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Two-dimensional attractors of A-flows and fibred links on three-manifolds

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Abstract

Let f^t be a flow satisfying Smale's Axiom A (in short, A-flow) on a closed orientable three-manifold M^3, and Ω a two-dimensional basic set of f^t. First, we prove that Ω is either an expanding attractor or contracting repeller. Next, one considers an A-flow f^t with a two-dimensional non-mixing attractor Λ_a. We construct a casing $M(\Lambda_a)$ of Λ_a that is a special compactification of the basin of Λ_a by a collection of circles $L(\Lambda_a) = \{l_1, \ldots, l_k\}$ such that $M(\Lambda_a)$ is a closed three-manifold and $L(\Lambda_a)$ is a fibre link in $M(\Lambda_a)$. In addition, f^t is extended on $M(\Lambda_a)$ to a nonsingular structurally stable flow with the non-wandering set consisting of the attractor Λ_a and the repelling periodic trajectories $l_1, \ldots, l_k$. We show that if a closed orientable three-manifold M^3 has a fibred link $L = \{l_1, \ldots, l_k\}$ then M^3 admits an A-flow f^t with the non-wandering set containing a two-dimensional non-mixing attractor and the repelling isolated periodic trajectories $l_1, \ldots, l_k$. This allows us to prove that any closed orientable n-manifold, $n \geqslant 3$, admits an A-flow with a two-dimensional attractor. We prove that the pair $(M(\Lambda_a); L(\Lambda_a))$ consisting of the casing $M(\Lambda_a)$ and the corresponding fibre link $L(\Lambda_a)$ is an invariant of conjugacy of the restriction $f^t|_{W^s(\Lambda_a)}$ of the flow f^t on the basin of the attractor Λ_a.

Keywords: attractor, A-flow, fibred link
Mathematics Subject Classification numbers: 37D20, 37C70, 37C15, 58C30.

Introduction

Dynamical systems satisfying Axiom A (in short, A-systems) were introduced by Smale [43]. By definition, the non-wandering set of A-system is the topological closure of periodic orbits and is endowed with a hyperbolic structure (see basic notation of dynamical systems

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in the books [4, 24, 39] and surveys [16, 43]). According Robinson [38] and Mañé [25], the set of A-systems contains all structurally stable systems including Anosov systems and Morse–Smale systems. Later on, we consider A-flows that are A-systems with continuous time.

Due to Smale's spectral decomposition theorem, the non-wandering set of any A-system is a disjoint union of closed, invariant, and topologically transitive sets called *basic sets* [37, 39, 43]. A basic set is called *trivial* if it is either an isolated singularity or an isolated periodic trajectory. Otherwise, a basic set is *nontrivial*. Any nontrivial basic set of A-flow has the topological dimension no less than one, and a supporting manifold admitting a nontrivial basic set has the dimension no less than three. Zeeman [47] proved that any n-manifold, $n \geqslant 3$, supporting nonsingular flows supports an A-flow with a one-dimensional nontrivial basic set. Bowen [7] gave a complete characterization of nontrivial one-dimensional basic sets showing that up to homeomorphism they are suspensions of basic subshifts of finite type. Pugh and Shub [36] considered the problem of realization for subshifts of finite type as a one-dimensional basic set in a three-sphere. Franks [11] proved that suspended subshifts of finite type can be realized as a one-dimensional basic set of a structurally stable nonsingular flow on some three-manifold. Note that a nontrivial one-dimensional basic set is of saddle type while one-dimensional attractors and repellers on a three-manifold are always trivial.

It is natural to consider the existence and properties of two-dimensional (automatically non-trivial) basic sets beginning with closed three-manifolds M^3. First, we prove that a two-dimensional basic set on M^3 is either an attractor or repeller, see lemma 1. Moreover, two-dimensional attractors and repellers are exactly expanding attractors and contracting repellers respectively introduced by Williams [46], see lemma 2. The first example of A-flow that is a nontransitive Anosov flow was constructed by Franks and Williams [12]. This nontransitive Anosov flow has the spectral decomposition consisting of one two-dimensional expanding attractor and one two-dimensional contracting repeller. The both basic sets are non-mixing. Recall that a flow g^t on a topological manifold M is called *mixing* if given any open subsets U, $V \subset M$ there is $t_0 \in \mathbb{R}$ such that $U \cap g_t(V) \neq \emptyset$ for all $t \geqslant t_0$ where $g_t(V)$ is the t-time shift of V along the trajectories of the flow g^t. Following Bowen [8], we say that a basic set Ω of an A-flow f^t is *mixing* if the restriction $f^t|_\Omega$ is a mixing flow.

