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ON DECOMPOSITION OF AMBIENT SURFACES ADMITTING A-DIFFEOMORPHISMS WITH NON-TRIVIAL ATTRACTORS AND REPELLERS

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Abstract. It is well-known that there is a close relationship between the dynamics of diffeomorphisms satisfying the axiom A and the topology of the ambient manifold. In the given article, this statement is considered for the class $\mathbb{G}(M^2)$ of A-diffeomorphisms of closed orientable connected surfaces, the non-wandering set of each of which consists of $k_f \geq 2$ connected components of one-dimensional basic sets (attractors and repellers). We prove that the ambient surface of every diffeomorphism $f \in \mathbb{G}(M^2)$ is homeomorphic to the connected sum of k_f closed orientable connected surfaces and l_f two-dimensional tori such that the genus of each surface is determined by the dynamical properties of appropriating connected component of a basic set and l_f is determined by the number and position of bunches, belonging to all connected components of basic sets. We also prove that every diffeomorphism from the class $\mathbb{G}(M^2)$ is Ω-stable but is not structurally stable.

1. Introduction and statement of results.

An overview of the main concepts related to the topic of the article can be found in the monograph [9] and in the following surveys: [1],[2],[7], [16], [19].

Let M^n be a closed smooth connected manifold of dimension $n \geq 1$, $f : M^n \to M^n$ be a diffeomorphism, and $NW(f)$ be a non-wandering set of f.

A closed f-invariant set $H \subset M^n$ is said to be hyperbolic if there exists continuous Df-invariant decomposition of the tangent subbundle $T_H M^n$ into the direct sum $E^s_H \oplus E^u_H$ such that $||Df^k(v)|| \leq C\lambda^k||v||$, $||Df^{-k}(w)|| \leq C\lambda^k||w||$, $\forall\, v \in E^s_H, \forall\, w \in E^u_H, \forall\, k \in \mathbb{N}$, for some fixed numbers $C > 0$ and $0 < \lambda < 1$. According to [19] (Theorem 7.3), for each point $x \in H$ there exists a stable manifold $W^s_x = J^s_x(E^s_x)$, where $J^s_x : E^s_x \to M^n$ is an injective immersion with the following properties: $W^s_x = \{y \in M^n : d(f^k(x), f^k(y)) \to 0 \text{ for } k \to +\infty\}$, where d is the metric on M^n induced by the Riemannian metric on TM^n; if $x, y \in H$, then W^s_x and W^s_y either

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coincide or they are disjoint; $f(W^s_x) = W^s_{f(x)}$; the tangent space for W^s_x at every point $y \in \Lambda$ is E^s_y. The unstable manifold W^u_x of x is the stable manifold of x for the diffeomorphism f^{-1}.

The diffeomorphism f is said to be A-diffeomorphism (diffeomorphism satisfying the axiom A) if its non-wandering set $NW(f)$ is hyperbolic and the periodic points are everywhere dense in $NW(f)$. According to [19] (Theorem 6.2), the non-wandering set of A-diffeomorphism can be uniquely expressed as the finite union of mutually disjoint subsets, called basic sets, each of which is compact, invariant and topologically transitive. A basic set which is a periodic trajectory is called trivial. Otherwise, a basic set is called non-trivial. The set $W^s_\Omega = \{y \in M^n : f^k(y) \to \Omega, k \to +\infty\}$, where Ω is basic set of the diffeomorphism f, is said to be stable manifold of the basic set Ω. The unstable manifold W^u_Ω of Ω is the stable manifold of Ω for the diffeomorphism f^{-1}. In compliance with [9] (Statement 1.5.), $W^s_\Omega = \bigcup\limits_{x\in\Omega} W^s_x$ ($W^u_\Omega = \bigcup\limits_{x\in\Omega} W^u_x$) and $\dim W^s_x = \dim W^s_y$ ($\dim W^u_x = \dim W^u_y$) for any points $x, y \in \Omega$.

According to [4] (Theorem 2.7), any basic set Ω of A-diffeomorphism $f : M^n \to M^n$ is uniquely expressed as the finite union of mutually disjoint compact subsets

$$\Omega = \Lambda_1 \cup ... \cup \Lambda_q,\ q \geq 1,$$

called periodic components of the set Ω[1], such that $f^q(\Lambda_j) = \Lambda_j, f(\Lambda_j) = \Lambda_{j+1}, j \in \{1, ..., q\}$ ($\Lambda_{q+1} = \Lambda_1$). For every point x belonging to the periodic component Λ_j, the set $W^s_x \cap \Lambda_j$ ($W^u_x \cap \Lambda_j$) is dense in Λ_j.

A basic set Ω is called an attractor (repeller) if it has a closed neighborhood $U_\Omega \subset M^n$ such that $f(U_\Omega) \subset int\ U_\Omega$, $\bigcap\limits_{k\in\mathbb{N}} f^k(U_\Omega) = \Omega$ ($f^{-1}(U_\Omega) \subset int\ U_\Omega$, $\bigcap\limits_{k\in\mathbb{N}} f^{-k}(U_\Omega) = \Omega$). The neighborhood U_Ω in this case is said to be trapping. For an attractor (repeller) Ω, the following equality holds: $\Omega = W^u_\Omega$ ($\Omega = W^s_\Omega$) ([9], Theorem 8.2.). A non-trivial basic set Ω, which is an attractor (repeller), is called an expanding attractor (contracting repeller) if $\dim \Omega = \dim W^u_x$ ($\dim \Omega = \dim W^s_x$)[2], where $x \in \Omega$. It follows from [14] (Theorem 2) that every expanding attractor (contracting repeller) Ω has the local structure of the direct product of the r-dimensional Euclidean space and the Cantor set, where r is the topological dimension of Ω.

