}{\omega} \in \mathbb{Z}$, $A \neq 0$ and $\rho = \rho(B, A; \omega) \equiv \ell (\mathrm{mod}\ 2\mathbb{Z})$ that lies in the boundary of the phase-lock area $L_\rho = L_\rho(\omega)$ and that is not a constriction [30, definition 1.16]; they exist only for $\ell \neq 0$.

Proposition 4.17. *A constriction $C = (B, A;\ \omega)$ cannot be a limit of generalized simple intersections with some $\omega_k \to \omega$.*

Proof. One has $\ell = \frac{B}{\omega} \in \mathbb{Z}$. Without loss of generality we can and will consider that $\ell \geqslant 1$ (symmetry). Generalized simple intersections correspond to special double confluent Heun equations (1.9) having polynomial solution [13, theorem 1.15]. If, to the contrary, the constriction C were a limit of generalized simple intersections, then it would also corresponds to equation (1.9) having polynomial solution. But this is impossible, by [12, theorems 3.3 and 3.10]. The contradiction thus obtained proves the proposition. □

Let a constriction $C(x_0) \in \mathrm{Constr}_\ell$ be negative. Let us show that for every x close to x_0 the constriction $C(x) = (B(x), A(x);\ \omega(x))$ is also negative: the case of positive constriction is treated analogously. (Note that each constriction is either positive, or negative, by [29, theorem 1.8].) Let $\rho \in \mathbb{Z}$ denote the rotation number of the constriction $C(x_0)$. Set $\omega_0 := \omega(x_0)$, $\Lambda_\ell(\omega) := \{B = \ell\omega\} \subset \mathbb{R}^2_{B,A}$. For every $r > 0$ let $U_r \subset \mathbb{R}^2$ denote the disk of radius r centred at $(B(x_0), A(x_0))$. Fix an $r > 0$ such that $(B(x_0), A(x_0))$ is the only point of intersection $\partial L_\rho(\omega_0) \cap \Lambda_\ell(\omega_0)$ lying in U_{2r}. Such an r exists, since the latter intersection is discrete, by analyticity of the graphs $\partial L_{\rho,0}$, $\partial L_{\rho,\pi}$ forming ∂L_ρ, and since none of these graphs is a vertical line.

Case (1). Let for every x close enough to x_0 the point $(B(x), A(x))$ be the only point of intersection $\partial L_\rho(\omega(x)) \cap \Lambda_\ell(\omega(x))$ lying in U_r. Then all the above constrictions $C(x)$ have the same, negative type, by definition.

Case (2). Let now the unique point $C(x_0)$ of intersection $\partial L_\rho(\omega(x_0)) \cap \Lambda_\ell(\omega(x_0))$ split into several intersection points, as we perturb $x = x_0$ slightly. Then all these points are constrictions, by proposition 4.17 and since $\rho \equiv \ell(\mathrm{mod}\, 2)$, see [28, theorem 3.17]. Their number is finite, and they split the intersection $\Lambda_\ell(\omega(x)) \cap U_r$ into a finite number of intervals. Any two adjacent division intervals either both lie outside the phase-lock area $L_\rho(\omega(x))$, or both lie inside $L_\rho(\omega(x))$, since the constriction separating them is either negative, or positive (see [29, theorem 1.8] and remark 1.8). The division intervals adjacent to ∂U_r should lie outside, since this is true for $x = x_0$ and by continuity. Therefore, all the above intervals lie outside. Hence, all the constrictions bounding them are negative. Theorem 1.12 is proved.

5. Slow–fast methods. Absence of ghost constrictions for small ω

We prove theorem 1.13 in subsections 5.1–5.5. Theorem 1.10 will be proved in subsection 5.6.

It suffices to prove absence of ghost constrictions with $B = \omega\ell$, $\ell \in \mathbb{N}$, and $A > 0$, by symmetry and since the constrictions with $\ell = 0$ are positive and lie in L_0. Thus, everywhere below without loss of generality we consider that $\ell \in \mathbb{N}$. It is already known that

$$\text{there are no constrictions}\, (\ell\omega, A)\, \text{with}\, A \in (0, 1 - \ell\omega], \tag{5.1}$$

since all the points (B, A) with $|B| + |A| \leqslant 1$ lie in the phase-lock area L_0 [13, proposition 5.22], and all the constrictions in L_0 lie in the A-axis.

First in subsection 5.1 for small ω we prove absence of ghost constrictions in the semiaxis Λ_ℓ with ordinate greater than

$$A_{\frac{1}{2}} = A_{\frac{1}{2}}(\omega) := 1 + \left(\ell - \frac{1}{2}\right)\omega. \tag{5.2}$$

Their absence follows from results of [29, 30], which imply that the whole ray $\{\ell\omega\} \times [A_{\frac{1}{2}}, +\infty)$ lies in the phase-lock area L_ℓ for small ω. In subsection 5.5 we show that there are no constrictions $(\ell\omega, A)$ with $A \in (1 - \ell\omega, A_{\frac{1}{2}}(\omega))$, whenever ω is small enough. This is done by studying family of systems (1.4) modelling Josephson junction as a slow–fast family of dynamical systems, with small ω and $A = A_\alpha(\omega) = 1 + (\ell - \alpha)\omega + o(\omega)$. The corresponding background material on slow-fast systems is given in subsection 5.2. The key lemma used in the proof of absence of the above-mentioned constrictions is the monotonicity lemma stated and proved in subsection 5.4. It concerns a pair of slow–fast families (1.4) corresponding to two families of ordinates A_{α_1} and A_{α_2} as above with $0 < \alpha_1 < \alpha_2$. It deals with their Poincaré maps of the cross-section $\{\tau = 0\}$ lifted to the universal cover as maps of the line $\{\tau = 0\}$ to $\{\tau = 2\pi\}$. The monotonicity lemma states that the Poincaré map of the system (1.4) corresponding to A_{α_2} is less than the analogous Poincaré map for A_{α_1}, whenever ω is small enough. Its proof is based on the comparison lemma on arrangement and disjointness of slow flowboxes of the systems in question (stated and proved in subsection 5.3).

5.1. Absence of ghost constrictions with big ordinates

Lemma 5.1. *For every $\ell \in \mathbb{N}$ and every $\omega > 0$ small enough dependently on ℓ the ray*

$$\Lambda_{\ell,\frac{1}{2}} := \Lambda_\ell \cap \left\{A \geqslant A_{\frac{1}{2}}\right\} \subset \Lambda_\ell,$$

see (5.2), lies in the phase-lock area with the rotation number ℓ. It contains no ghost constrictions.

Proof. The intersection of the phase-lock area L_ℓ with the semiaxis $\Lambda_\ell^+ := \Lambda_\ell \cap \{A > 0\}$ contains a ray $S\ell$ bounded by a point $\mathcal{P}_\ell$, the so-called higher generalized simple intersection [29, theorem 1.12]. Therefore, for the proof of the inclusion $\Lambda_{\ell,\frac{1}{2}} \subset L_\ell$ it suffices to show that $A(\mathcal{P}_\ell) < A_{\frac{1}{2}}$ whenever ω is small enough. Let us show that

$$A(\mathcal{P}_\ell) = 1 + (\ell - 1)\omega + o(\omega), \quad \text{as } \omega \to 0. \tag{5.3}$$

To do this, let us recall the definition of the point $\mathcal{P}_\ell$. Set

$$\mu := \frac{A}{2\omega}, \quad \lambda := \frac{1}{4\omega^2} - \mu^2 = \frac{1 - A^2}{4\omega^2}.$$

Consider the corresponding Heun equation (1.9). Fix an $\omega > 0$. The value $\mu(\mathcal{P}_\ell) = \frac{A(\mathcal{P}_\ell)}{2\omega}$ is the maximal number $\mu > 0$ for which equation (1.9) has a polynomial solution, see [29, definition 1.9], [13, theorem 1.15]. It was shown in [19] that existence of polynomial solution is equivalent to the condition that the point $(\lambda, \ \mu)$ lies in a remarkable algebraic curve $\Gamma_\ell \subset \mathbb{R}^2$, the so-called *spectral curve*. Thus, for every $\omega > 0$ the point $(\lambda(\mathcal{P}_\ell), \mu(\mathcal{P}_\ell))$ lies in Γ_ℓ, and it is the point in Γ_ℓ with the biggest coordinate μ. As $\omega \to 0$, one has $\frac{1}{\omega} = \sqrt{4(\lambda + \mu^2)} \to \infty$, thus, $(\lambda, \ \mu) \to \infty$. It is known that the complexified curve Γ_ℓ intersects the complex infinity line in $\mathbb{CP}^2$ at ℓ distinct regular real points. Their asymptotic directions correspond to the ratios $\frac{\lambda}{\mu}$ equal to $\ell - 1, \ell - 3, \ldots, -(\ell - 1)$, and the corresponding local branches are real. This was proved by Netay [30, proposition 1.10]. Therefore, as a point of the curve Γ_ℓ tends to its infinite point, one has $\mu \to \infty$,

$$\lambda = O(\mu) = o(\mu^2), \qquad \frac{1}{4\omega^2} = \lambda + \mu^2 \simeq \mu^2, \qquad 2\omega\mu = A \simeq 1,$$

$$\frac{\lambda}{\mu} = \frac{1 - A^2}{4\omega^2\mu} \simeq k, \quad k \in \{\ell - 1, \ell - 3, \ldots, -(\ell - 1)\}.$$