The simplest example of non-mixing two-dimensional attractor on a closed three-manifold can be constructed as follows. Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism of two-torus $\mathbb{T}^2$, and $f : \mathbb{T}^2 \to \mathbb{T}^2$ a DA-diffeomorphism constructed by Smale's surgery of the diffeomorphism A [39, 43]. We know that the non-wandering set of f consists of an orientable one-dimensional expanding attractor Λ_1 and isolated source. Then the dynamical suspension $\mathrm{sus}^t(f)$ over f is the A-flow such that the non-wandering set of $\mathrm{sus}^t(f)$ consists of the two-dimensional orientable non-mixing expanding attractor arising from Λ_1 and isolated repelling periodic trajectory arising from the source.

For definiteness, we will consider two-dimensional basic sets that are attractors. Our main attention concerns to an embedding of two-dimensional non-mixing attractors and its basins (stable manifolds) in supporting manifolds M^3. We prove that A-flows with two-dimensional attractors exist on every closed orientable n-manifold, $n \geqslant 3$. We also study the conjugacy problem of restrictions on basins for two-dimensional non-mixing attractors. Let us formulate the main results.

Let f^t be an A-flow on a closed orientable three-manifold M^3 and Λ_a a two-dimensional non-mixing attractor of f^t. The stable manifold (in short, basin) $W^s(\Lambda_a)$ of Λ_a is an open subset of M^3 consisting of the trajectories whose ω-limit sets belong to Λ_a. First, we construct a casing of Λ_a that is a special compactification of $W^s(\Lambda_a)$ by a collection of circles that form a fibred link (see section 1 for a precise definition).

Theorem 1. *Let f^t be an A-flow on an orientable closed three-manifold M^3 such that the non-wandering set $\mathrm{NW}(f^t)$ contains a two-dimensional non-mixing attractor Λ_a. Then there is a compactification $M(\Lambda_a)$ of the basin $W^s(\Lambda_a)$ by the family of circles $l_1, \ldots, l_k$ such that*

- *$M(\Lambda_a)$ is a closed orientable three-manifold;*
- *The restriction $f^t|_{W^s(\Lambda_a)}$ is extended continuously to the structurally stable nonsingular flow $\tilde{f}^t$ on $M(\Lambda_a)$ with the non-wandering set $NW(\tilde{f}^t) = \Lambda_a \cup_{i=1}^k l_i$ where $l_1, \ldots, l_k$ are the repelling isolated periodic trajectories of $\tilde{f}^t$;*
- *The family $L = \{l_1, \ldots, l_k\} \subset M(\Lambda_a)$ is a fibred link in $M(\Lambda_a)$.*

We see that the flow $\tilde{f}^t$ on $M(\Lambda_a)$ is a structurally stable nonsingular flow of attractor–repeller type with the nontrivial attractor Λ_a and the trivial global repeller $\cup_{i=1}^k l_i$.

Remark. It follows from the proof of theorem 1 that there is a compactification $W(\Lambda_a)$ which is a fibred (closed) three-manifold.

Let us illustrate this result. Consider a DA-diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ obtained from the Anosov diffeomorphism $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ mod $1 : \mathbb{T}^2 \to \mathbb{T}^2$ by Smale's surgery near the fixed point $(0;\ 0)$. The suspension $\mathrm{sus}(f)$ is the A-flow on the mapping torus $M_f = \mathbb{T}^2 \times [0;1]/(x;1) \sim (f(x);0)$. The non-wandering set of $\mathrm{sus}(f)$ consists of a periodic trajectory l_0 and two-dimensional attractor Λ_a. Note that $M_f = l_0 \cup W^s(\Lambda_a)$, and l_0 is not a fibred knot in M_f. But there is a compactification $M(\Lambda_a) = l \cup W^s(\Lambda_a)$ of $W^s(\Lambda_a)$ by a closed curve l such that $M(\Lambda_a) = \mathbb{S}^3$ and l is the figure eight-knot [10].

The manifold $M(\Lambda_a)$ satisfying theorem 1 is called a *casing* of the attractor Λ_a. We denote the corresponding fibred link $L = \{l_1, \ldots, l_k\}$ by $L(\Lambda_a)$. The second result of the paper, in a sense, is reversal to the first one.

Theorem 2. *Let $\{l_1, \ldots, l_k\} \subset M^3$ be a fibred link in a closed orientable three-manifold M^3. Then there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $\mathrm{NW}(f^t)$ contains a two-dimensional non-mixing attractor and the repelling isolated periodic trajectories $l_1, \ldots, l_k$.*

Applying Alexander's statement on fibred links [1], one gets the following result.

