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AROUND THE INFINITE DIVISIBILITY OF THE DICKMAN DISTRIBUTION AND RELATED TOPICS

Abstract. There are two probability distributions related to the Dickman function from number theory, which are sometimes confused with each other. We give a careful exposition on the difference between the two. While one is known to be infinite divisible, we give a computational proof to show that the other is not. We apply this to get related results for self-decomposable distributions with so-called truncated Lévy measures. Further, we extend several results about the infinitely divisible Dickman distribution related to its role in the context of sums on independent random variables and perpetuities. Along the way, we discuss several approaches for checking if a distribution is or is not infinitely divisible.

To Ildar Abdullovich Ibragimov on the occasion of his 90th birthday, with great admiration and gratitude

§1. Introduction

The Dickman distribution first appeared in the context of number theory in the following setting. For a positive integer k, let $p_1(k)$ be the largest prime divisor of k. If ξ_n is a uniform random variable on the set $\{1, 2, \dots, n\}$, then for any $a > 0$,

$$\mathrm{P}(p_1(\xi_n) \leqslant n^{1/a}) = \frac{\#\{k \in \{1, 2, \dots, n\} : p_1(k) \leqslant n^{1/a}\}}{n} \to D(a), \qquad (1)$$

as $n \to \infty$, where D is a continuous and nonnegative function satisfying the differential-difference equation

$$aD'(a) + D(a-1) = 0, \quad a > 1, \qquad (2)$$

with initial condition $D(a) = 1$ for $a \in [0, 1]$. We further set $D(a) = 0$ for $a < 0$. This result was first published in 1930 by Dickman in the remarkable paper [9]. This paper gave heuristic arguments, which were later rigorized and extended in the work of Buchstab [5], Ramaswami [21], de Bruijn [3,4],

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and others. The function D has come to be called the Dickman function. Many properties of this function are discussed in [27], see also the recent review [19].

The function $G(x) = 1 - D(x)$ is the cumulative distribution function (cdf) of a distribution supported on the interval $[1, \infty)$ and the Dickman function D is the corresponding survival (or tail) function, i.e., $\bar{G} = D$. From (2) it follows that this distribution is absolutely continuous with probability density function (pdf)

$$g(x) = -D'(x) = x^{-1} D(x-1), \quad x \geqslant 1.$$

In this context, we can reformulate (1) as

$$\frac{\log n}{\log p_1(\xi_n)} \stackrel{d}{\to} G \text{ as } n \to \infty,$$

where $\stackrel{d}{\to}$ denotes convergence in distribution. The fact that $\log n \geqslant \log p_1(\xi_n)$ explains the support of G. We will call this the **Dickman Type A** distribution and denote it by DA. We choose this terminology to avoid confusion with another distribution that is related to the Dickman function, which we now introduce.

It can be verified that

$$\int_0^\infty D(x) \mathrm{d}x = e^\gamma,$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. Thus, the function

$$f(x) = e^{-\gamma} D(x), \quad x > 0 \tag{3}$$

is the pdf of a distribution supported on $[0, \infty)$. Note that f satisfies the differential-difference equation (2), but with initial condition $f(a) = e^{-\gamma}$ for $a \in [0, 1]$. In the probability literature, this distribution is often called the Dickman distribution, but, to avoid confusion, we call it the **Dickman Type B** distribution and denote it by DB. This distribution arises in a variety of applications, see the surveys [20] and [19]. Many applications stem from the fact that DB satisfies the following relation. If $X \sim \mathrm{DB}$ and $U \sim U(0, 1)$ are independent random variables, then

$$X \stackrel{d}{=} U(1 + X), \tag{4}$$

where $\stackrel{d}{=}$ denotes equality in distribution. This further implies that if $U_1, U_2, \ldots$ are independent and identically distributed (iid) $U(0,1)$ random variables and

$$X = U_1 + U_1 U_2 + U_1 U_2 U_3 + U_1 U_2 U_3 U_4 + \ldots, \tag{5}$$

then $X \sim \mathrm{DB}$. Such sums of products are important in the study of perpetuities, see, e.g., [14, 28], and the references therein. From another perspective, the Dickman distribution is the limiting distribution in a number of limit theorem, see, e.g., [6, 13], or [2]. One simple example is the fact that

$$U_1^n + U_2^n + \cdots + U_n^n \xrightarrow{d} \mathrm{DB}. \tag{6}$$

From (3) it follows that the cdf G of the DA distribution and the pdf f of the DB distribution satisfy the relationship

$$G(x) = 1 - \frac{1}{f(0)} f(x), \quad x \geqslant 0. \tag{7}$$

In fact, given any bounded and monotonely decreasing pdf f on $[0, \infty)$, (7) defines a valid cdf so long as we use a version of f that is right continuous with left limits. In this case, it is easy to check that G satisfies $G(0) = 0$ and that it has a finite mean. In this paper, we introduce the transform $\mathcal{T}$ between such pdfs and cdfs given by (7) and discuss various relationships between the distribution with pdf f and that with cdf G. We are particularly interested in the question of whether this transform preserves infinite divisibility. We will see that there are cases where both f and G are infinitely divisible and cases where one is infinitely divisible, but the other is not. A full characterization is beyond the scope of this paper. To the best of our knowledge this problem was previously studied only in the context of distribution with log convex pdfs. In this case both f and G are infinitely divisible, see Proposition III.10.10 in the monograph [26].

Our interest in the question of infinite divisibility is motivated, in part, by the situation with the two types of Dickman distributions. It is well-known that the DB distribution is infinitely divisible. However, the infinite divisibility of the DA distribution has not been studied. We give a computational proof to show that, in fact, the DA distribution is not infinitely divisible. This has some important implications. In particular, it implies that for large classes of self-decomposable distributions with so-called truncated Lévy measures the image of the distribution under transform $\mathcal{T}$ is not infinitely divisible.

The rest of this paper is organized as follows. In Section 2 we formally introduce the transform $\mathcal{T}$ and give many properties. We also review some basic properties of infinitely divisible distributions. In Section 3 we give several example where both f and G are infinitely divisible. In Section 4 we give two computational proofs that the DA distribution is not infinitely divisible. In Section 5 we discuss the implications of this for certain self-decomposable distributions with truncated Lévy measures. In Section 6 we discuss some extensions of the result in (6) and, in particular, correct a mistake in [19]. In Section 7 we discuss the relationship between the Dickman distribution and perpetuities and give some results about generalizations of (4) and (5). The proof of the main result of Section 5 is postponed to Section 8.

Before proceeding we introduce some notation. We write cdf, pdf, and pmf for cumulative distribution function, probability density function, and probability mass function, respectively. We write a.s. for "almost surely" and 1_A to denote the indicator function on A. For a distribution μ we write $X \sim \mu$ to denote that X is a random variable with distribution μ and $X_1, X_2, \ldots \stackrel{\text{iid}}{\sim} \mu$ to denote that $X_1, X_2, \ldots$ are independently and identically distributed (iid) random variables with distribution μ. For simplicity, instead of μ, we often write the corresponding cdf or pdf. We write $U(a, b)$ to denote a uniform distribution on (a, b), $\mathrm{Exp}(\lambda)$ to denote the exponential distribution with rate λ, $\mathrm{gamma}(\alpha, \lambda)$ to denote a gamma distribution with pdf

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0,$$

where Γ is the gamma function, and $\mathrm{beta}(\alpha, \beta)$ to denote a beta distribution with pdf

$$\frac{1}{\mathrm{B}(a, b)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1,$$

where B is the beta function. We write $\vee$ and $\wedge$ to denote the maximum and minimum, respectively. Further, we use the convention that $\sum_{n=1}^0$ is 0. We write $:=$, $\stackrel{d}{=}$, $\stackrel{d}{\to}$, and $\stackrel{w}{\to}$ to denote a defining equality, equality in distribution, convergence in distribution, and weak convergence, respectively. For two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, we write $a_n \sim b_n$ to denote $a_n/b_n \to 1$ as $n \to \infty$.

