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Some Cases of Polynomial Solvability of the Edge Coloring Problem That Are Generated by Forbidden 8-Edge Subcubic Forests

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Abstract—The edge-coloring problem is to minimize the number of colors sufficient to color all the edges of a given graph so that any adjacent edges receive distinct colors. The complexity status of this problem is known for all the classes defined by the sets of forbidden subgraphs with 7 edges each. In this paper, we consider the case of prohibitions with 8 edges. It can readily be seen that the edge-coloring problem is NP-complete for such a class if there are no subcubic forests among its 8-edge prohibitions. We prove that forbidding any subcubic 8-edge forest generates a class with polynomial-time solvability of the edge-coloring problem, except for the cases formed by the disjoint sum of one of four forests and an empty graph. For all the remaining cases, we prove a similar result for the intersection with the set of graphs with a maximum degree of at least four.

Keywords: *monotone class, edge-coloring problem, computational complexity*

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INTRODUCTION

In the present paper, we consider only *ordinary graphs*, i.e., undirected acyclic graphs without multiple edges. A graph class is said to be *hereditary* if it is closed under the removal of vertices. Any hereditary class $\mathcal{X}$ (and only a hereditary class) of graphs can be defined by the set $\mathcal{Y}$ of its own *forbidden generated subgraphs*, and one writes $\mathcal{X} = \mathrm{Free}(\mathcal{Y})$. *Strongly hereditary* (or *monotone*) class of graphs is a hereditary class which is also closed under the removal of edges. Any monotone class $\mathcal{X}$ can be defined by the set $\mathcal{Y}$ of its own *forbidden subgraphs*, and one writes $\mathcal{X} = \mathrm{Free}_s(\mathcal{Y})$.

A *k-edge coloring* of a graph $G = (V, E)$ is any mapping $c\colon E \to \{1, 2, \dots, k\}$ such that $c(e_1) \neq c(e_2)$ for any adjacent edges e_1 and e_2. The minimum k for which there exists a k-edge-coloring of the graph G is called the *chromatic index of* G and is denoted by $\chi'(G)$.

For a given graph G, the *k-edge-coloring problem* (briefly, the k-EC *problem*) is to recognize whether the inequality $\chi'(G) \le k$ holds. For a given graph G and a number k, the *edge-coloring problem* (briefly, the EC *problem*) is to recognize whether the inequality $\chi'(G) \le k$ holds. The 3-EC and EC problems are NP-complete [1].

According to the well-known Vizing theorem in [2], the inequality $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$ holds for any graph G, where $\Delta(G)$ is the maximum power of the vertices of G. Thus, the EC problem for a graph G is equivalent to recognizing whether $\chi'(G) = \Delta(G)$ or not.

In [3], for any k a complete complexity dichotomy (i.e., a complete classification of complexity) was obtained for the k-EC problem and all classes of the form $\mathrm{Free}(\{H\})$. In [4], a complete classification of the complexity of the 3-EC problem was obtained for sets of forbidden generated subgraphs, each with no more than 6 vertices of which no more than two subgraphs have exactly 6 vertices. In [5], the EC problem and a family of monotone classes defined by the prohibition of subgraphs

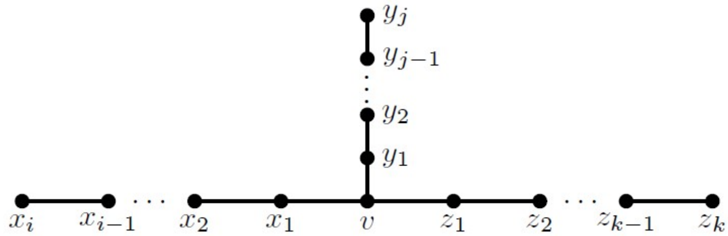

Fig. 1. Graph $T_{i,j,k}$.

each of which has no more than 6 edges or no more than 7 vertices were considered, and a complete classification of the complexity of the EC problem for the classes of graphs from this family was obtained. In [6], a complete classification of the complexity of the EC problem was obtained for monotone classes defined by the prohibition of subgraphs each of which has at most 7 edges.

In the present paper, we consider the case of prohibitions with 8 edges. It was proved in [7] that for any g the EC problem is NP-hard in the set of subcubic graphs of width $\geq g$, and so the EC problem will be NP-hard for any monotone class with 8-edge prohibitions if there is no subcubic forest among these prohibitions. In the present paper, we prove that the prohibition of any subcubic 8-edge forest generates a class with polynomial solvability of the edge coloring problem except for the cases formed by the disjoint sum of one of the four forests and the empty graph. For all remaining cases, a similar result is proved for the intersection with the set of graphs of maximum degree ≥ 4.

1. NOTATION

Let G be a graph, and let x be a vertex of G. The *open neighborhood of* x, i.e., the set of its neighbors, is denoted by $N(x)$. The *closed neighborhood of* x, i.e., the set $N(x) \cup \{x\}$, is denoted by $N[x]$. The degree of x is denoted by $\deg(x)$, and the maximum degree of vertices of G is denoted by $\Delta(G)$. If $\Delta(G) \leq 3$, then G is said to be *subcubic*. If the degrees of all vertices of a graph are equal to 3, then it is said to be *cubic*.

Let G be a graph, and let $V' \subseteq V(G)$. Then $G[V']$ is the subgraph of G generated by V', and $G \setminus V'$ is obtained by removing all elements of V' from G.

Let G_1 and G_2 be graphs. We write $G_1 \cong G_2$ if G_1 and G_2 are isomorphic. If $V(G_1) \cap V(G_2) = \emptyset$, then the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ will be denoted by $G_1 + G_2$. For a graph G and a number k, the notation kG means the graph $\underbrace{G + G + \cdots + G}_{k \text{ times}}$.

Let $G, H_1, H_2, \ldots, H_k$ be graphs. Then $\langle G; H_1, H_2, \ldots, H_k \rangle$ is a shorthand for the statement that G contains each of the graphs $H_1, H_2, \ldots, H_k$ as a subgraph.

As usual, by P_n, O_n, and $K_{p,q}$ we denote a simple path with n vertices, an empty graph with n vertices, and a complete bipartite graph with p vertices in one part and q vertices in the other part, respectively. By $K_4 - e$ we denote the graph obtained by removing an edge from the complete graph with 4 vertices.

By $T_{i,j,k}$, where $i \geq 0$, $j \geq 0$, and $k \geq 0$, we denote the tree, known as a *triode*, which is obtained by the simultaneous identification of the endpoints of the three simple paths

$$(v = x_0, x_1, \ldots, x_i),$$
$$(v = y_0, y_1, \ldots, y_j),$$
$$(v = z_0, z_1, \ldots, z_k)$$

by the vertex v (Fig. 1). In the subsequent proofs, the vertices of $T_{i,j,k}$ will be denoted as in this definition.

By $\mathcal{T}$ we denote the class of all forests each of whose connected components is a triode. Figure 2 lists all possible subcubic trees that do not belong to $\mathcal{T}$ and have at most 8 edges.

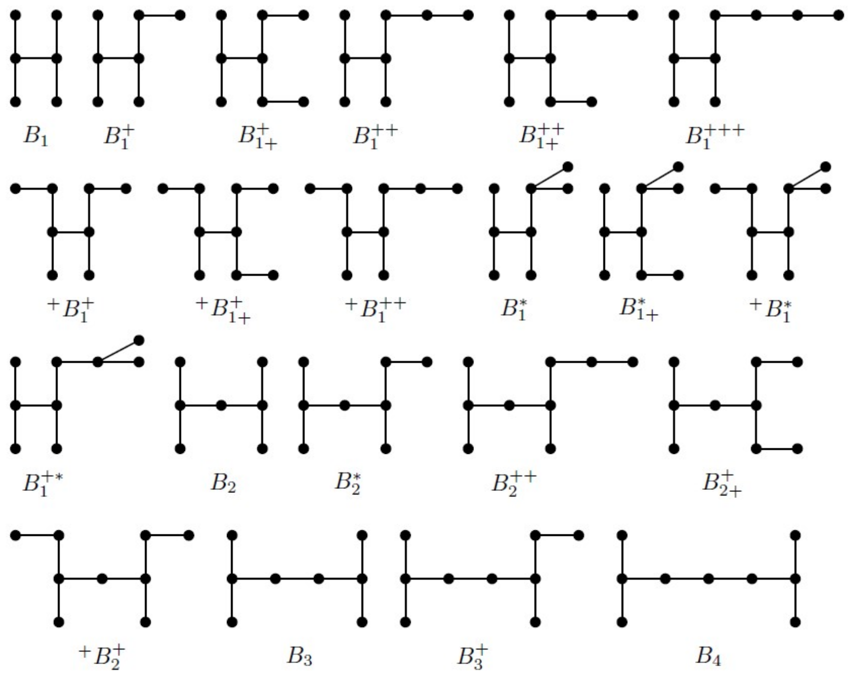

Fig. 2.

The set

$$\{B_1+3P_2, B_1+P_2+P_3, B_1+P_4, B_1+K_{1,3}, B_1^+ + 2P_2, B_1^+ + P_3, B_{1+}^+ + P_2, B_1^{++} + P_2, B_{1+}^{++}, B_1^{+++}, {}^+B_1^+ + P_2, {}^+B_{1+}^+, {}^+B_1^{++}, B_1^* + P_2, B_{1+}^*, {}^+B_1^*, B_1^{+*}, B_2 + 2P_2, B_2 + P_3, B_2^+ + P_2, B_2^{++}, B_{2+}^+, {}^+B_2^+, B_3 + P_2, B_3^+, B_4\}$$

will be denoted by $\mathcal{S}$. Note that $\mathcal{S}$ coincides with the set of all possible subcubic forests without isolated vertices each of which has exactly 8 edges and does not belong to the class $\mathcal{T}$.