But $\frac{1-A^2}{4\omega^2\mu} = \frac{(1-A)(1+A)}{2\omega A} \simeq \frac{1-A}{\omega}$. The latter ratio should tend to a number k as above. Therefore, as a point in Γ_ℓ tends to infinity, one of the following asymptotics takes place:

$$A = 1 + m\omega + o(\omega), \quad m = -k \in \{\ell - 1, \ell - 3, \ldots, -(\ell - 1)\}.$$

The asymptotics corresponding to points with the maximal possible A is given by $m = \ell - 1$. This proves (5.3). Hence, $A(\mathcal{P}_\ell) < A_{\frac{1}{2}} = 1 + (\ell - \frac{1}{2})\omega$, whenever ω is small enough, by (5.3). The inclusion $\Lambda_{\ell,\frac{1}{2}} \subset L_\ell$ is proved. It implies that all the constrictions in $\Lambda_{\ell,\frac{1}{2}}$ are positive, lie in L_ℓ, and hence, are not ghost. The lemma is proved. □

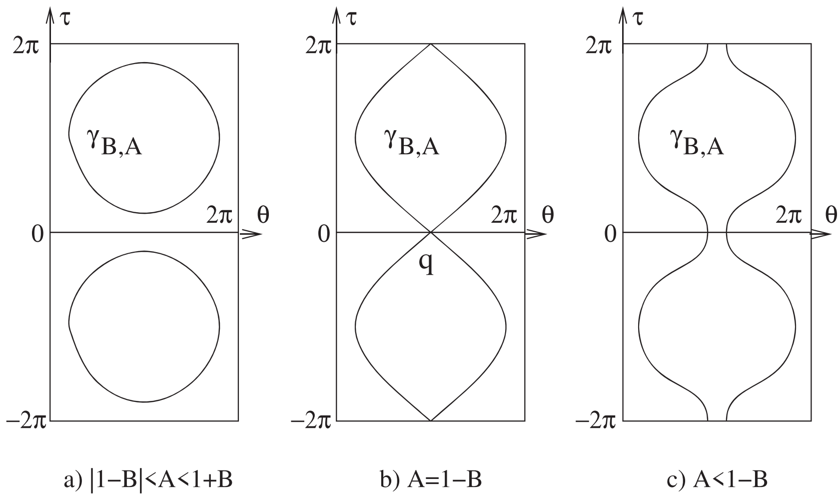

Figure 6. Different topological types of the curve $\gamma_{B,A} = \{f(\theta,\tau) = 0\}$ for $B, A > 0$, $A < 1 + B$. We present its liftings to the universal covering $\mathbb{R}^2_{\theta,\tau}$.

5.2. Model of Josephson junction with small ω as slow–fast system

We study one-parameter subfamilies of vector fields (1.4) on $\mathbb{T}^2$ parametrized by small ω as slow–fast families of dynamical systems, where $\ell = \frac{B}{\omega} \equiv \text{const}$ and A depends on ω. To do this, we recall the following results on topology of the zero level curve of the θ-component in (1.4): the so-called slow curve

$$\gamma = \gamma_{B,A} := \{f(\theta,\tau) = 0\}, \quad f(\theta,\tau) := \cos\,\theta + B + A\,\cos\,\tau.$$

Proposition 5.2 (see [43, proposition 2]). *For every $(A, B) \in \mathbb{R}^2_+$ with $|1 - B| < A < 1 + B$ the curve γ is a regular strictly convex contractible curve lying in the interior of the fundamental square $[0, 2\pi]^2$ of the torus $\mathbb{T}^2$. See figure 6(a)).*

Remark 5.3. The curve γ is always symmetric with respect to the horizontal and vertical lines through the centre of the latter square.

For completeness of presentation we give the proof of proposition 5.2.

Proof of proposition 5.2. Let $1 - B < A < 1 + B$. Let us now show that the curve γ does not intersect the boundary of the above fundamental square. Indeed, on the boundary either $\cos\theta = 1$, or $\cos\tau = 1$. If $\cos\theta = 1$, then $f(\theta,\tau) = \cos\theta + B + A\cos\tau \geqslant 1 + B - A > 0$. If $\cos\tau = 1$, then $f(\theta,\tau) \geqslant -1 + B + A > 0$. Therefore, $f(\theta,\tau) \neq 0$ on the boundary of the fundamental square, and γ lies in its interior. For the proof of strict convexity it suffices to show that the value[9] of the Hessian form of the function f on its skew gradient tangent to its level curves is positive on γ. That is,

$$\frac{\partial^2 f}{\partial\theta^2}\left(\frac{\partial f}{\partial\tau}\right)^2 + \frac{\partial^2 f}{\partial\tau^2}\left(\frac{\partial f}{\partial\theta}\right)^2 - 2\frac{\partial^2 f}{\partial\theta\partial\tau}\left(\frac{\partial f}{\partial\tau}\right)\left(\frac{\partial f}{\partial\theta}\right) > 0 \quad \text{on } \gamma. \tag{5.4}$$

[9] The value of the Hessian form of a function f on its skew gradient, i.e., the expression in the left-hand side in (5.4) was introduced by Tabachnikov in [64].

Substituting $u := \cos\theta$, $v := \cos\tau$ to the latter left-hand side and dividing it by A yields the following equivalent inequality:

$$-Au(1-v^2) - v(1-u^2) > 0, \quad \text{whenever } u + B + Av = 0 \text{ and } |u|, |v| \leqslant 1. \tag{5.5}$$

Substituting $u = -B - Av$ to the left-hand side in (5.5) transforms it to the polynomial

$$P(v) = ABv^2 + v(A^2 + B^2 - 1) + AB.$$

One has $P(v) > 0$ for every $v \in \mathbb{R}$, since its discriminant is negative, i.e., $-2AB < A^2 + B^2 - 1 < 2AB$. Indeed, the latter inequality can be rewritten as $|A - B| < 1 < A + B$, which is equivalent to the system of inequalities of the proposition for positive A and B. The proposition is proved. □

Proposition 5.4. *In the case, when $A, B > 0$ and $A = 1 - B$, the curve γ is regular, except for one singular point $q = (\pi, 0)$ of type 'transversal double self-intersection'. Its intersection with the interior of the fundamental square $[0, 2\pi]^2$ is a convex curve. In the case, when $B > 0$ and $0 < A < 1 - B$, the curve γ is regular and consists of two non-contractible closed connected components of homological type $(0, 1)$ in the standard basis in $H_1(\mathbb{T}^2_{\theta,\tau})$. See figures 6(b) and (c).*

Proof. Consider the first the case: $A = 1 - B$. Convexity is preserved under passing to limit, as $A > 1 - B$ tends to $1 - B$. The singular point statement and uniqueness of singular point follow by straightforward calculation, the implicit function theorem and Morse lemma. In more detail, γ being a level curve of an analytic function $f(\theta, \tau)$, its singular points (if any) are the critical points of the function f contained in γ. The critical points are those with $\cos\theta$, $\cos\tau = \pm 1$. The only critical point in γ is the one with $\cos\theta = -1$, $\cos\tau = 1$, i.e., $q = (\pi, 0)$. This is a Morse critical point with index -1, i.e., the Hessian form of the function f at q has eigenvalues of opposite signs: $\frac{\partial^2 f}{\partial\theta\partial\tau} = 0$, $\frac{\partial^2 f}{\partial\theta^2} = 1$, $\frac{\partial^2 f}{\partial\tau^2} = -A < 0$. Hence, it is a transversal self-intersection singular point of the curve γ (Morse lemma). See figure 6(b)).

