Mathematical Social Sciences xxx (xxxx) xxx

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss

The equal split-off set for NTU-games

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ARTICLE INFO

Article history:
Received 23 March 2022
Received in revised form 27 October 2022
Accepted 28 October 2022
Available online xxxx

Keywords:
NTU-games
equal split-off set
exact partition games
consistency

ABSTRACT

This paper introduces and studies the equal split-off set for cooperative games with nontransferable utility. We illustrate the new solution for the well-known Roth-Shafer examples and present two axiomatic characterizations based on different consistency properties on the class of exact partition games, i.e. the class of games where it intersects the core.

1. Introduction

Nontransferable utility games (NTU-games) arise when players in a cooperative game face the problem of allocating joint profits while having nonlinear utility functions over money. Also situations where the underlying infinitely divisible endowment is not of a monetary nature are accommodated. The opportunities of coalitions are represented by a set of attainable utility payoff allocations and the issue is to select payoff allocations for the grand coalition while taking these opportunities into account.

We focus on egalitarianism in the context of NTU-games. This requires the assumption that utility is not only comparable in an intrapersonal way, but also in an interpersonal way. In other words, we assume that utility is normalized in such a way that equating utilities for different players has a meaningful interpretation. This was also implied by the approach of Kalai and Samet (1985), who introduced and characterized an egalitarian solution for NTU-games which recursively assigns equal payoffs to members of coalitions in the form of dividends. Samet (1985) and de Clippel et al. (2004) provided other axiomatic characterizations of this solution. On the class of transferable utility games (TU-games), the Kalai and Samet (1985) solution coincides with the Shapley (1953) value. On the class of bargaining problems (cf. Nash, 1950), it coincides with the egalitarian Kalai (1977) solution.

We introduce and study the equal split-off set for NTU-games. Inspired by the computational algorithm underlying the egalitarian Dutta and Ray (1989) solution, Branzei et al. (2006) introduced the equal split-off set for TU-games. We generalize that definition to NTU-games in the following way. We start by proposing the maximal attainable equal payoff allocation for the grand coalition. However, the maximal attainable equal payoff allocation for some other coalition may give higher payoffs. We select one coalition with highest maximal attainable equal payoffs, assign those payoffs to the members, and let them leave the game. Coalitions of remaining players are able to attain all original payoff allocations in cooperation with the departed players. Again, one coalition with highest maximal attainable equal payoffs is selected, those payoffs are assigned to the members, and these members leave the game. The process is repeated until all players have left the game, leading to a payoff allocation for the grand coalition. The equal split-off set consists of all payoff allocations generated by this procedure, where the multiplicity arises from selecting distinct coalitions with highest maximal attainable equal payoffs.

We illustrate the equal split-off set for NTU-games using the well-known Roth-Shafer examples and compare it with the Kalai and Samet (1985) solution for these games. We show that the equal split-off set intersects the core if and only if the underlying game is an exact partition game, i.e. there exists a core element for which all players with highest payoffs, all players with highest and second highest payoffs, and so on, are able to attain their payoffs jointly by themselves. This generalizes the class of TU-games introduced by Llerena and Mauri (2017) and the corresponding result of Dietzenbacher and Yanovskaya (2021). On the class of exact partition games, we present two axiomatic characterizations of the equal split-off set based on weak versions of consistency properties that were employed by Peleg (1985) and Tadenuma (1992) to characterize the core. These axiomatic characterizations generalize the corresponding results of Dietzenbacher and Yanovskaya (2021) for TU-games. It turns out

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https://doi.org/10.1016/j.mathsocsci.2022.10.004
0165-4896/

Please cite this article as: B. Dietzenbacher and E. Yanovskaya, The equal split-off set for NTU-games. Mathematical Social Sciences (2022), https://doi.org/10.1016/j.mathsocsci.2022.10.004.

that NTU-games induced by bargaining problems and bankruptcy problems are exact partition games. On the class of bargaining problems, the equal split-off set coincides with the Kalai (1977) solution. On the class of NTU-bankruptcy problems, the equal split-off set coincides with the constrained equal awards rule (cf. Dietzenbacher, 2022), also called the constrained Kalai solution (cf. Albizuri et al., 2020).

This paper is organized in the following way. Section 2 presents preliminary notions and notations for NTU-games. Section 3 introduces the equal split-off set as a solution for all NTU-games and presents some elementary results. Section 4 introduces the class of exact partition games, shows that this class consists of all games where the equal split-off set intersects the core, and presents two axiomatic characterizations based on different consistency properties.

2. Preliminaries

Let N be a nonempty and finite set. Denote $2^N = \{S \mid S \subseteq N\}$. An *ordered partition* of N is an ordered set $\{T_1, \dots, T_m\} \subseteq 2^N \setminus \{\emptyset\}$ such that $\bigcup_{k=1}^{m} T_k = N$ and $T_k \cap T_\ell = \emptyset$ for all $k, \ell \in \{1, \dots, m\}$ with $k \neq \ell$. Let $e \in \mathbb{R}^N_+$ denote the vector of all ones, i.e. $e_i = 1$ for all $i \in N$. For all $x, y \in \mathbb{R}^N_+$, $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$, and $x < y$ denotes $x_i < y_i$ for all $i \in N$. For all $x \in \mathbb{R}^N_+$ and all $S \in 2^N$, denote $x_S = (x_i)_{i \in S}$. For all $A \subseteq \mathbb{R}^N_+$,

- the *Pareto set* is $\mathrm{P}(A) = \{x \in A \mid \nexists y \in A : y \geq x, y \neq x\}$;
- the *weak Pareto set* is $\mathrm{WP}(A) = \{x \in A \mid \nexists y \in A : y > x\}$.

Note that $\mathrm{P}(A) \subseteq \mathrm{WP}(A)$.