An *independent set* in a graph is any subset consisting of pairwise nonadjacent vertices.

2. INCOMPRESSIBLE GRAPHS

It is well known (see [8, p. 465]) that a graph G containing a vertex x such that $|\{y \in N(x) \mid \deg(y) = \Delta(G)\}| \le 1$ has an edge coloring of $\Delta(G)$ colors if and only if so does the graph $G \setminus \{x\}$.

Recall that a *hinge* in a graph is a vertex whose removal increases the number of connected components of the graph. Obviously, for any graph G and a hinge x in G the relation $\chi'(G) = \Delta(G)$ holds if and only if

$$\chi'\Big(G\big[V(H) \cup \{x\}\big]\Big) \le \Delta(G)$$

for each connected component H of the graph $G \setminus \{x\}$.

A connected hinge-free graph G is said to be *incompressible* if any vertex G has at least two neighbors of degree $\Delta(G)$. The EC problem for graphs in an arbitrary monotone class is polynomially reducible to the same problem for incompressible graphs in this monotone class.

3. CLIQUE-WIDTH OF GRAPHS AND CONSEQUENCES OF ITS BOUNDEDNESS

Clique-width is an important parameter of graphs. For a graph G it is denoted by $cw(G)$ and is defined as the minimum number of labels needed to construct G using the following four operations:

(1) Creating a new vertex with a given label i.

(2) Taking the disjoint union of two labeled graphs H_1 and H_2 with disjoint vertex sets.

(3) Connecting each vertex with label i with each vertex with label j by an edge.

(4) Renaming the label i into j.

For any number C, many graph problems (including the EC problem) are polynomially solvable for graphs whose clique-width does not exceed C (see, e.g., [9]). It follows from the results in [10, 11] (see the proof of Lemma 4 in [5]) that the following statement is true.

Lemma 1. *For any $C > 0$, the problem* EC *is polynomially solvable in the class $\{G \mid cw(G) \leq C\}$ of graphs.*

The following assertion was proved in [12].

Lemma 2. *For any monotone class $\mathcal{X}$ not containing the entire $\mathcal{T}$, there exists a number $C(\mathcal{X})$ such that $cw(G) < C(\mathcal{X})$ for any $G \in \mathcal{X}$.*

Lemma 3. *Let $H' \in \mathcal{T}$, and let $\mathcal{X}$ be a class of graphs such that $\mathcal{X} \subseteq \mathrm{Free}_s(\{H + H'\})$ for some graph H. Then the* EC *problem in the class $\mathcal{X}$ is polynomially reducible to the same problem in the class $\mathcal{X} \cap \mathrm{Free}_s(\{H\})$.*

Proof. We will need the concept of the tree width of a graph. A *tree decomposition* of a graph $G = (V, E)$ is a tree T whose vertices $X_1, \dots, X_n$ are subsets of V with the following properties:

(1) The union of all sets X_i is equal to V.

(2) For any vertex $v \in V$, the vertices of the tree containing v form a subtree of the tree T.

(3) For any edge (v, u) of the graph G there exists a subset X_i containing both v and u.

The *decomposition width* T is the number $\max\limits_i |X_i| - 1$. The *tree width* $tw(G)$ of a graph G is the minimum width of all possible decompositions of G. It can readily be seen that for each graph G and any of its vertices v one has $tw(G) \leq tw(G \setminus \{v\}) + 1$; to achieve this, it suffices to include v into all X_i of the optimal tree decomposition of the graph $G \setminus \{v\}$.

There is a relationship between the clique-width and the tree width of a graph. For example, the inequality $cw(G) \leq 3 \cdot 2^{tw(G)-1}$ holds for any graph G (see [13]), and one has $tw(G) \leq 3cw(G) \cdot (t-1) - 1$ for any graph G with the property $\neg\langle G; K_{t,t}\rangle$ (see [14]).

Let $G = (V, E)$ be an arbitrary graph in $\mathcal{X}$. If G contains a subgraph $H = (V_H, E_H)$, then $G \backslash V_H \in \mathrm{Free}_s(\{H'\})$ and there exists a $t^* = t^*(H, H')$ such that $\neg\langle G; K_{t^*,t^*}\rangle$, because $H' \in \mathcal{T}$ and $\mathcal{X} \subseteq \mathrm{Free}_s(\{H + H'\})$. Hence it follows from Lemma 2 and the remarks in the last two paragraphs that there exists a $C^* = C^*(H, H')$ such that for any $G \in \mathcal{X}$ one has $\langle G; H\rangle \Rightarrow cw(G) < C^*$; hence the EC problem is polynomially solvable in this class by Lemma 1. The proof of Lemma 3 is complete. $\square$

4. MONOTONE CASES OF POLYNOMIAL SOLVABILITY OF THE EC PROBLEM

Lemma 4. *For any $H \in \{B_{1+}^{++}, B_1^{+++}, {}^{+}B_{1+}^{+}, {}^{+}B_1^{++}\}$, the* EC *problem is polynomially solvable for graphs of the class* $\mathrm{Free}_s(\{H\})$.

Proof. Let us show that for graphs in $\mathrm{Free}_s(\{H\})$ the EC problem is polynomially reducible to the same problem in the class $\mathrm{Free}_s(\{H, T_{5,5,5}\})$. Based on this and Lemma 2, it follows that the assertion of this lemma is true. It suffices to consider incompressible graphs in $\mathrm{Free}_s(\{H\})$ containing the subgraph $T_{5,5,5}$. Let $G = (V, E)$ be such a graph.

If $N(x_1) \setminus V(T_{5,5,5}) \neq \emptyset$, then $\langle G; B_{1+}^{++}, B_1^{+++}, {}^{+}B_{1+}^{+}, {}^{+}B_1^{++}\rangle$. The same is true if $(N(x_1) \cap V(T_{5,5,5})) \setminus \{v, x_2, y_1, z_1\} \neq \emptyset$ or if $N(x_1) = \{v, x_2, y_1, z_1\}$. The same reasoning can also be carried out with respect to the vertices y_1 and z_1, and so we can assume that these cases are not realized. Thus, for any vertex $u \in \{x_1, y_1, z_1\}$ either $\deg(u) = 2$ or $\deg(u) = 3$, and in $\{x_1, y_1, z_1\} \setminus \{u\}$ there exists a neighbor of the vertex u. The same reasoning shows that the vertex v is not adjacent to the vertices in $T_{5,5,5}$ other than $x_1, y_1, z_1, x_5, y_5, z_5$. Moreover, if v is adjacent, say, to x_5, then x_5 is similar to x_1 (these vertices are interchangeable) and $\deg(x_5) = \deg(x_1) = 2$.

Assume that $\Delta(G) \geq 4$. Since G is incompressible, it follows that $N(v)$ contains at least two vertices of degree $\Delta(G)$. Let u be such a vertex of arbitrary choice. It is clear that $u \notin V(T_{5,5,5})$ and u is not adjacent to any of the vertices x_1, y_1, and z_1. Since the graph G is incompressible, we see that there exist distinct vertices $u_1, u_2 \in N(u)\setminus\{v\}$ such that $\deg(u_1) = \Delta(G)$. It is easily seen that if at least one of the vertices u_1 and u_2 belongs to $V(T_{5,5,5})$, then $\langle G; B_{1+}^{++}, B_1^{+++}, {}^+B_{1+}^{+}, {}^+B_1^{++}\rangle$. We can assume that $u_1, u_2 \notin V(T_{5,5,5})$. Then $\langle G; B_{1+}^{++}, B_1^{+++}\rangle$. Since $\deg(u_1) = \Delta(G) \geq 4$, it follows that there exists a neighbor u' of u_1 that is different from each of v, u, and u_2. Then $\langle G; {}^+B_{1+}^{+}, {}^+B_1^{++}\rangle$.

Assume that $\Delta(G) = 3$. In view of the incompressibility of G and symmetry, we can assume that $x_1y_1 \in E$. Let $\deg(z_1) = 2$; otherwise, $\langle G; B_{1+}^{++}, B_1^{+++},\ {}^+B_{1+}^{+}, {}^+B_1^{++}\rangle$. Let us shrink the triangle (v, x_1, y_1) to the vertex v^* to obtain the graph G^*. It is clear that $\chi'(G) = 3 \Leftrightarrow \chi'(G^*) = 3$. If there exists at most one vertex of degree 3 among x_2 and y_2, then

$$\chi'(G^*) \leq 3 \Leftrightarrow \chi'\big(G^* \setminus \{v^*\}\big) \leq 3, \quad \text{with} \quad G^* \setminus \{v^*\} \cong G \setminus \{v, x_1, y_1\}.$$

Hence $\deg(x_2) = \deg(y_2) = 3$ and $\langle G; B_{1+}^{++}, B_1^{+++}, {}^+B_{1+}^{+}, {}^+B_1^{++}\rangle$. The proof of Lemma 4 is complete. $\square$

Lemma 5. *For any $H \in \{B_2^{++}, B_{2+}^{+}, {}^+B_2^{+}\}$, the* EC *problem is polynomially solvable for the graphs in the class* $\mathrm{Free}_s(\{H\})$.

Proof. Let us show that the EC problem for the graphs in $\mathrm{Free}_s(\{H\})$ is polynomially reducible to the same problem in the class $\mathrm{Free}_s(\{H, T_{7,7.7}\})$. Hence it will follow from this and Lemma 2 that the desired assertion is true. It suffices to consider incompressible graphs in $\mathrm{Free}_s(\{H\})$ containing the subgraph $T_{7,7,7}$. Let $G = (V, E)$ be such a graph.