Let $Diff(M^n)$ be the space of C^1 diffeomorphisms on M^n endowed with the uniform C^1 topology [11]. A diffeomorphism $f : M^n \to M^n$ is said to be structurally stable if there is a neighborhood $\mathcal{U}$ of the diffeomorphism f in $Diff(M^n)$ such that every diffeomorphism $g \in \mathcal{U}$ is topologically conjugate to f. A diffeomorphism $f : M^n \to M^n$ is said to be Ω-stable if there is a neighborhood $\mathcal{U}$ of the diffeomorphism f in $Diff(M^n)$ such that for any $g \in \mathcal{U}$ restrictions $g|_{NW(g)}$ and $f|_{NW(f)}$ are topologically conjugate.

Let us introduce the relation $\prec$ for basic sets as follows: $\Omega_i \prec \Omega_j \Leftrightarrow W^s_{\Omega_i} \cap W^u_{\Omega_j} \neq \varnothing$. A k-cycle ($k \geq 1$) is a collection of mutually disjoint basic sets $\Omega_0, \Omega_1, ..., \Omega_k$ that satisfy the condition $\Omega_0 \prec \Omega_1 \prec ... \prec \Omega_k \prec \Omega_0$. It follows from [13], [20] that the diffeomorphism $f : M^n \to M^n$ is Ω-stable if and only if it satisfies the axiom A and has no cycles (for the formulation, see [9], Theorem 1.9.).

Let M^2 be a closed smooth orientable connected surface, $f : M^2 \to M^2$ be a diffeomorphism satisfying the axiom A.

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[1] R. Bowen called these components C-dense (see [4]). In this paper, following [9], we call them periodic (by analogy with periodic points of a periodic orbit).

[2] Here and throughout the article, dim denotes the topological dimension.

Let Ω be a non-trivial basic set of the diffeomorphism f. Let us note that stable and unstable manifolds of each point $x \in \Omega$ are one-dimensional and that all periodic points of the set Ω are saddle. For $\sigma \in \{s, u\}$, let us put $\overline{\sigma} = s$ if $\sigma = u$, and $\overline{\sigma} = u$ if $\sigma = s$. A periodic point p belonging to the set Ω is called a boundary periodic point of type σ (σ-boundary periodic point) if one of the connected components of the set $W^\sigma_p \backslash p$ does not intersect with Ω (we will denote this connected component by μ^σ_p), and both connected components of the set $W^{\overline{\sigma}}_p \backslash p$ intersect with Ω. A periodic point p belonging to the set Ω is called a boundary periodic point of type (s, u) if one of the connected components of each of the sets $W^s_p \backslash p$, $W^u_p \backslash p$ does not intersect with Ω. If the set Ω is zero-dimensional or one-dimensional, then the set of boundary periodic points is non-empty and is finite ([5], Lemma 2.4. and Lemma 2.5.; see also [15], Theorem 1.7.).

If a basic set Ω of the diffeomorphism f is one-dimensional, then by virtue of [14] (Theorem 3) it is an attractor or a repeller. If a one-dimensional basic set is an attractor (repeller), then it contains only s-boundary (u-boundary) periodic points.

Let x be an arbitrary point belonging to a periodic component Λ of a one-dimensional attractor (repeller) Ω. Then the set W^u_x (W^s_x) belongs to the set Λ and is dense in this set. Due to this fact and the fact that the closure of a connected set is connected, it follows that the periodic component Λ is connected. Thus, each connected component of the one-dimensional attractor (repeller) Ω coincides with one of its periodic components.

It is known ([5], Lemma 3.3.; see also [16], Theorem 2.1.) that for a one-dimensional attractor (repeller) Ω accessible from inside boundary[3] of the set $M^2 \backslash \Omega$ consists of a finite number of bunches. In compliance with [10] (Definition 3)[4], a bunch b of the one-dimensional attractor Ω is the union of the maximum number h_b of the unstable manifolds $W^u_{p_1}, \ldots, W^u_{p_{h_b}}$ of the s-boundary periodic points $p_1, \ldots, p_{h_b}$ of the set Ω whose stable separatrices[5] $\mu^s_{p_1}, \ldots, \mu^s_{p_{h_b}}$ belong to the same connected component of the set $W^s_\Omega \backslash \Omega$. The number h_b is called the degree of the bunch. Similarly, the concept of a bunch can be defined for a one-dimensional repeller.

In [3] (Theorem 1, Theorem 2), for A-diffeomorphisms of compact connected surfaces (orientable and nonorientable), estimates of the maximum number of their one-dimensional basic sets are given, and the estimates are precise. It follows from [3] that the maximum number of one-dimensional basic sets of the diffeomorphisms under consideration depends on the topological properties of the ambient surface (the genus of the surface and the number of connected components of the boundary) and on the geometric properties of one-dimensional basic sets (the number of thorn-type bunches in one-dimensional basic sets, see the definition in [3]).