An *NTU-game* is a pair (N, V) where N is a nonempty and finite set of *players* and V assigns to each *coalition* $S \in 2^N \setminus \{\emptyset\}$ a set of attainable payoff allocations $V(S) \subseteq \mathbb{R}^S_+$ such that

- $V(S)$ is nonempty, closed, and bounded;
- $V(S)$ is comprehensive, i.e. $\{y_S \in \mathbb{R}^S_+ \mid y_S \leq x_S\} \subseteq V(S)$ for all $x_S \in V(S)$;
- (N, V) is monotonic, i.e. $V(S) \subseteq \{x_S \mid x_T \in V(T)\}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$.

The nonnegativity condition on the attainable payoff allocations was also assumed by e.g. Asscher (1976), Asscher (1977), Greenberg (1985), and Lejano (2011). The nonemptiness, closedness, boundedness, and comprehensiveness conditions are standard. The monotonicity condition was also assumed by e.g. Otten et al. (1998) and Hendrickx et al. (2002). Note that we do not assume that the sets of attainable payoff allocations are convex in order to allow for utility functions that are not necessarily of the Von Neumann-Morgenstern type. In line with Kalai and Samet (1985), we assume that utility is normalized in such a way that it is interpersonally comparable.

Let Γ_{all} denote the class of all NTU-games. A *solution* σ on a class of games $\Gamma \subseteq \Gamma_{all}$ assigns to each $(N, V) \in \Gamma$ a nonempty set of payoff allocations $\sigma(N, V) \subseteq V(N)$. Throughout this paper, Γ is the generic notation for a class of games and σ is the generic notation for a solution on Γ. The *core* is the solution that assigns to each game (N, V) where it is nonempty the set of payoff allocations

$$C(N, V) = \left\{x \in V(N) \mid \forall S \in 2^N \setminus \{\emptyset\} \nexists y_S \in V(S) : y_S > x_S\right\}.$$

3. The equal split-off set

In this section, we introduce the equal split-off set as a solution for all NTU-games and present some elementary results. The equal split-off set for TU-games was introduced by Branzei et al. (2006). We generalize this solution to NTU-games in the following way. Consider an arbitrary NTU-game for which we face the problem of selecting payoff allocations for the grand coalition. One of the coalitions with highest maximal attainable equal payoff allocation is selected and the members leave with these payoffs. The remaining players determine the attainable payoff allocations for each subgroup in coalition with the departed players. One of the subgroups with highest maximal attainable equal payoff allocation is selected and the members leave with these payoffs. This process continues and results in a payoff allocation for the players. The equal split-off set consists of all payoff allocations generated by this procedure.

Equal split-off set Let $(N, V) \in \Gamma_{all}$. Define $N_0 = N$, $V_0 = V$, and $T_0 = \emptyset$. The *equal split-off set* $ESOS(N, V)$ consists of all payoff allocations $x \in \mathbb{R}^N_+$ for which there exists an ordered partition $\{T_1, \dots, T_m\}$ of N such that for all $k \in \{1, \dots, m\}$,

$$T_k \in \operatorname*{argmax}_{S \in 2^{N_k} \setminus \{\emptyset\}} \max \{t \in \mathbb{R}_+ \mid te_S \in V_k(S)\}$$

and $$x_i = \max_{S \in 2^{N_k} \setminus \{\emptyset\}} \max \{t \in \mathbb{R}_+ \mid te_S \in V_k(S)\} \text{ for all } i \in T_k,$$

where (N_k, V_k) is the game defined by $N_k = N_{k-1} \setminus T_{k-1}$ and

$$V_k(S) = \left\{y_S \in \mathbb{R}^S_+ \;\middle|\; (y_S, x_{T_{k-1}}) \in V_{k-1}(S \cup T_{k-1})\right\} \text{ for all } S \in 2^{N_k}.$$

Note that the equal split-off set is well-defined due to the assumptions on NTU-games.

Let (N, V) be a game. For each $x \in ESOS(N, V)$ with corresponding ordered partition $\{T_1, \dots, T_m\}$ the following holds:

- $x_i = x_j$ for all $i, j \in T_k$ with $k \in \{1, \dots, m\}$;
- $x_{\bigcup_{\ell=1}^{k} T_\ell} \in \mathrm{WP}(V(\bigcup_{\ell=1}^{k} T_\ell))$ for all $k \in \{1, \dots, m\}$.

We illustrate the new solution by means of the examples introduced by Roth (1980) and Shafer (1980). These examples initiated an interesting and extensive discussion on the interpretation of solutions for NTU-games. For details, we refer to Harsanyi (1980), Aumann (1985b), Hart (1985b), Roth (1986), and Aumann (1986). Along the lines of this discussion, we compare the equal split-off set with the egalitarian Kalai and Samet (1985) solution.

Example 1 (*cf. Roth, 1980*)**.** Let $(N, V_p) \in \Gamma_{all}$ with $N = \{1, 2, 3\}$ and $p \in [0, \frac{1}{2}]$ be the game given by

$$V_p(S) = \begin{cases} \{x_i \in \mathbb{R}^{\{i\}}_+ \mid x_i \leq 0\} & \text{if } S = \{i\} \text{ and } i \in N; \\ \{x_{\{1,2\}} \in \mathbb{R}^{\{1,2\}}_+ \mid (x_1, x_2) \leq (\frac{1}{2}, \frac{1}{2})\} & \text{if } S = \{1, 2\}; \\ \{x_{\{1,3\}} \in \mathbb{R}^{\{1,3\}}_+ \mid (x_1, x_3) \leq (p, 1-p)\} & \text{if } S = \{1, 3\}; \\ \{x_{\{2,3\}} \in \mathbb{R}^{\{2,3\}}_+ \mid (x_2, x_3) \leq (p, 1-p)\} & \text{if } S = \{2, 3\}; \\ comp(conv(\{(\frac{1}{2}, \frac{1}{2}, 0), & \\ \quad (p, 0, 1-p), (0, p, 1-p)\})) & \text{if } S = N. \end{cases}$$

Here, *comp* denotes the comprehensive hull and *conv* denotes the convex hull, i.e. the smallest containing comprehensive set and the smallest containing convex set, respectively.