Suppose that among x_2, y_2, and z_2 there are at least two vertices of degree ≥ 3, say, x_2 and y_2. If it is not true that

$$(x_2v \in E \vee x_2y_1 \in E \vee x_2z_1 \in E) \wedge (y_2v \in E \vee y_2x_1 \in E \vee y_2z_1 \in E),$$

then $\langle G; B_2^{++}, B_{2+}^{+}, {}^+B_2^{+}\rangle$. If this condition is satisfied, then only six cases are generated due to symmetry,

$$\begin{gathered} x_2z_1 \in E,\ y_2z_1 \in E; \quad x_2v \in E,\ y_2x_1 \in E; \quad x_2v \in E,\ y_2z_1 \in E; \\ x_2y_1 \in E,\ y_2z_1 \in E; \quad x_2v \in E,\ y_2v \in E; \quad x_2y_1 \in E,\ y_2x_1 \in E. \end{gathered}$$

In the first four cases, $\langle G; B_2^{++}, B_{2+}^{+}, {}^+B_2^{+}\rangle$.

In the fifth case, $\langle G; B_2^{++}, {}^+B_2^{+}\rangle$. We can assume that $\deg(x_2) = \deg(y_2) = 3$. Since G is incompressible, we have $(\deg(x_1) = \Delta(G)) \vee (\deg(x_3) = \Delta(G))$, where $\Delta(G) \geq 5$. If $\deg(x_1) \geq 5$, then there exists a neighbor x_1 not belonging to $\{v, x_2, x_3, y_1\}$, and $\langle G; B_{2+}^{+}\rangle$. If $\deg(x_3) \geq 5$, then there exists a neighbor x_3 not belonging to $\{v, x_1, x_2, x_4\}$, and $\langle G; B_{2+}^{+}\rangle$.

In the sixth case, we can assume that $\deg(x_2) = \deg(y_2) = 3$. Then either $\deg(x_3) \geq 3$ or $\deg(x_4) \geq 3$ owing to the incompressibility of G; in each of these cases, we have $\langle G; B_2^{++}, B_{2+}^{+}, {}^+B_2^{+}\rangle$. Assume that among x_2, y_2 and z_2 there exists at most one vertex of degree ≥ 3. Then, owing to the incompressibility of G, we can assume that $\deg(x_2) = \deg(y_2) = 2$ and

$$\deg(x_1) = \deg(y_1) = \deg(x_3) = \deg(y_3) = \Delta(G).$$

We can assume that none of the vertices x_1 and y_1 is adjacent to any of the vertices $V(T_{7,7,7}) \setminus \{v, x_1, y_1, z_1, z_2, x_3, y_3, z_3, x_7, y_7, z_7\}$; otherwise, $\langle G; B_2^{++}$ and $B_{2+}^{+}, {}^+B_2^{+}\rangle$. If $x_1x_3 \in E$ or $y_1y_3 \in E$, then $\langle G; B_2^{++}, B_{2+}^{+}, {}^+B_2^{+}\rangle$. If $x_1y_3 \in E$ or $x_1z_3 \in E$, then $\langle G; B_2^{++}, {}^+B_2^{+}\rangle$. At the same time, $\langle G; B_{2+}^{+}\rangle$ in these cases; to prove this, it suffices to recall that $\deg(x_3) \geq 3$. Thus, we can assume that none of the vertices x_1, y_1 is adjacent to any of the vertices x_3, y_3, z_3.

If $x_1y_1 \in E$ or $x_1z_1 \in E$, then $\langle G; B_2^{++}, B_{2+}^{+}, {}^+B_2^{+}\rangle$, and if $vx_3 \in E$, then $\deg(x_3) \geq 4$. Further, we assume that $x_1y_1 \notin E$ and $x_1z_1 \notin E$. In a similar way, using the vertex y_3, one can show that $y_1z_1 \notin E$. In view of the incompressibility of G, we have

$$\exists\, x_1' \in N(x_1) \setminus \{v, x_2\}\ \exists\, y_1' \in N(y_1) \setminus \{v, y_2\}\colon\ \deg(x_1') = \deg(y_1') = \Delta(G).$$

Additionally, assume that $x'_1 \neq y'_1$. Then $\langle G; B_2^{++}, {}^{+}B_2^{+}\rangle$. If $x'_1 z_1 \in E$ or $y'_1 z_1 \in E$, then $\langle G; B_{2+}^{+}\rangle$. Moreover, let none of the vertices x'_1 and y'_1 be adjacent to z_1. If $x'_1 z_2 \in E$, then $\langle G; B_{2+}^{+}\rangle$. The case of $x'_1 y_2 \in E$ is impossible, because y_2 is adjacent only to y_1 and y_3, which cannot coincide with x'_1. If $x'_1 z_2 \notin E$, then $\langle G; B_{2+}^{+}\rangle$; to prove this, it suffices to use $N[x'_1] \cup \{v, y_1, y_2, z_1, z_2\}$ and also note that $\Delta(G) \geq 4$ if $x'_1 v \in E$.

Additionally, assume that $x'_1 = y'_1$. Recall that $\deg(x_3) = \Delta(G) \geq 3$. Then $\langle G; B_2^{++}, B_{2+}^{+}, {}^{+}B_2^{+}\rangle$, a fact that can readily be verified by separately considering three cases in which x_3 is adjacent to at least one of the vertices v and x'_1 and in which it is not adjacent to any of them. The proof of Lemma 5 is complete. $\square$

Lemma 6. *The* EC *problem is polynomially solvable for graphs in the class* $\mathrm{Free}_s(\{B_3^+\})$.

Proof. Let us show that the EC problem for graphs in $\mathrm{Free}_s(\{B_3^+\})$ is polynomially reducible to the same problem in the class $\mathrm{Free}_s(\{B_3^+, T_{7,7,7}\})$. It follows from this and Lemma 2 that the desired assertion holds. It suffices to consider incompressible graphs in $\mathrm{Free}_s(\{B_3^+\})$ containing the subgraph $T_{7,7,7}$. Let $G = (V, E)$ be such a graph.

Assume that among x_3, y_3, z_3 there are at least two vertices of degree ≥ 3, say, x_3 and y_3. If it is not true that

$$(x_3 v \in E \vee x_3 x_1 \in E \vee x_3 y_1 \in E \vee x_3 z_1 \in E) \wedge (y_3 v \in E \vee y_3 y_1 \in E \vee y_3 x_1 \in E \vee y_3 z_1 \in E),$$

then $\langle G; B_3^+\rangle$. If this condition is true, then exactly 10 pairwise nonequivalent cases are generated owing to symmetry. It is easily seen that one has $\langle G; B_3^+\rangle$ in all of these cases except for $x_3 v \in E$, $y_3 v \in E$; $x_3 x_1 \in E$, $y_3 x_1 \in E$; and $x_3 z_1 \in E$, $y_3 z_1 \in E$.

Consider the case in which $x_3 v \in E$ and $y_3 v \in E$. We can assume that $\deg(x_3) = \deg(y_3) = 3$; otherwise, $\langle G; B_3^+\rangle$. Since G is incompressible, we have $\deg(x_2) = \Delta(G) \vee \deg(x_4) = \Delta(G)$, where $\Delta(G) \geq 5$. If $\deg(x_2) \geq 5$, then there exists a neighbor of x_2 not belonging to $\{v, x_1, x_3, y_2, y_4\}$, and $\langle G; B_3^+\rangle$. If $\deg(x_4) \geq 5$, then there exists a neighbor of x_4 not belonging to $\{v, x_3, x_5, y_2, y_4\}$, and $\langle G; B_3^+\rangle$.

Consider the case in which $x_3 x_1 \in E$ and $y_3 x_1 \in E$. It is clear that

$$y_2 v \notin E, \quad y_2 y_4 \notin E, \quad y_4 x_2 \notin E, \quad y_4 x_4 \notin E;$$

otherwise, $\langle G; B_3^+\rangle$. We can assume that $\deg(x_3) = \deg(y_3) = 3$; otherwise, $\langle G; B_3^+\rangle$. Since G is incompressible, we have $(\deg(y_2) = \Delta(G)) \vee (\deg(y_4) = \Delta(G))$, where $\Delta(G) \geq 4$. If $\deg(y_2) = \Delta(G)$, then there exists a neighbor of y_2 not belonging to $\{y_1, y_3, x_1, x_2\}$, $x_4 y_2 \notin E$, and $\langle G; B_3^+\rangle$. If $\deg(y_4) = \Delta(G)$, then there exists a neighbor of y_4 not belonging to $\{y_3, y_5, x_1\}$, and $\langle G; B_3^+\rangle$.

Consider the case in which $x_3 z_1 \in E$ and $y_3 z_1 \in E$. It is clear that

$$\begin{gathered} y_2 v \notin E, \quad y_2 x_i \notin E,\ i = 1, \ldots, 5, \quad y_2 y_4 \notin E, \quad y_2 y_5 \notin E, \\ y_2 z_j,\ j = 2, \ldots, 5, \quad y_4 x_i \notin E,\ i = 1, \ldots, 5, \quad y_4 y_1 \notin E, \quad y_4 z_j,\ j = 2, \ldots, 5; \end{gathered}$$

otherwise, $\langle G; B_3^+\rangle$. Since G is incompressible, we have $(\deg(y_2) = \Delta(G)) \vee (\deg(y_4) = \Delta(G))$, where $\Delta(G) \geq 4$. If $\deg(y_2) = \Delta(G)$, then there exists a neighbor of y_2 not belonging to $V(T_{5,5,5})$, and $\langle G; B_3^+\rangle$. If $\deg(y_4) = \Delta(G)$, then there exists a neighbor of y_4 not belonging to $V(T_{5,5,5})$, and $\langle G; B_3^+\rangle$.