As A and B vary, the topological type of the curve γ may change only near those parameter values, for which γ is a critical level curve of the function $f(\theta, \tau)$. It follows from the above critical point description that γ is a critical level curve, if and only if $\pm 1 + B \pm A = 0$ for some of the four possible sign choices. Therefore, the topological type is constant in the domain $\{B > 0, 0 < A < 1 - B\}$ in the parameter space. To find this topological type, fix a point $(B_0, A_0) \in \mathbb{R}^2_+$ with $A_0 = 1 - B_0$. We show that as $A > A_0$ decreases and crosses the value A_0, a connected contractible curve $\gamma_{B_0,A}$ given by proposition 5.2 is transformed to two disjoint curves isotopic to the τ-circle. For A close to A_0 the complement of each $\gamma_{B_0,A}$ to a small disk U centred at the singular point q is a regular curve depending analytically on the parameter A. It consists of two connected components $\gamma_{B_0,A;\pm}(U)$ disjoint from the circle $\{\theta = \pi\}$ and projected diffeomorphically to an interval $(\varepsilon, 2\pi - \varepsilon)$ of the τ-circle; here $\varepsilon = \varepsilon(U)$ is small. (The projection interval is the same for both components, since the symmetries $\theta \mapsto -\theta$, $\tau \mapsto -\tau$ preserve each curve $\gamma_{B,A}$.) The curves $\gamma_{B_0,A} \cap U$ form a singular foliation in U by level curves of the function $g(\theta, \tau) := -\frac{1}{\cos\tau}(\cos\theta + B_0)$ with critical value A_0 corresponding to a Morse critical point q of index 1. The union of local branches of the singular curve γ_{B_0,A_0} at q is invariant under the above symmetries, and the local branches intersect transversally. Therefore, they are transversal to the circles $\{\theta = \pi\}$, $\{\tau = 0\}$. For $A > A_0$ close to A_0 the curve $\gamma_{B_0,A}$ is strictly convex, and its intersection with U is a union of two connected components separated by the circle $\{\tau = 0\}$, by proposition 5.2. This implies that for $A < A_0$ close to A_0

the local level curve $\gamma_{B_0,A} \cap U$ consists of two components intersecting the circle $\{\tau = 0\}$, diffeomorphically projected to an interval in the τ-circle and disjoint from the circle $\{\theta = \pi\}$. Adding the latter components to $\gamma_{B_0,A;\pm}(U)$ results in two closed curves in $\mathbb{T}^2$ disjoint from the circle $\{\theta = \pi\}$ and projected diffeomorphically onto the τ-circle. See figure 6(c). Thus, they are isotopic to the τ-circle. This proves the last statement of proposition 5.4. □

Consider family (1.4) with a fixed $\ell \in \mathbb{N}$ and $\mu = \frac{A(\omega)}{2\omega}$ where

$$A(\omega) = A_\alpha(\omega) = 1 + (\ell - \alpha)\omega + o(\omega), \quad \text{as } \omega \to 0;\ \alpha > 0 \text{ is a constant.} \tag{5.6}$$

Multiplying family (1.4) by ω yields a slow-fast family of dynamical systems

$$\begin{cases} \dot{\theta}_t = f_\alpha(\theta, \tau; \omega) \\ \dot{\tau}_t = \omega, \end{cases} \qquad t = \omega^{-1}\tau,\ \ f_\alpha(\theta, \tau; \omega) = \cos\theta + \ell\omega + A_\alpha(\omega)\cos\tau \tag{5.7}$$

on $\mathbb{T}^2$ with $\omega \to 0$. According to the commonly used terminology in the theory of slow-fast systems, see e.g. [35], we will call the curve

$$\gamma_\alpha(\omega) := \{f_\alpha(\theta, \tau; \omega) = 0\} \subset \mathbb{T}^2 = \mathbb{R}^2_{(\theta,\tau)}/2\pi\mathbb{Z}$$

the *slow curve* of family (5.7). Propositions 5.2 and 5.4 imply the following.

Corollary 5.5. *For every fixed $\ell, \alpha \in \mathbb{R}_+$ with $\alpha \neq 2\ell$, for every ω small enough dependently on ℓ and α*

(*a*) *If $0 < \alpha < 2\ell$, then the slow curve of system (5.7) is convex, regular, contractible and lies in the interior of the fundamental square $[0, 2\pi]^2$;*

(*b*) *If $\alpha > 2\ell$, then the slow curve is regular and consists of two non-contractible closed connected components of homological type $(0, 1)$.*

Remark 5.6. Fix an arbitrary $\alpha > 0$. As $\omega \to 0$, the slow curve tends to the square with vertices $(0, \pi)$, $(\pi, 2\pi)$, $(2\pi, \pi)$, $(\pi, 0)$, whose sides lie in the lines $\{\theta + \tau = 2\pi \pm \pi\}$, $\{\tau - \theta = \pm\pi\}$. The corresponding vector fields converge to a vector field with zero τ-component and whose θ-component has simple zeros on the edges of the above square (with vertices deleted).

We deal with the liftings to the universal cover $\mathbb{R}^2$ over $\mathbb{T}^2$ of vector fields (5.7) and their phase portraits. The lifted fields will be denoted by the same symbol (5.7). The slow curve $\gamma_\alpha = \gamma_\alpha(\omega) \subset \mathbb{T}^2$ will be identified with its lifting γ_α^0 to the square $[0, 2\pi]^2 \subset \mathbb{R}^2$. Its other lifting, obtained from the latter one by translation by the vector $(2\pi, 0)$ will be denoted by γ_α^1.

Definition 5.7. The *interior component* of the complement $\mathbb{T}^2 \setminus \gamma_\alpha$ is its connected component containing the point (π, π). Its liftings to the squares $[0, 2\pi]^2$ and $[2\pi, 4\pi] \times [0, 2\pi]$ will be called the interior components of the complements of the latter squares to the curves γ_α^0 and γ_α^1 respectively.

Fix constants h_0, h_1, h_2 such that

$$\frac{3\pi}{2} < h_0 < h_1 < h_2 < 2\pi.$$

For example, one can take, $h_0 = \frac{6.5\pi}{4}$, $h_1 = \frac{7\pi}{4}$, $h_2 = \frac{15\pi}{8}$.

Proposition 5.8. *For every $\omega > 0$ small enough the restriction of the function $f_\alpha(\theta, \tau) := f_\alpha(\theta, \tau;\ \omega)$ to the rectangle $[0, 4\pi] \times [0, 2\pi]$ is negative exactly in the interior*

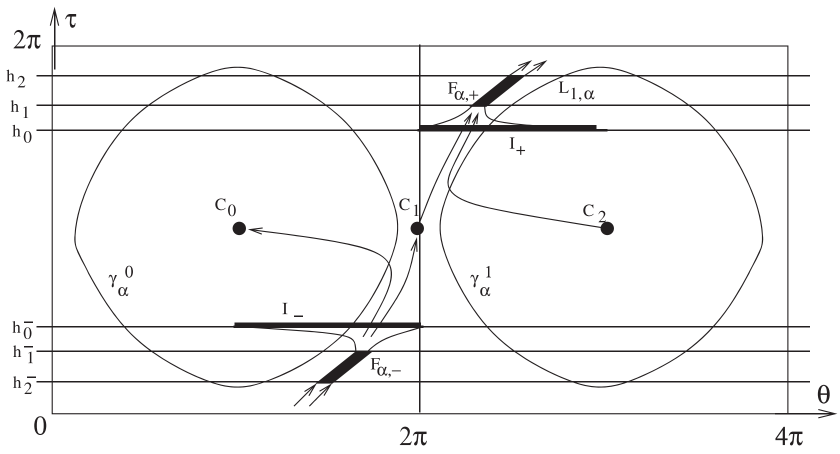

Figure 7. The slow flowboxes $F_{\alpha,\pm}$ (black) and orbits of points C_j.

components of complements of the curves γ_α^j, $j = 0, 1$, *and positive outside the closure of the latter components. The strip*

$$\Pi := \{h_1 \leqslant \tau \leqslant h_2\}$$

intersects the curve γ_α^1 *by two disjoint graphs* (*called left and right*)

$$L_{1,\alpha} := \{\theta = \psi_1(\tau)\}, \qquad L_{2,\alpha} := \{\theta = \psi_2(\tau)\}, \quad \tau \in [h_1, h_2],\ \psi_1 < \psi_2.$$

The latter graphs converge uniformly in the C^1*-norm to segments parallel to the lines* $\{\tau = \theta\}$ *and* $\{\tau = -\theta\}$ *respectively, as* $\omega \to 0$.

Proposition 5.8 follows from remark 5.6.

Proposition 5.9. *Let* $\alpha > 0$. *Let* $I_+ \subset \mathbb{R}^2$ *denote the horizontal segment connecting the points* $(2\pi, h_0)$ *and* $(3\pi, h_0)$. *The intersection of the strip* Π *with the orbit of the segment* I_+ *by flow of vector field* (5.7) *is a flowbox denoted by*

$$F_{\alpha,+} = F_{\alpha,+}(\omega).$$

It will be called a ***slow flowbox***. *Its flow lines are uniformly* ω*-close to* $L_{1,\alpha}$ *in the* C^1*-norm. The intersections* $F_{\alpha,+} \cap \{\tau = h\}$ *with* $h \in [h_1, h_2]$ *are segments whose lengths are uniformly bounded* (*in* h, ω) *by an exponentially small quantity* $\exp(-\frac{c}{\omega})$; $c > 0$ *is independent on* h *and* ω. *See figure* 7.

