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Finite-Dimensional Reduction of Systems of Nonlinear Diffusion Equations

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Abstract—We present a class of one-dimensional systems of nonlinear parabolic equations for which the phase dynamics at large time can be described by an ODE with a Lipschitz vector field in $\mathbb{R}^n$. In the considered case of the Dirichlet boundary value problem, the sufficient conditions for a finite-dimensional reduction turn out to be much wider than the known conditions of this kind for a periodic situation.

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1. INTRODUCTION

One of the main problems in the study of evolution equations is related to the description of the final (long-time) behavior of their solutions. We consider systems of diffusion equations with the Dirichlet boundary condition

$$\partial_t u = D\,\partial_{xx} u + f(x,u)\,\partial_x u + g(x,u), \qquad u(0) = u(1) = 0 \tag{1.1}$$

on the interval $J = [0,1]$. Here $u = (u_1, \dots, u_m)$, and f and g are sufficiently regular matrix functions and vector functions, respectively. We assume that the constant coefficient matrix D is similar to a diagonal matrix with positive eigenvalues. In the case of $D = \operatorname{diag}\{d_1, \dots, d_m\}$ with $d_j > 0$, we deal with reaction-diffusion-convection equations. Under appropriate conditions on f and g, system (1.1) induces a smooth dissipative semiflow $\{\Phi_t\}_{t\ge0}$ in the phase space $X^\alpha \subset C^1(J,\mathbb{R}^m)$ with an appropriate $\alpha > 0$, where $\{X^\alpha\}_{\alpha\ge0}$ is the Hilbert half-scale [1] generated by the linear sectorial operator $u \to -Du_{xx}$ in $X = L^2(J,\mathbb{R}^m)$. In this situation, there exists a global attractor [2]–[4] (simply an attractor below), i.e., a connected compact invariant set $\mathscr{A} \subset X^\alpha$ of finite Hausdorff dimension which uniformly attracts bounded subsets X^α as $t \to +\infty$.

Our goal is to find conditions under which the dynamics on attractor (final dynamics) of the parabolic system (1.1) is finite-dimensional in the sense of [5]. This means that, for some ODE $\partial_t \xi = h(\xi)$ in $\mathbb{R}^N$ with Lipschitz vector field h, resolving flow $\{\Theta_t\}$, and invariant compact set $\mathscr{K} \subset \mathbb{R}^N$, the phase semiflows $\{\Phi_t\}_{t\ge0}$ on $\mathscr{A}$ and $\{\Theta_t\}_{t\ge0}$ on $\mathscr{K}$ are Lipschitz conjugate. In this connection, one can speak [6] about a finite reduction of the evolution problem (1.1).

The main result of the paper (Theorem 4.3) ensures that the final phase dynamics of system (1.1) is finite under the *matching condition*

$$Df(x,u) = f(x,u)D, \qquad (x,u) \in J \times \operatorname{co}\mathscr{A}, \tag{1.2}$$

where $\operatorname{co}\mathscr{A}$ is the convex hull of $\mathscr{A}$.

It is known [7] that, in the case of scalar diffusion ($D = dE$ with unit matrix E) and sufficiently regular $f = f(u)$ and $g = g(u)$, there exists an inertial manifold (IM) which is a finite-dimensional

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invariant C^1-surface in the phase space containing an attractor and exponentially attracting (with an asymptotic phase) all trajectories of the system as $t \to +\infty$. The presence of IM implies the finite-dimensionality of the final dynamics; an extensive literature is devoted to the existence of such manifolds (see, e.g., [3], [4], [6], [8]). An original approach to such problems is presented in recent Anikushin's works (see [9] and the references therein).

For the periodic case (J is a circle of length 1), conditions for the finite-dimensionality of the final dynamics of system (1.1) with $D = \mathrm{diag}$ were obtained by the author in [10, p. 13409]. We note that, in the class of periodic systems (1.1) with scalar diffusion, the first example of semilinear parabolic equation of mathematical physics that does not exhibit such a dynamics was constructed in [11, Theorem 1.2].