Assume that among x_3, y_3, and z_3 there is at most one vertex of degree ≥ 3. Then, owing to the incompressibility of G, we can assume that $\deg(x_3) = \deg(y_3) = 2$, $\deg(x'_2) = \deg(y'_2) = \Delta(G)$, and

$$\begin{gathered} \deg(x_2) = \deg(y_2) = \deg(x_4) = \deg(y_4) = \Delta(G), \\ N(x_2) \supseteq \{x'_2, x_1, x_3\}, \quad N(y_2) \supseteq \{y'_2, y_1, y_3\}. \end{gathered}$$

It is clear that none of the vertices x_2, y_2, and z_2 is adjacent to any vertex in

$$\{x_4, x_5, x_6, y_4, y_5, y_6, z_3, z_4, z_5, z_6\};$$

otherwise, $\langle G; B_3^+\rangle$. The same argument shows that $x_4 z_3 \notin E$ and $y_4 z_3 \notin E$.

Additionally, assume that $x_2y_2 \in E$. Then $x_4y_1 \in E$, $y_4x_1 \in E$, and $\Delta(G) = 3$; otherwise, $\langle G; B_3^+\rangle$, but then $\langle G; B_3^+\rangle$.

Additionally, assume that $x_2y_1 \in E$. Then $x_4y_1 \in E$, $x_4v \in E$, and $\Delta(G) = 4$; otherwise, $\langle G; B_3^+\rangle$. The vertex x_2 has a neighbor distinct from each of the vertices $y_1, v, x_1, x_3, x_4, x_5, x_6$; i.e., $\langle G; B_3^+\rangle$.

In addition, assume that $x_2v \in E$. It is clear that $x_4x_1 \notin E$; otherwise, $\langle G; B_3^+\rangle$. Then $x_4y_1 \in E, x_4z_1 \in E$; otherwise, $\langle G; B_3^+\rangle$. Thus, $\langle G; B_3^+\rangle$.

Thus, x_2 is not adjacent to any of the vertices in $V(T_{7,7,7})\setminus\{x_1, x_3, x_7, y_7, z_1, z_2, z_7\}$. By symmetry, we can assume that y_2 is not adjacent to any of the vertices in $V(T_{7,7,7}) \setminus \{y_1, y_3, x_7, y_7, z_1, z_2, z_7\}$. It is easily seen that each of the sets $N(x_2') \setminus \{v, x_2\}$ and $N(y_2') \setminus \{v, y_2\}$ contains at least two elements.

Additionally, assume that $x_2' \neq y_2'$. Consider the edges issuing from x_2'. In each of the cases $x_2'x_1 \in E$, $x_2'y_1 \in E$, and $x_2'z_1 \in E$, it readily turns out that $\langle G; B_3^+\rangle$. The same is true in the case where there are none of the specified edges in G. In this case, we use only the relation $\deg(x_2') = \Delta(G)$, but we do not use the relation $\deg(y_2') = \Delta(G)$.

Additionally, assume that $x_2' = y_2'$. Then $\Delta(G) = 3$, because for $\Delta(G) \geq 4$ one can take x_2' and y_2' so that $x_2' \neq y_2'$ and $\deg(x_2') = \Delta(G)$. Since the graph G is incompressible, it follows that $\deg(x_1) = \deg(y_1) = 3$. We can assume that $x_1x_2' \notin E$. Then $x_1y_1 \in E$; otherwise, $\langle G; B_3^+\rangle$. Since $\deg(x_4) = 3$, we have $\langle G; B_3^+\rangle$. The proof of Lemma 6 is complete. $\square$

Lemma 7. *The* EC *problem is polynomially solvable for graphs in the class* $\mathrm{Free}_s(\{B_4\})$.

Proof. Let us show that the EC problem for graphs in $\mathrm{Free}_s(\{B_4\})$ is polynomially reducible to the same problem in the class $\mathrm{Free}_s(\{B_4, T_{7,7,7}\})$. It follows from this and Lemma 2 that the assertion of the present lemma is true. It suffices to consider incompressible graphs in $\mathrm{Free}_s(\{B_4\})$ containing the subgraph $T_{7,7,7}$. Let $G = (V, E)$ be such a graph.

Assume that among x_4, y_4, and z_4 there exist at least two vertices of degree ≥ 3, say, x_4 and y_4. Then $\langle G; B_4\rangle$ unless

$$(x_4v \in E \vee x_4x_1 \in E \vee x_4x_2 \in E \vee x_4y_1 \in E \vee x_4z_1 \in E)$$
$$\wedge\, (y_4v \in E \vee y_4y_1 \in E \vee y_4y_2 \in E \vee y_4x_1 \in E \vee y_4z_1 \in E).$$

If the last condition holds, then exactly 15 pairwise nonequivalent cases are generated, and in all of them except for

$$y_4v \in E,\, x_4v \in E; \quad y_4y_1 \in E,\, x_4v \in E; \quad y_4x_1 \in E,\, x_4x_1 \in E;$$
$$y_4x_1 \in E,\, x_4x_2 \in E; \quad x_4z_1 \in E,\, y_4z_1 \in E,$$

one has $\langle G; B_4\rangle$.

Assume additionally that $y_4v \in E$ and $x_4v \in E$. Then either $\deg(x_4) = 3$ or $\deg(x_4) = 4$ and $x_4x_1 \in E$, otherwise $\langle G; B_4\rangle$. If $\deg(x_4) = 4$ and $x_4x_1 \in E$, then $\deg(x_1) < \Delta(G)$. Indeed, we have $\Delta(G) \geq 5$, and if $\deg(x_1) = \Delta(G)$, then $\langle G; B_4\rangle$, so that $(\deg(x_3) = \Delta(G)) \vee (\deg(x_5) = \Delta(G))$ owing to the incompressibility of G, regardless of the degree of the vertex x_4. Owing to symmetry, we have $(\deg(y_3) = \Delta(G)) \vee (\deg(y_5) = \Delta(G))$.

Assume additionally that $x_4z_1 \in E$ and $y_4z_1 \in E$. This case can be treated in exactly the same way as the preceding one.

Assume additionally that $y_4y_1 \in E$ and $x_4v \in E$. Then either $\deg(y_4) = 3$ or $\deg(y_4) = 4$ and $y_4v \in E$; otherwise, $\langle G; B_4\rangle$. By analogy with the reasoning in the first case, we can show that $(\deg(y_3) = \Delta(G)) \vee (\deg(y_5) = \Delta(G))$. Since $\langle G; B_4\rangle$ does not hold, it follows that either $\deg(x_4) = 3$ or $\deg(x_4) = 4$ and $x_4y_1 \in E$. If $\deg(x_4) = 3$, then $(\deg(x_3) = \Delta(G)) \vee (\deg(x_5) = \Delta(G))$. If $\deg(x_4) = 4$ and $x_4y_1 \in E$, then $\deg(y_4) = 3$, otherwise, $\langle G; B_4\rangle$. Then $\deg(y_3) = \Delta(G) \geq 4$ owing to the incompressibility of G; therefore, $\langle G; B_4\rangle$.

Assume additionally that $y_4x_1 \in E$ and $x_4x_1 \in E$. Then either $\deg(y_4) = 3$ or $\deg(y_4) = 4$ and $y_4v \in E$; otherwise, $\langle G; B_4\rangle$. If $\deg(y_4) = 4$ and $y_4v \in E$, then this variant has been analyzed in the third case. If $\deg(y_4) = 3$, then $(\deg(y_3) = \Delta(G)) \vee (\deg(y_5) = \Delta(G))$ owing to the incompressibility of G. Since $\langle G; B_4\rangle$ is not satisfied, we have either $\deg(x_4) = 3$ or $\deg(x_4) = 4$ and $x_4x_2 \in E$. If $\deg(x_4) = 4$ and $x_4x_2 \in E$, then $\langle G; B_4\rangle$, because $\deg(y_3) = \Delta(G)$. If $\deg(x_4) = 3$, then $(\deg(x_3) = \Delta(G)) \vee (\deg(x_5) = \Delta(G))$ owing to the incompressibility of G.

Assume additionally that $y_4x_1 \in E$ and $x_4x_2 \in E$. Then either $\deg(x_4) = 3$ or $\deg(x_4) = 4$ and $x_4x_1 \in E$; otherwise, $\langle G; B_4\rangle$. The case of $\deg(x_4) = 4$ and $x_4x_1 \in E$ has been analyzed in the previous case. If $\deg(x_4) = 3$, then $(\deg(x_3) = \Delta(G)) \vee (\deg(x_5) = \Delta(G))$ owing to the incompressibility of G. Since $\langle G; B_4\rangle$ is not satisfied, we have $\deg(y_4) = 3$. Since the graph G is incompressible, we have $(\deg(y_3) = \Delta(G)) \vee (\deg(y_5) = \Delta(G))$.

Thus, we obtain

$$\big(\deg(x_3) = \Delta(G) \vee \deg(x_5) = \Delta(G)\big) \wedge \big(\deg(y_3) = \Delta(G) \vee \deg(y_5) = \Delta(G)\big).$$

Then it is easily seen that the subgraph B_4 arises in each of the four possible cases.

Suppose that among x_4, y_4, and z_4 there is at most one vertex of degree ≥ 3. Then, by the incompressibility of G, we can assume that $\deg(x_4) = \deg(y_4) = 2$, $\deg(x_3') = \deg(y_3') = \Delta(G)$, and

$$\begin{gathered}\deg(x_3) = \deg(y_3) = \deg(x_5) = \deg(y_5) = \Delta(G),\\ N(x_3) \supseteq \{x_3', x_2, x_4\}, \quad N(y_3) \supseteq \{y_3', y_2, y_4\}.\end{gathered}$$

It is clear that x_3 is not adjacent to any of the vertices in the set

$$V(T_{7,7,7}) \setminus \{v, x_1, y_1, z_1, x_2, y_2, z_2, x_4, x_7, y_7, z_7\};$$

otherwise, $\langle G; B_4\rangle$. In a similar way, the vertex y_3 is not adjacent to any of the vertices in the set $V(T_{7,7,7}) \setminus \{v, x_1, y_1, z_1, x_2, y_2, z_2, y_4, x_7, y_7, z_7\}$.