Corollary 1. *Given any closed orientable three-manifold M^3, there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $\mathrm{NW}(f^t)$ contains a two-dimensional attractor.*

This corollary was proved by another method in [27]. This assertion for the dimension $n = 3$ implies the following statement for $n \geqslant 4$.

Theorem 3. *Given any closed orientable n-manifold M^n, $n \geqslant 4$, there is an A-flow f^t on M^n such that the non-wandering set $\mathrm{NW}(f^t)$ contains a two-dimensional attractor.*

Note that a casing of two-dimensional attractor is not unique. Denote by $\mathbb{F}(\Lambda_a)$ the set of casings of Λ_a satisfying the conditions of theorem 1.

Theorem 4. *Let Λ_a be a two-dimensional non-mixing attractor of an A-flow f^t on orientable closed three-manifold M^3 and $M(\Lambda_a) = W^s(\Lambda_a) \cup_{i=1}^k l_i \in \mathbb{F}(\Lambda_a)$ a casing of Λ_a with the fibred link $L = \{l_1, \ldots, l_k\}$. Then any other casing from $\mathbb{F}(\Lambda_a)$ can be obtained by a surgery along the link $L = \{l_1, \ldots, l_k\}$.*

Note that there are infinitely many casings $M(\Lambda_a)$ with $M(\Lambda_a)\setminus L(\Lambda_a)$ endowed with a hyperbolic structure [44].

Fibred links and casings of attractors allow to consider classification problems. Solving classification problems for flows, one considers mainly two relations, a conjugacy and a topological equivalence. Recall that two flows f_1^t, f_2^t on a manifold M are *conjugate* provided there exists a homeomorphism $\psi : M \to M$ such that the mappings f_1^t, f_2^t are conjugate for any $t \in \mathbb{R}$. Here, f_i^t means the t-time shift along the trajectories of the flow f_i^t, $i = 1, 2$. One says that flows f_1^t, f_2^t are (topologically) *equivalent* provided there exists a homeomorphism $\psi : M \to M$ taking the trajectories of one flow to the trajectories of the other flow. A non-mixing property depends on speeds along the trajectories of a flow, and this property can be changed by a deformation of speeds. Therefore, it is natural to consider the conjugacy studying the classification problem in the frame of the class of non-mixing attractors.

Let $(M(\Lambda_a); L(\Lambda_a))$ be a pair consisting of the casing $M(\Lambda_a)$ of the attractor Λ_a and the corresponding fibred link $L(\Lambda_a)$. We will say that the pairs $(M(\Lambda_1); L(\Lambda_1))$, $(M(\Lambda_2); L(\Lambda_2))$ are *homeomorphic* if there is a homeomorphism $M(\Lambda_1) \to M(\Lambda_2)$ taking the fibred link $L(\Lambda_1)$ to the fibred link $L(\Lambda_2)$. The following result says that a pair is the invariant of conjugacy for the restriction of A-flow on the basin of two-dimensional non-mixing attractor.

Theorem 5. *Let f_i^t be an A-flow on an orientable closed three-manifold M_i^3 such that the non-wandering set $NW(f_i^t)$ contains a two-dimensional non-mixing attractor Λ_i, $i = 1, 2$. If the restrictions $f_1^t|_{W^s(\Lambda_1)}$, $f_2^t|_{W^s(\Lambda_2)}$ are conjugate, then there are homeomorphic pairs $(M(\Lambda_1); L(\Lambda_1))$, $(M(\Lambda_2); L(\Lambda_2))$.*

Let us mention some results concerning the subject. Christly [10] constructed a so-called tidy swaddled graph for a two-dimensional hyperbolic attractor. In [10], using tidy swaddled graphs, it was obtained the classification under the equivalence relation of restrictions $f^t|_{\Lambda_a}$ of A-flows f^t on attractors Λ_a. Note that the classification of restrictions on attractors does not imply the classification of restrictions on basins of attractors. Robinson and Williams [40] constructed two diffeomorphisms f and g with attractors Λ_f and Λ_g respectively such that $f : \Lambda_f \to \Lambda_f$ is conjugate to $g : \Lambda_g \to \Lambda_g$ but there is not even a homeomorphism from a neighbourhood of Λ_f to a neighbourhood of Λ_g taking Λ_f to Λ_g. Passing to dynamical suspensions, one can get a similar example for flows. Another examples see in [48]. Morales [28] studied a transitivity of A-flows with a transverse torus on three-manifolds. In [5], one got the classification of codimension one non-mixing orientable attractors on closed n-manifolds for $n \geqslant 4$. Béguin *et al* [6] constructed nontransitive Anosov flows with two-dimensional attractors with prescribed entrance foliation (in particular, with some incoherent attractors). Recall that Margulis [26] proved that the fundamental group $\pi_1(M^3)$ of a supporting manifold M^3 for Anosov flows has an exponential growth. Due to corollary 1, we see that the realm of A-flows is much richer than the realm of Anosov flows with two-dimensional attractors.