2. PRELIMINARIES

In what follows, if necessary, we will use the technique developed in [10]. All preliminary constructions in Secs. 2, 3 are performed for the case $D = \mathrm{diag}$. We write system (1.1) as a semilinear parabolic equation (SPE)

$$\partial_t u = -Au + F(u) \tag{2.1}$$

in the *real* Hilbert space $X = L^2(J, \mathbb{R}^m)$ with norm $\|\cdot\|$. Here $A\colon u \to -Du_{xx}$ with the Dirichlet boundary condition and nonlinearity $F\colon u \to f(x,u)\,\partial_x u + g(u)$. For a linear positive definite operator A, we set $X^\alpha = \mathscr{D}(A^\alpha)$ with $\alpha \ge 0$ and $X_0 = X$; then $\|u\|_\alpha = \|A^\alpha u\|$. We will say that a function F belongs to the *class* $W^2(X^\alpha, X)$ if

$$F \in C^2(X^\alpha, X) \cap \mathrm{Lip}(X^\alpha, X) \qquad \text{and} \qquad \|F(u)\| \le M \qquad \text{for } u \in X^\alpha \tag{2.2}$$

for some $\alpha \in [0,1)$. In this case, SPE (2.1) generates [1] a smooth compact resolving semiflow $\{\Phi_t\}_{t\ge 0}$ in the phase space X^α. Assumption (2.2) implies [8, Lemma 1.1] the X^α-dissipativity of (2.1):

$$\limsup_{t\to+\infty} \|\Phi_t u\|_\alpha \le r$$

for some $r > 0$ uniformly in $u \in$ balls in X^α. Under these conditions, there exists [2]–[4] a compact attractor $\mathscr{A} \subset X^\alpha$ consisting of all bounded complete trajectories $\{u(t)\}_{t\in\mathbb{R}} \subset X^\alpha$. In fact, $\mathscr{A} \subset X^1$ owing to the *smoothing action* of the parabolic equation [1]. Simple reasoning [10, p. 13410] shows that, in all constructions related to SPE (2.1), the nonlinearity exponent α can be replaced by any value $\alpha_1 \in (\alpha, 1)$, and if condition (2.2) is satisfied in the two spaces $(X^\theta, X^{\theta+\alpha})$ with $\theta > 0$ instead of (X, X^α), then all the above-listed properties of the dynamics remain valid for the phase space $X^{\theta+\alpha}$. In what follows, we will use functions $Y_1 \to Y_2$ of class (2.2) for some Banach spaces Y_1 and Y_2.

As in [10], we will use sufficient conditions for the final finite-dimensionality of the dynamics [12]. Let $G(u) = F(u) - Au$ be the vector field (2.1), and let $\mathscr{N} = \mathscr{A} \times \mathscr{A}$ and Y be Banach spaces.

Definition 2.1 [12]. A continuous field $\Pi\colon \mathscr{N} \to Y$ is said to be *regular* if, for any $u, v \in \mathscr{A}$, the function $\Pi(\Phi_t u, \Phi_t v)\colon [0,+\infty) \to Y$ is of class C^1 with the derivative $\partial_t \Pi(u,v)$ at zero uniformly bounded in $(u,v) \in \mathscr{N}$.

The smoothness of the semiflow $\{\Phi_t\}$ and the invariance of the compact set $\mathscr{A} \subset X^\alpha$ imply the regularity of the identical embedding $\mathscr{N} \to X^\alpha \times X^\alpha$, and hence of any field $\Pi\colon \mathscr{N} \to Y$ that can be continued to a C^1-mapping in the $(X^\alpha \times X^\alpha)$-neighborhood of the set $\mathscr{N}$. In this situation, $\partial_t \Pi(u,v) = \Pi'(u,v)(G(u), G(v))$, where $(\,\cdot\,)'$ is the Fréchet differentiation. Under condition (2.2) on the nonlinearity F, the function $u \to G(u)$ on $\mathscr{A}$ is continuous and even Hölder [5] in the X^α-metric. The regular fields $\mathscr{N} \to Y$ form a linear structure as well as a multiplicative one if Y is a Banach algebra. In the last case, if all elements $\Pi(u,v) \in Y$ are invertible, then the field Π^{-1} also turns out to be regular.