Additionally, assume that $x_3y_2 \in E$. Then either $x_2y_5 \in E$ and $\Delta(G) = 3$ or $y_5x_2 \in E$, $y_5y_2 \in E$, and $\Delta(G) = 4$; otherwise, $\langle G; B_4\rangle$. In the first case, $x_5y_6 \in E$ is obligatory; otherwise, $\langle G; B_4\rangle$. Then $\langle G; B_4\rangle$. In the second case, $x_5x_2 \in E$ and $x_5y_6 \in E$, and so $\langle G; B_4\rangle$. Thus $x_3y_2 \notin E$. In a similar way, we can prove that $y_3x_2 \notin E$.

Additionally, assume that $x_3y_1 \in E$. Then $y_5v \notin E$; otherwise, $\langle G; B_4\rangle$. By the same argument, $y_5y_2 \in E$ and $\Delta(G) = 3$ or $y_5y_2 \in E$, $y_5y_1 \in E$, and $\Delta(G) = 4$. In both cases, $x_5x_1 \in E$ or $x_5z_1 \in E$; otherwise, $\langle G; B_4\rangle$, but then $\langle G; B_4\rangle$. In a similar way, one can prove that $y_3x_1 \notin E$.

Additionally, assume that $x_3v \in E$. There exists a neighbor of y_3 that is distinct from y_1 and v at the same time. Then $\langle G; B_4\rangle$. In what follows, we assume everywhere that $x_3v \notin E$ and $y_3v \notin E$.

Additionally, assume that $x_3' = y_3'$. It is clear that $x_3' \neq z_1$. Then $x_3x_1 \notin E$ and $y_3y_1 \notin E$; otherwise, $\langle G; B_4\rangle$. Each element of the set $N(x_3') \setminus \{x_3, y_3\}$ must belong to the set $\{v, x_1, y_1, z_1\}$; otherwise, $\langle G; B_4\rangle$. It is true that $\Delta(G) = 3$; otherwise, $\deg(x_3) = \deg(y_3) \geq 4$ and $\langle G; B_4\rangle$. Thus, $N(x_3') = \{x_3, y_3, t\}$, where $t \in \{x_1, y_1, z_1\}$ and $\langle G; B_4\rangle$; this can be verified by using one of the sets $N[x_5]$ or $N[y_5]$.

Additionally, assume that $x_3' \neq y_3'$. It is easy to see that each of the sets $N(x_3') \setminus \{v, x_3\}$ and $N(y_3') \setminus \{v, y_3\}$ contains at least two elements. If it is not true that

$$(x_3'x_1 \in E \vee x_3'x_2 \in E \vee x_3'y_1 \in E \vee x_3'z_1 \in E) \wedge (y_3'y_1 \in E \vee y_3'y_2 \in E \vee y_3'x_1 \in E \vee y_3'z_1 \in E),$$

then $\langle G; B_4\rangle$. If this condition is true, then exactly 9 nonequivalent cases are generated with $\langle G; B_4\rangle$ in all of them. The proof of Lemma 7 is complete. $\square$

5. POLYNOMIAL SOLVABILITY OF THE EC PROBLEM FOR SOME CLASSES OF GRAPHS OF MAXIMUM DEGREE NOT LESS THAN 4

Lemma 8. *For any $H \in \{B_1^* + P_2, {}^+B_1^*, B_1^{+*}\}$, the* EC *problem is polynomially solvable on the set*

$$\Big\{G \mid G \in \mathrm{Free}_s\big(\{H\}\big),\ \Delta(G) \geq 4\Big\}$$

of graphs.

Proof.

I. Let $H = B_1^* + P_2$. Lemma 8 in [6] proves that the EC problem is polynomially solvable on the set

$$\Big\{G \mid G \in \mathrm{Free}_s\big(\{B_1^*\}\big),\, \Delta(G) \geq 4\Big\}.$$

This, together with Lemma 3, implies the desired assertion for $H = B_1^* + P_2$.

II. Let $H = {}^+B_1^*$. It suffices to consider incompressible graphs in

$$\Big\{G \mid G \in \mathrm{Free}_s\big(\{H\}\big),\, \Delta(G) \geq 4\Big\}$$

containing the subgraph B_1^*. Let G be a graph in which there exists a subgraph B_1^*, where x, y, z are vertices of degree 3 of this subgraph B_1^*, $xy, yz \in E(B_1^*)$, and x', x'', y', z', z'' are the leaves of B_1^* adjacent to x, y, z, respectively. Since $\neg\langle G; {}^+B_1^*\rangle$, it follows that each neighbor of the vertices x', x'', z', z'' belongs to $V(B_1^*)$. The same argument shows that $N(u) \subseteq V(B_1^*)$, and so there are no paths (y, y_1, y_2) and (y, y_1', y_2') in which $\{y_1, y_2\} \cap \{y_1', y_2'\} = \emptyset$ and $y_1, y_2, y_1', y_2' \notin V({}^+B_1^*) \setminus \{y, y'\}$; otherwise y would be a hinge of the graph G.

Let us show that the graph $G \setminus V(B_1^*)$ is empty. Assume it contains an edge e. Since y is not a hinge of G, in G there exists a simple path

$$\big(v_1 \in \{x, z\}, v_2, \dots, v_k, v_{k+1}\big), \quad v_k v_{k+1} = e,$$

not passing through y. Up to renaming, we can assume that any such path passes through y'; otherwise, $\langle G; {}^+B_1^*\rangle$. If $v_i = y'$, $i \neq 2$, then $v_{i-1}, v_i, v_{i+1}, v_{i+2}$, together with x, x', x'', y, z, generate a supergraph of the graph ${}^+B_1^*$, and so $v_2 = y'$. Then $N(y) = \{v_2, x, z\}$ or $N(y) = \{v_2, v_3, x, z\}$; otherwise, $\langle G; {}^+B_1^*\rangle$. Thus, either v_2 or v_3 is a hinge of G.

We see that $G \setminus V(B_1^*)$ is empty. Obviously, the clique-width of any empty graph is equal to 1 and the clique-width of any graph does not exceed the number of its vertices. Then

$$cw(G) \leq cw\big(G \setminus V(B_1^*)\big) + |B_1^*| + 1 \leq 10.$$

This and Lemma 1 imply that the desired assertion holds for $H = {}^+B_1^*$.

III. Let $H = B_1^{+*}$. By Lemma 7, it suffices to consider incompressible graphs in

$$\Big\{G \mid G \in \mathrm{Free}_s\big(\{H\}\big),\, \Delta(G) \geq 4\Big\}$$

containing the subgraph B_4. Let $G = (V, E)$ be a graph that contains a subgraph B_4, where (x, y_1, y_2, y_3, z) is the central 4-path of this subgraph B_4 and x_1, x_2 and z_1, z_2 are the leaves of B_4 adjacent to x and z, respectively. It is clear that

$$y_1y_3 \notin E, \quad y_1z_1 \notin E, \quad y_1z_2 \notin E,$$
$$y_3x_1 \notin E, \quad y_3x_2 \notin E, \quad y_2x \notin E, \quad y_2z \notin E;$$

otherwise, $\langle G; B_1^{+*}\rangle$. In view of the incompressibility of the graph G, either y_2 has a neighbor y_2' of degree $\Delta(G)$ distinct from y_1 and y_3 or $\deg(y_1) = \deg(y_3) = \Delta(G)$.

Consider the first case. The situation with $y_2' \notin \{x_1, x_2, z_1, z_2\}$ is impossible, otherwise $\langle G; B_1^{+*}\rangle$, which can be verified by considering all possible values for $|N(y_2') \cap \{y_1, y_3\}| \in \{0, 1, 2\}$. If $y_2' \in \{x_1, x_2, z_1, z_2\}$, then, owing to symmetry, we can assume that $y_2' = x_1$. Then $x_1x_2 \notin E$, $x_1z_1 \notin E$, and $x_1z_2 \notin E$; otherwise, $\langle G; B_1^{+*}\rangle$. By the same reasoning, x_1 has no neighbor outside $V(B_4)$, and so $x_1y_1 \in E$, $x_1z \in E$, and $\Delta(G) = 4$. It is obvious that $\deg(y_3) = 2$; otherwise, $\langle G; B_1^{+*}\rangle$, but then $\deg(y_2) = 4$, because G is incompressible. Since $\neg\langle G; B_1^{+*}\rangle$, we see that y_2 is adjacent to at least one of the vertices z_1 and z_2, say, to z_1. By the same reasoning and since $\Delta(G) = 4$, we have $\deg(y_1) = 3$. Then $\deg(z_1) = 4$ owing to the incompressibility of G. Since $\neg\langle G; B_1^{+*}\rangle$, we have $z_1z_2 \notin E$ and $z_1x_2 \notin E$; i.e., z_1 has a neighbor outside $V(B_4)$, but then $\langle G; B_1^{+*}\rangle$.