We use the terms 'Theorem', 'Proposition', and 'Lemma' to denote results that have been proved rigorously. We use the term 'Result' to denote results that were either proved computationally or whose proof uses a result that was proved computationally. While we have faith in our computational results, we acknowledge that such proofs are not completely rigorous.

§2. Setup and preliminaries

We begin this section by defining the transform between certain pdfs and cdfs given by (7). This generalizes the relationship between the Dickman Type A and Dickman Type B distributions. First, we define the domain and range of this transform. Let $\mathfrak{A}$ be the collection of bounded pdfs that are vanishing on $(-\infty, 0)$ and monotonely decreasing on $[0, \infty)$. Without loss of generality we assume that these are right continuous with left limits. Let $\mathfrak{B}$ be the collection of cdfs G with

$$G(0) = 0 \tag{8}$$

and having a finite mean. A standard application of Fubini's theorem implies that if $G \in \mathfrak{B}$ and $X \sim G$, then

$$\mathrm{E}[X] = \int_0^\infty x G(\mathrm{d}x) = \int_0^\infty (1 - G(x))\,\mathrm{d}x < \infty.$$

We define a bijection $\mathcal{T} : \mathfrak{A} \mapsto \mathfrak{B}$ as follows. If $f \in \mathfrak{A}$, then $G = \mathcal{T}(f)$ is the cdf given by

$$G(x) = 1 - cf(x), \quad x \geqslant 0. \tag{9}$$

where $c = 1/f(0)$. Equivalently, the corresponding survival function is given by

$$\bar{G}(x) = 1 - G(x) = cf(x), \quad x \geqslant 0.$$

If f is differentiable, then G has pdf

$$g(x) = -cf'(x).$$

The inverse transform $\mathcal{T}^{-1}$ is as follows. If $G \in \mathfrak{B}$, then $f = \mathcal{T}^{-1}(G)$ is the pdf given by

$$f(x) = \frac{1 - G(x)}{\int_0^\infty (1 - G(x))\,\mathrm{d}x}, \quad x \geqslant 0.$$

Note that, if $G = \mathcal{T}(f)$, then

$$c = \frac{1}{f(0)} = \int_0^\infty (1 - G(x))\,\mathrm{d}x = \int_0^\infty x\mathrm{d}G(x).$$

Clearly, if f is the pdf of the Dickman Type B distribution and G is the cdf of the Dickman Type A distribution, then $G = \mathcal{T}(f)$. In this case $c = e^\gamma$.

Remark 1. The formula in (9) can be generalized slightly. For any $f \in \mathfrak{A}$ and any $0 \leqslant c' \leqslant 1/f(0)$ we can define a cdf G by $G(0-) = 0$ and

$$G(x) = 1 - c'f(x), \quad x \geqslant 0.$$

In this case $G(0) = 1 - c'f(0)$ and, if $c' \neq 1/f(0)$, then we have a mass at 0. In order to avoid such masses, to have a well-defined transform, and to focus on a clear extension of the situation with the two types of Dickman distributions, we do not consider such cases here.

We now gather several useful facts about $\mathcal{T}$.

Proposition 1. *Fix $f \in \mathfrak{A}$ and let $X \sim f$, $G = \mathcal{T}(f)$, and $Y \sim G$. The following hold:*
1. Fix $a > 0$. If $f_a(\cdot) = f(\cdot/a)/a$ is the pdf of aX, then $f_a \in \mathfrak{A}$ and $G_a = \mathcal{T}(f_a)$ is of the form

$$G_a(x) = G(x/a).$$

2. Let $h : [0, \infty) \mapsto \mathbb{R}$ be a differentiable function with $|h(0)| < \infty$. We have

$$\mathrm{E}|h(Y)| < \infty \text{ if and only if } \mathrm{E}|h'(X)| < \infty.$$

Further, when these are finite, we have

$$\mathrm{E}[h(Y)] = c\mathrm{E}[h'(X)] + h(0).$$

3. For the moments we have

$$\mathrm{E}[Y^\beta] = c\beta\mathrm{E}[X^{\beta-1}], \quad \beta > 0 \tag{10}$$

and for the Laplace transforms we have

$$\mathrm{E}[e^{-sY}] = 1 - cs\mathrm{E}[e^{-sX}], \quad s \geqslant 0. \tag{11}$$

4. Let $f_1, f_2, \dots \in \mathfrak{A}$, $X_n \sim f_n$ for each n, and let $Y_n \sim G_n = \mathcal{T}(f_n)$ for each n. If $f_n(0) \to f(0)$ and $X_n \xrightarrow{d} X$, then $Y_n \xrightarrow{d} Y$.

Proof. The first part follows immediately from the definition of $\mathcal{T}$. To see the second part, note that

$$\begin{aligned}\mathrm{E}[h(Y)] &= \int_0^\infty \left(\int_0^y h'(t)\mathrm{d}t + h(0) \right) \mathrm{d}G(y) \\ &= \int_0^\infty h'(t) \int_0^\infty 1_{[t<y]} \mathrm{d}G(y)\mathrm{d}t + h(0) \\ &= \int_0^\infty h'(t) \left(1 - G(t)\right) \mathrm{d}t + h(0) \\ &= c \int_0^\infty h'(t) f(t)\mathrm{d}t + h(0) = c\mathrm{E}[h'(X)] + h(0),\end{aligned}$$

where the second line follows by Fubini's theorem and the fourth by the definition of $\mathcal{T}$. The third part follows immediately from the second. For the fourth part, note that (11) implies

$$\mathrm{E}[e^{-sY_n}] = 1 - \frac{1}{f_n(0)} s\mathrm{E}[e^{-sX_n}], \quad s \geqslant 0$$

and the result follows by the fact that convergence of Laplace transforms is equivalent to convergence in distribution. □

In the context of the fourth part of Proposition 1, we note that it is important that the limiting pdf $f \in \mathfrak{A}$. This is because, in general, $\mathfrak{A}$ is not closed under weak convergence since distributions with pdfs can converge weakly to ones without pdfs. Similarly, note that $\mathfrak{B}$ is not closed under weak convergence since distributions with finite means can converge weakly to ones with infinite means and distributions satisfying (8) may converge to ones that do not satisfy this.

In this paper, we are interested in the following question: If $f \in \mathfrak{A}$ and $G = \mathcal{T}(f)$, under what conditions will both distributions be infinitely divisible? While we do not have a complete characterization, we give several illustrative examples in the next section. We then give our results about the two types of Dickman distributions. Before proceeding, we recall some facts about infinitely divisible distributions.

A distribution μ on $\mathbb{R}$ is called infinitely divisible if for any positive integer n, there exists a distribution μ_n on $\mathbb{R}$ such that if $X \sim \mu$ and

$Y_1, Y_2, \ldots, Y_n \overset{\text{iid}}{\sim} \mu_n$, then

$$X \overset{d}{=} Y_1 + Y_2 + \cdots + Y_n.$$

For a wealth of information on infinitely divisible distributions, see the classic text [15] or the more recent monographs [23] and [26]. We are specifically interested in infinitely divisible distributions on the positive half line. Such distributions are sometimes called subordinators. If μ is a subordinator, then its Laplace transform is of the form

$$\phi_\mu(s) = \int_{[0,\infty)} e^{-sx}\mu(\mathrm{d}x) = \exp\left\{-s\eta - \int_0^\infty (1-e^{-sx})\nu(\mathrm{d}x)\right\}, \quad s \geqslant 0, \quad (12)$$

where $\eta \geqslant 0$ is called the drift and ν is called the Lévy measure. It is a Borel measure satisfying $\nu((-\infty, 0]) = 0$ and $\int_0^\infty (x \wedge 1)\nu(\mathrm{d}x) < \infty$. We denote this distribution by $\mathrm{ID}_+(\nu, \eta)$. It is well-known that the support of $\mathrm{ID}_+(\nu, \eta)$ is contained in $[\eta, \infty)$. For simplicity of terminology, if a distribution is infinitely divisible, we will refer to its cdf as infinitely divisible as well. Further, if the distribution has a pdf, we will also refer to the pdf as infinitely divisible.