If $p < \frac{1}{2}$, the equal split-off set is $ESOS(N, V_p) = \{(\frac{1}{2}, \frac{1}{2}, 0)\}$ corresponding to ordered partition $\{\{1, 2\}, \{3\}\}$ and the Kalai and Samet (1985) solution is $(\frac{1}{2} - \frac{1}{3}p, \frac{1}{2} - \frac{1}{3}p, \frac{2}{3}p)$. Note that, in contrast to the Kalai and Samet (1985) solution, the equal split-off set assigns the unique core element to this game. Moreover, Roth (1980) claims that $(\frac{1}{2}, \frac{1}{2}, 0)$ is the unique outcome of this game consistent with the hypothesis that the players are rational utility maximizers, because this outcome is strictly preferred by both players 1 and 2 over all other feasible outcomes, and it can be achieved without the cooperation of player 3.

If $p = \frac{1}{2}$, the game is completely symmetric with respect to the players and it is no longer the case that cooperation with player 3 offers strictly less to players 1 or 2 than cooperation with one another. The equal split-off set is $ESOS(N, V_{\frac{1}{2}}) = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$ corresponding to ordered partitions $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, and $\{\{2, 3\}, \{1\}\}$. The Kalai and Samet (1985) solution is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Note that the equal split-off set coincides with the core in this case, whereas the Kalai and Samet (1985) solution is not in the core. △

Example 2 (*cf. Shafer, 1980 and Hart and Kurz, 1983*)**.** Let $(N, V_\varepsilon) \in \Gamma_{all}$ with $N = \{1, 2, 3\}$ and $\varepsilon \in [0, \frac{1}{5}]$ be the game given by

$$V_\varepsilon(S) = \begin{cases} \{x_i \in \mathbb{R}^{\{i\}}_+ \mid x_i \le 0\} & \text{if } S = \{i\} \\ & \text{and } i \in \{1, 2\}; \\ \{x_3 \in \mathbb{R}^{\{3\}}_+ \mid x_3 \le \varepsilon\} & \text{if } S = \{3\}; \\ \{x_{\{1,2\}} \in \mathbb{R}^{\{1,2\}}_+ \mid x_1 + x_2 \le 1 - \varepsilon\} & \text{if } S = \{1, 2\}; \\ \{x_{\{1,3\}} \in \mathbb{R}^{\{1,3\}}_+ \mid x_1 \le \varepsilon, x_1 + x_3 \le \frac{1}{2} + \frac{1}{2}\varepsilon\} & \text{if } S = \{1, 3\}; \\ \{x_{\{2,3\}} \in \mathbb{R}^{\{2,3\}}_+ \mid x_2 \le \varepsilon, x_2 + x_3 \le \frac{1}{2} + \frac{1}{2}\varepsilon\} & \text{if } S = \{2, 3\}; \\ \{x \in \mathbb{R}^N_+ \mid x_1 + x_2 + x_3 \le 1\} & \text{if } S = N. \end{cases}$$

The equal split-off set is $ESOS(N, V_\varepsilon) = \{(\frac{1}{2} - \frac{1}{2}\varepsilon, \frac{1}{2} - \frac{1}{2}\varepsilon, \varepsilon)\}$ corresponding to ordered partition $\{\{1, 2\}, \{3\}\}$ and the Kalai and Samet (1985) solution is $(\frac{1}{2} - \frac{5}{6}\varepsilon, \frac{1}{2} - \frac{5}{6}\varepsilon, \frac{5}{3}\varepsilon)$. Note that, in contrast to the Kalai and Samet (1985) solution, the equal split-off set assigns a core element to this game. △

In contrast to what is the case for TU-games, equal split-off set allocations make players not necessarily leave with their payoffs in nonincreasing order, i.e. for each $x \in ESOS(N, V)$ with corresponding ordered partition $\{T_1, \dots, T_m\}$, it does not generally hold that $x_i \ge x_j$ for all $i \in T_k$ and $j \in T_\ell$ with $k, \ell \in \{1, \dots, m\}$ and $k \le \ell$. This is shown by the following example.

Example 3. Let $(N, V) \in \Gamma_{all}$ with $N = \{1, 2\}$ be the game given by

$$V(S) = \begin{cases} \{x_1 \in \mathbb{R}^{\{1\}}_+ \mid x_1 \le 0\} & \text{if } S = \{1\}; \\ \{x_2 \in \mathbb{R}^{\{2\}}_+ \mid x_2 \le 1\} & \text{if } S = \{2\}; \\ \{x \in \mathbb{R}^N_+ \mid (x_1, x_2) \le (2, 1)\} & \text{if } S = N. \end{cases}$$

The equal split-off set is $ESOS(N, V) = \{(2, 1), (1, 1)\}$ corresponding to ordered partitions $\{\{2\}, \{1\}\}$ and $\{\{1, 2\}\}$. Note that both equal split-off set allocations belong to the core of this game. For the allocation $(2, 1)$, the payoff to player 2 is assigned first, while the higher payoff to player 1 is assigned last. △

Observations change significantly if we slightly restrict the domain of NTU-games. Let $\widetilde{\Gamma}_{all}$ denote the class of all NTU-games (N, V) where for all $S \in 2^N \setminus \{\emptyset\}$,

- $V(S)$ is nonleveled, i.e. $\mathrm{P}(V(S)) = \mathrm{WP}(V(S))$.

Note that all NTU-games in Γ_{all} can be approximated by games in $\widetilde{\Gamma}_{all}$. The nonlevelness condition was also assumed by e.g. Aumann (1985a), Hart (1985a), Peleg (1985), Tadenuma (1992), and Hart and Mas-Colell (1996).