Proof. The proposition follows from proposition 5.8 and the classical theory of slow–fast systems. See, e.g., [35, theorem 3 and proposition 4]. □

Remark 5.10. The phase-portrait of vector field (5.7) is symmetric with respect to the points

$$C_0 := (\pi, \pi), \qquad C_1 := (2\pi, \pi), \qquad C_2 := (3\pi, \pi);$$

the symmetry changes the sign (i.e., orientation) of the field. Let I_- denote the horizontal segment symmetric to I_+ with respect to the point C_1, see figure 7. The above construction applied to the inverse vector field, the segment I_- and the heights $h_j^- := 2\pi - h_j$ yields the slow flowbox

$$F_{\alpha,-} \text{ symmetric to } F_{\alpha,+} \text{ with respect to the point } C_1.$$

5.3. The comparison lemma

Lemma 5.11 ***(comparison lemma)***. *Let $0 < \alpha_1 < \alpha_2$. Consider two families $(5.7)_j$, $j = 1, 2$, of dynamical systems (5.7) with $A = A_{\alpha_j}(\omega)$ satisfying (5.6). For every $\omega > 0$ small enough the corresponding flowboxes F_{α_1+} and $F_{\alpha_2,+}$ are disjoint and $F_{\alpha_2,+}$ lies on the left from the flowbox $F_{\alpha_1,+}$. Similarly, the flowboxes F_{α_1-} and $F_{\alpha_2,-}$ are disjoint and $F_{\alpha_2,-}$ lies on the right from the flowbox $F_{\alpha_1,-}$.*

It suffices to prove the statement of the lemma for the flowboxes $F_{\alpha_j,+}$, by symmetry (remark 5.10). Here and below we use the next proposition.

Proposition 5.12. *For every ω small enough the following statements hold. The vectors of the fields $(5.7)_1$ and $(5.7)_2$ form a positively oriented basis at each point of the union of two strips*

$$W := \left\{0 \leqslant \tau < \frac{\pi}{2}\right\} \cup \left\{\frac{3\pi}{2} < \tau \leqslant 2\pi\right\}.$$

At each point in the $\frac{\omega}{8}$-neighbourhood of the flowbox $F_{\alpha_1,+}$ the angles between the vectors of the fields are greater than $\sigma := \arctan(2 + b) - \arctan 2$, $b = \frac{\cos h_0}{2}(\alpha_2 - \alpha_1)$. The image of the flowbox $F_{\alpha_1,+}$ under the unit time flow map of the field $(5.7)_2$ is disjoint from $F_{\alpha_1,+}$, and its intersection with the strip Π lies on the left from $F_{\alpha_1,+}$.

Proof. The vectors of the fields $(5.7)_1$ and $(5.7)_2$ have the same τ-component equal to ω. The difference of their θ-components is $f_{\alpha_1}(\theta, \tau; \omega) - f_{\alpha_2}(\theta, \tau; \omega) = (\alpha_2 - \alpha_1)\omega(1 + o(1)) \cos \tau > 0$ on W, whenever ω is small enough, since $\cos \tau > 0$ on W. Therefore, the vectors of the field $(5.7)_2$ are directed to the left from the vectors of the field $(5.7)_1$ on W, that is, the orientation statement of the proposition holds. For every ω small enough one has

$$f_{\alpha_1}(\theta, \tau; \omega) - f_{\alpha_2}(\theta, \tau; \omega) > b\omega, \quad b := \frac{\cos h_0}{2}(\alpha_2 - \alpha_1), \quad \text{if } \tau \in [h_0, h_2], \tag{5.8}$$

by the above asymptotics, and also

$$\frac{\omega}{4} < f_{\alpha_1}(\theta, \tau; \omega) < 2\omega \quad \text{on the } \frac{\omega}{8} - \text{neighbourhood of } F_{\alpha_1,+}. \tag{5.9}$$

Indeed, the flow lines of the field $(5.7)_1$ in $F_{\alpha_1,+}$ C^1-converge to the line $\tau = \theta - \pi$ (propositions 5.8 and 5.9), hence $f_{\alpha_1}(\theta, \tau; \omega) \simeq \omega$ on $F_{\alpha_1,+}$. This together with (5.6), (5.7) and the obvious inequality $|\cos' x| = |\sin x| \leqslant 1$ implies (5.9). The angle lower bound statement of proposition 5.12 follows from (5.8) and (5.9). Its last statement on image of the flowbox $F_{\alpha_1,+}$ under the unit time flow map of the field $(5.7)_2$ follows from the above angle bound and the fact that the vectors of the field $(5.7)_2$ have length no less than ω, while the width of the flowbox $F_{\alpha_1,+}$ is exponentially small (the last statement of proposition 5.9). Proposition 5.12 is proved. □

Proof of the comparison lemma. **Claim**. For every $\omega > 0$ small enough for every $p \in W$ the positive flow line of the field $(5.7)_2$ through p in W lies on the left from the corresponding flow line of the field $(5.7)_1$.

The claim follows from the orientation statement of proposition 5.12.

Fix an intermediate number $h_1' \in (h_0, h_1)$. Consider the flowbox $F'_{\alpha_1,+}$ constructed as in proposition 5.9 with Π replaced by $\Pi' := \{h_1' \leqslant \tau \leqslant h_2\}$. One obviously has $\Pi \cap F'_{\alpha_1,+} = F_{\alpha_1,+}$. The lengths of horizontal sections of the flowbox $F'_{\alpha_1,+}$ are uniformly bounded by a quantity $\exp(-\frac{d}{\omega})$, with $d > 0$ independent on ω (proposition 5.9). Take the lower horizontal base of the flowbox $F'_{\alpha_1,+}$, which is a segment in the line $\{\tau = h_1'\}$ with length bounded by the above exponent. Let $q_1 := (\chi_1, h_1')$ denote its right boundary point, which lies in the $(5.7)_1$-orbit of the end $(3\pi, h_0)$ of the segment I_+.

Consider the analogous flowbox $F'_{\alpha_2,+}$ and point $q_2 := (\chi_2, h_1')$ for the field $(5.7)_2$. One has $\chi_2 < \chi_1$, by the claim. First suppose that $q_2 \notin F'_{\alpha_1,+}$. Then the lower base of the flowbox $F'_{\alpha_2,+}$ is disjoint from the flowbox $F'_{\alpha_1,+}$ and lies on its left. This together with the above claim implies that the flowboxes are disjoint. In the case, when $q_2 \in F'_{\alpha_1,+}$, the image q_2' of the point q_2 under the time 1 flow map of the field $(5.7)_2$ would lie strictly to the left from the flowbox $F'_{\alpha_1,+}$, by proposition 5.12. Therefore, the positive orbit of the point q_2' also lies on its left, by the claim. Note that

$$\tau(q_2') = \tau(q_2) + \omega = h_1' + \omega < h_1,$$

whenever ω is small enough. Therefore, the above positive orbit intersects the strip $\Pi = \{h_1 \leqslant \tau \leqslant h_2\}$ by an arc of curve going from its lower base to its upper base and lying on the left from the flowbox $F_{\alpha_1,+}$. The latter curve bounds $F_{\alpha_2,+}$ from the right, by construction. Hence, $F_{\alpha_2,+}$ is disjoint from $F_{\alpha_1,+}$ and lies on its left. The comparison lemma is proved. □

5.4. The monotonicity lemma

Consider two families of vector fields $(5.7)_j$, $j = 1, 2$ (treated as fields lifted to $\mathbb{R}^2$), as in the comparison lemma, corresponding to $\alpha_1 > 0$ and $\alpha_2 > \alpha_1$. We study their *Poincaré maps* $P_j^{\tau_1,\tau_2}$: the time $\frac{\tau_2-\tau_1}{\omega}$ flow maps from the line $\{\tau = \tau_1\}$ to the line $\{\tau = \tau_2\}$ considered as functions of the coordinate θ. For simplicity, we denote

$$P_j(\theta) := P_j^{0,2\pi}(\theta).$$

Lemma 5.13 ***(monotonicity lemma)***. *For every $\omega > 0$ small enough*

$$P_2(\theta) < P_1(\theta) \quad \text{for every } \theta \in \mathbb{R}. \tag{5.10}$$

Lemma 5.13 is proved below. In its proof we use the following proposition.

Proposition 5.14. *Let $C_0, C_1, C_2, h_k^-, F_{\alpha_j,-}$ be the same, as in remark 5.10. The intersection of the positive orbit of the segment $[C_1, C_2]$ under the flow of the field $(5.7)_j$ with the strip $\Pi = \{h_1 \leqslant \tau \leqslant h_2\}$ lies in the flowbox $F_{\alpha_j,+}$. The intersection of the negative orbit of the segment $[C_0, C_1]$ with the strip $\Pi_- := \{h_2^- \leqslant \tau \leqslant h_1^-\}$ lies in $F_{\alpha_j,-}$. See figure 7.*

Proof. It suffices to prove the first statement of the proposition, due to symmetry (remark 5.10). The segment I_+ defining the flowbox $F_{\alpha_j,+}$ is horizontal and is obtained from the segment $[C_1, C_2]$ by vertical shift up. The shift length is fixed and equal to $h_0 - \pi > 0$. Let J_l and J_r denote respectively the segment connecting $C_1(C_2)$ to the left (respectively, right) endpoint of the segment I_+. One has $f_{\alpha_j} > 0$ on J_l and $f_{\alpha_j} < 0$ on J_r, which follows from remark 5.6 and proposition 5.8. Thus, on the segment J_l (J_r) the vectors of the field $(5.7)_j$ are