We now discuss the moments and cumulants of subordinators. The cumulant generating function of $\mu = \mathrm{ID}_+(\nu, \eta)$ is given by

$$C_\mu(s) = \log \phi_\mu(s) = -\eta s - \int_0^\infty (1-e^{-sx})\nu(\mathrm{d}x), \quad s \geqslant 0.$$

The first cumulant of μ is given by

$$\kappa_1 = -C'_\mu(0) = \eta + \int_0^\infty x\nu(\mathrm{d}x) \quad (13)$$

and for integer $k \geqslant 2$, the kth cumulant is given by

$$\kappa_k = (-1)^k C^{(k)}_\mu(0) = \int_0^\infty x^k \nu(\mathrm{d}x), \quad (14)$$

where $C^{(k)}_\mu$ is the kth derivative of the cumulant generating function. The cumulants can be easily converted into moments. The relationship between these is well-known, see e.g. [25]. If m_k is the kth moment, then

$$m_k = \sum_{i=0}^{k-1} \binom{k-1}{i} \kappa_{k-i} m_i$$

and conversely

$$\kappa_k = m_k - \sum_{i=1}^{k-1} \binom{k-1}{i} \kappa_{k-i} m_i. \tag{15}$$

In particular, this means that, if $X \sim \mathrm{ID}_+(\nu, \eta)$, then

$$\mathrm{E}[X] = \eta + \int_0^\infty x\nu(\mathrm{d}x) \quad \text{and} \quad \mathrm{var}(X) = \int_0^\infty x\nu(\mathrm{d}x).$$

We conclude this section with four facts that will be foundational to our results. They are useful for checking whether a distribution is or is not infinitely divisible. Another method is described in Section 4 below. The first fact is given in Corollary III.7.2 of [26], the second in Theorem 8.7 of [23], the third follows by combining Theorem III.4.1 in [26] with (12), and the fourth by Corollary 24.4 in [23].

Fact 1: The cumulants of a subordinator are nonnegative. Thus, if a distribution has its support contained in $[0, \infty)$ and it has at least one negative cumulant, then it is not infinitely divisible.

Fact 2: If $\{\mu_n\}$ is a sequence of infinitely divisible distributions and $\mu_n \stackrel{w}{\to} \mu$, then μ is infinitely divisible. Equivalently, if $\mu_n \stackrel{w}{\to} \mu$ and μ is not infinitely divisible, then μ_n is not infinitely divisible for large enough n.

Fact 3: A positive and differentiable function ϕ on $[0, \infty)$ with $\phi(0+) = 1$ is the Laplace transform of an infinitely divisible subordinator if and only if the function

$$\rho(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \log \phi(s), \quad s \geqslant 0$$

is of the form

$$\rho(s) = \eta + \int_0^\infty x e^{-xs} \nu(\mathrm{d}x)$$

for some Lévy measure ν and $\eta \geqslant 0$. In this case, ϕ is the Laplace transform of $\mathrm{ID}_+(\nu, \eta)$.

Fact 4: If the support of a distribution is bounded and not concentrated at a point, then the distribution is not infinitely divisible.

§3. Examples

In this section we give several examples of $f \in \mathfrak{A}$ and $G \in \mathfrak{B}$ with $G = \mathcal{T}(f)$, where both f and G are infinitely divisible.

Example 1. From (11) and the fact that Laplace transforms uniquely determine distributions of $[0, \infty)$, it follows that $f \in \mathfrak{A}$ and $G = \mathcal{T}(f)$ correspond to the same distribution if and only if the Laplace transform is given by

$$\frac{1}{1 + s/\lambda}, \quad s \geqslant 0,$$

which holds if and only if f is the pdf of an $\mathrm{Exp}(\lambda)$ distribution. Since $\mathrm{Exp}(\lambda) = \mathrm{ID}_+(\nu_\lambda, 0)$ with $\nu_\lambda(\mathrm{d}x) = \frac{e^{-\lambda x}}{x} 1_{[x>0]} \mathrm{d}x$, see Example 4.8 in [26], it follows that both f and $\mathcal{T}(f)$ are infinitely divisible in this case.

Example 2. Let g be a pdf on $[0, \infty)$ and let G be the corresponding cdf. Assume that g is a completely monotone function. This means that for every positive integer n

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} g(x) \geqslant 0.$$

In this case, a version of Bernstein's theorem (see Proposition A.3.11 in [26] or Section XIII.4 in [11]) implies that g is a scale mixture of exponentials. Specifically, that

$$g(x) = \int_0^\infty \lambda e^{-\lambda x} \mathrm{d}H(\lambda),$$

where H is some cdf on $(0, \infty)$. It is well known that all distributions with completely monotone pdfs are infinitely divisible, see, e.g., Theorem III.10.7 in [26]. Let $X \sim G$ and note that, by Fubini's theorem, we have

$$c := \mathrm{E}[X] = \int_0^\infty \int_0^\infty \lambda x e^{-\lambda x} \mathrm{d}x \mathrm{d}H(\lambda) = \int_0^\infty \frac{1}{\lambda} \mathrm{d}H(\lambda).$$

Henceforth, assume that this is finite and set $f = \mathcal{T}^{-1}(G)$. Fubini's Theorem implies that

$$f(x) = \frac{1}{c}\left(1 - G(x)\right) = \frac{1}{c}\int_x^\infty \int_0^\infty \lambda e^{-\lambda t} \mathrm{d}H(\lambda)\mathrm{d}t$$
$$= \frac{1}{c}\int_0^\infty e^{-\lambda x}\mathrm{d}H(\lambda) = \int_0^\infty \lambda e^{-\lambda x}\mathrm{d}H_1(\lambda)$$

where $\mathrm{d}H_1(\lambda) = \lambda^{-1}\mathrm{d}H(\lambda)/c$. Thus, f is again completely monotone and, thus, the corresponding distribution is infinitely divisible.

Example 3. We now give an example, which can be found in Proposition III.10.10 of [26]. Assume that pdf f is log convex with a finite mean. In this case, both f and $\mathcal{T}(f)$ are infinitely divisible. It is readily checked that the density of the exponential distribution is log convex. Further, all completely monotone functions are log convex. Thus, Examples 1 and 2 are special cases of this result.

Example 4. In this example we consider convolutions of exponential distributions. Let $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ be independent exponential random variables with $\lambda_1 \neq \lambda_2$ and let $Y = X_1 + X_2$. The distribution of Y is the so-called hypoexponential distribution. It is also sometimes called the generalized Erlang distribution. Its pdf is given by

$$g(x) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 x} - e^{-\lambda_2 x}\right), \quad x > 0.$$

This is an infinitely divisible distribution since it is the convolution of two infinitely divisible distributions. Integrating we find that the survival function is given by

$$\bar{G}(x) = 1 - G(x) = \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{-\lambda_1 x} - \lambda_1 e^{-\lambda_2 x}\right), \quad x > 0.$$

This is integrable and, after rescaling, we get the pdf $f = \mathcal{T}^{-1}(G)$ given by

$$f(x) = \frac{\lambda_1\lambda_2}{\lambda_2^2 - \lambda_1^2}\left(\lambda_2 e^{-\lambda_1 x} - \lambda_1 e^{-\lambda_2 x}\right), \quad x > 0.$$

The corresponding Laplace transform is

$$\begin{aligned}\phi(s) &= \frac{\lambda_1\lambda_2}{\lambda_2^2-\lambda_1^2}\left(\frac{\lambda_2}{\lambda_1+s}-\frac{\lambda_1}{\lambda_2+s}\right) = \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_1}\frac{\lambda_1+\lambda_2+s}{(\lambda_1+s)(\lambda_2+s)}\\ &= \left(\frac{\lambda_2}{\lambda_2+s}\right)\left(\frac{\lambda_1}{\lambda_2+\lambda_1}\frac{\lambda_1+\lambda_2+s}{(\lambda_1+s)}\right) = \left(\frac{\lambda_2}{\lambda_2+s}\right)\left(\frac{\frac{\lambda_1}{\lambda_1+s}}{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+s}}\right)\end{aligned}$$

The first term is the Laplace transform of $\mathrm{Exp}(\lambda_2)$ and the second term is the ratio of the Laplace transform of $\mathrm{Exp}(\lambda_1)$ and the Laplace transform of $\mathrm{Exp}(\lambda_1+\lambda_2)$. Such ratios are Laplace transforms of infinitely divisible distributions by Example III.11.8 in [26]. Thus, this distribution is infinitely divisible. In fact it is not difficult to show that a random variable with this second term as its Laplace transform is compound geometric. Toward this end, we write $\mathrm{geo}(p)$ to denote the distribution with pmf

$$p(n) = (1-p)^n p, \quad n = 0, 1, 2, \ldots.$$

Let $N \sim \mathrm{geo}(p)$ with $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, let $Z_1, Z_2, \ldots \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda_1+\lambda_2)$, and set

$$S = \sum_{i=1}^{N} Z_i.$$

A standard conditioning argument shows that the distribution of S has the required Laplace transform. One can try to extend this example to consider the case of more than two exponential random variables. Unfortunately, our approach does not seem to scale and it does not seem feasible even for the sum of three independent exponential random variables with different means. In the next example we consider the case where the means are equal.