In [8], a class $\mathbb{G}(M^2)$ of A-diffeomorphisms of closed orientable connected surfaces such that their non-wandering sets consist of one-dimensional basic sets is introduced, and necessary and sufficient conditions for the existence of a homeomorphism of the entire surface conjugating the restrictions of these diffeomorphisms on

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[3]Let A be a subset of a topological space X. A point $y \in \partial A$ is called accessible from a point $x \in int\ A$ if there exists a path $c : [0; 1] \to X$ such that $c(0) = x, c(1) = y$ and $c(t) \in int\ A$ for every $t \in (0; 1)$. The union of all points accessible from the points of the set $int\ A$ is called the boundary accessible from inside of the set A.

[4]The definition of a bunch given in [10] is equivalent to the definition of a bunch from [16] (Definition 2.4.).

[5]Stable (unstable) separatrix of a hyperbolic periodic point p is a connected component of the set $W^s_p \backslash p$ ($W^u_p \backslash p$).

their non-wandering sets are found. The results of [15] (Theorem 2.2. and Corollary to Theorem 3.2.) imply that the two-dimensional sphere and the two-dimensional torus do not admit diffeomorphisms from the class $\mathbb{G}(M^2)$. The first example of a diffeomorphism from the class $\mathbb{G}(M^2)$ was constructed in [18]. Specifically, based on the DA-diffeomorphism of the two-dimensional torus (see [12], [17], [19], [21]) and the diffeomorphism inverse to it, a diffeomorphism of closed orientable surface of genus 2 (pretzel) such that its non-wandering set consists of a one-dimensional attractor and a one-dimensional repeller was constructed. It is proved in [18] (Theorem 1) that the constructed diffeomorphism is not finitely C^2-stable (see the definition in [18]), but is Ω-stable. The primary mission of the given article is to research the interrelation between the dynamical properties of diffeomorphisms from the class $\mathbb{G}(M^2)$ and the topology of the ambient surface M^2, as well as to research the stability of diffemorphisms from the class $\mathbb{G}(M^2)$.

Let $f : M^2 \to M^2$ be a diffeomorphism from the class $\mathbb{G}(M^2)$ such that its non-wandering set consists of k_f periodic components $\Lambda_1, ..., \Lambda_{k_f}$. It follows from the Lemma 3.1 that the non-wandering set of the diffeomorphism f contains at least one attractor and at least one repeller, that is, $k_f \geq 2$. Let us denote by m_{Λ_i} the number of bunches belonging to Λ_i, by h_{Λ_i} the sum of the degrees of these bunches. Let us denote by m_f the number of all bunches belonging to periodic components of the diffeomorphism f, by h_f the sum of the degrees of these bunches. For the number $g \geq 0$, let us denote by M^2_g a closed orientable connected surface of genus g.

Theorem 1. *Let $f \in \mathbb{G}(M^2)$. Then the surface M^2 is homeomorphic to the connected sum:*

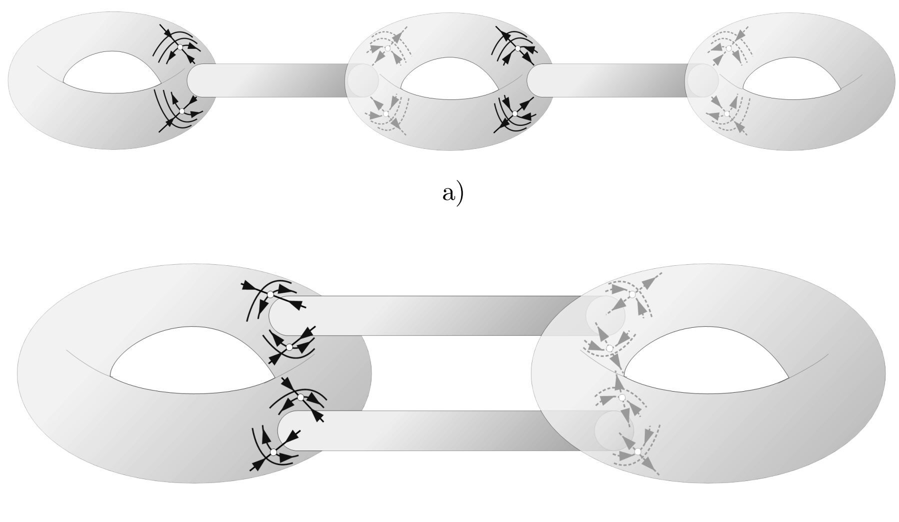
$$M^2_{g_1}\#...\#M^2_{g_{k_f}}\#\underbrace{\mathbb{T}^2\#...\#\mathbb{T}^2}_{l_f},$$

where $g_i = 1 + \frac{h_{\Lambda_i}}{4} - \frac{m_{\Lambda_i}}{2}$ $(i \in \{1, ..., k_f\})$, $l_f = \frac{m_f}{2} - k_f + 1$.

Remark 1. It follows from the Lemma 2.1 and the proof of the Theorem 1 that for every surface $M^2_{g_i}$, $i \in \{1, ..., k_f\}$, there exists a compact orientable connected submanifold $N_{\Lambda_i} \subset M^2$ of the genus $g_i = 1 + \frac{h_{\Lambda_i}}{4} - \frac{m_{\Lambda_i}}{2}$ with m_{Λ_i} boundary components which contains the periodic component Λ_i. Herewith, $N_{\Lambda_i} \cap N_{\Lambda_j} = \varnothing$ for $i \neq j$.