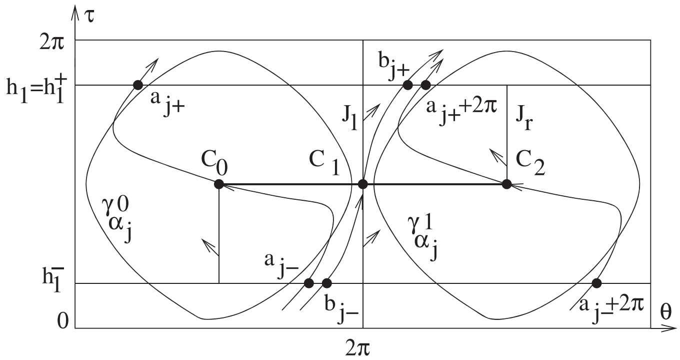

Figure 8. Orbits of segments $[C_0, C_1]$, $[C_1, C_2]$ and points $a_{j\pm}$, $b_{j\pm}$.

directed to the right (respectively, left). This implies that the time $\frac{h_0-\pi}{\omega}$ flow map of the field sends the segment $[C_1, C_2]$ strictly inside the segment I_+. This together with the definition of the flowbox $F_{\alpha_j,\omega}$ implies the first statement of the proposition. $\square$

Proof of the monotonicity lemma. Set $h_j^+ := h_j$. One has

$$P_j = P_j^{h_1^+,2\pi} \circ \widetilde{P}_j \circ P_j^{0,h_1^-}, \qquad \widetilde{P}_j := P_j^{h_1^-,h_1^+}. \tag{5.11}$$

Claim 1. Whenever ω is small enough, one has $P_2^{0,h_1^-}(\theta) < P_1^{0,h_1^-}(\theta)$, $P_2^{h_1^+,2\pi}(\theta) < P_1^{h_1^+,2\pi}(\theta)$ for every $\theta \in \mathbb{R}$.

Proof. Vector fields $(5.7)_2$ and $(5.7)_1$ have the same τ-components. On the set $\{\tau \in [0, h_1^-] \cup [h_1^+, 2\pi]\}$ the θ-component of the former vector field is less than that of the latter, since $\cos\tau > 0$ on this set. This implies the inequalities of the claim. $\square$

Taking into account claim 1 and (5.11), for the proof of the monotonicity lemma it suffices to prove the above inequality for the middle Poincaré maps in (5.11) for all ω small enough:

$$\widetilde{P}_2(\theta) < \widetilde{P}_1(\theta) \quad \text{for every } \theta \in \mathbb{R}. \tag{5.12}$$

Consider the horizontal lines $L_\pm := \{\tau = h_1^\pm\}$, which are the cross-sections for the Poincaré maps in question. We identify each their point with its θ-coordinate. For every $j = 1, 2$ let $b_{j\pm}$ denote the point of intersection of the line $L_\pm$ with the orbit of vector field $(5.7)_j$ through the point $C_1 = (2\pi, \pi)$. Let $a_{j\pm}$ denote the analogous intersection points with the orbit through the point $C_0 = (\pi, \pi)$. See figure 8.

Claim 2. *One has*

$$a_{1-} < b_{1-} < a_{2-} < b_{2-} < a_{1-} + 2\pi,$$

$$a_{2+} < a_{1+} < b_{2+} < a_{2+} + 2\pi < b_{1+} < a_{1+} + 2\pi. \tag{5.13}$$

Proof. The points a_{j-} and b_{j-} are the images of the points C_0 and C_1 respectively under the Poincaré map P_j^{π,h_1^-}, and $\theta(C_0) < \theta(C_1)$, by definition. Hence, $a_{j-} < b_{j-}$. The segment $[a_{j-}, b_{j-}]$ lies in the flowbox $F_{\alpha_j,-}$, by proposition 5.14. The flowbox $F_{\alpha_1,-}$ is disjoint from the flowbox $F_{\alpha_2,-}$ and lies on the left from it, by the comparison lemma. Therefore, the same is true for the corresponding segments $[a_{1-}, b_{1-}]$ and $[a_{2-}, b_{2-}]$. The four endpoints of the latter segments are $O(\omega)$-close to each other. Indeed the flowboxes in question are $O(\omega)$-close to the right arcs of the corresponding intersections $\gamma^0_{\alpha_j} \cap \{h_2^- \leqslant \tau \leqslant h_1^-\}$ (proposition 5.9). The latter arcs are ω-close, which follows from the implicit function theorem for the equations defining the curves $\gamma^0_{\alpha_j}$. This together with the above discussion proves $O(\omega)$-closeness of the four points a_{j-} and b_{j-}, $j = 1, 2$. This proves the first part of inequality (5.13). The proof of its second part is analogous. □

Proof of inequality (5.12). It suffices to prove it on the segment $K := [a_{1-}, a_{1-} + 2\pi] \subset L_-$, by periodicity. The segment K is split into four subsegments by points a_{j-}, b_{j-}. We check inequality (5.12) on each splitting subsegment.

(a) Let us start with the segment $[b_{2-}, a_{1-} + 2\pi]$. One has

$$\widetilde{P}_1([b_{2-}, a_{1-} + 2\pi]) \subset \widetilde{P}_1([b_{1-}, a_{1-} + 2\pi]) = [b_{1+}, a_{1+} + 2\pi],$$

$$\widetilde{P}_2([b_{2-}, a_{1-} + 2\pi]) \subset \widetilde{P}_2([b_{2-}, a_{2-} + 2\pi]) = [b_{2+}, a_{2+} + 2\pi],$$

by (5.13). The latter segment-image in the right-hand side is disjoint from the former one and lies on the left from it, by (5.13). This proves inequality (5.12) on the segment $[b_{2-}, a_{1-} + 2\pi]$.

(b) Now we consider the segment $[a_{2-}, b_{2-}]$. One has

$$\widetilde{P}_1([a_{2-}, b_{2-}]) \subset \widetilde{P}_1([b_{1-}, a_{1-} + 2\pi]) = [b_{1+}, a_{1+} + 2\pi],$$

$$\widetilde{P}_2([a_{2-}, b_{2-}]) = [a_{2+}, b_{2+}], \ b_{2+} < b_{1+},$$

by (5.13). This proves inequality (5.12) on $[a_{2-}, b_{2-}]$.

(c) The segment $[b_{1-}, a_{2-}]$. One has

$$\widetilde{P}_1([b_{1-}, a_{2-}]) \subset \widetilde{P}_1([b_{1-}, a_{1-} + 2\pi]) = [b_{1+}, a_{1+} + 2\pi],$$

$$\widetilde{P}_2([b_{1-}, a_{2-}]) \text{ lies on the left from the point } a_{2+} = \widetilde{P}_2(a_{2-}) < b_{1+},$$

by (5.13). This proves inequality (5.12) on $[b_{1-}, a_{2-}]$.

(d) The segment $[a_{1-}, b_{1-}]$. One has

$$\widetilde{P}_1([a_{1-}, b_{1-}]) = [a_{1+}, b_{1+}],$$

$$\widetilde{P}_2([a_{1-}, b_{1-}]) \text{ lies on the left from the point } a_{2+} = \widetilde{P}_2(a_{2-}) < a_{1+},$$

since $b_{1-} < a_{2-}$, see (5.13). This proves inequality (5.12) on $[a_{1-}, b_{1-}]$. Inequality (5.12) is proved on all of the segment K, and hence, on the whole horizontal line L_-. □

The statement of the monotonicity lemma follows from (5.11), claim 1 and inequality (5.12). □

5.5. Absence of constrictions with small ordinates

Here we prove the following theorem and then theorem 1.13.

Theorem 5.15. *For every $\ell \in \mathbb{N}$, $\beta > 0$ and every $\omega > 0$ small enough dependently on ℓ and β there are no constrictions (B, A) with $B = \ell\omega$ and $A \in [1 - \ell\omega, 1 + (\ell - \beta)\omega]$.*

Remark 5.16. Absence of constrictions with $B - 1 < A < B + 1$, $B = \ell\omega$, for small ω was numerically observed in [43, figures 2 and 3]. Theorem 5.15 confirms a part of this experimental result theoretically.

Proof of theorem 5.15. It suffices to prove the statement of the theorem for arbitrarily small β, e.g., $\beta < \frac{1}{2}$. Set $B = \ell\omega$. For every $\alpha > 0$ set $A_\alpha = A_\alpha(\omega) = 1 + (\ell - \alpha)\omega$. The family of systems (5.7) defined by this ordinate family A_α will be denoted by $(5.7)_\alpha$.

Suppose the contrary: there exists a sequence $\omega_k \to 0$ such that there exists a sequence of constrictions (B_k, A_{α_k}) with

$$B_k = \ell\omega_k, \qquad A_{\alpha_k} = 1 + (\ell - \alpha_k)\omega_k, \quad \beta \leqslant \liminf \alpha_k \leqslant \limsup \alpha_k \leqslant 2\ell.$$

Passing to a subsequence, without loss of generality we can and will consider that α_k converge to some $\alpha^* \geqslant \beta$. Thus, the sequence of dynamical systems corresponding to the above (B_k, A_{α_k}) can be embedded into a continuous family of systems (5.7) with α replaced by α^*. The latter new family of systems (5.7) will be denoted by $(5.7)_{\alpha*}$.