Consider the second case. Assume that there exists a vertex $y_1' \notin V(B_4)$ adjacent to y_1. Then $y_2x_1 \notin E$, $y_2x_2 \notin E$, and $y_3x \notin E$; otherwise, $\langle G; B_1^{+*}\rangle$. Since $\deg(y_3) \geq 4$, it follows that $\langle G; B_1^{+*}\rangle$. Thus, $N(y_1) \setminus V(B_4) = N(y_3) \setminus V(B_4) = \emptyset$. Therefore, we can assume that $y_1x_1 \in E$ and $y_3z_1 \in E$. Then $y_1z \notin E$ and $y_3x \notin E$; otherwise, $\langle G; B_1^{+*}\rangle$. Thus, $y_1x_2 \in E$, $y_3z_2 \in E$, and $\Delta(G) = 4$. Since $\neg\langle G; B_1^{+*}\rangle$, we have $\deg(y_2) = 2$. By the same reasoning, none of the vertices x, x_1, x_2, z, z_1, z_2 has a neighbor outside $V(B_4)$. Thus, $V = V(B_4)$. The proof of the lemma is complete. $\square$

Lemma 9. *The* EC *problem is polynomially solvable on the set of graphs*

$$\Big\{G \mid G \in \mathrm{Free}_s\big(\{B_{1+}^*\}\big),\ \Delta(G) \geq 4\Big\}.$$

Proof. Let us show that the EC problem for the indicated class of graphs is polynomially reducible to the same problem in the class $\mathrm{Free}_s(\{B_{1+}^*,\ T_{7,7,7}\})$. It follows from this and Lemma 2 that the desired assertion is true. It suffices to consider incompressible graphs in

$$\Big\{G \mid G \in \mathrm{Free}_s\big(\{B_{1+}^*\}\big),\ \Delta(G) \geq 4\Big\}$$

containing the subgraph $T_{7,7,7}$. Let $G = (V, E)$ be such a graph. The vertex x_1 is adjacent to none of the vertices y_3–y_6 and z_3–z_6, the vertex y_1 is adjacent to none of the vertices x_3–x_6 and z_3–z_6, and the vertex z_1 is adjacent to none of the vertices x_3–x_6 and y_3–y_6; otherwise, $\langle G; B_{1+}^*\rangle$. In what follows, two important cases, which we denote by I and II, will be considered. The set $\{x_1, y_1, z_1\}$ will be denoted by V_1, the set $\{x_2, y_2, z_2\}$, by V_2, and the set $V(T_{7,7,7}) \setminus \{x_7, y_7, z_7\}$, by $\widetilde{V}$.

I. Assume that

$$\begin{aligned}\big((x_1y_2 \in E \vee x_1z_2 \in E) \wedge \deg(x_1) \geq 4\big)\\ \vee \big((y_1x_2 \in E \vee y_1z_2 \in E) \wedge \deg(y_1) \geq 4\big)\\ \vee \big((z_1x_2 \in E \vee z_1y_2 \in E) \wedge \deg(z_1) \geq 4\big).\end{aligned}$$

Without loss of generality, we can assume that $x_1y_2 \in E$ and $\deg(x_1) \geq 4$. Then

$$x_1x_i \notin E,\ i = 3, \dots, 6, \quad x_1z_1 \notin E, \quad x_1z_2 \notin E;$$

otherwise, $\langle G; B_{1+}^*\rangle$. By the same reasoning, either $\deg(v) = 3$, or $vy_2 \in E$ and

$$\deg(v) = \deg(x_1) = \deg(y_2) = 4,$$

or $x_1y_1 \in E$, $vx_2 \in E$, and

$$\deg(v) = \deg(x_1) = 4, \quad \deg(y_2) = 3.$$

I.a. Additionally, assume that $\deg(v) = 3$. Since $\langle G; B_{1+}^*\rangle$ is not satisfied, we have $\deg(y_2) = 3$ or

$$x_1y_1 \in E, \quad x_2y_2 \in E, \quad \deg(x_1) = \deg(y_2) = 4.$$

I.a.1. Let $\deg(y_2) = 3$. Since the graph G is incompressible, it follows that $\deg(y_1) \geq 4$. Since $\neg\langle G; B_{1+}^*\rangle$, we have $x_1y_1 \in E$. By the same reasoning (due to the incompressibility of G and the fact that $\neg\langle G; B_{1+}^*\rangle$), we conclude that $\deg(x_1) = \deg(y_1) = 4 = \Delta(G)$. Thus, $N(y_1) = \{v, x_1, y_2, y_1'\}$, $\deg(y_1') = 4$ owing to the incompressibility of G. Since $\neg\langle G; B_{1+}^*\rangle$, we have $y_1'x_2 \notin E$, and so $y_1' = x_2$; otherwise, $N(y_1') \setminus \{x_2, y_1, y_3\}$ contains at least two elements and $\langle G; B_{1+}^*\rangle$. Since G is incompressible, $\deg(x_2) = 4$, which implies that $\langle G; B_{1+}^*\rangle$.

I.a.2. Now let $x_1y_1 \in E$, $x_2y_2 \in E$, and $\deg(x_1) = \deg(y_2) = 4$. Since $\neg\langle G; B_{1+}^*\rangle$, we conclude that either $x_2y_1 \in E$, $\deg(x_2) = \deg(y_1) = 4$, or $\deg(x_2) = 3$. In the first case, by the same reasoning and in view of the incompressibility of G, we have $\deg(x_3) = \deg(y_3) = 2$, $\Delta(G) = 4$. An arbitrary edge 4-coloring c of the graph $G \setminus \{v, x_1, x_2, y_1, y_2\}$ extends to an edge 4-coloring of G. To this end, regardless of the colors of the edges x_3x_4 and y_3y_4 and the edges incident to z_1, we can set, up to a permutation of colors,

$$c(vx_1) = c(x_2x_3) = 1,$$
$$c(vy_1) = c(y_2y_3) = 2.$$

Let

$$c(x_1x_2) = 2, \qquad c(y_1y_2) = 1,$$
$$c(x_1y_1) = c(x_2y_2) = 3, \quad c(x_1y_2) = c(x_2y_1) = 4.$$

Then we obtain an edge 4-coloring of G, and so

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, x_2, y_1, y_2\}\big) \le 4.$$

In the second case, owing to the incompressibility of G and the fact that $\langle G; B^*_{1+}\rangle$, we have $\deg(y_1) = 4$, $y_1y_3 \in E$, and $\Delta(G) = 4$. Since $\langle G; B^*_{1+}\rangle$, we conclude that

$$\deg(y_3) = 3, \quad \deg(x_3) = \deg(y_4) = 2.$$

An arbitrary edge 4-coloring c of the graph $G \setminus \{v, x_1, x_2, y_1, y_2, y_3\}$ extends to an edge 4-coloring of G. To this end, regardless of the colors of the edges x_3x_4 and y_4y_5 and the edges incident to z_1, we can set, up to a permutation of colors,

$$c(vx_1) = 1, \quad c(x_2x_3) = c(vy_1) = c(y_3y_4) = 2.$$

Let

$$c(x_2y_2) = c(y_1y_3) = 1, \quad c(x_1y_2) = 2,$$
$$c(x_1x_2) = c(y_1y_2) = 3, \quad c(x_1y_1) = c(y_2y_3) = 4.$$

Then we obtain an edge 4-coloring of G; therefore,

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, x_2, y_1, y_2, y_3\}\big) \le 4.$$

I.b. In addition, assume that

$$vy_2 \in E, \quad \deg(v) = \deg(x_1) = \deg(y_2) = 4.$$

Since $\neg\langle G; B^*_{1+}\rangle$, we have $x_1y_1 \in E$. By the same reasoning, for each vertex $u \in \{x_2, y_3, z_1\}$ either $\deg(u) = 2$ or $uy_1 \in E$ and $\deg(y_1) \le 4$. If y_1 is adjacent to the vertex $u \notin V(T_{7,7,7})$, then $\deg(u) = 2$ owing to the incompressibility of G and the fact that $\langle G; B^*_{1+}\rangle$. It can readily be seen that

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, y_1, y_2\}\big) \le 4.$$

I.c. Additionally, assume that

$$x_1y_1 \in E, \quad vx_2 \in E, \quad \deg(v) = \deg(x_1) = 4, \quad \deg(y_2) = 3.$$

Since G is incompressible and $\neg\langle G; B^*_{1+}\rangle$, we have

$$\big(y_1x_2 \in E,\ \Delta(G) = 4\big) \vee \big(y_1z_1 \in E,\ \deg(y_1) = 4,\ \deg(z_1) = \deg(x_2) = 3,\ \Delta(G) = 4\big).$$

In the first case, we have $\deg(x_3) = \deg(y_3) = \deg(z_2) = 2$, because $\neg\langle G; B^*_{1+}\rangle$. It can readily be seen that

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, x_2, y_1, y_2\}\big) \le 4.$$

In the second case, we have $\deg(x_3) = \deg(y_3) = \deg(z_3) = 2$, because $\neg\langle G; B^*_{1+}\rangle$. It can readily be seen that

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, x_2, y_1, y_2, z_1\}\big) \le 4.$$

The analysis of case I is complete.

II. Assume that

$$\big((x_1y_1 \in E \vee x_1z_1 \in E) \wedge \deg(x_1) \geq 4\big) \\ \vee \big((y_1x_1 \in E \vee y_1z_1 \in E) \wedge \deg(y_1) \geq 4\big) \\ \vee \big((z_1x_1 \in E \vee z_1y_1 \in E) \wedge \deg(z_1) \geq 4\big).$$

Without loss of generality, we can assume that $x_1y_1 \in E$, $\deg(x_1) \geq 4$, and case I does not occur. Since $\neg\langle G; B^*_{1+}\rangle$, it follows that the vertex v is not adjacent to any of the vertices in $\widetilde{V} \setminus (V_1 \cup V_2)$.

If $z_1y_1 \in E$, then $x_1z_1 \in E$, because $\neg\langle G; B^*_{1+}\rangle$. If $z_1x_1 \in E$, then, owing to the incompressibility of G, at least one of the vertices v, y_1, z_1 has degree at least 4. If $\deg(y_1) \geq 4$ or $\deg(z_1) \geq 4$, then $y_1z_1 \in E$, because $\neg\langle G; B^*_{1+}\rangle$. If $\deg(y_1) = \deg(z_1) = 3$, then $vu \in E$ and $\deg(u) = \Delta(G)$. If $u \notin \widetilde{V}$, then u is adjacent to at most one element in V_2, and if $ux_1 \in E$, then $\Delta(G) \geq 5$. Then $N(u) \setminus \{v\}$ contains two elements not simultaneously belonging to $\widetilde{V}$, and $\langle G; B^*_{1+}\rangle$. This is true if $u \in V_2$, because $\deg(u) \geq 4$.