Corollary 1. *Let $f \in \mathbb{G}(M^2)$. Then the surface M^2 has the genus $g = 1 + \frac{h_f}{4}$.*

Making use of the idea of constructing of the example from [18], one can construct examples of diffeomorphisms from the class $\mathbb{G}(M^2)$ on any closed orientable connected surface of genus $g \geq 2$. In the Figure 1 a), it is shown a phase portrait of A-diffeomorphism f_1 of a closed orientable connected surface such that its non-wandering set consists of two one-dimensional attractors (each attractor has one bunch of degree two) and a one-dimensional repeller (which has two bunches of degree two). It follows from the Theorem 1 that $l_{f_1} = 0$ and the ambient surface M^2 of the diffeomorphism f_1 is homeomorphic to the connected sum $M^2_{g_1}\#M^2_{g_2}\#M^2_{g_3}$, where $g_1 = g_2 = g_3 = 1$. In the Figure 1 b), it is shown a phase portrait of A-diffeomorphism f_2 of a closed orientable connected surface such that its non-wandering set consists of a one-dimensional attractor (which has two bunches of degree two) and a one-dimensional repeller (which also has two bunches of degree two). It follows from the Theorem 1 that $l_{f_2} = 1$ and the ambient surface M^2 of the diffeomorphism f_2 is homeomorphic to the connected sum $M^2_{g_1} \# M^2_{g_2} \# \mathbb{T}^2$, where $g_1 = g_2 = 1$.



a)

b)

FIGURE 1. Phase portrait of the diffeomorphism a) f_1; b) f_2.

Theorem 2. *Let $f \in \mathbb{G}(M^2)$. Then f is Ω-stable, but is not structurally stable.*

2. Auxiliary information and results.

2.1. **Surfaces with boundary.** Let us recall that a compact two-dimensional manifold with a non-empty boundary is called a surface with boundary. The boundary of a compact surface is the union of finitely many mutually disjoint simple closed curves. Let us denote by $M^2(Q)$ a compact surface with boundary, where Q is the union of all simple closed curves belonging to the boundary of this surface. By gluing a finite number of closed two-dimensional disks (along their boundaries) to all the components of the boundary of the surface $M^2(Q)$, one obtains a compact surface without boundary (a closed surface), which we denote by M^2. A surface $M^2(Q)$ is said to be orientable if the corresponding surface M^2 without boundary is orientable. If the surface $M^2(Q)$ is connected, then the genus of $M^2(Q)$ is defined to be the genus of M^2, and the Euler characteristic of $M^2(Q)$, by virtue of [9] (Statement 10.40), is equal to the difference between the Euler characteristic of M^2 and the number of curves in the set Q.

2.2. **Saddle singularities.** Let $k \in \mathbb{N}$ and $k \neq 2$. The foliation W_k on $\mathbb{R}^2$ with the standard saddle singularity at the point O (coordinate origin) and k separatrices is the image of the horizontal lines $\{Im\, z = c, c \in \mathbb{R}\}$ under the map $w = z^{\frac{k}{2}}$ in the case of odd k and under the map $w^2 = z^k$ in the case of even k. For $k = 2$ all the leaves of the foliation W_2 are straight lines $y = c$, but the axis Ox is artificially split into three parts: the origin and two the half-axes, the latter called the separatrices.

Let M^2 be a closed connected surface, $\mathcal{F}$ be a foliation on the surface M^2. The foliation $\mathcal{F}$ is said to be a foliation with saddle singularities if the set $\mathcal{S}$ of the singularities of the foliation $\mathcal{F}$ consists of a finite number of points and for any point $s \in \mathcal{S}$ there is a neighborhood $U_s \subset M^2$, the homeomorphism $\psi_s : U_s \to \mathbb{R}^2$ and the number $k_s \in \mathbb{N}$ such that $\psi_s(s) = O$ and $\psi_s(\mathcal{F} \cap U_s) = W_{k_s} \backslash O$. The point s is called the saddle singularity with k_s separatrices. Index $I(s)$ of each saddle singularity $s \in \mathcal{S}$ can be calculated via the number of separatrices k_s by the following formula (see [9], formula (10.17)):

$$I(s) = 1 - \frac{k_s}{2}. \tag{1}$$

Let $\chi(M^2)$ be the Euler characteristic of the surface M^2. The next formula follows from the Poincaré-Hopf theorem (see [9], Statement 10.100):

$$\chi(M^2) = \sum_{s \in \mathcal{S}} I(s). \tag{2}$$

2.3. **Auxiliary results.** Let M^2 be a closed smooth orientable connected surface, $f : M^2 \to M^2$ be an A-diffeomorphism such that its non-wandering set contains a one-dimensional attractor (repeller). Let Λ be a periodic component of this attractor (repeller), $b_1, ..., b_{m_\Lambda}$ be the bunches belonging to Λ (m_Λ bunches in total), h_Λ be the sum of the degrees of these bunches.

The proof of the following lemma uses the ideas from [1], [6], [7], as well as the proof scheme from [9] (Theorem 9.6.).