Fix an arbitrary $\alpha \in (0, \beta)$. Let P and P^* denote respectively the Poincaré maps $P^{0,2\pi}$ of the line $\{\tau = 0\}$ to the line $\{\tau = 2\pi\}$ defined by vector fields $(5.7)_\alpha$ and $(5.7)_{\alpha*}$. For every ω small enough the point $(\ell\omega, 1 + (\ell - \alpha)\omega)$ lies in the phase-lock area L_ℓ, by lemma 5.1 and since $\alpha < \beta < \frac{1}{2}$. Therefore, the corresponding system $(5.7)_\alpha$ has a periodic orbit with rotation number ℓ. This means that there exists a point a in the θ-axis with $P(a) = a + 2\pi\ell$. On the other hand,

$$P^* < P, \quad P^*(a) < P(a) = a + 2\pi\ell, \quad \text{whenever}\, \omega \,\text{is small enough},$$

by the monotonicity lemma and since $\alpha^* \geqslant \beta > \alpha$. Therefore, the rotation number of system $(5.7)_{\alpha*}$ is no greater than ℓ and a cannot be its periodic point with rotation number at least ℓ for small ω. In particular, the latter statements holds for the systems corresponding to the above constrictions $(B_k, A_{\alpha_k}) \in \Lambda_\ell$. On the other hand, the dynamical system (1.4) corresponding to a constriction lying in Λ_ℓ should have rotation number at least ℓ and all its orbits should be periodic with rotation number at least ℓ, see [28, theorem 1.2 and proposition 2.2]. The contradiction thus obtained proves theorem 5.15. □

Proof of theorem 1.13. Fix an $\ell \in \mathbb{N}$. For every $\omega > 0$ small enough all the constrictions lying in Λ_ℓ with ordinates $A \geqslant A_{\frac{1}{2}} = 1 + (\ell - \frac{1}{2})\omega$ are not ghost (lemma 5.1). There are no constrictions in Λ_ℓ with smaller positive ordinates (theorem 5.15 and statement (5.1)). Theorem 1.13 is proved. □

5.6. Proof of theorem 1.10

Let, to the contrary, there exist a ghost constriction $(B, A;\, \omega)$. Then $\ell = \frac{B}{\omega} \in \mathbb{Z} \setminus \{0\}$, and without loss of generality we can and will consider that $\ell \geqslant 1$ (see the beginning of section 5). Let $\mathcal{C}$ denote the connected component of the submanifold $\mathrm{Constr}_\ell \subset (\mathbb{R}^2_+)_{\mu,\eta}$ containing the corresponding point $(\frac{A}{2\omega}, \omega^{-1})$. The restriction to $\mathcal{C}$ of the function $\omega = \eta^{-1}$ is unbounded from below, while all the constrictions in $\mathcal{C}$ are ghost (theorem 1.12). Thus, there exist ghost constrictions with given ℓ and arbitrarily small ω. This yields a contradiction to theorem 1.13 and proves absence of ghost constrictions. The proof of theorem 1.10, and hence, theorems 1.4 and 1.7 is complete.

6. Some applications and open problems

6.1. Geometry of phase-lock areas

For every $\ell \in \mathbb{Z}_{\neq 0}$ let $\mathcal{P}_\ell = (\ell\omega, A(\mathcal{P}_\ell)) \subset \mathbb{R}^2_{B,A}$ denote the higher generalized simple intersection lying in $\Lambda_\ell := \{B = \ell\omega\}$, see subsection 5.1. Recall that

$$S\ell := \Lambda_\ell \cap \{A \geqslant A(\mathcal{P}_\ell)\} \subset L^+_\ell := L_\ell \cap \{A > 0\}, \quad \mathcal{P}_\ell \in \partial L^+_\ell.$$

The **connectivity conjecture**, see [29, conjecture 1.14], states that *the intersection $L^+_\ell \cap \Lambda_\ell$ coincides with the ray $S\ell$, and thus, is connected.*

Theorem 1.7 implies the following corollary

Corollary 6.1. *Let, to the contrary to the above conjecture, the intersection $L\Lambda_\ell := L^+_\ell \cap \Lambda_\ell \cap \{0 < A < A(\mathcal{P}_\ell)\}$ be non-empty. Then its lowest point* (*i.e., its point with minimal ordinate A*) *is a generalized simple intersection.*

Proof. The lowest point $P \in L\Lambda_\ell$ is well-defined, has positive ordinate and lies in ∂L_ℓ, since the growth point in L_ℓ, i.e., its intersection point with the abscissa axis, has abscissa $\sqrt{\ell^2\omega^2 + 1} > \ell\omega$. Hence, it is either a constriction, or a generalized simple intersection, by definition. If P were a constriction, it would be negative, since its lower adjacent interval $\Lambda_\ell \cap \{0 < A < A(P)\}$ lies outside the phase-lock area L_ℓ. But there are no negative constrictions, by theorem 1.7. Therefore, P is a generalized simple intersection. $\square$

Remark 6.2. It is known that the generalized simple intersections $(\ell\omega, A)$ correspond to the parameters $(\lambda,\ \mu)$, $\mu = \frac{A}{2\omega}$, $\lambda = \frac{1}{4\omega^2} - \mu^2$, of those special double confluent Heun equations (1.9) that have polynomial solutions. The set of the latter parameters $(\lambda,\ \mu)$ is a remarkable algebraic curve: the so-called *spectral curve* $\Gamma_\ell \subset \mathbb{R}^2_{(\lambda,\mu)}$ introduced in [19] and studied in [19, 30]. It is the zero locus of the polynomial from [19, formula (21)], which is the determinant of a three-diagonal matrix formed by diagonal terms of type $\lambda +$ const and linear functions in μ at off-diagonal places. See also [30, formula (1.4)]. (The complexification of the spectral curve is known to be irreducible, see [30, theorem 1.3].) For every given $\omega > 0$ the curve Γ_ℓ contains at most ℓ points $(\lambda,\ \mu)$ corresponding to the given ω with $\mu > 0$; the point with the biggest μ corresponds to the higher generalized simple intersection $\mathcal{P}_\ell$. This follows from Bézout theorem and the fact that the spectral curve Γ_ℓ is the zero locus of a polynomial of degree ℓ in $(\lambda,\ \mu^2)$, see [19, p 937].

Corollary 6.1 and the above remark reduce the connectivity conjecture to the following equivalent, algebro-geometric conjecture.

Conjecture 6.3. For every $\omega > 0$ the above real spectral curve Γ_ℓ contains a unique point $(\lambda,\ \mu)$ with $\lambda = \frac{1}{4\omega^2} - \mu^2$ (up to change of sign at μ) for which the corresponding rotation number $\rho = \rho(\ell\omega, 2\mu\omega)$ equals ℓ. (The point $(B, A) = (\ell\omega, 2\mu\omega)$ coincides with $\mathcal{P}_\ell$, see the above remark.)

Theorem 6.4. *For every $\ell \in \mathbb{Z}_{\neq 0}$ and every positive $\omega < \frac{1}{|\ell|}$ the connectivity conjecture holds.*

Proof. Let, say, $\ell > 0$, and let $0 < \omega < \frac{1}{\ell}$. Then for every $r \in \mathbb{N}$, $0 < r < \ell$, the boundary ∂L_r intersects $\Lambda_\ell^+ := \Lambda_\ell \cap \{A > 0\}$ in at least two points. Indeed, the abscissa $\sqrt{r^2\omega^2 + 1}$ of the growth point of the phase-lock area L_r is greater than $\ell\omega < 1$. On the other hand, each boundary curve of the area L_r contains constrictions, which lie in the axis Λ_r, and hence, on the left from the axis Λ_ℓ. Hence, each boundary curve intersects Λ_ℓ^+ in at least one point (this statement is given by [30, theorem 1.18] for all ω small enough). It cannot be a common intersection point for both boundary curves, i.e., it cannot be a constriction, since $r = \rho < \ell$ and by theorem 1.4. Therefore, the intersection $\partial L_r \cap \Lambda_\ell^+$ contains at least two distinct points. Analogously, ∂L_0 intersects Λ_ℓ^+ in at least one point, since the point $(1, 0) \in \partial L_0$ lies on the right from the point $(\ell\omega, 0) \in \Lambda_\ell$. If $0 \leqslant r < \ell$ and $r \equiv \ell (\mathrm{mod}\, 2)$, then each point of intersection $\partial L_r \cap \Lambda_\ell^+$ is a generalized simple intersection. Taking these intersections for all latter r yields $\ell - 1$ distinct generalized simple intersections lying in Λ_ℓ^+. But the total number of generalized simple intersections in Λ_ℓ^+ is no greater than ℓ, see the above remark. Therefore, at most one of them may correspond to the rotation number ℓ, and hence, is reduced to the known generalized simple intersection $\mathcal{P}_\ell$ with $\rho = \ell$. In particular, there are no generalized simple intersections in $\Lambda_\ell \cap L_\ell$ with $0 < A < A(\mathcal{P}_\ell)$. This together with corollary 6.1 implies that $L_\ell^+ \cap \Lambda_\ell = S\ell$ and proves the connectivity conjecture for $0 < \omega < \frac{1}{|\ell|}$. □

Problem 6.5 [13, *subsection 5.8*]. *What is the asymptotic behaviour of the phase-lock area portrait in family* (1.4)*, as* $\omega \to 0$*?*

This problem is known and motivated by physics applications. Buchstaber, Tertychnyi and later by Filimonov, Kleptsyn, Schurov performed numerical experiences studying limit behaviour of the phase-lock area portrait after appropriate rescaling of the variables (B, A). Their experiences have shown that the interiors of the phase-lock areas tend to open subsets (the so-called *limit rescaled phase-lock areas*) whose connected components form a partition of the plane. In some planar region, the latter partition looks like a chess table turned by $\frac{\pi}{4}$. It would be interesting to prove this mathematically and to find the boundaries of the limit phase-lock areas.