We start from the decomposition

$$G(u) - G(v) = (T_0(u,v) - T(u,v))(u - v), \qquad (u,v) \in \mathscr{N}, \tag{2.3}$$

where $T_0 \in \mathscr{L}(X^\alpha)$ and $T \in \mathscr{L}(X^1, X)$ are unbounded linear operators in X similar to positive definite ones. By

$$\Sigma_T = \bigcup_{u,v\in\mathscr{A}} \operatorname{spec} T(u,v)$$

we denote the total spectrum of the operators T.

We will need the special case of [12, Theorem 2.8] in the situation $\Sigma_T \subset \mathbb{R}^+$.

Theorem 2.2. *Assume that $F \in W^2(X^\alpha, X)$ and*

$$T(u,v) = S^{-1}(u,v)H(u,v)S(u,v) \tag{2.4}$$

on $\mathscr{N}$, where the unbounded self-adjoint linear operators $H(u,v)$ are positive definite in X, the fields $S, S^{-1}\colon \mathscr{N} \to \mathscr{L}(X)$ and $T_0\colon \mathscr{N} \to \mathscr{L}(X^\alpha, X)$ are regular, and the field $T_0\colon \mathscr{N} \to \mathscr{L}(X^\alpha)$ is bounded. Moreover, if the set $\mathbb{R}^+ \setminus \Sigma_T$ contains intervals $(a_k - \xi_k, a_k + \xi_k)$ with $a_k > \xi_k > 0$ such that

$$\xi_k \to \infty, \qquad a_k^{\alpha/2} = o(\xi_k) \tag{2.5}$$

as $k \to +\infty$, then the final X^α-dynamics of SPE (2.1) *is finite-dimensional.*

We further assume that the matrix function $f = f(x,u)$ and the vector function $g = g(x,u)$ in (1.1) satisfy the following regularity conditions.

Condition (H). The functions f and g of the class C^∞ on $J \times \mathbb{R}^m$ are compactly supported in u, and $f(x,0) = g(x,0) = 0$ for $x = 0, 1$.

By $\mathscr{H}^s = \mathscr{H}^s(J)$ we denote generalized Sobolev L^2-spaces (spaces of Bessel potentials [1], [13]) of scalar functions on J with arbitrary $s \geq 0$. If $s > 1/2$, then $\mathscr{H}^s \subset C(J)$ and $\mathscr{H}^s$ is a Banach algebra [13, Sec. 2.8.3]. The differentiation operator acts in the spaces $\partial_x \in \mathscr{L}(\mathscr{H}^{s+1}, \mathscr{H}^s)$. In fact, the X^s are closed subspaces (with equivalent norm) in the spaces $\mathscr{H}^{2s}(J, \mathbb{R}^m)$ of vector-functions, and $X^s = \mathscr{H}^{2s}(J, \mathbb{R}^m)$ for $s \leq 1/4$. For $s > 1/4$, the space X^s consists of elements $u \in \mathscr{H}^{2s}(J, \mathbb{R}^m)$ with $u(0) = u(1) = 0$.

Now fix an arbitrary $\alpha \in (3/4, 1)$; then $\mathscr{H}^{2\alpha} \hookrightarrow C^1(J)$ and $X^\alpha \hookrightarrow C^1(J, \mathbb{R}^m)$, where the symbol $\hookrightarrow$ denotes a linear continuous embedding of function spaces. Let us use necessary embedding theorems [1], [13]. For an arbitrary C^∞-function $z\colon J \times \mathbb{R}^m \to \mathbb{R}$, the mapping $\psi\colon u \to z(x,u)$ is a function of class W^2 (see (2.2)) from $C^s(J, \mathbb{R}^m)$ to $C^s(J)$ for all $s \in \mathbb{N}$. This implies that $\psi \in W^2(\mathscr{H}^{2\alpha}(J, \mathbb{R}^m), C^1(J))$. Using the embedding $\mathscr{H}^{s+1} \hookrightarrow C^s(J) \hookrightarrow \mathscr{H}^s$, we can conclude that $\psi \in W^2(\mathscr{H}^s(J, \mathbb{R}^m), \mathscr{H}^s(J))$. So $F \in W^2(X^1, X^{1/2})$ for the nonlinear part $F\colon u \to f(x,u)\, \partial_x u + g(u)$ of system (1.1). Moreover, $X^\alpha \hookrightarrow C^1(J, \mathbb{R}^m) \hookrightarrow C(J, \mathbb{R}^m) \hookrightarrow X$, and hence $F \in W^2(X^\alpha, X)$. We also note that $X^{3/2} \hookrightarrow C^2(J, \mathbb{R}^m)$.