Lemma 1 (*cf. Yanovskaya, 2010*)**.** *Let $(N, V) \in \widetilde{\Gamma}_{all}$ and let $x \in ESOS(N, V)$ with corresponding ordered partition $\{T_1, \dots, T_m\}$. Then $x_i \ge x_j$ for all $i \in T_k$ and $j \in T_{k+1}$ with $k \in \{1, \dots, m - 1\}$.*

Let $(N, V) \in \widetilde{\Gamma}_{all}$. For all $x \in \mathbb{R}^N_+$, we define $R^x_0 = \emptyset$ and for all $k \in \mathbb{N}$,

$$R^x_k = \left\{i \in N \mid \forall j \in N \setminus R^x_{k-1} : x_j \le x_i\right\}$$

and $a^x_k = x_i$ for all $i \in R^x_k \setminus R^x_{k-1}$.

Note that $R^x_{k-1} \subseteq R^x_k$ for all $k \in \mathbb{N}$ and $R^x_{|N|} = N$. Coalition $R^x_1 \in 2^N \setminus \{\emptyset\}$ consists of all players with highest payoffs in x, coalition $R^x_2 \in 2^N \setminus \{\emptyset\}$ consists of all players with highest and second highest payoffs in x, and so on. Lemma 1 implies that for each $x \in ESOS(N, V)$ the following holds:

- $x_{R^x_k} \in \mathrm{WP}(V(R^x_k))$ for all $k \in \mathbb{N}$.

In Example 3, the equal split-off set consists of multiple core allocations. Another consequence of assuming nonleveled attainable sets of payoff allocations is that the equal split-off set is single-valued when it intersects the core, generalizing the corresponding result of Dietzenbacher and Yanovskaya (2021) for TU-games.

Lemma 2. *Let $(N, V) \in \widetilde{\Gamma}_{all}$. If $ESOS(N, V) \cap C(N, V) \neq \emptyset$, then $|ESOS(N, V)| = 1$.*

Proof. Let $x \in ESOS(N, V) \cap C(N, V)$. We show that for all $k \in \mathbb{N}$ with $R^x_{k-1} \neq N$,

$$R^x_k \setminus R^x_{k-1} = \bigcup_{S \in 2^{N \setminus R^x_{k-1}} \setminus \{\emptyset\}} \operatorname{argmax} \max \left\{t \in \mathbb{R}_+ \;\middle|\; \left(te_S, x_{R^x_{k-1}}\right) \in V(S \cup R^x_{k-1})\right\}.$$

Let $k \in \mathbb{N}$ be such that $R^x_{k-1} \neq N$. Note that $x_i \le a^x_k$ for all $i \in N \setminus R^x_{k-1}$. Suppose for the sake of contradiction that there exist

$$T \in \operatorname*{argmax}_{S \in 2^{N \setminus R^x_{k-1}} \setminus \{\emptyset\}} \max \left\{t \in \mathbb{R}_+ \middle| \left(te_S, x_{R^x_{k-1}}\right) \in V(S \cup R^x_{k-1})\right\} \quad \text{and} \quad i \in T$$

such that $x_i < a^x_k$. By comprehensiveness, $x_{T \cup R^x_{k-1}} \in V(T \cup R^x_{k-1}) \setminus \mathrm{P}(V(T \cup R^x_{k-1}))$. By nonlevelness, $x_{T \cup R^x_{k-1}} \in V(T \cup R^x_{k-1}) \setminus \mathrm{WP}(V(T \cup R^x_{k-1}))$. This contradicts that $x \in C(N, V)$. □

4. Exact partition games

In this section, we introduce the class of exact partition games, we show that this class consists of all games where the equal split-off set intersects the core, and we present two axiomatic characterizations based on different consistency properties. Llerena and Mauri (2017) introduced exact partition games in the TU context. We generalize this definition to NTU-games. A game is an exact partition game if there exists a core allocation for which all players with highest payoffs, all players with highest and second highest payoffs, and so on, are able to attain their payoffs jointly by themselves.

Exact partition games A game $(N, V) \in \widetilde{\Gamma}_{all}$ is an *exact partition game* if there exists $x \in C(N, V)$ such that $x_{R^x_k} \in V(R^x_k)$ for all $k \in \mathbb{N}$.

Example 4. Let $(N, V) \in \widetilde{\Gamma}_{all}$ with $N = \{1, 2, 3\}$ be the game given by

$$V(S) = \begin{cases} \{x_1 \in \mathbb{R}^{\{1\}}_+ \mid x_1 \le 6\} & \text{if } S = \{1\}; \\ \{x_i \in \mathbb{R}^{\{i\}}_+ \mid x_i \le 0\} & \text{if } S = \{i\} \text{ and } i \in \{2, 3\}; \\ \{x_{\{1,2\}} \in \mathbb{R}^{\{1,2\}}_+ \mid \min\{2x_1 + x_2, \\ x_1 + 2x_2\} \le 12\} & \text{if } S = \{1, 2\}; \\ \{x_S \in \mathbb{R}^S_+ \mid x_S \le 0_S\} & \text{if } S \in \{\{1, 3\}, \{2, 3\}\}; \\ \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \le 10\} & \text{if } S = N. \end{cases}$$

The equal split-off set is $ESOS(N,V) = \{(6,3,1)\}$ corresponding to ordered partition $\{\{1\},\{2\},\{3\}\}$. Denote $x = (6,3,1)$. Then $x \in C(N,V)$. Moreover, $R_1^x = \{1\}$, $x_1 \in V(\{1\})$, $R_2^x = \{1,2\}$, $x_{\{1,2\}} \in V(\{1,2\})$, $R_3^x = N$, and $x \in V(N)$. This means that (N,V) is an exact partition game. △

Let $\widetilde{\Gamma}_{ep}$ denote the class of all exact partition games. In Example 4, we observe that the equal split-off set of an exact partition game intersects the core. We show that the equal split-off set of a game intersects the core if and only if it is an exact partition game, generalizing the corresponding result of Dietzenbacher and Yanovskaya (2021) for TU-games. Then Lemma 2 implies that the equal split-off set of an exact partition game is single-valued.