Example 5. Let $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda)$ and set $Y = X_1+X_2+\cdots+X_n$. The distribution of Y is $\mathrm{gamma}(n,\lambda)$, such distributions are also sometimes called Erlang distributions. Since Y is the sum of independent infinitely divisible random variables, it is infinitely divisible. Its pdf is given by

$$g_n(x) = \frac{\lambda^n}{(n-1)!}x^{n-1}e^{-\lambda x}, \quad x \geqslant 0.$$

and the survival function is given by

$$\bar{G}_n(x) = 1 - G_n(x) = \frac{\lambda^n}{(n-1)!}\int_x^\infty t^{n-1}e^{-\lambda t}\mathrm{d}t = e^{-\lambda x}\sum_{k=0}^{n-1}\frac{(\lambda x)^k}{k!}, \quad x \geqslant 0,$$

where the last equality can be verified using substitution, induction on n, and integration by parts. It is easily checked that $\int_0^\infty \bar{G}_n(x)\mathrm{d}x = n/\lambda$. Thus, $f_n = \mathcal{T}^{-1}(G_n)$ is given by

$$f_n(x) = \frac{1}{n} e^{-\lambda x} \sum_{k=0}^{n-1} \frac{\lambda^{k+1} x^k}{k!} = \frac{1}{n} \sum_{k=1}^{n} \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x}.$$

This is a mixture of gamma(k, λ) distributions with $k = 1, 2, \ldots, n$, where the mixing distribution is discrete uniform on the set $\{1, 2, \ldots, n\}$. We can equivalently write the pdf as

$$f_n(x) = \frac{\lambda^{n+1}}{n!} \int_x^\infty t^{n-1} e^{-\lambda t} \mathrm{d}t = \frac{\lambda}{n!} \int_{\lambda x}^\infty t^{n-1} e^{-t} \mathrm{d}t.$$

Applying 6.5.36 in [1] shows that the Laplace transform is

$$\phi_n(s) = \frac{\lambda}{sn} \left(1 - \frac{\lambda^n}{(\lambda+s)^n}\right) = \frac{\lambda}{sn} \frac{(\lambda+s)^n - \lambda^n}{(\lambda+s)^n}$$

We can check the infinite divisibility of this distribution using Fact 3. Let

$$\rho_n(s) := -\frac{\mathrm{d}}{\mathrm{d}s} \log \phi_n(s) = \frac{1}{s} + \frac{n}{\lambda+s} - \frac{n(\lambda+s)^{n-1}}{(\lambda+s)^n - \lambda^n}.$$

When $n = 1$

$$\rho_1(s) = \frac{1}{\lambda+s} = \int_0^\infty x e^{-xs} e^{-\lambda x} x^{-1} \mathrm{d}x$$

and the distribution is $\mathrm{ID}_+(\nu_1, 0)$, where $\nu_1(\mathrm{d}x) = \frac{e^{-\lambda x}}{x} 1_{[x>0]} \mathrm{d}x$. This was already discussed in Example 1. When $n = 2$

$$\rho_2(s) = \frac{2}{\lambda+s} - \frac{1}{s+2\lambda} = \int_0^\infty x e^{-sx} \left(2e^{-\lambda x} - e^{-2\lambda x}\right) x^{-1} \mathrm{d}x.$$

and the distribution is $\mathrm{ID}_+(\nu_2, 0)$, where $\nu_2(\mathrm{d}x) = \frac{e^{-\lambda x}}{x}(2 - e^{-\lambda x})1_{[x>0]}\mathrm{d}x$. When $n = 3$

$$\begin{aligned}\rho_3(s) &= \frac{3}{\lambda + s} - \frac{3\lambda + 2s}{3\lambda^2 + 3\lambda s + s^2} \\ &= \int_0^\infty x e^{-sx}\left(3e^{-\lambda x} - 2e^{-3\lambda x/2}\cos(\lambda\sqrt{3}x/2)\right)x^{-1}\mathrm{d}x,\end{aligned}$$

where we use the fact that

$$\int_0^\infty \cos(bx)e^{-(s+a)x}\mathrm{d}x = \frac{s+a}{(s+a)^2 + b^2},$$

see 29.3.27 in [1]. It follows that the distribution is $\mathrm{ID}_+(\nu_3, 0)$, where $\nu_3(\mathrm{d}x) = \frac{e^{-\lambda x}}{x}\left(3 - 2e^{-\lambda x/2}\cos(\lambda\sqrt{3}x/2)\right)1_{[x>0]}\mathrm{d}x$. When $n = 4$ we can write

$$\begin{aligned}\rho_4(s) &= \frac{4}{\lambda + s} - \frac{1}{s + 2\lambda} - \frac{2(\lambda + s)}{s^2 + 2s\lambda + 2\lambda^2} \\ &= \int_0^\infty x e^{-sx}\left(4e^{-\lambda x} - e^{-2\lambda x} - 2e^{-\lambda x}\cos(x\sqrt{\lambda})\right)x^{-1}\mathrm{d}x\end{aligned}$$

and the distribution is $\mathrm{ID}_+(\nu_4, 0)$, where

$$\nu_4(\mathrm{d}x) = \frac{e^{-\lambda x}}{x}\left(4 - e^{-\lambda x} - 2\cos(x\sqrt{\lambda})\right)1_{[x>0]}\mathrm{d}x.$$

We conjecture that the distribution is infinitely divisible for every n, but are unable to show this for $n \geqslant 5$.

§4. Results for the Dickman type A distribution

It is well-known that the Dickman Type B distribution is infinitely divisible. In fact $\mathrm{DB} = \mathrm{ID}_+(\nu, 0)$, where $\nu(\mathrm{d}x) = x^{-1}1_{[0<x<1]}\mathrm{d}x$. In this section we show, computationally, that the Dickman Type A distributions is not infinitely divisible. We do this in two ways. In the first approach, we will show that some of the cumulants of the Dickman Type A distributions are negative, which implies that the distribution is not infinitely divisible by Fact 1. The second approach will use the cumulants in a more intricate manner.

For $k = 1, 2, \dots$, let m_k^A, C_k^A and m_k^B, κ_k^B represents the kth moment and kth cumulant of the Dickman Type A and Dickman Type B distributions, respectively. It is readily checked that

$$\kappa_k^B = \int_0^1 x^{k-1} \mathrm{d}x = \frac{1}{k}, \quad k = 1, 2, \dots$$

and by Proposition 3 in [20] we have $m_0^B = 1$ and

$$m_k^B = \frac{1}{k} \sum_{i=0}^{k-1} \binom{k}{i} m_i^B, \quad k = 1, 2, \dots. \tag{16}$$

From (10) we get

$$m_k^A = e^{\gamma} k m_{k-1}^B, \quad k = 1, 2, \dots.$$

Applying (16) gives

$$m_k^A = \frac{1}{k-1} \sum_{i=1}^{k-1} \binom{k}{i} m_i^A, \quad k = 2, 3, \dots,$$

where $m_1^A = e^{\gamma}$.