We take X^α as the phase space of system (1.1). Following [7], we can show that the phase dynamics of (1.1) in X^α is dissipative and there exists a global attractor $\mathscr{A} \subset X^\alpha$. Since $F \in W^2(X^1, X^{1/2})$, system (1.1) also generates a smooth dissipative phase semiflow in the space X^1, and the attractor $\mathscr{A}$ is compact in $X^{3/2}$. As above, we denote $\mathscr{N} = \mathscr{A} \times \mathscr{A}$.

Remark 2.3. The phase dynamics of system (1.1) has the following property: if Y is a Banach space, then each vector field $\Pi\colon \mathscr{N} \to Y$ continuous in the $(X^\alpha \times X^\alpha)$-metric and extendable to a C^1-mapping $X^1 \times X^1 \to Y$ is regular in the sense of Definition 2.1.

Indeed, the smoothness of a semiflow in X^1 means the smoothness of the mapping

$$(t,u) \to \Phi_t u\colon (0, +\infty) \times X^1 \to X^1.$$

This ensures the regularity of the identity mapping $\mathscr{N} \to X^1 \times X^1$ and hence the regularity of the field Π on $\mathscr{N}$.

3. DECOMPOSITION OF THE VECTOR FIELD ON THE ATTRACTOR

We will apply Theorem 2.2 to SPE (1.1) with $D = \operatorname{diag}$ and the phase spaces X^α, $\alpha \in (3/4, 1)$. By $\mathbb{M}^m$ we denote the algebra of numerical $(m \times m)$-matrices with Euclidean norm, and by $Y(J, \mathbb{M}^m)$, the linear spaces of such matrices with elements from some Banach space Y of scalar functions on $J = [0, 1]$. Following [10, pp. 13412–13413], we set

$$B_0(x; u, v) = \int_0^1 (f_u(x, w(x))w_x(x) + g_u(x, w(x))\, d\tau, \tag{3.1}$$

$$B(x; u, v) = \int_0^1 f(x, w(x))\, d\tau \tag{3.2}$$

for $u, v \in X^\alpha$, where $w(x) = \tau u(x) + (1-\tau)v(x)$, $x \in J$. The entries of the matrices B_0 and B are continuous functions, and for $u, v \in \mathscr{A}$ they are functions of the class C^2 on J. Using the C^1-smoothness of the mappings $(u, v) \to f_u(x, w)w_x + g_u(x, w)$, $(u, v) \to f(x, w)$, $X^\alpha \times X^\alpha \to C(J, \mathbb{M}^m)$ for a fixed $\tau \in [0, 1]$ and differentiating the expressions for B_0 and B with respect to the parameter (u, v), we conclude that the mappings

$$(u, v) \to B_0(\,\cdot\,; u, v), \qquad (u, v) \to B(\,\cdot\,; u, v) \tag{3.3}$$

are of the class $C^1(X^\alpha \times X^\alpha, C(J, \mathbb{M}^m))$. We use the integral mean value theorem for nonlinear operators to write the decomposition of the vector field (1.1) on the attractor $\mathscr{A} \subset X^\alpha$ in the form

$$\begin{aligned} G(u) - G(v) &= -Ah + \left(\int_0^1 F'(\tau u + (1-\tau)v)\, d\tau\right) h \\ &= Dh_{xx} + B_0(x; u, v)h + B(x; u, v)h_x, \qquad u, v \in \mathscr{A}, \end{aligned}$$