Lemma 2.1. *For the periodic component Λ of a one-dimensional attractor (repeller) of the diffeomorphism $f : M^2 \to M^2$, there are a submanifold N_Λ and a natural number n with the following properties:*

1. *N_Λ is a trapping neighborhood of the set Λ with respect to the diffeomorphism f^n;*
2. *N_Λ is a compact orientable connected surface of the genus $g = 1 + \frac{h_\Lambda}{4} - \frac{m_\Lambda}{2}$ with m_Λ boundary components.*

Proof. For definiteness, we will assume that Λ is a periodic component of the attractor of the diffeomorphism f (if Λ is a periodic component of the repeller, it is sufficient to consider the diffeomorphism f^{-1}).

The finiteness of the number of periodic components of a basic set and the finiteness of the set of boundary periodic points of a one-dimensional attractor imply that there exists a number $n \in \mathbb{N}$ such that $f^n(\Lambda) = \Lambda$ and all boundary periodic points of the set Λ are fixed with respect to the diffeomorphism f^n.

Further for any points $x, y \in W^u_z$ (W^s_z), where $x \neq y$ and z is any point from the set Λ, we will denote by $(x, y)^u$ ($(x, y)^s$) connected open arc on the manifold W^u_z (W^s_z) with boundary points x, y. Let us denote by b an arbitrary bunch belonging to the set Λ, by h_b the degree of this bunch. It follows from the definition of the bunch that $b = W^u_{p_1} \cup ... \cup W^u_{p_{h_b}}$, where $p_j, j \in \{1, ..., h_b\}$, is s-boundary periodic point of the set Λ. By virtue of [5] (Lemma 3.3), there exists a sequence of points $x_1, ..., x_{2h_b}$ such that:

1. x_{2j-1}, x_{2j} belong to different connected components of the set $W^u_{p_j} \backslash p_j$;
2. $x_{2j+1} \in W^s_{x_{2j}}$ (we assume $x_{2h_b+1} = x_1$);
3. $(x_{2j}, x_{2j+1})^s \cap \Lambda = \varnothing, j = 1, ..., h_b$.

For each $j \in \{1, ..., h_b\}$, let us choose a pair of points $\tilde{x}_{2j-1}, \tilde{x}_{2j}$, and a simple curve l_j with the boundary points $\tilde{x}_{2j-1}, \tilde{x}_{2j}$ such that:

1. $(\tilde{x}_{2j}, \tilde{x}_{2j+1})^s \subset (x_{2j}, x_{2j+1})^s$ $(x_{2h_b+1} = x_1)$;
2. the curve l_j transversally intersects with the stable manifold of any point belonging to the arc $(x_{2j-1}, x_{2j})^u$ at exactly one point;
3. $L_b = \bigcup\limits_{j \in \{1,...,h_b\}} [l_j \cup (\tilde{x}_{2j}, \tilde{x}_{2j+1})^s]$ is a simple closed piecewise smooth curve and the set $L_\Lambda = \bigcup\limits_{t \in \{1,...,m_\Lambda\}} L_{b_t}$ has the properties:
   (a) $f^n(L_\Lambda) \cap L_\Lambda = \varnothing$;
   (b) for every curve $L_{b_t}, t \in \{1, ..., m_\Lambda\}$, there exists a curve from the set $f^n(L_\Lambda)$ such that these two curves are the boundary of the two-dimensional closed annulus K_{b_t};
   (c) the annuli $\{K_{b_t}, t \in \{1, ..., m_\Lambda\}\}$ are pairwise disjoint (see Figure 2).

For an arbitrary bunch b, we will call the curve L_b the characteristic curve of the bunch b.

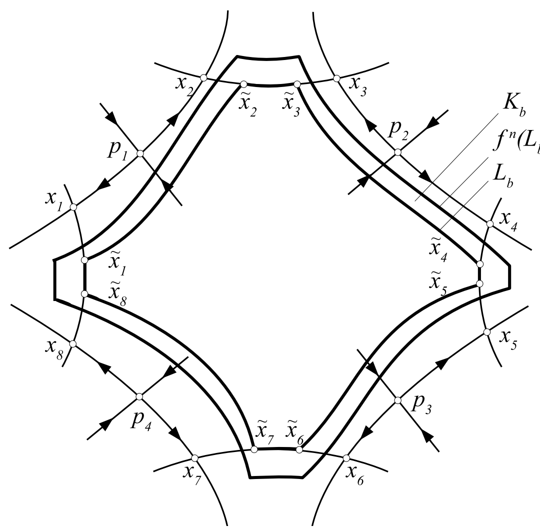


FIGURE 2. Construction of the characteristic curve for the bunch of degree 4.

Let us put $N_\Lambda = \Lambda \cup \bigcup\limits_{k \geq 1} f^{kn}(\bigcup\limits_{t \in \{1,...,m_\Lambda\}} K_{b_t})$. By construction, the annuli $\{K_{b_t}, t \in \{1, ..., m_\Lambda\}\}$ consist of wandering points of the diffeomorphism f^n and N_Λ is a compact orientable surface with non-empty boundary (consisting of m_Λ components) such that $f^n(N_\Lambda) \subset int\, N_\Lambda$ and $\Lambda = \bigcap\limits_{k \geq 0} f^{kn}(N_\Lambda)$. Thus, N_Λ is the trapping neighborhood of the set Λ with respect to the diffeomorphism f^n. Since the set Λ is connected and N_Λ is its trapping neighborhood, then N_Λ is connected.