1. Basic definitions

Hyperbolic invariant sets. Let f^t be a smooth flow on a closed n-manifold M^n, $n \geqslant 3$. A subset $\Lambda \subset M^n = M$ is *invariant* provided Λ consists of trajectories of f^t. An invariant nonsingular set $\Lambda \subset M$ is called *hyperbolic* if the sub-bundle $T_\Lambda M$ of the tangent bundle TM can be represented as a Df^t-invariant continuous splitting $E^{ss}_\Lambda \oplus E^t_\Lambda \oplus E^{uu}_\Lambda$ such that

(a) $\dim E^{ss}_\Lambda + \dim E^t_\Lambda + \dim E^{uu}_\Lambda = n$;
(b) E^t_Λ is the line bundle tangent to the trajectories of the flow f^t;
(c) There are $C_s > 0$, $C_u > 0$, $0 < \lambda < 1$ such that

$$\|df^t(v)\| \leqslant C_s\lambda^t\|v\|, \quad v \in E^{ss}_\Lambda; \quad \|df^{-t}(v)\| \leqslant C_u\lambda^t\|v\|, v \in E^{uu}_\Lambda, \quad t > 0.$$

A singular point x is hyperbolic if x is an isolated hyperbolic equilibrium state. The topological structure of flow near x is described by Grobman–Hartman theorem, see for example [39]. In this case $E^t_x = 0$ and $\dim E^{ss}_\Lambda + \dim E^{uu}_\Lambda = n$.

If Λ does not contain fixed points, then the bundles

$$E^{uu}_\Lambda \oplus E^t_\Lambda = E^u_\Lambda, \quad E^{ss}_\Lambda \oplus E^t_\Lambda = E^s_\Lambda, \quad E^{uu}_\Lambda, \quad E^{ss}_\Lambda,$$

are uniquely integrable [22, 43]. The corresponding leaves $W^u(x)$, $W^s(x)$, $W^{uu}(x)$, $W^{ss}(x)$ through a point $x \in \Lambda$ are called *unstable*, *stable*, *strongly unstable*, and *strongly stable manifolds* respectively.

A-flows. Given a set $U \subset M^n$, denote by $f_{t_0}(U)$ the shift of U along the trajectories of f^t on the time t_0. Recall that a point x is non-wandering if given any neighbourhood U of x and a number $T_0 > 0$, there is $t_0 \geqslant T_0$ such that $U \cap f_{t_0}(U) \neq \emptyset$. The *non-wandering set* $\mathrm{NW}(f^t)$ of f^t is the union of all non-wandering points.

Denote by $\mathrm{Fix}(f^t)$ the set of fixed points of f^t. Following Smale [43], we will call f^t an *A-flow* provided its non-wandering set $\mathrm{NW}(f^t)$ is hyperbolic and the periodic trajectories are dense in $\mathrm{NW}(f^t)\setminus\mathrm{Fix}(f^t)$. According to Smale's spectral decomposition theorem [37, 43], the non-wandering set of A-flow is a disjoint union of closed, and invariant, and transitive sets called *basic sets*.

Expanding attractors. Following Williams [46], we will call a basic set Ω an *expanding attractor* provided Ω is an attractor and its topological dimension equals the dimension of unstable manifold $W^u(x)$ for every point $x \in \Omega$. A basic set Λ is called *orientable* provided the both fibre bundles E^{ss}_Λ and E^{uu}_Λ are orientable. Note that if E^{ss}_Λ and E^{uu}_Λ are one-dimensional, then the orientability of Λ means that the both E^{ss}_Λ and E^{uu}_Λ can be embedded in vector fields on M^n.

For expanding attractors of A-diffeomorphisms, there is a deep theory developed in the papers [13, 15, 18, 35], see also the book [19]. This theory is based on the paper [29]. Similar theory based on the paper [45] holds for A-flows. We shortly recall some results of this theory.