where $h = u - v$, $\tau u + (1-\tau)v \in \operatorname{co}\mathscr{A}$, and $(\,\cdot\,)'$ is the Frechet differentiation. To eliminate the dependence on h_x, we (following [14]) apply the transformation $h = U\eta$, where the $(m \times m)$-matrix function $U(x) = U(x; u, v)$, $x \in [0, 1]$, is the solution of the Cauchy problem

$$U_x = -\frac{1}{2} D^{-1}B(x)U, \qquad U(0) = E. \tag{3.4}$$

As a result, we obtain relation (2.3) with linear operators

$$T_0(u, v)h = \left(B_0(x) - \frac{1}{2} B_x(x) - \frac{1}{4} B(x)D^{-1}B(x)\right)h, \tag{3.5}$$

$$T(u, v)h = -DU\,\partial_{xx}U^{-1}h. \tag{3.6}$$

Note that, under the change of variable $h = U\eta$, the Dirichlet boundary conditions for the linear part of (1.1) are preserved. When writing the matrices B_0, B, U, U^{-1}, we often omit the dependence on u, v, and sometimes, on x.

Lemma 3.1. *The field of the operator T_0 on $\mathscr{N}$ is regular with values in $\mathscr{L}(X^\alpha, X)$ and bounded with values in $\mathscr{L}(X^\alpha)$.*

Proof. We set $T_0h = Q(x; u, v)h$ in (3.5) with $h \in \mathscr{A} - \mathscr{A} \subset X^\alpha$. The convex hull of the attractor $\mathscr{A}$ is bounded in the $X^{3/2}$-norm equivalent to the norm in $\mathscr{H}^3(J, \mathbb{R}^m)$, and therefore, the matrix functions B, $BD^{-1}B$, and B_0 are bounded uniformly in $(u, v) \in \mathscr{N}$ in $\mathscr{H}^3(J, \mathbb{M}^m)$ and $\mathscr{H}^2(J, \mathbb{M}^m)$, respectively. Thus, the matrix functions B_x and Q are bounded on $\mathscr{N}$ in the norm of $\mathscr{H}^2(J, \mathbb{M}^m)$, and T_0 is the operator of multiplication of vector functions in $X^\alpha \subset \mathscr{H}^{2\alpha}(J, \mathbb{R}^m)$ by the matrix $Q \in \mathscr{H}^{2\alpha}(J, \mathbb{M}^m)$ with $2\alpha \in (3/2, 2)$. Since $\mathscr{H}^{2\alpha}(J)$ is a Banach algebra, we see that $T_0(u, v) \in \mathscr{L}(X^\alpha)$ and $\|T_0(u, v)\|_\alpha \le \text{const}$ on $\mathscr{N}$.

In view of Remark 2.3 and the above-noted smoothness of the mapping (3.3), the regularity of the field of the operator $T_0\colon \mathscr{N} \to \mathscr{L}(X^\alpha, X)$ can be established exactly just as for the case of periodic boundary conditions in [10, Lemma 3.3]. □

The matrix function $U(x)$ in the Cauchy problem (3.4) can be treated as a bounded linear operator in X.

Lemma 3.2. *The fields of the operators $U, U^{-1}\colon \mathscr{N} \to \mathscr{L}(X)$ are regular.*

Proof. For the field U, this can proved as the similar assertion in the periodic case [10, Lemma 3.4]. At the same time, the regularity of U implies the regularity of the field of inverse operators U^{-1}.