We used this formula to find the first 16 moments of the Dickman Type A distribution. We then converted these into cumulants using (15). This was done computationally using the statistical software R. We obtained the following values for the first 16 cumulants:

[1]	1.781072	0.3899259	0.2814155	0.2399205
[5]	0.2241395	0.2150961	0.1900575	0.1073168
[9]	−0.1027234	−0.4956145	−0.8315754	0.5344953
[13]	9.871239	43.49430	116.5033	81.20428

Since some of these are negative, we can conclude the following:

Result 1. *The Dickman Type A distribution is not infinitely divisible.*

The above approach required us to compute 9 cumulants before we obtained one that was negative. We now describe an alternate approach that only requires 7 cumulants. This is a general approach and we will use it later for another distribution of interest.

We begin by describing this methodology for showing that a distribution is not infinitely divisible. First, consider the distribution $\mathrm{ID}_+(\nu, \eta)$ and assume that $\int_0^\infty x^{2k-1} \nu(\mathrm{d}x) < \infty$ for some integer $k \geqslant 1$. Let $\kappa_1, \kappa_2, \dots, \kappa_{2k-1}$

be the first $2k-1$ cumulants of this distribution, let $z=(z_0,z_1,\dots,z_{k-1})\in\mathbb{R}^d$ and let

$$M_k=\begin{pmatrix}\kappa_1 & \kappa_2 & \kappa_3 & \dots & \kappa_k\\ \kappa_2 & \kappa_3 & \kappa_4 & \dots & \kappa_{k+1}\\ \kappa_3 & \kappa_4 & \kappa_5 & \dots & \kappa_{k+2}\\ \dots & \dots & \dots & \dots & \dots\\ \kappa_k & \kappa_{k+1} & \kappa_{k+2} & \dots & \kappa_{2k-1}\end{pmatrix}. \tag{17}$$

We have

$$\begin{aligned}0 &\leqslant \int_0^\infty x\left(z_0+z_1x+\dots+z_{k-1}x^{k-1}\right)^2\nu(\mathrm{d}x)\\ &=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} z_iz_j\int_0^\infty x^{i+j+1}\nu(\mathrm{d}x)\\ &\leqslant \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} z_iz_j\kappa_{i+j+1}=z^TM_kz\end{aligned}$$

where z^T is the transpose of vector z and the third line follows from (13), (14) and the fact that $\eta\geqslant 0$. Thus, the matrix M_k is nonnegative definite for every $k\geqslant 1$. This leads to the following lemma.

Lemma 1. *Given a distribution whose support is contained in* $[0,\infty)$ *with cumulants* $\kappa_1,\kappa_2,\dots,\kappa_{2k-1}$*, let* M_k *be as in* (17)*. If there exists at least one* $k\geqslant 1$ *such that* $\det(M_k)<0$*, then the distribution is not infinitely divisible.*

We now apply this to the Dickman Type A distribution. Evaluating the cumulants and determinants computationally, we get the following results

$$\det(M_1)\approx 1.78,\quad \det(M_2)\approx 0.34,\quad \det(M_3)\approx 0.006,\quad \det(M_4)\approx -0.0001.$$

The fact that $\det(M_4)<0$ means that the Dickman Type A distribution is not infinitely divisible. This gives an alternate verification of Result 1. We note that, in this case, we only use the first 7 cumulants, which is less than the 9 cumulants that we needed in the previous approach.

It may be interesting to ask if a result corresponding to Result 1 holds for $G=\mathcal{T}(f)$ when f is not the pdf of the Dickman Type B distribution, but of a distribution from a slightly more general class. Specifically, consider distributions of the form $\mathrm{ID}_+(\nu_\theta,0)$, where $\nu_\theta(\mathrm{d}x)=\theta x^{-1}1_{[0<x<1]}\mathrm{d}x$

for some $\theta > 0$. Such distributions are called generalized Dickman distributions in [20]. We note that this term was used to denote a different class of distributions in [19]. For more on this difference, see Section 7 below. While these distributions always have pdfs, the only case where the pdf belongs to $\mathfrak{A}$, the domain of the transform $\mathcal{T}$, is when $\theta = 1$, which corresponds to the Dickman Type B distribution. In all other cases, it does not belong to $\mathfrak{A}$ because, in those cases, the pdf is not bounded and monotonely decreasing. To see this, note that, by (13) in [20], the cdf of a generalized Dickman distribution on the interval $[0, 1]$ is given by

$$\kappa_\theta t^\theta$$

for some $\kappa_\theta > 0$. It follows that the pdf on $[0, 1]$ is given by

$$\kappa_\theta \theta t^{\theta-1}.$$

As $t \to 0^+$, this approaches a finite and strictly positive constant if and only if $\theta = 1$. Thus we cannot apply transform $\mathcal{T}$ to these distributions.

§5. Truncated self-decomposable distributions

A distribution μ is called self-decomposable if for any $b > 1$ there exists a distribution ρ_b such that if $X \sim \mu$ and $Y_b \sim \rho_b$ are independent, then

$$X \stackrel{d}{=} b^{-1}X + Y_b. \tag{18}$$

Such distributions are important in the study of autoregressive processes and limit theorems for sums of independent random variables, see [23]. All self-decomposable distributions are infinitely divisible. Further, a self-decomposable distribution μ is a subordinator if and only if $\mu = \mathrm{ID}_+(\nu, \eta)$ with $\eta \geqslant 0$ and the Lévy measure is of the form

$$\nu(\mathrm{d}x) = \frac{k(x)}{x} 1_{[x>0]} \mathrm{d}x, \tag{19}$$

where k is a monotonely decreasing function on $[0, \infty)$ satisfying

$$\int_0^\infty k(x) \left(1 \wedge x^{-1}\right) \mathrm{d}x < \infty. \tag{20}$$

This distribution has a pdf so long as $k(0+) > 0$. Further, the pdf is bounded and monotonely decreasing on $[0,\infty)$ if and only if $\eta = 0$

$$k(0+) = 1 \text{ and } \int_0^1 \frac{1-k(x)}{x}\mathrm{d}x < \infty, \tag{21}$$

see Corollary V.2.18 and (2.28) in [26]. Thus, under these conditions, the pdf of $\mathrm{ID}_+(\nu,\eta)$ belongs to $\mathfrak{A}$, the domain of $\mathcal{T}$.

Henceforth, let $\mu = \mathrm{ID}_+(\nu, 0)$, where ν is of the form (19), k satisfies the appropriate conditions and (21) holds. Let f and ϕ be the pdf and Laplace transform of μ, respectively. By Proposition A.3.4 in [26]

$$f(0+) = \lim_{s\to\infty} s\phi(s). \tag{22}$$

We now introduce the truncated version of this distribution. Fix $\epsilon > 0$ and let $\mu_\epsilon = \mathrm{ID}_+(\nu_\epsilon, 0)$ where

$$\nu_\epsilon(\mathrm{d}x) = 1_{[0<x<\epsilon]}\nu(\mathrm{d}x).$$

We call this the truncated Lévy measure. In the context of Lévy processes, this corresponds to focusing only on small jumps. For more on distributions with truncated Lévy measures see [7] or [8]. Note that μ_ϵ is still self-decomposable and hence it has a pdf, which we denote by f_ϵ. Further, (21) is still satisfied and thus $f_\epsilon \in \mathfrak{A}$. Next, we apply an important limit theorem due to [6]. Specifically, Corollary 2.1 in that paper implies that, if $X_\epsilon \sim \mu_\epsilon$, then

$$\frac{X_\epsilon}{\epsilon} \xrightarrow{d} \mathrm{DB} \text{ as } \epsilon \downarrow 0, \tag{23}$$

where DB is the Dickman Type B distribution. This leads to the following.

Result 2. *There exists a $\delta > 0$ such that for every $\epsilon \in (0,\delta)$, the distribution $\mathcal{T}(f_\epsilon)$ is not infinitely divisible.*

The proof follows by showing that (23) implies that the distributions $\mathcal{T}(f_\epsilon)$ (after rescaling) converge to the DA distribution as $\epsilon \downarrow 0$. We then use the fact that the DA distribution is not infinitely divisible together with Fact 2. While the basic idea of the proof is straightforward, the proof is not trivial and is given in Section 8.