Some results on smallness of gaps between rescaled phase-lock areas for small ω were obtained in [43].

To our opinion, methods elaborated in [43] and in the present paper could be applied to study problem 6.5.

6.2. The dynamical isomonodromic foliation

Let us consider family (1.4) modelling overdamped Josephson junction as a three-dimensional family, with variable frequency ω. Its *three-dimensional phase-lock areas* in $\mathbb{R}^3_{B,A,\omega}$ are defined in the same way, as in definition 1.1. Each three-dimensional phase-lock area is fibred by two-dimensional phase-lock areas in $\mathbb{R}^2_{B,A}$ corresponding to different fixed values of ω.

Linear systems (1.7) corresponding to (1.4) form a transversal hypersurface to the isomonodromic foliation of the four-dimensional manifold $\mathbf{J}^N(\mathbb{R}_+)$ (lemma 3.17). It appears that there is another four-dimensional manifold with the latter property that has the following advantage: it consists of linear systems on $\overline{\mathbb{C}}$ coming from a family of dynamical systems on two-torus. Namely, consider the following four-dimensional family of dynamical systems on $\mathbb{T}^2$ containing (1.4):

$$\frac{d\theta}{d\tau} = \nu + a\cos\theta + s\cos\tau + \psi\cos(\theta - \tau); \quad \nu, a, \psi \in \mathbb{R},\ s > 0,\ (a, \psi) \neq (0, 0). \tag{6.1}$$

The variable changes $\Phi = \mathrm{e}^{\mathrm{i}\theta}$, $z = \mathrm{e}^{\mathrm{i}\tau}$ transform (6.1) to the Riccati equation

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{1}{z^2}\left(\frac{s}{2}\Phi + \frac{\psi}{2}\Phi^2\right) + \frac{1}{z}\left(\nu\Phi + \frac{a}{2}(\Phi^2+1)\right) + \left(\frac{s}{2}\Phi + \frac{\psi}{2}\right).$$

A function $\Phi(z)$ is a solution of the latter Riccati equation, if and only if $\Phi(z) = \frac{Y_2(z)}{Y_1(z)}$, where $Y = (Y_1, Y_2)(z)$ is a solution of the linear system

$$Y' = \left(-s\frac{\mathbf{K}}{z^2} + \frac{\mathbf{R}}{z} + s\mathbf{N}\right)Y, \tag{6.2}$$

$$\mathbf{K} = \begin{pmatrix} \frac{1}{2} & \chi \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} -b & -\frac{a}{2} \\ \frac{a}{2} & \chi a \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} -\frac{1}{2} & 0 \\ \chi & 0 \end{pmatrix};$$

$$\chi = \frac{\psi}{2s}, \qquad b = \nu - \frac{\psi}{2s}a = \nu - \chi a.$$

The residue matrix of the formal normal forms of system (6.2) at 0 and at ∞ is the same and equal to

$$\mathrm{diag}(-\ell, 0), \quad \ell := b - \chi a = \nu - 2\chi a = \nu - \frac{\psi a}{s}. \tag{6.3}$$

Theorem 6.6. *The four-dimensional family of linear systems (6.2) is analytically foliated by one-dimensional isomonodromic families defined by the following non-autonomous system of differential equations:*

$$\begin{cases} \chi'_s = \dfrac{a - 2\chi(\ell + 2\chi a)}{2s} \\ a'_s = -2s\chi + \dfrac{a}{s}(\ell + 2\chi a) \\ \ell'_s = 0 \end{cases}. \tag{6.4}$$

For every $\ell \in \mathbb{R}$ the function

$$w(s) := \frac{a(s)}{2s\chi(s)} = \frac{a(s)}{\psi(s)} \tag{6.5}$$

satisfies Painlevé 3 equation (3.14) along solutions of (6.4).

Proof. The composition of variable rescalings $z = s^{-1}\zeta$ and gauge transformations

$$Y = \begin{pmatrix} 1 & 0 \\ -2\chi & 1 \end{pmatrix}\widetilde{Y} \tag{6.6}$$

sends family (6.2) to the following family of linear systems:

$$Y'_\zeta = \left(-\frac{t}{\zeta^2}K + \frac{R}{\zeta} + \mathrm{diag}\left(-\frac{1}{2}, 0\right)\right)Y, \tag{6.7}$$

$$t = s^2, \qquad K = \begin{pmatrix} \frac{1}{2} - 2\chi^2 & \chi \\ \chi(1-4\chi^2) & 2\chi^2 \end{pmatrix}, \qquad R = \begin{pmatrix} -\ell & -\frac{a}{2} \\ -2\chi(\ell + \chi a) + \frac{a}{2} & 0 \end{pmatrix}.$$

Let $\mathcal{J}$ denote the space of special Jimbo type systems, see (3.9), (3.10), with real matrices. Systems (6.7) lie in $\mathcal{J}$, since the formal normal forms of a system (6.2) at $0, \infty$ have common residue matrix $\mathrm{diag}(-\ell, 0)$. Every system $\mathcal{L}_0$ of type (6.7) with $\chi \neq 0$ has a neighbourhood $W = W(\mathcal{L}) \subset \mathcal{J}$ where family (6.7) forms a hypersurface $\mathcal{X} = \{K_{22} = 2K_{12}^2\} \cap W$ so that each system $\mathcal{L} \in W$ can be projected to a system $\mathcal{L}^* \in \mathcal{X}$ by a diagonal gauge transformation $(Y_1, Y_2) \mapsto (Y_1, \lambda Y_2)$, $\lambda = \lambda(\mathcal{L})$. Projecting to $\mathcal{X}$ a Jimbo isomonodromic family given by (3.11) yields an isomonodromic family of systems (6.7). The differential equation satisfies by the projected isomonodromic families is found analogously to the proof of equation (3.17). To do this, fix a $t_0 > 0$ and matrices $K(t_0)$, $R(t_0)$ as in (6.7). Let $\widetilde{K}(t)$, $\widetilde{R}(t)$ be solutions of (3.11) with initial conditions $K(t_0)$ and $R(t_0)$ at t_0, and let $(Y_1, Y_2) \mapsto (Y_1, \lambda(t) Y_2)$ be the family of the above normalizing gauge transformations: the matrices $K(t) = \mathrm{diag}(1, \lambda(t))\widetilde{K}(t)\mathrm{diag}(1, \lambda^{-1}(t))$, $R(t) = \mathrm{diag}(1, \lambda(t))\widetilde{R}(t)\mathrm{diag}(1, \lambda^{-1}(t))$ are the same, as in (6.7), that is $K_{22}(t) = 2K_{12}^2(t)$; $\lambda(t_0) = 1$. Set $\xi = \lambda'(t_0)$.

The equation on the matrix function $\widetilde{K}(t)$ given by (3.11) yields

$$K_{12}'(t_0) = -\xi K_{12}(t_0) + \frac{1}{t_0}[R(t_0), K(t_0)]_{12}, \qquad K_{22}'(t_0) = \frac{1}{t_0}[R(t_0), K(t_0)]_{22}. \tag{6.8}$$

Substituting $K_{22} = 2\chi^2$, $K_{12} = \chi$ to the second equation in (6.8) and changing the time parameter t to $s = \sqrt{t}$ yields formula for the derivative $\chi_t' = (K_{12})_t'$ and the first equation in (6.4). Substituting thus found derivative K_{12}' to the first formula in (6.8) yields a linear equation on ξ, whose solution is $\xi = -\frac{\ell + 2\chi a}{2t_0}$. The differential equation on the matrix $\widetilde{R}(t)$ in (3.11) yields the differential equation on R_{12} analogous to the first equation in (6.8), which also includes the above already found value ξ. Substituting $R_{12} = -\frac{a}{2}$ there yields the second equation in (6.4). Painlevé 3 equation (3.14) on $w(s)$ along isomonodromic families thus constructed follows from theorem 3.9, since diagonal gauge transformations do not change the ratio $\frac{R_{12}}{K_{12}}$. Equation (3.14) can be also deduced directly from (6.4). □