Now let $d_- = \min_{1\le j\le m} d_j$ and $d_+ = \max_{1\le j\le m} d_j$ for $D = \operatorname{diag}\{d_j\}$. Also let $\{\lambda_n : \lambda_1 < \lambda_2 < \cdots\}$ be the eigenvectors of the linear operator $A = -D\,\partial_{xx}$. Since

$$\operatorname{spec}(A) = \{d_j\pi^2\nu^2,\ \nu \in \mathbb{N},\ j \in 1, \dots, , m\}, \tag{3.7}$$

we have $\lambda_n \le \pi^2 d_+ n^2$. Using the counting function for spec(A), we obtain

$$n \le \sum_{j=1}^{m} \frac{\sqrt{\lambda_n}}{\pi\sqrt{d_j}} \le \frac{m}{\pi\sqrt{d_-}}\sqrt{\lambda_n}\,,$$

and hence

$$\frac{\pi^2 d_-}{m^2}\, n^2 \le \lambda_n \le \pi^2 d_+ n^2, \qquad n \in \mathbb{N}. \tag{3.8}$$

□

Lemma 3.3. *The following estimate holds*:

$$\limsup_{n\to\infty} n^{-1}(\lambda_{n+1} - \lambda_n) > 0.$$

Proof. If, on the contrary, $\lambda_{n+1} - \lambda_n = \beta_n n$ with $\beta_n \overset{n\to\infty}{\longrightarrow} 0$, then

$$\begin{aligned} n^{-2}\lambda_n &= n^{-2}\biggl(\lambda_1 + \sum_{k=1}^{n-1}(\lambda_{k+1} - \lambda_k) = n^{-2}\biggl(\lambda_1 + \sum_{k=1}^{n-1}\beta_k k\biggr) \\ &\le n^{-2}\biggl(\lambda_1 + \sum_{k=1}^{n-1}\beta_k n\biggr) \le n^{-2}\lambda_1 + n^{-1}\sum_{k=1}^{n}\beta_k. \end{aligned}$$

But this implies the relation $\lambda_n = o(n^2)$, which contradicts the left inequality in (3.8). □

4. MAIN RESULTS

By the assumptions of Theorem 2.2, it is necessary, for the operators $T(u, v)$ in (3.6), to establish "uniform" similarity of the positive definite form (2.4) as well as the required sparsity (2.5) of their total spectrum Σ_T. We assume that the regularity Condition (H) is satisfied for the functions f and g in (1.1).

Theorem 4.1. *If the matrix D has the form $D = \operatorname{diag}\{d_j\}$ with $d_j > 0$ and the matching condition* (1.2) *is satisfied, then the phase dynamics on the attractor is finite-dimensional.*

Proof. The operator $A = -D\,\partial_{xx}$ with the Dirichlet condition is self-adjoint and positive definite in X. Assumption (1.2) implies (for any $x \in J$ and $u, v \in \mathscr{A}$) the relation $DB(x) = B(x)D$ for the matrices $B(x) = B(x; u, v)$ in (3.2). Thus, the matrices $B(x)$ and $D^{-1}B(x)$ inherit the block (with respect to the same d_j) structure of the diffusion matrix $D = \operatorname{diag}\{d_1, \dots, d_m\}$. Therefore, this is also true for the solutions $U(x)$ of the Cauchy problem (3.4), and hence $DU(x) = U(x)D$, $x \in J$, and

$$T(u, v) = U(u, v)(-D\,\partial_{xx})U^{-1}(u, v)$$

in (3.6). Thus, for $T(u,v)$ we have the representation (2.4) with $S(u,v)=U^{-1}(u,v)$ and $H(u,v)\equiv A$. The total spectrum Σ_T coincides with spec(A) in (3.7). By Lemma 3.3, there exists an $\varepsilon>0$ and an increasing sequence of indices $n(k)$ such that $\lambda_{n(k)+1}-\lambda_{n(k)}>\varepsilon n(k)$ for $k\ge k_0$. Set

$$a_k=\frac{\lambda_{n(k)+1}+\lambda_{n(k)}}{2}, \qquad \xi_k=\frac{\lambda_{n(k)+1}-\lambda_{n(k)}}{3}, \qquad M=\pi^2 d_+.$$