We now focus to the special case of scale mixtures of exponential distributions, which we discussed in Example 2. A distribution of this type has a pdf of the form

$$f(x) = \int_0^\infty \lambda e^{-\lambda x} \mathrm{d}H(\lambda),$$

where H is a cdf on $(0, \infty)$. In this case

$$f(0) = \int_0^\infty \lambda \mathrm{d}H(\lambda),$$

is the mean of H. The Lévy measure of this distribution is given by

$$\nu(\mathrm{d}x) = 1_{[x>0]} q(x)\mathrm{d}x, \tag{24}$$

where $q(x) = \int_0^\infty e^{-\lambda x} v(\lambda)\mathrm{d}\lambda$ for some measurable function v with $0 \leqslant v \leqslant 1$. In particular, if v is a cdf on the positive half-line, then integration by parts gives

$$q(x) = x^{-1} \int_0^\infty e^{-\lambda x} \mathrm{d}v(\lambda). \tag{25}$$

Such distributions are self-decomposable and satisfy (21).

Let $\mu = \mathrm{ID}_+(\nu, 0)$, where ν is given by (24) and q is given by (25), and let f be the pdf of μ. In this case, $\mathcal{T}(f)$ is infinitely divisible by Example 2. However, for the truncated case, Result 2 implies that $\mathcal{T}(f_\epsilon)$ is not infinitely divisible for small enough ϵ. We conjecture that this will, in fact, hold for all $\epsilon > 0$, but are unable to prove this. **It is interesting to note that the infinite divisibility of $\mathcal{T}(f)$ is not preserved under truncation of the Lévy measure.**

§6. Dickman distribution as a limit

In (6) we saw that the Dickman Type B distribution is the limiting distribution, as $n \to \infty$, of the sum of n iid $U(0, 1)$ random variables each raised to the power of n. In this section we consider a more general situation where the limit is the Dickman Type B or a closely related distribution. Such limit theorems can be used for simulation. Further, they help to explain the mechanism by which these distributions may arise in applications. Our result extends the results given in Theorem 1 of [13] and Theorem 4.1 in [19]. We note that the result in [19] has several misprints, specifically in the form of the Lévy measure. Thus, our result should be used in place of the one given there.

We consider the following generalization of the situation in (6). Assume that the random variables $T_1, T_2, \dots$ are iid with cdf F and that there exists a nonrandom constant $0 < B < \infty$ such that $0 \leqslant T_1 \leqslant B$ a.s. This implies that $F(0-) = 0$ and $F(B) = 1$. Assume that $X_1, X_2, \dots$ are iid random variables with support contained in $[0,1]$ and

$$P(X_1 > x) = (1-x)^\alpha L(1-x), \quad x \in (0,1),$$

where $\alpha > 0$ and L is a slowly varying at 0 function. The slow variation of L means that for every $t > 0$

$$\lim_{x \to 0^+} \frac{L(xt)}{L(x)} = 1.$$

Theorem 1. *Assume that the T_i's and the X_i's are independent of each other and have distributions as described above. Let $N(n)$ be a sequence of integers with $N(n)n^{-\alpha}L(1/n) \to c$ for some $c \in (0,\infty)$ and set*

$$A_n = \sum_{i=1}^{N(n)} T_i X_i^n.$$

Then $A_n \xrightarrow{d} A_\infty$, where the Laplace transform of A_∞ is equal to

$$\mathrm{E} e^{-\lambda A_\infty} = \exp\left\{ -\int_0^B \left(1 - e^{-\lambda x}\right) k(x) \frac{1}{x} \mathrm{d}x \right\},$$

with

$$k(x) = c\alpha \int_{(x,B]} \left(\log(t/x)\right)^{\alpha-1} \mathrm{d}F(t).$$

When $\alpha \geqslant 1$ the distribution of A_∞ is self-decomposable.

Note that, when $L(0) = c \in (0,\infty)$ we can take $N(n) = \lfloor n^\alpha \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Note further that, when $\alpha = 1$ and F is a point mass at 1, in the limit we get the Dickman Type B distribution.

Proof. To prove this result, it suffices to verify the conditions for convergence of sums of infinitesimal triangular arrays. Such conditions can be found in, e.g., [18] or [17]. We use the conditions given in Proposition 3 of [12] as these are fine-tuned for convergence to subordinators. We begin by noting that, by L'Hôpital's rule, for any $t > s > 0$ we have

$$n\left(1-(s/t)^{1/n}\right) \sim \log(t/s).$$

Let $L_0(t) = L(1/t)$ and note that L_0 is slowly varying at ∞, i.e. for every $t > 0$

$$\lim_{x\to\infty} \frac{L_0(xt)}{L_0(x)} = 1.$$

Proposition 2.6 in [22] implies that for $t > s > 0$ we have

$$L\left(1-(s/t)^{1/n}\right) = L_0\left(\frac{n}{n(1-(s/t)^{1/n})}\right) \sim L_0\,(n) = L(1/n).$$

Hence

$$\begin{aligned}\lim_{n\to\infty} N(n)P(X_1 > (s/t)^{1/n}) &= \lim_{n\to\infty} cn^{\alpha}(1-(s/t)^{1/n})^{\alpha}\frac{L(1-(s/t)^{1/n})}{L(1/n)}\\ &= c(\log(t/s))^{\alpha}.\end{aligned}$$

Next, note that for $s \geqslant B$

$$\lim_{n\to\infty} N(n)P(X_1^n T_1 > s) = 0$$

and for $0 < s < B$

$$\begin{aligned}\lim_{n\to\infty} N(n)P(X_1^n T_1 > s) &= \lim_{n\to\infty} \int\limits_{(s,B]} N(n)P(X_1 > (s/t)^{1/n})\mathrm{d}F(t)\\ &= \int\limits_{(s,B]} \lim_{n\to\infty} N(n)P(X_1 > (s/t)^{1/n})\mathrm{d}F(t)\\ &= c\int\limits_{(s,B]} (\log(t/s))^{\alpha}\,\mathrm{d}F(t)\\ &= c\int\limits_{(s,B]}\int\limits_{s}^{t} \alpha\,(\log(t/u))^{\alpha-1}\,u^{-1}\mathrm{d}u\mathrm{d}F(t)\\ &= \int\limits_{s}^{B} k(u)u^{-1}\mathrm{d}u,\end{aligned}$$

where the last line follows by Fubini's theorem and in the second line we interchange limit and integration using bounded convergence. Specifically, let N_1 be such that, for $n \geqslant N_1$ we have

$$N(n)P(X_1 > (s/B)^{1/n}) \leqslant c\left(\log(B/s)\right)^{\alpha} + 1.$$

For all such n

$$N(n)P(X_1 > (s/t)^{1/n}) \leqslant N(n)P(X_1 > (s/B)^{1/n}) \leqslant c\left(\log(B/s)\right)^{\alpha} + 1$$

and we can use bounded convergence. Next consider

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} N(n) \mathrm{E}\left[X_1^n T_1 1_{[X_1^n T < \epsilon]}\right] \\
&\quad = \lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} N(n) \int_0^\infty P(X_1^n T_1 1_{[X_1^n T<\epsilon]} > s)\mathrm{d}s \\
&\quad \leqslant \lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} N(n) \int_0^\epsilon P(X_1^n T_1 > s)\mathrm{d}s \\
&\quad \leqslant \lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \int_0^\epsilon N(n) P(X_1 > (s/B)^{1/n})\mathrm{d}s \\
&\quad = \lim_{\epsilon \downarrow 0} \int_0^\epsilon \limsup_{n\to\infty} N(n) P(X_1 > (s/B)^{1/n})\mathrm{d}s \\
&\quad \leqslant \lim_{\epsilon \downarrow 0} c \int_0^\epsilon \left(\log(B/s)\right)^{\alpha} \mathrm{d}s = \lim_{\epsilon \downarrow 0} cB \int_{\log(B/\epsilon)}^\infty s^\alpha e^{-s}\mathrm{d}s = 0,
\end{aligned}$$

where we interchange lim sup and integration similar to the above. From here, the result follows by Proposition 3 in [12]. For $\alpha \geqslant 1$, the self-decomposability follows from the fact that k satisfies (20) and is monotonely decreasing in this case. □