We will prove that the genus g of the surface N_Λ is equal to $1 + \frac{h_\Lambda}{4} - \frac{m_\Lambda}{2}$. Remove the set $int(\bigcup\limits_{t \in \{1,...,m_\Lambda\}} K_{b_t})$ from the surface M^2. As a result, the surface

M^2 decomposes into a finite number of connected components, one of which is the set N_Λ. In this case, the set $\bigcup\limits_{t\in\{1,\dots,m_\Lambda\}} f^n(L_{b_t})$ is the boundary of the set N_Λ. To each curve $f^n(L_{b_t})$ $(t \in \{1, ..., m_\Lambda\})$ let us glue a closed two-dimensional disk D_{b_t} (along its boundary) and denote the obtained manifold by M_Λ. Let us construct a homeomorphism $F : M_\Lambda \to M_\Lambda$ such that $F|_{N_\Lambda} = f|_{N_\Lambda}$ and the non-wandering set of $F|_{D_{b_t}}$ (for all $t \in \{1, ..., m_\Lambda\}$) consists of exactly one hyperbolic periodic source point α_{b_t}. By construction, α_{b_t} belongs to the closure $W^s_{p_j}$ for each $j \in \{1, ..., h_{b_t}\}$ (see Figure 3).

Let us put $S_\Lambda = \bigcup\limits_{t\in\{1,\dots,m_\Lambda\}} \alpha_{b_t}$. The surface M_Λ admits a foliation

$$\mathcal{F}_{M_\Lambda} = \{W^s_x, x \in (\Lambda \cup S_\Lambda)\},$$

which has m_Λ singularities (points $\alpha_{b_t}, t \in \{1, ..., m_\Lambda\}$), and all these singularities are saddle. The formula (1) implies that the index $I(\alpha_{b_t})$ of each saddle singularity α_{b_t} is equal to $(1 - \frac{h_{b_t}}{2})$. From here and from the formula (2) one gets:

$$\chi(M_\Lambda) = \sum_{t\in\{1,\dots,m_\Lambda\}} I(\alpha_{b_t}) = m_\Lambda - \frac{h_\Lambda}{2}, \tag{3}$$

where $\chi(M_\Lambda)$ is the Euler characteristic of the surface M_Λ.

Since M_Λ is closed orientable connected surface, its genus g is related to the Euler characteristic $\chi(M_\Lambda)$ by the following formula: $\chi(M_\Lambda) = 2 - 2g$. This fact and the formula (3) imply that the genus of the surface M_Λ is calculated by the formula $g = 1 + \frac{h_\Lambda}{4} - \frac{m_\Lambda}{2}$.

It follows from the construction of the surface M_Λ that $N_\Lambda = M_\Lambda \backslash (\bigcup\limits_{t\in\{1,\dots,m_\Lambda\}} int\, D_{b_t})$. Hence, the surface N_Λ has the same genus as M_Λ. □

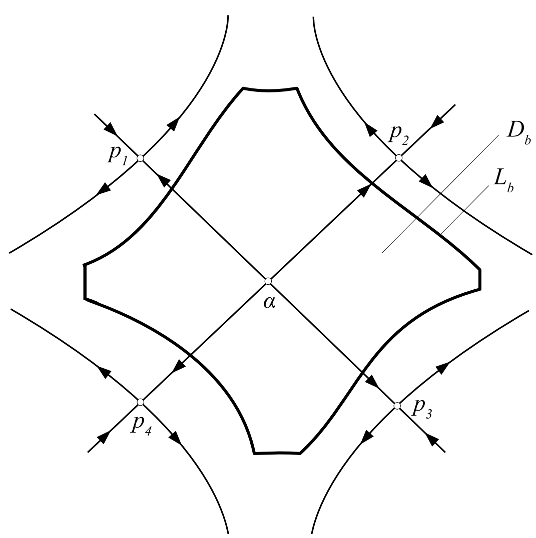


FIGURE 3. Construction of the surface M_Λ.

3. **Proof of the main results.** Throughout this section, $f : M^2 \to M^2$ is a diffeomorphism from the class $\mathbb{G}(M^2)$. Let us denote the periodic components of this diffeomorphism by Λ_i $(i \in \{1, ..., k_f\})$. One can choose the number $n \in \mathbb{N}$ such that $f^n(\Lambda_i) = \Lambda_i$ for all $i \in \{1, ..., k_f\}$ and all boundary periodic points of all one-dimensional basic sets are fixed with respect to the diffeomorphism f^n. Therefore, without loss of generality, throughout this section we will assume that every basic set has a unique periodic component and all boundary periodic points of all one-dimensional basic sets are fixed with respect to the diffeomorphism f. We will call each set Λ_i a basic set.