From the right inequality in (3.8), we obtain

$$a_k\le M\left(n^2(k)+n(k)+\frac{1}{2}\right)\le 3Mn^2(k)\le \frac{3M}{\varepsilon^2}(\lambda_{n(k)+1}-\lambda_{n(k)})^2\le \frac{27M}{\varepsilon^2}\,\xi_k^2$$

for $k\ge k_0$; i.e., $a_k=O(\xi_k^2)$ as $k\to\infty$. Since $a_k^{\alpha/2}=o(\xi_k)$ for $\alpha\in(3/4,1)$ as $k\to\infty$, we see that the desired assertion follows from Lemmas 3.1 and 3.2 and Theorem 2.2. □

Remark 4.2. Parabolic systems (1.1) with $D=$ diag demonstrate a finite-dimensional dynamics on the attractor for any admissible nonlinearities f and g in the case of scalar diffusion and under the condition $f=$ diag in the case of m distinct diffusion coefficients d_j. In the case of s distinct diffusion coefficients with $1<s<m$, the dynamics on the attractor is finite-dimensional under the condition that the matrix function f inherits the block structure (with the same d_j) of the matrix $D=\operatorname{diag}\{d_j\}$.

Now we state the main result. We assume that the matrix D in system (1.1) has the form $D=C\overline{D}C^{-1}$, where the matrix C is nonsingular and $\overline{D}=\operatorname{diag}\{d_1,\dots,d_m\}$ with $d_j>0$. The linear operator $-D\,\partial_{xx}=-C(\overline{D}\,\partial_{xx})C^{-1}$ is sectorial in $X=L^2(J,\mathbb{R}^m)$. The linear change of variables $u=Cv$ reduces (1.1) to the system of equations

$$\begin{gathered}\partial_t v=\overline{D}\,\partial_{xx}v+\overline{f}(x,v)\partial_x v+\overline{g}(x,v), \qquad v(0)=v(1)=0,\\ \overline{f}(x,v)=C^{-1}f(x,Cv)C, \qquad \overline{g}(x,v)=C^{-1}g(x,Cv).\end{gathered} \tag{4.1}$$

The matrix function $\overline{f}$ and the vector function $\overline{g}$ inherit the regularity properties (H) of the original functions f and g. The phase semiflows of systems (4.1) and (1.1) are linearly conjugate. System (4.1) is dissipative in X^α, and hence this is also true for system (1.1). The attractors $\mathscr{A}$ of system (1.1) and $\overline{\mathscr{A}}$ of system (4.1) are related by the expression $\mathscr{A}=C\overline{\mathscr{A}}$. According to the definition of the finite-dimensionality of the phase dynamics (Sec. 1), systems (4.1) and (1.1) demonstrate this property simultaneously.

Theorem 4.3 (main). *If the matrix D is similar to* $\operatorname{diag}\{d_j\}$ *with $d_j>0$ and the matching condition* (1.2) *is satisfied, then the phase dynamics of system* (1.1) *is finite-dimensional.*

Proof. Since $Df(x,u)=f(x,u)D$ on $J\times\operatorname{co}\mathscr{A}$, it follows that

$$\begin{aligned}\overline{D}\,\overline{f}(x,v)&=C^{-1}DC\cdot C^{-1}f(x,Cv)C=C^{-1}Df(x,u)C\\ &=C^{-1}f(x,u)DC=C^{-1}f(x,Cv)C\overline{D}=\overline{f}(x,v)\overline{D}\end{aligned}$$

on $J\times\operatorname{co}\overline{\mathscr{A}}$. Here $u\in\operatorname{co}\mathscr{A}$ and $v\in\overline{\mathscr{A}}$. As we can see, condition (1.2) is satisfied for the matrix function $\overline{f}$, and the dynamics of system (4.1) on the attractor $\overline{\mathscr{A}}\subset X^\alpha$ is finite-dimensional by Theorem 4.1. This also implies the finite-dimensionality of the dynamics of system (1.1) on the attractor $\mathscr{A}\subset X^\alpha$. □

Remark 4.4. Under the matching condition (1.2), the final dynamics of system (1.1) is finite-dimensional if all eigenvalues of the matrix D are distinct and positive. Condition (1.2) is satisfied, in particular, for $f=D_1\varphi$, where the numerical matrix D_1 commutes with D and $\varphi=\varphi(x,u)$ is a smooth scalar function compactly supported in u.

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