Assume that $z_1x_1 \in E$ and $z_1y_1 \in E$. Since $\neg\langle G; B^*_{1+}\rangle$, we have $\Delta(G) = 4$. Since $\neg\langle G; B^*_{1+}\rangle$, for each vertex $u \in \{x_2, y_2, z_2\}$ we have either $\deg(u) = 2$ or $uv \in E$ and $\deg(u) = 3$. If $uv \in E$, $u \notin \widetilde{V}$, then $\deg(u) = 2$, because G is incompressible and $\neg\langle G; B^*_{1+}\rangle$. It is easily seen that

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus (V_1 \cup \{v\})\big) \leq 4.$$

Further, assume that $z_1x_1 \notin E$ and $z_1y_1 \notin E$. Since $\neg\langle G; B^*_{1+}\rangle$, this yields $vy_2 \notin E$ and $vz_2 \notin E$.

By N we denote the set of vertices that do not belong to $V_1 \cup V_2 \cup \{v\}$ and are adjacent to at least one of the vertices v, y_1, and z_1. Since $\neg\langle G; B^*_{1+}\rangle$, each element of N is adjacent to x_1. By the same reasoning, N contains at most one element.

In view of the incompressibility of the graph G and the fact that $\neg\langle G; B^*_{1+}\rangle$, the equality $N = \emptyset$ is possible only if

$$\big(\deg(y_1) = \deg(z_1) = 3,\ \deg(v) = \deg(x_1) = 4, vx_2 \in E,\ z_1x_2 \in E,\ x_1x_3 \in E\big) \\ \vee \big(\deg(y_1) = 3,\ \deg(z_1) = 2, \deg(v) = \deg(x_1) = \deg(x_2) = 4,\ vx_2 \in E,\ x_2y_2 \in E,\ x_1x_3 \in E\big).$$

In the first case, since $\neg\langle G; B^*_{1+}\rangle$ and $\deg(x_3) = 3$, we have $\deg(x_4) = 2$, and so G is not incompressible. In the second case, since $\neg\langle G; B^*_{1+}\rangle$, we have $\deg(y_3) = 3$ and $\deg(y_4) = 2$, and so G is not incompressible.

Assume additionally that $N = \{u'\}$. Since $\neg\langle G; B^*_{1+}\rangle$, we have

$$\deg(z_1) \leq 3, \\ \deg(y_1) \leq 4.$$

Since v contains at least two vertices of degree $\Delta(G)$ (because G is incompressible), by the same reasoning we have

$$\big(\deg(v) = \deg(x_1) = 5,\ vx_2 \in E,\ \exists\, u'' \notin \widetilde{V} : u''v \in E,\ u''x_1 \in E\big) \\ \vee \big(\deg(v) \leq 4,\ \deg(x_1) = 4 = \Delta(G),\ vx_2 \notin E\big).$$

The first case is impossible. Indeed, $\deg(x_1) = 4$, because $\neg\langle G; B^*_{1+}\rangle$, and so $\deg(x_2) = 5$ owing to the incompressibility of G. Then $\langle G; B^*_{1+}\rangle$.

Assume that $\deg(v) \leq 4$, $\deg(x_1) = 4 = \Delta(G)$, and $vx_2 \notin E$. Since G is incompressible, it follows that each of the vertices v, y_1, y_2, z_1 is adjacent to at least two vertices of degree 4, and so $vu' \in E$, $y_1u' \in E$, and $x_1u' \in E$, because $\neg\langle G; B^*_{1+}\rangle$. By the same reasoning, for each vertex $u \in \{x_2, y_2, z_1\}$ either $\deg(u) = 2$ or $uu' \in E$ and $\deg(u) = 3$, and if u is adjacent to $w \notin V_1 \cup \{v, x_2, y_2\}$, then $\deg(w) \leq 1$. It is easily seen that

$$\chi'(G) = 4 \Leftrightarrow \chi'\Big(G \setminus \big(V_1 \cup \{v, u'\}\big)\Big) \leq 4.$$

Consider the case in which $vu' \in E$ and $y_1u' \notin E$. Then $\deg(y_1) = 3$, because $\neg\langle G; B^*_{1+}\rangle$. Since G is incompressible, we have $\deg(u') = 4$. Then $\langle G; B^*_{1+}\rangle$ in all cases except for $u'z_1 \in E$ and $u'x_2 \in E$. If $u'z_1 \in E$ and $u'x_2 \in E$, then $\deg(x_2) = 3$ and $\deg(x_3) = 2$, because $\neg\langle G; B^*_{1+}\rangle$, and so G is not incompressible.

Consider the case in which $vu' \notin E$ and $y_1u' \in E$. Then $\deg(v) = 3$. Since G is incompressible, we have $\deg(u') = 4$. Then $\langle G; B^*_{1+}\rangle$ in all cases except for $u'x_2 \in E$ and $u'y_2 \in E$. Since $\neg\langle G; B^*_{1+}\rangle$, we have $\deg(x_2) = 3$ and $\deg(x_3) = 2$, and therefore, G is not incompressible.

The analysis of case II is complete.

In what follows, we assume that options I and II are not realized. Since the graph G is incompressible, it follows that the set $N(v)$ contains at least two vertices v_1 and v_2 of degree $\Delta(G)$. It is clear that $|\{v_1, v_2\} \cap V_1| \le 1$; otherwise, $\langle G; B^*_{1+}\rangle$.

Assume that there exists a vertex $v_i \notin \widetilde{V}$, say, v_1, adjacent to at least one of the vertices of the set V_1, say, x_1. Then $v_2 \notin \{y_1, z_1\}$; otherwise, $\langle G; B^*_{1+}\rangle$.

First, consider the case in which $v_1x_2 \in E$ and $v_1x_3 \in E$. It is easily seen that $\langle G; B^*_{1+}\rangle$ if $(v_2 \ne x_1) \vee (\Delta(G) \ge 5)$. If $v_2 = x_1$ and $\Delta(G) = 4$, then $x_1x_3 \in E$ and either $\deg(x_2) = 3$ or $x_2x_4 \in E$. In the first case,

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v_1, x_1, x_2\}\big) \le 4.$$

In the second case, since $\neg\langle G; B^*_{1+}\rangle$, we have either $\deg(y_1) = \deg(z_1) = 2$ or $\deg(y_1) = \deg(z_1) = 3$ and $x_1y_1 \in E$. The second case is impossible, because then $\deg(y_2) = 4$ owing to the incompressibility of G, and so $\langle G; B^*_{1+}\rangle$. Using the incompressibility of G and the fact that $\neg\langle G; B^*_{1+}\rangle$, one can readily verify that

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v_1, x_1, x_2, x_3\}\big) \le 4$$

except for the case in which there is a copy of the subgraph $G[\{v, v_1, x_1, x_2, x_3, x_4\}]$ intersecting with it at the vertex x_4. Let us shrink these two subgraphs to a vertex and denote the resulting graph by G^*. It is easily seen that $G^* \in \mathrm{Free}_s(\{B^*_{1+}\})$ and $\chi'(G) = 4 \Leftrightarrow \chi'(G^*) = 4$.

Assume that at least one of the edges vx_2 and vx_3 does not belong to E. Since $\neg\langle G; B^*_{1+}\rangle$, if $\Delta(G) \ge 5$, then v_1 is adjacent to every vertex in V_1, and $v_2 \notin \widetilde{V}$. Then v_2 has no neighbor in V_1; otherwise, $\langle G; B^*_{1+}\rangle$. Recalling that $\deg(v_2) \ge 5$, we conclude that $\langle G; B^*_{1+}\rangle$.

Additionally, assume that $\Delta(G) = 4$. Since $\neg\langle G; B^*_{1+}\rangle$, we conclude that $v_1y_1 \notin E, v_1z_1 \notin E$, and v_1 has a neighbor in $\{x_2, x_3\}$, so $v_2 = x_1$. Using the incompressibility of G and the fact that $\langle G; B^*_{1+}\rangle$, one can readily see that

$$\big(v_1x_2 \in E,\, \exists\, v', v'' \notin \widetilde{V}\colon v_1v' \in E,\, v_2v'' \in E\big) \vee \big(v_1x_2 \in E,\, \exists\, v' \notin \widetilde{V}\colon v_1v' \in E,\, v_2v' \in E\big).$$

In the first case, by virtue of $\neg\langle G; B^*_{1+}\rangle$ we have

$$\deg(y_1) = \deg(z_1) = 2, \quad \deg(x_2) = 3, \quad \max\big(\deg(v'), \deg(v'')\big) \le 1;$$

therefore,

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v_1, x_1, x_2\}\big) \le 4.$$

Consider the second case. Since $\neg\langle G; B^*_{1+}\rangle$, we have $N(v') \subseteq \{v_1, v_2, x_2, x_3\}$. By the same reasoning,

$$\big(\deg(v') = 2 \Rightarrow \deg(x_2) = 3\big) \wedge \big(v'x_2 \in E,\, v'x_3 \notin E \Rightarrow \deg(v') = 3\big)$$
$$\wedge \big(v'x_2 \notin E,\, v'x_3 \in E \Rightarrow \deg(x_2) = \deg(v') = 3\big).$$

In view of the incompressibility of G and the fact that $\langle G; B^*_{1+}\rangle$, we obtain $\deg(y_1) = \deg(z_1) = 2$. One can readily verify that

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v_1, v_2, v'', x_2\}\big) \le 4,$$

where $v'' = v'$ or $v'' = x_3$. Thus, we can assume that if $v_i \notin \widetilde{V}$, then v_i is not adjacent to any of the elements in V_1.