Lemma 3.1. *Let $f \in \mathbb{G}(M^2)$. Then its non-wandering set contains at least one attractor and at least one repeller.*

Proof. Assume the opposite. Let the non-wandering set of the diffeomorphism f consist of one-dimensional attractors $\Lambda_1, ..., \Lambda_{k_f}$ (if it consists of one-dimensional repellers, then it is sufficient to consider the diffeomorphism f^{-1}). According to [14] (Theorem 2), the set Λ_i $(i \in \{1, ..., k_f\})$ has the local structure of the direct product of the interval and the Cantor set. Thus, every set Λ_i is nowhere dense. The properties of basic set that is an attractor imply that $\Lambda_i = W^u_{\Lambda_i}$ $(i \in \{1, ..., k_f\})$. It follows from [19] (Corollary 6.3) and aforesaid that $M^2 = \bigcup\limits_{i\in\{1,...,k_f\}} W^u_{\Lambda_i} = \bigcup\limits_{i\in\{1,...,k_f\}} \Lambda_i$. That contradicts Baire category theorem which states that a non-empty complete metric space cannot be represented as a countable union of nowhere dense subsets. $\square$

Let us denote the set $M^2 \setminus \bigcup\limits_{i\in\{1,...,k_f\}} \Lambda_i$ by V. Since every set Λ_i $(i \in \{1, ..., k_f\})$ is nowhere dense, then the set V is non-empty.

Lemma 3.2. *The set V consists of a finite number of mutually disjoint open connected sets such that the boundary accessible from inside of each such set consists of two bunches, one of which belongs to some attractor, and the other belongs to some repeller of the diffeomorphism f.*

Proof. Let us note that in compliance with Lemma 3.1, the non-wandering set $NW(f)$ of the diffeomorphism f contains at least one attractor and at least one repeller.

Let Λ^a be a one-dimensional attractor of the diffeomorphism f. From the definition of a bunch and the fact that the boundary accessible from inside of the set $M^2 \setminus \Lambda^a$ consists of a finite number of bunches (see section 1), it follows that the set $W^s(\Lambda^a) \setminus \Lambda^a$ consists of a finite number of mutually disjoint connected sets. Moreover, by virtue of the definition of a bunch, for each such set there exists a single bunch of the attractor Λ^a, which belongs to the boundary accessible from inside of this set.

Let us denote by $\Lambda^a_1, ..., \Lambda^a_{m^a}$ $(m^a \geq 1)$ all one-dimensional attractors of the diffeomorphism f. The aforesaid and the fact that the sets $W^s_{\Lambda^a_1} \setminus \Lambda^a_1$,...,$W^s_{\Lambda^a_{m^a}} \setminus \Lambda^a_{m^a}$ are mutually disjoint imply that the set $\bigcup\limits_{j\in\{1,...,m^a\}} W^s_{\Lambda^a_j} \setminus \Lambda^a_j$ consists of a finite number of mutually disjoint connected sets. Herewith, for each such set there exists a single bunch of some attractor of the diffeomorphism f, which belongs to the boundary accessible from inside of this set. According to [19] (Corollary 6.3), $M^2 = \bigcup\limits_{i\in\{1,...,k_f\}} W^s_{\Lambda_i}$. Considering this, the fact that the non-wandering set of the diffeomorphism f consists of one-dimensional attractors and one-dimensional repellers, and the fact that one-dimensional repeller coincides with its stable manifold, one obtains the following: $V = \bigcup\limits_{j\in\{1,\dots,m^a\}} W^s_{\Lambda^a_j}\setminus\Lambda^a_j$.

Let us denote by $\Lambda^r_1, \dots, \Lambda^r_{m^r}$ ($m^r \geq 1$) all one-dimensional repellers of the diffeomorphism f. Applying the same reasoning for these repellers, one obtains the following: the set $\bigcup\limits_{j\in\{1,\dots,m^r\}} W^u_{\Lambda^r_j}\setminus\Lambda^r_j$ consists of a finite number of mutually disjoint connected sets; for each such set there exists a single bunch of some repeller of the diffeomorphism f, which belongs to the boundary accessible from inside of this set; $V = \bigcup\limits_{j\in\{1,\dots,m^r\}} W^u_{\Lambda^r_j}\setminus\Lambda^r_j$.

Thus, the set V consists of a finite number of mutually disjoint connected sets such that the boundary accessible from inside of each such set consists of two bunches, one of which belongs to some attractor, and the other belongs to some repeller of the diffeomorphism f. Since the set $V = M^2\setminus \bigcup\limits_{i\in\{1,\dots,k_f\}} \Lambda_i$ is open and is a subset of closed manifold, then each its connected component is an open set. $\square$

The proof of the Lemma 3.2 implies the following Corollary.

Corollary 2. *The number of bunches of all attractors of the diffeomorphism f is equal to the number of bunches of all its repellers.*

Proof of Theorem 1. Let Λ^a be a one-dimensional attractor of the diffeomorphism f, b^a be one of its bunches, L_{b^a} be the characteristic curve of the bunch b^a (see the proof of the Lemma 2.1). Let $S = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1^2 + x_2^2 = 1\}$ be a unit circle, $\varphi : S \times [0; 1] \to M^2$ be an embedding[6] such that $\varphi(S \times \{0\}) = L_{b^a}, \varphi(S \times \{1\}) = f(L_{b^a})$. Let us denote by U the image of the set $S \times (0; 1)$ with respect to the map φ. It follows from the proof of the Lemma 3.2 that the curve L_{b^a} belongs to some connected component J of the set V. Hence, $U \subset J$. Remove the set U from the surface M^2. The boundary of $M^2\setminus U$ consists of two connected components, each of which is homeomorphic to a circle. Let us glue closed two-dimensional disks B_1 and B_2 (along their boundaries) to these components and denote the resulting surface by $\tilde{M}^2$. There are two possible cases:

1. the surface $\tilde{M}^2$ is connected. Then the surface M^2 is homeomorphic to the connected sum $\tilde{M}^2\#\mathbb{T}^2$;
2. the surface $\tilde{M}^2$ is disconnected and is a union of two closed orientable connected surfaces P_1 and P_2. Then the surface M^2 is homeomorphic to the connected sum $P_1\#P_2$.