Lemma 3. *Let $(N,V) \in \widetilde{\Gamma}_{all}$. Then $ESOS(N,V) \cap C(N,V) \neq \emptyset$ if and only if $(N,V) \in \widetilde{\Gamma}_{ep}$.*

Proof. *Step 1: sufficiency.* Assume that there exists $x \in ESOS(N,V) \cap C(N,V)$. Then $x_{R_k^x} \in \mathrm{WP}(V(R_k^x))$ for all $k \in \mathbb{N}$. Hence, $(N,V) \in \widetilde{\Gamma}_{ep}$.

Step 2: necessity. Assume that $(N,V) \in \widetilde{\Gamma}_{ep}$. Let $x \in C(N,V)$ be such that $x_{R_k^x} \in V(R_k^x)$ for all $k \in \mathbb{N}$. Then $x_{R_k^x} \in \mathrm{WP}(V(R_k^x))$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $R_{k-1}^x \neq N$. Then

$$a_k^x = \max\left\{t \in \mathbb{R}_+ \middle| \left(te_{R_k^x \setminus R_{k-1}^x}, x_{R_{k-1}^x}\right) \in V(R_k^x)\right\}.$$

Note that $x_i \leq a_k^x$ for all $i \in N \setminus R_{k-1}^x$. Suppose for the sake of contradiction that there exists $S \in 2^{N \setminus R_{k-1}^x} \setminus \{\emptyset\}$ such that

$$a_k^x < \max\left\{t \in \mathbb{R}_+ \middle| \left(te_S, x_{R_{k-1}^x}\right) \in V(S \cup R_{k-1}^x)\right\}.$$

By comprehensiveness, $x_{S \cup R_{k-1}^x} \in V(S \cup R_{k-1}^x) \setminus \mathrm{P}(V(S \cup R_{k-1}^x))$. By nonlevelness, $x_{S \cup R_{k-1}^x} \in V(S \cup R_{k-1}^x) \setminus \mathrm{WP}(V(S \cup R_{k-1}^x))$. This contradicts that $x \in C(N,V)$, which implies that

$$R_k^x \setminus R_{k-1}^x \in \underset{S \in 2^{N \setminus R_{k-1}^x} \setminus \{\emptyset\}}{\operatorname{argmax}} \max\left\{t \in \mathbb{R}_+ \,\middle|\, \left(te_S, x_{R_{k-1}^x}\right) \in V(S \cup R_{k-1}^x)\right\}.$$

Hence, $x \in ESOS(N,V)$. □

Example 5. Let $(N,V) \in \widetilde{\Gamma}_{all}$ with $N = \{1,2,3\}$ be the game given by

$$V(S) = \begin{cases} \{x_i \in \mathbb{R}_+^{\{i\}} \mid x_i \leq 0\} & \text{if } S = \{i\} \text{ and } i \in \{1,2,3\}; \\ \{x_{\{1,i\}} \in \mathbb{R}_+^{\{1,i\}} \mid \min\{2x_1 + x_i, & \\ \quad x_1 + 2x_i\} \leq 12\} & \text{if } S = \{1,i\} \text{ and } i \in \{2,3\}; \\ \{x_{\{2,3\}} \in \mathbb{R}_+^{\{2,3\}} \mid x_{\{2,3\}} \leq 0_{\{2,3\}}\} & \text{if } S = \{2,3\}; \\ \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq 10\} & \text{if } S = N. \end{cases}$$

The equal split-off set is $ESOS(N,V) = \{(4,4,2),(4,2,4)\}$ corresponding to ordered partitions $\{\{1,2\},\{3\}\}$ and $\{\{1,3\},\{2\}\}$. Since $|ESOS(N,V)| \neq 1$, Lemmas 2 and 3 imply that $(N,V) \notin \widetilde{\Gamma}_{ep}$. △

The following punctual properties are satisfied by the equal split-off set on the class of exact partition games.

Feasible highest payoffs for all $(N,V) \in \Gamma$ and all $x \in \sigma(N,V)$, it holds that $x_{R_1^x} \in V(R_1^x)$.

Feasible high payoffs for all $(N,V) \in \Gamma$, all $x \in \sigma(N,V)$, and all $k \in \mathbb{N}$, it holds that $x_{R_k^x} \in V(R_k^x)$.

Equal payoff stability for all $(N,V) \in \Gamma$, all $x \in \sigma(N,V)$, and all $S \in 2^N \setminus \{\emptyset\}$, there exists $i \in S$ such that $x_i \geq \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}$.

Core selection for all $(N,V) \in \Gamma$, it holds that $\sigma(N,V) \subseteq C(N,V)$.

Feasible highest payoffs requires that the players with highest payoffs are able to attain their payoffs by themselves. Feasible high payoffs requires that the players with highest payoffs, the players with highest and second highest payoffs, and so on, are able to attain their payoffs jointly by themselves. Equal payoff stability requires that no coalition is better off by an attainable equal payoff allocation, i.e. for all coalitions there exists a member whose allocated payoff is at least the maximal attainable equal payoff. Core selection requires that only core elements are assigned. Note that feasible high payoffs implies feasible highest payoffs, and core selection implies equal payoff stability. Clearly, the equal split-off set satisfies feasible high payoffs. By Lemmas 2 and 3, the equal split-off set satisfies core selection on the class of exact partition games. In fact, an axiomatic characterization of the equal split-off set is directly obtained.

Theorem 1. *The equal split-off set is the unique solution for exact partition games satisfying feasible high payoffs and core selection.*

Proof. *Step 1: sufficiency.* The equal split-off set satisfies feasible high payoffs. By Lemmas 2 and 3, the equal split-off set satisfies core selection on $\widetilde{\Gamma}_{ep}$.