§7. Dickman distribution and perpetuities

Recall that the Dickman Type B distribution satisfies the relation given in (4). We can consider the more general relation

$$X \stackrel{d}{=} AX + T, \tag{26}$$

where (A, T) and X are independent. Equivalently,

$$X = T_1 + T_2 A_1 + T_3 A_1 A_2 + T_4 A_1 A_2 A_3 + \dots, \tag{27}$$

where $(A_1, T_1), (A_2, T_2), (A_3, T_3), \dots$ are iid random vectors having the same distribution as (A, T). Note that we do not require A and T to be independent. In fact when $A = T$ and $A \sim U(0, 1)$, (4) holds and X has a Dickman Type B distribution. When $A = T$ and $A \sim \text{beta}(\alpha, 1)$ for some $\alpha > 0$, the distribution of X is what was called a generalized Dickman distribution in [20]. We adopt this terminology here. In [19] the term "generalized Dickman distribution" was used to refer to the distribution of any solution X to (26). However, in this paper, we refer to the latter class of distributions by the more commonly used term: "perpetuities." Perpetuities are important for a variety of application areas. They are particularly useful in applications to actuarial science and economics, see e.g. [14] and the references therein. For other applications, see [19].

In principle, (26) places no restriction on the distribution of A. No matter what A is, if we take $T = c(1 - A)$ for some $c \in \mathbb{R}$, then a solution to (26) is given by $X = c$ a.s. Of course, such situations are not interesting. Conditions for (26) to have a solution, where X is not a point mass are given in [28]. A necessary condition is

$$\mathrm{E}[\log |A|] < 0 \tag{28}$$

A simple sufficient condition is

$$-\infty < \mathrm{E}[\log |A|] < 0 \text{ and } \mathrm{E}[\log(|T| \vee 1)] < \infty.$$

Theorem 5.1 in [28] says that so long as there exists an integer k with $\mathrm{E}[|T|^k] < \infty$ and $\mathrm{E}[|A|^k] < 1$, a unique solution to (26) exists. Further, in this case, $\mathrm{E}[|X|^k] < \infty$ and

$$\mathrm{E}[X^k] = \sum_{j=0}^{k} \binom{k}{j} \mathrm{E}\big[A^j T^{k-j}\big] \mathrm{E}[X^j]$$

If $\mathrm{E}[A^k] \neq 1$,

$$\mathrm{E}[X^k] = \frac{1}{1 - \mathrm{E}\big[A^k\big]} \sum_{j=0}^{k-1} \binom{k}{j} \mathrm{E}\big[A^j T^{k-j}\big] \mathrm{E}[X^j].$$

If, in addition, $A = cT$ for some $c \in \mathbb{R}$, then this simplifies to

$$\mathrm{E}[X^k] = \frac{\mathrm{E}\big[T^k\big]}{1 - c^k \mathrm{E}\big[T^k\big]} \sum_{j=0}^{k-1} \binom{k}{j} c^j \mathrm{E}[X^j].$$

An important question is: Under what conditions are perpetuities infinitely divisible? This question goes back, at least, to [28], see also [14]. The problem of giving a full characterization has proven to be a tough nut to crack. However, many results are known. We now give some interesting examples. We begin with the case where A and T are independent:

- If there is a constant $c > 0$ with $A = c$ a.s., then (28) implies that $c \in (0,1)$. In this case every self-decomposable distribution satisfies (26) for an appropriate choice of T, since (18) holds. More generally, in this case (26) holds if and only if the distribution of X has a so-called semi-self-decomposable distribution. See Section 15 in [23] for more on these distribution.
- If for some $\epsilon \in (0,1)$ we have $|A| \leqslant (1-\epsilon)$ a.s. and for some nonrandom constant $B > 0$ we have $|T| \leqslant B$ a.s., then (27) implies that $|X| \leqslant \sum_{i=0}^{\infty} B(1-\epsilon)^i = B/\epsilon < \infty$ a.s. It follows that the distribution of X has a bounded support and is thus not infinitely divisible by Fact 4.
- When $T = 1$ a.s. and $A \sim U(0.5, 1)$ then X is not infinitely divisible. This was verified computationally in [14] using Fact 1.
- If the characteristic function of T has zeros, then X is not infinitely divisible. This holds, in particular, if $T \sim U(0,1)$, see Section 5.4 in [19].
- In [16] it is shown that if $T \sim \mathrm{Exp}(\lambda)$ and $A \sim \mathrm{beta}(\alpha, 1)$, then $X \sim \mathrm{gamma}(\alpha{+}1, \lambda)$, which is well-known to be infinitely divisible.

The proof of this last example as given in [16] is fairly complicated. We now give a simple derivation, which further illustrates a standard approach to these kinds of problems. Let $T \sim \mathrm{Exp}(\lambda)$ and $A \sim \mathrm{beta}(\alpha, 1)$ be independent random variables. The Laplace transform of X for $u > 0$ is given by

$$\begin{aligned}\phi(u) = \mathrm{E}\big[e^{-uX}\big] &= \mathrm{E}\big[e^{-uT}\big]\mathrm{E}\big[e^{-uAX}\big]\\ &= \frac{\lambda}{u+\lambda}\int_0^1 \phi(ux)\alpha x^{\alpha-1}\mathrm{d}x\\ &= \frac{\lambda}{(u+\lambda)u^\alpha}\int_0^u \phi(x)\alpha x^{\alpha-1}\mathrm{d}x.\end{aligned}$$

We rewrite this equality as

$$(u+\lambda)u^\alpha\phi(u) = \lambda\int_0^u \phi(x)\alpha x^{\alpha-1}\mathrm{d}x$$

and differentiate both parts with respect to u. After simplification, we get the differential equation

$$\phi'(u) = -\frac{1+\alpha}{u+\lambda}\phi(u)$$

with initial condition $\phi(0) = 1$. It follows that

$$\log\phi(u) = -\int_0^u \frac{1+\alpha}{t+\lambda}\mathrm{d}t = -(\alpha+1)\log\left(\frac{u+\lambda}{\lambda}\right)$$

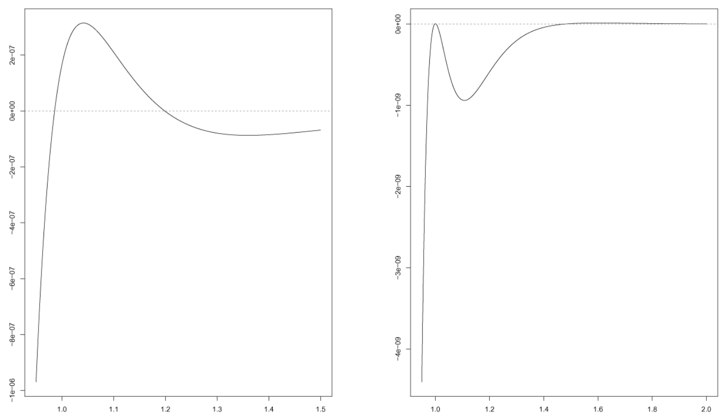
and hence

$$\phi(u) = \Big(\frac{\lambda}{u+\lambda}\Big)^{\alpha+1},$$

which is the Laplace transform of gamma$(\alpha+1,\lambda)$, as required.