It follows from the Lemma 3.2 that the boundary accessible from inside of the set J consists of the bunch b^a of the attractor Λ^a and a bunch b^r of some repeller Λ^r of the diffeomorphism f. We denote by $p_1, \dots, p_{h_{b^a}}$ the boundary periodic points belonging to the bunch b^a, and by $q_1, \dots, q_{h_{b^r}}$ the boundary periodic points belonging to the bunch b^r.

Let us define a homeomorphism $F : \tilde{M}^2 \to \tilde{M}^2$ such that:

1. $F|_{\tilde{M}^2\setminus(int\, B_1\cup int\, B_2)} = f|_{\tilde{M}^2\setminus(int\, B_1\cup int\, B_2)}$;

[6]The map $\varphi : X \to Y$, where X, Y are topological spaces, is said to be an embedding if $\varphi : X \to \varphi(X) \subset Y$ is a homeomorphism, where $\varphi(X)$ carries the subspace topology inherited from Y.

2. the non-wandering set of $F|_{B_1}$ consists of exactly one hyperbolic fixed source point α (by construction, this point belongs to the closure $W^s_{p_j}$ for each $j \in \{1, ..., h_{b^a}\}$);
3. the non-wandering set of $F|_{B_2}$ consists of exactly one hyperbolic fixed sink point ω (by construction, this point belongs to the closure $W^u_{q_j}$ for each $j \in \{1, ..., h_{b^r}\}$).

Let us consistently perform the procedure described above for all bunches belonging to attractors of the diffeomorphism f. As a result, one gets a disconnected manifold, which is the union of k_f closed orientable connected surfaces. Indeed, each of these surfaces contains a single non-trivial basic set Λ_i (for some $i \in \{1, ..., k_f\}$) and, in fact, is the surface M_{Λ_i} constructed in the proof of Lemma 2.1. It follows from the proof of Lemma 2.1 that every such surface has genus $g_i = 1 + \frac{h_{\Lambda_i}}{4} - \frac{m_{\Lambda_i}}{2}$ ($i \in \{1, ..., k_f\}$) (see the notation in the condition of the Theorem 1). Since the number of all bunches belonging to attractors of the diffeomorphism f is equal to $\frac{m_f}{2}$ (see Corollary 2), then the procedure described above is performed $\frac{m_f}{2}$ times. Among them, there are $k_f - 1$ steps, as a result of each of which the manifold splits into two disconnected manifolds, and $l_f = \frac{m_f}{2} - k_f + 1$ steps, as a result of each of which the manifold remains connected. Thus, one obtains that the original surface M^2 is homeomorphic to the connected sum:

$$M^2_{g_1} \# ... \# M^2_{g_{k_f}} \# \underbrace{\mathbb{T}^2 \# ... \# \mathbb{T}^2}_{l_f},$$

where $g_i = 1 + \frac{h_{\Lambda_i}}{4} - \frac{m_{\Lambda_i}}{2}$ ($i \in \{1, ..., k_f\}$), $l_f = \frac{m_f}{2} - k_f + 1$. □

Proof of Corollary 1. Since the surface M^2 is homeomorphic to the connected sum of k_f closed orientable connected surfaces of the genus g_i ($i \in \{1, ..., k_f\}$) and l_f two-dimensional tori, then the genus g of the surface M^2 is calculated by the following formula:

$$g = \sum_{i \in \{1,...,k_f\}} (1 + \frac{h_{\Lambda_i}}{4} - \frac{m_{\Lambda_i}}{2}) + \frac{m_f}{2} - k_f + 1 = k_f + \frac{h_f}{4} - \frac{m_f}{2} + \frac{m_f}{2} - k_f + 1 = 1 + \frac{h_f}{4}.$$

□

Proof of Theorem 2. In [6] (Theorem 1), it is proved that if the non-wandering set of a structurally stable diffeomorphism of a closed smooth orientable surface contains a one-dimensional attractor (repeller), then it contains a source (sink) periodic point. This fact and the fact that the non-wandering set of the diffeomorphism f consists of one-dimensional attractors and one-dimensional repellers entail that the diffeomorphism f is not structurally stable.

We will prove that the diffeomorphism f is Ω-stable. Let us note that in compliance with Lemma 3.1, the non-wandering set $NW(f)$ of the diffeomorphism f contains at least one attractor and at least one repeller. Let Λ^a be an arbitrary one-dimensional attractor of the diffeomorphism f. Unstable manifold of this attractor coincides with it, and stable manifold of this attractor, by virtue of [19] (Corollary 6.3), intersects with the unstable manifolds of a finite number of repellers $\Lambda^r_1, ..., \Lambda^r_l$. Herewith, the stable manifold of each of the repellers $\Lambda^r_1, ..., \Lambda^r_l$ coincides with it. Conducting similar reasoning for an arbitrary repeller of the diffeomorphism f, one obtains that the diffeomorphism f has no cycles. Hence, according to [9] (Theorem 1.9.), the diffeomorphism f is Ω-stable. □

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