Step 2: necessity. Let σ be a solution on $\widetilde{\Gamma}_{ep}$ satisfying feasible high payoffs and core selection. Let $(N,V) \in \widetilde{\Gamma}_{ep}$ and let $x \in \sigma(N,V)$. By core selection and feasible high payoffs, $x \in C(N,v)$ and $x_{R_k^x} \in V(R_k^x)$ for all $k \in \mathbb{N}$. By the proof of Lemma 3, this implies that $x \in ESOS(N,V)$. By Lemma 2, $|ESOS(N,V)| = 1$. Hence, $\sigma(N,V) = ESOS(N,V)$. □

Theorem 1 generalizes the corresponding result of Calleja et al. (2021) for TU-games. To show that the corresponding properties are independent, we consider the following solutions. The core satisfies core selection, but does not satisfy feasible high payoffs. The equal payoff solution, which assigns to all exact partition games the maximal attainable equal payoff allocation of the grand coalition, satisfies feasible high payoffs, but does not satisfy core selection.

The equal split-off set is not the unique solution for exact partition games satisfying feasible highest payoffs and equal payoff stability. The solution which assigns $(6,2,2)$ to the game in Example 4, and the equal split-off set to all other exact partition games, also satisfies these properties. However, this solution does not apply feasible highest payoffs in a coherent way to NTU-games with variable population. In other words, it does not satisfy the relational property of consistency. Suppose that we apply a certain solution to select payoff allocations for the grand coalition and consider one such assigned payoff allocation. Some players leave with their payoffs and the remaining players reevaluate their payoffs on the basis of a reduced game. The solution is consistent if it assigns the same payoffs to the remaining players in the reduced game as in the original game.

Peleg (1985) axiomatically characterized the core for NTU-games using the consistency property where the attainable payoff allocations for the remaining players in the reduced game are the attainable payoff allocations in coalition with any subgroup of departed players in the original game when these departed players are assigned their original payoffs. This generalizes the consistency property for TU-games of Davis and Maschler (1965) and we refer to it as max-consistency, following the terminology of Thomson (2011). In order to axiomatically characterize the equal split-off set for exact partition games, and inspired by Thomson (1996), we use the weaker version which only requires consistent payoff allocations when all players with highest payoffs leave.

Max-consistency for highest payoffs for all $(N,V) \in \Gamma$ and all $x \in \sigma(N,V)$ with $R_1^x \neq N$, it holds that $(N \setminus R_1^x, V_{max}^x) \in \Gamma$ and $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, V_{max}^x)$, where

$$V^x_{max}(S) = \begin{cases} \{y_{N\setminus R^x_1} \in \mathbb{R}^{N\setminus R^x_1}_+ \mid (y_{N\setminus R^x_1}, x_{R^x_1}) \in V(N)\} & \text{if } S = N \setminus R^x_1; \\ \bigcup_{Q\subseteq R^x_1} \{y_S \in \mathbb{R}^S_+ \mid (y_S, x_Q) \in V(S \cup Q)\} & \text{if } \emptyset \subset S \subset N \setminus R^x_1. \end{cases}$$

Theorem 2. *The equal split-off set is the unique solution for exact partition games satisfying feasible highest payoffs, equal payoff stability, and max-consistency for highest payoffs.*

Proof. *Step 1: sufficiency.* The equal split-off set satisfies feasible highest payoffs. By Lemmas 2 and 3, the equal split-off set satisfies equal payoff stability on $\widetilde{\Gamma}_{ep}$. To show that the equal split-off set satisfies max-consistency for highest payoffs on $\widetilde{\Gamma}_{ep}$, let $(N, V) \in \widetilde{\Gamma}_{ep}$ and let $x \in ESOS(N, V)$ be such that $R^x_1 \neq N$. By Lemmas 2 and 3, $x \in C(N, V)$. Suppose for the sake of contradiction that $x_{N\setminus R^x_1} \notin C(N\setminus R^x_1, V^x_{max})$. Then there exist $S \in 2^N$ with $\emptyset \subset S \subset N \setminus R^x_1$ and $y_S \in V^x_{max}(S)$ such that $y_S > x_S$. This implies that there exists $Q \subseteq R^x_1$ such that $(y_S, x_Q) \in V(S \cup Q)$. By comprehensiveness, $x_{S\cup Q} \in V(S\cup Q)\setminus P(V(S\cup Q))$. By nonlevelness, $x_{S\cup Q} \in V(S \cup Q) \setminus WP(V(S \cup Q))$, which implies that there exists $z_{S\cup Q} \in V(S \cup Q)$ such that $z_{S\cup Q} > x_{S\cup Q}$. This contradicts that $x \in C(N, V)$, so $x_{N\setminus R^x_1} \in C(N\setminus R^x_1, V^x_{max})$. Note that $R_k^{x_{N\setminus R^x_1}} = R^x_{k+1}\setminus R^x_1$ for all $k \in \mathbb{N}$. For all $k \in \mathbb{N}$, $x_{R^x_{k+1}} \in V(R^x_{k+1})$, so $(x_{R_k^{x_{N\setminus R^x_1}}}, x_{R^x_1}) \in V(R_k^{x_{N\setminus R^x_1}} \cup R^x_1)$, which implies that $x_{R_k^{x_{N\setminus R^x_1}}} \in V^x_{max}(R_k^{x_{N\setminus R^x_1}})$. This means that $(N \setminus R^x_1, V^x_{max}) \in \widetilde{\Gamma}_{ep}$ and $x_{N\setminus R^x_1} \in ESOS(N \setminus R^x_1, V^x_{max})$. Hence, the equal split-off set satisfies max-consistency for highest payoffs.