We now turn to the case where A and T are dependent. Several known situations are as follows:

- If $A = T$ and $A \sim$ beta$(\alpha, 1)$ for $\alpha > 0$, then X has a generalized Dickman distribution, which is infinitely divisible. In particular, when $\alpha = 1$ the beta distribution reduces to a $U(0,1)$ distribution and X has a Dickman Type B distribution.
- Several examples involving gamma and beta distributions are collected in [10]. For instance, if $T = -A$ and $T \sim$ beta(α,β), then



$\det(M_4)$ $\det(M_5)$

Figure 1. Plots of $\det(M_4)$ on the left and $\det(M_5)$ on the right. The x-axis is the value of β, which ranges for $\beta \in [0.95, 1.5]$ on the left and for $\beta \in [0.95, 2]$ on the right. We can see that $\det(M_4)$ is only positive for $\beta \in [0.986, 1.198]$ and $\det(M_5)$ is only positive for $\beta = 1$ and $\beta \in [1.479, 2.992]$.

$X \sim \text{beta}(\alpha, \alpha+\beta)$. Since beta distributions have bounded supports, Fact 4 implies that they are not infinite divisible. This situation was used for modeling the distance between parked cars in [24].

- In [16] it is shown that every self-decomposable distribution that is not concentrated at a point, satisfies (28) for some A and T whose distributions are not concentrated at a point.

The first example suggests the following question. Let $A = T$ with $A \sim \text{beta}(1, \beta)$ for some $\beta > 0$. In this case, will the distribution of X be infinitely divisible? Of course when $\beta = 1$ the distribution reduces to $U(0, 1)$ and X has a Dickman Type B distribution. In the other cases, we are unable to get analytic results. However, computational results suggest that the distribution is not infinitely divisible. We use the approach described by Lemma 1. We now discuss our results.

Example 6. Let $A = T$ with $A \sim \text{beta}(1,\beta)$ for some $\beta > 0$. For a fixed value of β, we calculate the moments of X using (29). We then convert these into cumulants using (15) and, for several choices of k, we evaluate $\det(M_k)$, where M_k is given by (17). We did this for every β on the grid from 0.1 to 5 with step 0.001. We found that $\det(M_k) > 0$ for $k = 1, 2, 3$ and for all considered values of β. However, $\det(M_4)$ is positive only for $\beta \in [0.986, 1.198]$ and $\det(M_5)$ is positive only for $\beta = 1$ and $\beta \in [1.479, 2.992]$. These results are summarized in Figure 1, which plot the values of $\det(M_4)$ and $\det(M_5)$ near the point $\beta = 1$. Our results show, computationally, that this distribution is not infinitely divisible for all considered values of β except $\beta = 1$. We conjecture (but cannot prove rigorously) that the distribution is infinitely divisible only for $\beta = 1$, which is when X has a Dickman Type B distribution.

§8. Proof of result 2

Let $X \sim \mu = \mathrm{ID}_+(\nu, 0)$ and $X_\epsilon \sim \mu_\epsilon = \mathrm{ID}_+(\nu_\epsilon, 0)$. Let F_ϵ, f_ϵ, and ϕ_ϵ be the cdf, pdf, and Laplace transform of X_ϵ/ϵ, respectively, let F_∞, f_∞, and ϕ_∞ be the corresponding terms for the distribution of X, and let F_0, f_0, ϕ_0 be the corresponding terms for the Dickman Type B distribution, respectively. Let $G_\epsilon = \mathcal{T}(f_\epsilon)$ and $G_0 = \mathcal{T}(f_0)$. Note that G_0 is the cdf of the Dickman Type A distribution. Result 2 follows immediately from the the first part of Proposition 1 and the following result.

Result 3. *There exists a $\delta > 0$ such that for every $\epsilon \in (0, \delta)$ the distribution of G_ϵ is not infinitely divisible.*

Before proving this result, we give a lemma.

Lemma 2. *We have*

$$f_\infty(0+) = e^{-\gamma} e^{\int_0^1 \frac{1-k(x)}{x} \mathrm{d}x} e^{-\int_1^\infty \frac{k(x)}{x} \mathrm{d}x}, \tag{29}$$

$$\lim_{\epsilon \to 0} f_\epsilon(0+) = f_0(0+),$$

and

$$G_\epsilon \xrightarrow{w} G_0 \quad \textit{as } \epsilon \downarrow 0. \tag{30}$$

In (30) when we take the limit as $\epsilon \downarrow 0$ we mean the limit along any sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$.

Proof. First note that (22) and dominated convergence imply that

$$\begin{aligned}
f_\infty(0+) &= \lim_{s\to\infty} s e^{\int_0^\infty \left(e^{-sx}-1\right)\frac{k(x)}{x}\mathrm{d}x} \\
&= \lim_{s\to\infty} s\phi_0(s) e^{\int_0^\infty \left(e^{-sx}-1\right)\frac{k(x)}{x}\mathrm{d}x - \int_0^1 \left(e^{-sx}-1\right)\frac{1}{x}\mathrm{d}x} \\
&= \lim_{s\to\infty} s\phi_0(s) e^{\int_1^\infty \left(e^{-sx}-1\right)\frac{k(x)}{x}\mathrm{d}x - \int_0^1 \left(e^{-sx}-1\right)\frac{1-k(x)}{x}\mathrm{d}x} \\
&= e^{-\gamma} e^{\int_0^1 \frac{1-k(x)}{x}\mathrm{d}x} e^{-\int_1^\infty \frac{k(x)}{x}\mathrm{d}x}
\end{aligned}$$

and similarly for $\epsilon \in (0,1)$

$$\begin{aligned}
f_\infty(0+) &= \lim_{s\to\infty} s e^{\int_0^\infty \left(e^{-sx}-1\right)\frac{k(x)}{x}\mathrm{d}x} \\
&= \lim_{s\to\infty} \epsilon^{-1} s e^{\int_0^\infty \left(e^{-sx/\epsilon}-1\right)\frac{k(x)}{x}\mathrm{d}x} \\
&= \lim_{s\to\infty} \epsilon^{-1} s\phi_\epsilon(s) e^{\int_\epsilon^\infty \left(e^{-sx/\epsilon}-1\right)\frac{k(x)}{x}\mathrm{d}x} \\
&= \epsilon^{-1} f_\epsilon(0+) e^{-\int_\epsilon^\infty \frac{k(x)}{x}\mathrm{d}x} \\
&= f_\epsilon(0+) e^{\int_\epsilon^1 \frac{1-k(x)}{x}\mathrm{d}x} e^{-\int_1^\infty \frac{k(x)}{x}\mathrm{d}x}.
\end{aligned}$$

By dominated convergence, it follows that

$$\lim_{\epsilon\to 0} f_\epsilon(0+) = f_\infty(0+) e^{-\int_0^1 \frac{1-k(x)}{x}\mathrm{d}x} e^{\int_1^\infty \frac{k(x)}{x}\mathrm{d}x} = e^{-\gamma} = f_0(0).$$

From here the result follows by the fourth part of Proposition 1. □

Proof of Result 3. Assume for the sake of contradiction that there does not exist such a $\delta > 0$. Then there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that G_{ϵ_n} is infinitely divisible for every n. But $G_{\epsilon_n} \xrightarrow{w} G_0$, which is not infinitely divisible by Result 1. This is a contradiction by Fact 2. □

Example 7. We can illustrate (29) for the case where $X \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. In this case $k(x) = e^{-\lambda x}$ and $f_\infty(0+) = \lambda$. We have

$$e^{-\gamma} e^{\int_0^1 \frac{1-k(x)}{x} \mathrm{d}x} e^{-\int_1^\infty \frac{k(x)}{x} \mathrm{d}x} = e^{-\gamma} e^{\int_0^1 \frac{1-e^{-\lambda x}}{x} \mathrm{d}x} e^{-\int_1^\infty \frac{e^{-\lambda x}}{x} \mathrm{d}x}$$

$$= e^{-\gamma} e^{\int_0^\lambda \frac{1-e^{-x}}{x} \mathrm{d}x} e^{-\int_\lambda^\infty \frac{e^{-x}}{x} \mathrm{d}x} = \lambda = f_\infty(0+),$$

as required. Here the third equality follows from the fact that

$$\int_0^\lambda \frac{1-e^{-x}}{x} \mathrm{d}x = \int_\lambda^\infty \frac{e^{-x}}{x} \mathrm{d}x + \ln \lambda + \gamma,$$

see 5.1.39 in [1].

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