The foliation from theorem 6.6 given by (6.4) induces a one-dimensional foliation in the four-dimensional space of dynamical systems (6.1) given by the following non-autonomous system of equations obtained from (6.4) by change of the variable χ to $\psi = 2s\chi$:

$$\begin{cases} \psi_s' = a + (1 - \ell)\dfrac{\psi}{s} - \dfrac{a\psi^2}{s^2} \\ a_s' = -\psi + \ell\dfrac{a}{s} + \dfrac{\psi a^2}{s^2}. \end{cases} \tag{6.9}$$

The latter foliation of family (6.1) given by (6.9) will be denoted by $\mathcal{G}$ and called the *dynamical isomonodromic foliation.*

Lemma 6.7. *The conjugacy class of flow* (6.1) *under diffeomorphisms $\mathbb{T}^2 \to \mathbb{T}^2$ isotopic to identity, its rotation number and ℓ, see* (6.3)*, are constant on leaves of the dynamical isomonodromic foliation $\mathcal{G}$. The hypersurface of systems* (1.4) *modelling Josephson junction is transversal to $\mathcal{G}$. The function $w(s) = \frac{a(s)}{\psi(s)}$, see* (6.5)*, satisfies Painlevé 3 equation* (3.14) *along its leaves. A point (s, ψ, a, ℓ) corresponds to a system* (1.4)*, if and only if $\psi = 0$; this holds if and only if the function w has pole of order 1 at s with residue 1.*

Proof. The projectivized monodromy of linear system (6.2) is the complexification of the Poincaré map of the corresponding dynamical system (6.1). Therefore, constance of its conjugacy class along leaves implies constance of conjugacy class of the Poincaré map and hence, of the flow and of its rotation number; $\ell = \text{const}$, by theorem 6.6. Family of systems (1.4) coincides with the hypersurface $\{\psi = 0\} \cap \{a > 0\}$ in the parameter space. It is transversal to the vector field (6.9), since $\psi' = a > 0$ at all its points. The characterization of systems (1.4) in terms of poles follows from construction and lemma 3.16 and, on the other hand, immediately from (6.9): if $\psi(s_0) = 0$, then $\psi(s) \simeq a(s_0)(s - s_0)$, $w(s) = \frac{a(s)}{\psi(s)} \simeq \frac{1}{s-s_0}$, as $s \to s_0$, and vice versa. □

Recall that the growth point of a two-dimensional phase-lock area L_r has abscissa $\text{sign}(r)\sqrt{r^2\omega^2 + 1}$. For a given $r \in \mathbb{Z}_{\neq 0}$ the latter growth points form a curve bijectively parametrized by $\omega > 0$ in the three-dimensional parameter space, which will be called the rth *growth curve*. We already know that the family of constrictions in $\mathbb{R}^2_{B,A} \times (\mathbb{R}_+)_\omega$ is a one-dimensional submanifold, by theorem 1.12. Thus, it is a disjoint union of connected curves, which will be called the *constriction curves*.

In what follows the three-dimensional phase-lock area in $\mathbb{R}^2_{B,A} \times (\mathbb{R}_+)_\omega$ with a rotation number $r \in \mathbb{Z}$ will be denoted by $\widehat{L}_r$.

Conjecture 6.8. Each constriction curve is bijectively projected onto $(\mathbb{R}_+)_\omega$. Each three-dimensional phase-lock area of family (1.4) is a countable garland of domains, where any two adjacent domains are separated either by the corresponding growth curve, or by a constriction curve.

Remark 6.9. Conjecture 6.8 **does not** follow from known results on two-dimensional phase-lock area. *A priori*, a constriction curve may be not bijectively projected to the ω-axis, and the projection may have some critical value ω_0 that is a local maximum (minimum). In this case the corresponding two-dimensional phase-lock area with ω less (greater) than ω_0 has two constrictions that collide for $\omega = \omega_0$ and disappear, when ω crosses the critical value ω_0. Numerical experiences made by Tertychnyi, Filimonov, Kleptsyn, Schurov show that such a scenario does not arise. This can be viewed as a numerical confirmation of conjecture 6.8.

Remark 6.10. In each two-dimensional phase-lock area L_r the constrictions lying in the half-plane $\{A > 0\}$ are ordered by natural numbers k corresponding to their heights: the lowest constriction is ordered by 1, the second one by two, etc. If conjecture 6.8 is true, then along each constriction curve $\mathcal{C}$ lying in the upper quarter-space $\{A > 0\}$ the above height number is constant. In this case each constriction curve $\mathcal{C} = \mathcal{C}_{\ell,k} \subset \{A > 0\}$ is numerated by two integer numbers

$$\ell = \rho = \frac{B}{\omega}, \quad k := \text{the above height number.}$$

Studying of the following two problems, which are of independent interest, would have important applications to conjecture 6.8 and related problems.

Problem 6.11. *Study the Poincaré map of the dynamical isomonodromic foliation $\mathcal{G}$, see* (6.9)*, acting on the transversal hypersurface given by family of systems* (1.4)*. The Poincaré map sends the intersection of its definition domain with each three-dimensional phase-lock area in family* (1.4) *to the same phase-lock area, by constance of the rotation number along leaves. Study the action of the Poincaré map of the foliation $\mathcal{G}$ given by* (6.9) *on the three-dimensional phase-lock area portrait of family* (1.4)*.*

Problem 6.12. *Is it true that the above Poincaré map is well-defined on each constriction curve $\mathcal{C}_{\ell,k}$ and sends its diffeomorphically onto $\mathcal{C}_{\ell,k+1}$?*

If conjecture 6.8 is true, then for every $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}$ there is a unique connected component $\mathcal{O}_{\ell,k}$ of the interior of the three-dimensional phase-lock area $\widehat{L}_\ell$ that is adjacent to the constriction curves $\mathcal{C}_{\ell,k}$ and $\mathcal{C}_{\ell,k+1}$.

Problem 6.13. *Is it true that the Poincaré map of the foliation $\mathcal{G}$ is well-defined on each component $\mathcal{O}_{\ell,k}$ and sends it diffeomorphically onto $\mathcal{O}_{\ell,k+1}$? Is it a well-defined diffeomorphism on a neighbourhood of the closure $\overline{\mathcal{O}_{\ell,k}}$? What is the intersection of its definition domain with the component of the phase-lock area $\widehat{L}_\ell$ adjacent to $\mathcal{O}_{\ell,1}$ and to the corresponding growth curve? How does it act there?*

Remark 6.14. The Poincaré map of the foliation $\mathcal{G}$ (where it is defined) can be viewed as the suspension over the map sending a given simple pole $s_0 > 0$ with residue 1 of solution $w(s)$ of Painlevé 3 equation (3.14) to its next pole $s_1 > s_0$ of the same type (if any). Many solutions of (3.14) have an infinite lattice of simple poles with residue 1 converging to $+\infty$. Our Painlevé 3 equations (3.14) admit a one-dimensional family of Bessel type solutions, see [24], whose poles are zeros of solutions of Bessel equation and are known to form an infinite lattice. Victor Novokshenov's recent numerical experience has shown that their small deformations also have an infinite lattice of poles. Few solutions, e.g., the *tronquée solutions* [50], are bounded on some semi-interval $[C, +\infty)$, and hence, do not have poles there.

Problem 6.15. *Describe those parameter values of family (1.4) for which the corresponding solution $w(s)$ of (3.14) is tronquée. Is it true that this holds for some special points of boundaries of the three-dimensional phase-lock areas? Does this hold for the higher generalized simple intersections $\mathcal{P}_\ell$ discussed in subsection 6.1?*

Problem 6.16. *Study geometry of phase-lock areas*[10] *in four-dimensional family (6.1) of dynamical systems on $\mathbb{T}^2$. Study special points of boundaries of the phase-lock areas: analogues of growth points, constrictions and generalized simple intersections.*

Let Σ denote the subfamily in (6.1) consisting of dynamical systems with trivial Poincaré map. The value $\ell = \nu - \frac{\psi a}{s}$ corresponding to a system in Σ should be integer, as in proposition 4.6, and its rotation number ρ is also integer. For every $\ell, \rho \in \mathbb{Z}$ let $\Sigma_{\ell,\rho} \subset \Sigma$ denote the subset consisting of systems with given ℓ and ρ. Those systems (1.4) with given ℓ that correspond to constrictions are contained in $\Sigma_{\ell,\ell}$, by theorem 1.4.

Problem 6.17. *Is it true that systems (1.4) with given ℓ corresponding to constrictions lie in one connected component of the set $\Sigma_{\ell,\ell}$?*

Remark 6.18. One can show that a positive solution of conjecture 6.8 would imply positive answer to problem 6.17.

To our opinion, a progress in studying the above problems would have applications to problems on geometry of phase-lock areas, for example, to problems discussed in the previous subsection.

Studying conjectures 6.3, 6.8 and problems 6.11, 6.16, 6.17 is a work in progress.

[10] Recently it was observed by Buchstaber and the second author (Glutsyuk) that the rotation number quantization effect holds in family (6.1): phase-lock areas exist only for integer values of the rotation number. The proof is the same, as in [17].

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