Step 2: necessity. Let σ be a solution on $\widetilde{\Gamma}_{ep}$ satisfying feasible highest payoffs, equal payoff stability, and max-consistency for highest payoffs. We show by induction on the number of players that $\sigma(N, V)$ consists of a unique payoff allocation for all $(N, V) \in \widetilde{\Gamma}_{ep}$. By equal payoff stability, $\sigma(N, V) = \{WP(V(N))\}$ for all $(N, V) \in \widetilde{\Gamma}_{ep}$ with $|N| = 1$. Let $k \in \mathbb{N}$ and assume that $\sigma(N, V)$ consists of a unique payoff allocation for all $(N, V) \in \widetilde{\Gamma}_{ep}$ with $|N| \leq k$. Let $(N, V) \in \widetilde{\Gamma}_{ep}$ with $|N| = k+1$ and let $x \in \sigma(N, V)$. By equal payoff stability,

$$\begin{aligned} a^x_1 &\geq \max_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\} \\ &\geq \max\left\{t \in \mathbb{R}_+ \mid te_{R^x_1} \in V(R^x_1)\right\}. \end{aligned}$$

By feasible highest payoffs,

$$\begin{aligned} a^x_1 &\leq \max\left\{t \in \mathbb{R}_+ \mid te_{R^x_1} \in V(R^x_1)\right\} \\ &\leq \max_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}. \end{aligned}$$

This implies that

$$a^x_1 = \max_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}$$

$$\text{and} \quad R^x_1 \in \operatorname*{argmax}_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}.$$

Define $U^V = \bigcup \operatorname{argmax}_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}$. Since $(N, V) \in \widetilde{\Gamma}_{ep}$, Lemmas 2 and 3 imply that $U^V \in \operatorname{argmax}_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}$. Suppose for the sake of contradiction that $R^x_1 \neq U^V$. By max-consistency for highest payoffs, $x_{N\setminus R^x_1} \in \sigma(N \setminus R^x_1, V^x_{max})$.

By equal payoff stability, there exists $i \in U^V \setminus R^x_1$ such that

$$\begin{aligned} x_i &\geq \max\left\{t \in \mathbb{R}_+ \mid te_{U^V\setminus R^x_1} \in V^x_{max}(U^V \setminus R^x_1)\right\} \\ &\geq \max\left\{t \in \mathbb{R}_+ \mid \left(te_{U^V\setminus R^x_1}, x_{R^x_1}\right) \in V(U^V)\right\} \\ &= \max\left\{t \in \mathbb{R}_+ \mid \left(te_{U^V\setminus R^x_1}, a^x_1 e_{R^x_1}\right) \in V(U^V)\right\} \\ &= a^x_1, \end{aligned}$$

where the last equality follows from nonlevelness. This is a contradiction, so $R^x_1 = U^V$. If $R^x_1 = U^V = N$, then $\sigma(N, V) = \{(\max\{t \in \mathbb{R}_+ \mid te_N \in V(N)\})_{i\in N}\}$. Suppose that $R^x_1 = U^V \neq N$. By max-consistency for highest payoffs, $x_{N\setminus R^x_1} \in \sigma(N \setminus R^x_1, V^x_{max})$, where, by the induction hypothesis, $\sigma(N \setminus R^x_1, V^x_{max})$ consists of a unique payoff allocation since $|N \setminus R^x_1| \leq k$. Hence, $\sigma(N, V)$ consists of a unique payoff allocation. □

Theorem 2 generalizes the corresponding result of Dietzenbacher and Yanovskaya (2021) for TU-games. To show that the corresponding properties are independent, we consider the following solutions. The solution which assigns the core to all exact partition games with at most two players, and the equal split-off set to all exact partition games with more players, satisfies all axioms in the theorem except for feasible highest payoffs. The equal payoff solution, which assigns to all exact partition games the maximal attainable equal payoff allocation of the grand coalition, satisfies all axioms in the theorem except for equal payoff stability. The solution which assigns $(6, 2, 2)$ to the game in Example 4, and the equal split-off set to all other exact partition games, satisfies all axioms in the theorem except for max-consistency for highest payoffs.

Tadenuma (1992) axiomatically characterized the core for NTU-games using the alternative consistency property where the attainable payoff allocations for the remaining players in the reduced game are the attainable payoff allocations in coalition with all departed players in the original game when these departed players are assigned their original payoffs. This is inspired by the consistency property of Moulin (1985) and we refer to it as complement-consistency, following the terminology of Thomson (2011). In order to axiomatically characterize the equal split-off set for exact partition games, and inspired by Thomson (1996), we use the weaker version which only requires consistent payoff allocations when all players with highest payoffs leave.

Complement-consistency for highest payoffs for all $(N, V) \in \Gamma$ and all $x \in \sigma(N, V)$ with $R^x_1 \neq N$, it holds that $(N \setminus R^x_1, V^x_{comp}) \in \Gamma$ and $x_{N\setminus R^x_1} \in \sigma(N \setminus R^x_1, V^x_{comp})$, where for all $S \in 2^{N\setminus R^x_1} \setminus \{\emptyset\}$,

$$V^x_{comp}(S) = \left\{y_S \in \mathbb{R}^S_+ \,\middle|\, (y_S, x_{R^x_1}) \in V(S \cup R^x_1)\right\}.$$

Theorem 3. *The equal split-off set is the unique solution for exact partition games satisfying feasible highest payoffs, equal payoff stability, and complement-consistency for highest payoffs.*

Proof. *Step 1: sufficiency.* The equal split-off set satisfies feasible highest payoffs. By Lemmas 2 and 3, the equal split-off set satisfies equal payoff stability on $\widetilde{\Gamma}_{ep}$. To show that the equal split-off set satisfies complement-consistency for highest payoffs on $\widetilde{\Gamma}_{ep}$, let $(N, V) \in \widetilde{\Gamma}_{ep}$ and let $x \in ESOS(N, V)$ be such that $R^x_1 \neq N$. By Lemmas 2 and 3, $x \in C(N, V)$. Suppose for the sake of contradiction that $x_{N\setminus R^x_1} \notin C(N \setminus R^x_1, V^x_{comp})$. Then there exist $S \in 2^N$ with $\emptyset \subset S \subset N \setminus R^x_1$ and $y_S \in V^x_{comp}(S)$ such that $y_S > x_S$. This implies that $(y_S, x_{R^x_1}) \in V(S \cup R^x_1)$. By comprehensiveness, $x_{S\cup R^x_1} \in V(S\cup R^x_1)\setminus P(V(S\cup R^x_1))$. By nonlevelness, $x_{S\cup R^x_1} \in V(S\cup R^x_1)\setminus WP(V(S \cup R^x_1))$, which implies that there exists $z_{S\cup R^x_1} \in V(S \cup R^x_1)$ such that $z_{S\cup R^x_1} > x_{S\cup R^x_1}$. This contradicts that $x \in C(N, V)$, so $x_{N\setminus R^x_1} \in C(N \setminus R^x_1, V^x_{comp})$. Note that $R_k^{x_{N\setminus R^x_1}} = R^x_{k+1} \setminus R^x_1$ for all $k \in \mathbb{N}$. For all $k \in \mathbb{N}$, $x_{R^x_{k+1}} \in V(R^x_{k+1})$, so $(x_{R_k^{x_{N\setminus R^x_1}}}, x_{R^x_1}) \in V(R_k^{x_{N\setminus R^x_1}} \cup R^x_1)$, which implies that $x_{x_{N\setminus R_1^x}}{}_{R_k} \in V^x_{comp}(R_k^{x_{N\setminus R_1^x}})$. This means that $(N \setminus R_1^x, V^x_{comp}) \in \widetilde{\Gamma}_{ep}$ and $x_{N\setminus R_1^x} \in ESOS(N \setminus R_1^x, V^x_{comp})$. Hence, the equal split-off set satisfies complement-consistency for highest payoffs.