Since $\neg\langle G; B^*_{1+}\rangle$, we have $|N(v_i) \cap V_2| \leq 1$ if $v_i \notin \widetilde{V}$, and if $v_i \in \widetilde{V}$, then v_i is not adjacent to any vertex of branches $T_{7,7,7}$ other than leaf ones. Therefore, among $(x_1, x_2), (y_1, y_2), (z_1, z_2)$ there exists a pair such that none of its elements is adjacent to v_1 or v_2. If $(N(v_1) \cup N(v_2)) \setminus \{v, v_1, v_2\}$ contains at least four vertices, then there exist two elements $v'_1, v'_2 \in N(v_1) \setminus \{v, v_2\}$ such that $|N(v_2) \setminus \{v, v_1, v'_1, v'_2\}| \geq 2$. Hence $\langle G; B^*_{1+}\rangle$ holds for $\Delta(G) \geq 5$ except for the case in which

$$\Delta(G) = 5, \quad v_1v_2 \in E, \quad N(v_1) \setminus \{v_2\} = N(v_2) \setminus \{v_1\}.$$

In this case, since $\langle G; B^*_{1+}\rangle$, no vertex in $N(v_1) \setminus \{v, v_2\}$ has a neighbor outside $N(v_1)$. Then v is a hinge of the graph G.

Assume that $\Delta(G) = 4$. Then either exactly one of the vertices v_1, v_2 does not belong to $\widetilde{V}$ and the other one belongs to V_1, or they both belong to $\widetilde{V}$. In the first case, we can assume that $v_1 \notin \widetilde{V}$, $v_2 = x_1$, $v_1x_1 \notin E$, and $N(v_1) \setminus \{v\} = N(v_2) \setminus \{v\}$. Then $N(v_2) \cap \widetilde{V} = \{v, x_2\}$; otherwise, $\langle G; B^*_{1+}\rangle$. By the same reasoning, the set $N(v_2) \setminus \{v\}$ is independent. Since G is incompressible, it follows that either $\deg(x_2) = 4$ or the degree of at least one of the vertices in $N(v_2) \setminus \widetilde{V}$ is equal to 4, but then $\langle G; B^*_{1+}\rangle$.

Consider the second case. Since $\neg\langle G; B^*_{1+}\rangle$, it follows that the vertices v_1, v_2 simultaneously belong to exactly one of the sets $\{x_i\}_{i=1}^6$, $\{y_i\}_{i=1}^6$, and $\{z_i\}_{i=1}^6$, say, the first of them. Then we can assume that $v_1 = x_1$, because $\Delta(G) = 4$. Let us show that $\deg(y_1) = \deg(z_1) = 2$. Consider only the vertex y_1. Since $\neg\langle G; B^*_{1+}\rangle$, we have $\deg(y_1) \leq 3$, $y_1z_2 \notin E$, and $y_1x_2 \notin E$, and so $\deg(y_1) = 3$ only if $y_1z_1 \in E$. Then $\deg(z_1) = 3$. Consequently, $\deg(y_2) = 4$ owing to the incompressibility of G. Then $\langle G; B^*_{1+}\rangle$.

Since $\neg\langle G; B^*_{1+}\rangle$, we have $x_1x_5 \notin E$ and $x_1x_6 \notin E$. By the same reasoning, if $v_1v_2 \notin E$, then $N(v_1)\setminus\{v\} = N(v_2)\setminus\{v\}$. Then $v_2 \notin \{x_4, x_5, x_6\}$, and so $v_2 = x_3$, $\exists\, v' \notin \widetilde{V}\colon v'x_1 \in E$, $v'x_3 \in E$. Since $\neg\langle G; B^*_{1+}\rangle$, we have

$$\big(\deg(v') = 3,\, v'x_5 \in E,\, x_1x_4 \in E,\, \deg(x_4) = 3\big) \vee \big(\deg(v') = 2,\, \deg(x_4) = 3\big).$$

Since G is incompressible, it follows that $\deg(x_2) = 4$, but then $\langle G; B^*_{1+}\rangle$.

In addition, assume that $v_1v_2 \in E$. Then $v_2 = x_i$, where $i \in \{2, 3, 4\}$, and v_1 and v_2 have a common neighbor besides v. Since $\neg\langle G; B^*_{1+}\rangle$, we have $i \neq 4$, and $x_1x_4 \in E$ for $i = 3$. If $i = 3$, then

$$\begin{gathered} N(x_2) \subseteq \{x_1, x_3, x_4, x_5\}, \\ N(x_4) \subseteq \{x_1, x_2, x_3, x_5\}, \\ x_2x_5 \in E \Rightarrow \deg(x_5) = 3, \end{gathered}$$

because $\neg\langle G; B^*_{1+}\rangle$. It can readily be seen that

$$\begin{aligned} &\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, x_2, x_3, x_4\}\big) \leq 4 \qquad &&\text{if } x_2x_5 \notin E, \\ &\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, x_2, x_3, x_4, x_5\}\big) \leq 4 &&\text{otherwise.} \end{aligned}$$

Let $i = 2$, and let v' be an arbitrary common neighbor of the vertices v_1 and v_2 distinct from v. It is clear that $v' \notin \widetilde{V} \setminus \{x_3, x_4\}$. If $v' \notin \widetilde{V}$, then either $x_1x_3 \in E$ or $N(x_1) = \{v, v', v^*, x_2\}$, where $v^* \notin \widetilde{V}$. In the first case,

$$\begin{gathered} N(v') \subseteq \{x_1, x_2, x_3, x_4\}, \\ N(x_3) \subseteq \{x_1, x_2, v', x_4\}, \end{gathered}$$

because $\neg\langle G; B^*_{1+}\rangle$. If at least one of the edges $v'x_3$ and $v'x_4$ does not belong to E, then

$$\chi'(G) = 4 \Leftrightarrow \chi'\big(G \setminus \{v, v', x_1, x_2, x_3\}\big) \leq 4.$$

If $v'x_3 \in E$ and $v'x_4 \in E$, then, owing to the incompressibility of G and the fact that $\neg\langle G; B^*_{1+}\rangle$, one can readily verify that

$$\chi'(G) = 4 \Leftrightarrow \chi'\bigl(G \setminus \{v', x_1, x_2, x_3\}\bigr) \le 4$$

except for the case in which there exists a copy of the subgraph $G[\{v, v', x_1, x_2, x_3, x_4\}]$ meeting it at the vertex x_4. Let us shrink these two subgraphs to a vertex and denote the resulting graph by G^{**}. It is easily seen that $G^{**} \in \mathrm{Free}_s(\{B^*_{1+}\})$ and

$$\chi'(G) = 4 \Leftrightarrow \chi'(G^{**}) = 4.$$

In the second case, we have $\langle G; B^*_{1+}\rangle$. Further, we assume that no common neighbor of v_1 and v_2 belongs to $\widetilde{V}$.

If $v' = x_4$, then $x_1x_3 \in E$; otherwise, $\langle G; B^*_{1+}\rangle$. By the same reasoning, we have either $\deg(x_3) = 3$ or $x_3x_5 \in E$. It is easily seen that in the first case we have

$$\chi'(G) = 4 \Leftrightarrow \chi'\bigl(G \setminus \{v, x_1, x_2, x_3, x_4\}\bigr) \le 4.$$

In the second case, owing to the incompressibility of G and the fact that $\neg\langle G; B^*_{1+}\rangle$, one can readily verify that

$$\chi'(G) = 4 \Leftrightarrow \chi'\bigl(G \setminus \{x_1, x_2, x_3, x_4\}\bigr) \le 4$$

except for the case in which there exists a copy of the subgraph $G[\{v, x_1, x_2, x_3, x_4, x_5\}]$ meeting it at the vertex x_5. Let us shrink these two subgraphs to a vertex and denote the resulting graph by G^{***}. One can readily see that $G^{***} \in \mathrm{Free}_s(\{B^*_{1+}\})$ and

$$\chi'(G) = 4 \Leftrightarrow \chi'(G^{***}) = 4.$$

If $v' = x_3$, then either $x_1x_4 \in E$ or there exists a vertex $v'_1 \notin \widetilde{V}$ adjacent to v_1. In the first case, $v_2x_4 \in E$ is obligatory, as otherwise $\langle G; B^*_{1+}\rangle$ and we pass to the previous case $v' = x_4$ and obtain the same two subcases as earlier. In the second case, $N(v'_1) \subseteq \{x_1, x_4\}$ is satisfied owing to the incompressibility of G and the fact that $\neg\langle G; B^*_{1+}\rangle$. Then $\langle G; B^*_{1+}\rangle$, which is easy to verify by recalling that $\deg(x_2) = 4$. The proof of Lemma 9 is complete. $\square$

6. THE MAIN RESULT

The following assertion is the main result of the present paper.

Theorem 1. *Let F be an arbitrary 8-edge forest not belonging to the set*

$$\bigl\{B^*_1 + P_2 + O_n \mid n \ge 0\bigr\} \cup \bigl\{{}^+B^*_1 + O_n \mid n \ge 0\bigr\} \cup \bigl\{B^{+*}_1 + O_n \mid n \ge 0\bigr\} \cup \bigl\{B^*_{1+} + O_n \mid n \ge 0\bigr\}.$$

Then the EC *problem is polynomially solvable in the class* $\mathrm{Free}_s(\{F\})$. *If the forest F belongs to this set, then the* EC *problem is polynomially solvable in the class* $\{G \in \mathrm{Free}_s(\{F\}) \mid \Delta(G) \ge 4\}$.

Proof. If $F \in \mathcal{T}$, then the EC problem is polynomially solvable in the class $\mathrm{Free}_s(\{F\})$ by Lemmas 1 and 2. If $F \notin \mathcal{T}$, then $F = F' + O_n$ for some $F' \in \mathcal{S}$ and n, where the set $\mathcal{S}$ has been defined in Sec. 1. Then the assertion of the theorem follows from Lemmas 3–9. $\square$

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