Step 2: necessity. Let σ be a solution on $\widetilde{\Gamma}_{ep}$ satisfying feasible highest payoffs, equal payoff stability, and complement-consistency for highest payoffs. We show by induction on the number of players that $\sigma(N, V)$ consists of a unique payoff allocation for all $(N, V) \in \widetilde{\Gamma}_{ep}$. By equal payoff stability, $\sigma(N, V) = \{\mathrm{WP}(V(N))\}$ for all $(N, V) \in \widetilde{\Gamma}_{ep}$ with $|N| = 1$. Let $k \in \mathbb{N}$ and assume that $\sigma(N, V)$ consists of a unique payoff allocation for all $(N, V) \in \widetilde{\Gamma}_{ep}$ with $|N| \leq k$. Let $(N, V) \in \widetilde{\Gamma}_{ep}$ with $|N| = k + 1$ and let $x \in \sigma(N, V)$. By equal payoff stability,

$$\begin{aligned} a_1^x &\geq \max_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\} \\ &\geq \max\left\{t \in \mathbb{R}_+ \mid te_{R_1^x} \in V(R_1^x)\right\}. \end{aligned}$$

By feasible highest payoffs,

$$\begin{aligned} a_1^x &\leq \max\left\{t \in \mathbb{R}_+ \mid te_{R_1^x} \in V(R_1^x)\right\} \\ &\leq \max_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}. \end{aligned}$$

This implies that

$$a_1^x = \max_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}$$

$$\text{and} \quad R_1^x \in \operatorname*{argmax}_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}.$$

Define $U^V = \bigcup \operatorname{argmax}_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}$. Since $(N, V) \in \widetilde{\Gamma}_{ep}$, Lemmas 2 and 3 imply that $U^V \in \operatorname{argmax}_{S\in 2^N\setminus\{\emptyset\}} \max\{t \in \mathbb{R}_+ \mid te_S \in V(S)\}$.

Suppose for the sake of contradiction that $R_1^x \neq U^V$. By complement-consistency for highest payoffs, $x_{N\setminus R_1^x} \in \sigma(N \setminus R_1^x, V^x_{comp})$. By equal payoff stability, there exists $i \in U^V \setminus R_1^x$ such that

$$\begin{aligned} x_i &\geq \max\left\{t \in \mathbb{R}_+ \mid te_{U^V\setminus R_1^x} \in V^x_{comp}(U^V \setminus R_1^x)\right\} \\ &= \max\left\{t \in \mathbb{R}_+ \mid \left(te_{U^V\setminus R_1^x}, x_{R_1^x}\right) \in V(U^V)\right\} \\ &= \max\left\{t \in \mathbb{R}_+ \mid \left(te_{U^V\setminus R_1^x}, a_1^x e_{R_1^x}\right) \in V(U^V)\right\} \\ &= a_1^x, \end{aligned}$$

where the last equality follows from nonlevelness. This is a contradiction, so $R_1^x = U^V$. If $R_1^x = U^V = N$, then $\sigma(N, V) = \{(\max\{t \in \mathbb{R}_+ \mid te_N \in V(N)\})_{i\in N}\}$. Suppose that $R_1^x = U^V \neq N$. By complement-consistency for highest payoffs, $x_{N\setminus R_1^x} \in \sigma(N \setminus R_1^x, V^x_{comp})$, where, by the induction hypothesis, $\sigma(N \setminus R_1^x, V^x_{comp})$ consists of a unique payoff allocation since $|N \setminus R_1^x| \leq k$. Hence, $\sigma(N, V)$ consists of a unique payoff allocation. □

Theorem 3 generalizes the corresponding result of Dietzenbacher and Yanovskaya (2021) for TU-games. To show that the corresponding properties are independent, we consider the following solutions. The solution which assigns the core to all exact partition games with at most two players, and the equal split-off set to all exact partition games with more players, satisfies all axioms in the theorem except for feasible highest payoffs. The equal payoff solution satisfies all axioms in the theorem except for equal payoff stability. The solution which assigns $(6, 2, 2)$ to the game in Example 4, and the equal split-off set to all other exact partition games, satisfies all axioms in the theorem except for complement-consistency for highest payoffs.

It can be shown that both bargaining games and NTU-bankruptcy games are exact partition games. The equal split-off set for bargaining games coincides with the Kalai (1977) solution for bargaining problems (cf. Nash, 1950). The equal split-off set for NTU-bankruptcy games (cf. Dietzenbacher, 2018) coincides with the constrained equal awards rule (cf. Dietzenbacher, 2022) for NTU-bankruptcy problems (cf. Orshan et al., 2003), also called the constrained Kalai solution (cf. Albizuri et al., 2020) for bargaining problems with claims (cf. Chun and Thomson, 1992).

Data availability

No data was used for the research described in the article.

Acknowledgments

Support from the HSE University Basic Research Program is gratefully acknowledged.

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