Let Λ_a be an expanding attractor of A-flow on a closed three-manifold M^3. For a point $x \in \Lambda_a$, let us denote by $W^{+ss}(x)$ and $W^{-ss}(x)$ the components of $W^{ss}\setminus\{x\}$. A point $p \in \Lambda_a$ is called *boundary* if $W^{+ss}(p) \cap \Lambda_a = \emptyset$ or $W^{-ss}(p) \cap \Lambda_a = \emptyset$. Due to a hyperbolicity, if p is a boundary point then the trajectory $O(p)$ consists of boundary points, and $O(p)$ is said to be a *boundary trajectory*. An unstable manifold $W^u(\cdot) \subset \Lambda_a$ is called *boundary* if $W^u(\cdot)$ contains a boundary trajectory. If p is a boundary point, then there is a unique component of $W^{ss}(p)\setminus\{p\}$ denoted by $W^{ss}_\emptyset(p)$ such that $W^{ss}_\emptyset(p) \cap \Lambda_a = \emptyset$. Set $W^s_\emptyset(O(p)) = \cup_{x\in O(p)} W^{ss}_\emptyset(x)$.

Denote by $(x, y)^{ss}$ (respectively, $[x, y]^{ss}$) the open (respectively, closed) arc of $W^{ss}(z)$ with the endpoints $x, y \in W^{ss}(z)$ where $z \in \Lambda_a$. The following results hold:

- Expanding attractor Λ_a contains finitely many boundary trajectories;
- Any boundary trajectory is periodic;
- Given any boundary trajectory $O(p) \subset \Lambda_a$ and a point $x \in W^u(p)\setminus O(p)$, there is a unique arc $(x, y)^{ss} \overset{\text{def}}{=} (x, y)^{ss}_\emptyset$ such that $(x, y)^{ss} \cap \Lambda_a = \emptyset$ where $y \in \Lambda_a$, $y \overset{\text{def}}{=} \mathrm{opp}(x)$;
- $\mathrm{opp}(x)$ belongs to $W^u(q)\setminus O(q)$ where $O(q)$ is a some boundary trajectory (maybe, $p = q$).

The set $B(\Lambda_a)$ of the boundary unstable manifolds splits into disjoint bunches such that all unstable manifolds of a bunch can be consequently connected by arcs $[x, y]^{ss}_\emptyset$ called connecting arcs.

A^*-homeomorphisms. Let $f : X \to X$ be a homeomorphism of a compact metric space X endowed with a metric d. The homeomorphism f is an *A^*-homeomorphism* if the following conditions hold:

(a1) The periodic points of f are dense in X;

(a2) Given any $\delta > 0$, there is $\varepsilon(\delta) = \varepsilon > 0$ such that $W^s_\delta(x) \cap W^u_\delta(z) \neq \emptyset$ whenever $d(x, z) \leqslant \varepsilon$;

(a3) There are $\gamma > 0$, $0 < \lambda < 1$ and $c \geqslant 1$ such that for all $n \geqslant 0$ one holds $d(f^n(x), f^n(y)) \leqslant c\lambda^n d(x, y)$ if $y \in W^s_\gamma(x)$ and $d\left(f^{-n}(x), f^{-n}(y)\right) \leqslant c\lambda^n d(x, y)$ if $y \in W^u_\gamma(x)$.

Here, $W^s_\alpha(x) = \{y \in X \ : \ d(f^n(x), f^n(y)) \leqslant \alpha \text{ for all } n \geqslant 0\}$ and $W^u_\alpha(x) = \{y \in X \ : \ d(f^n(x), f^n(y)) \leqslant \alpha \text{ for all } n \leqslant 0\}$.

Fibred links. Recall that a *link* in a three-manifold M^3 is a collection of disjoint embedded circles $L = \{l_1, \dots, l_k\} \subset M^3$. The link $L = \{l_1, \dots, l_k\}$ is *fibred* if $M^3 \backslash \left(\cup_{i=1}^k l_i\right)$ is the total space of fibre bundle $p : \left(M^3 \backslash L\right) \to S^1$ and the boundary of the fibres $p^{-1}(\cdot)$ is L. In addition, the fibres $p^{-1}(\cdot)$ meet L nicely. To be precise, consider the solid torus $\mathbb{P}_0 = \mathbb{S}^1 \times \mathbb{D}^2$ called a *canonical solid torus*. Here, $\mathbb{S}^1$ is a circle endowed with the cyclic coordinate ϑ, and $\mathbb{D}^2$ an unit disk $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. Set $S^1 = \partial\mathbb{D}^2$. The mapping $p_0(\vartheta, z) = \frac{z}{|z|}$, $\vartheta \in \mathbb{S}^1$, $z \in \mathbb{D}^2 \backslash \{0\}$, is the fibre bundle

$$p_0 : \mathbb{S}^1 \times \left(\mathbb{D}^2 \backslash \{0\}\right) \to S^1,$$

over S^1 with the fibre an annulus denoted by A_0. There is a tubular neighbourhood $T(l_i)$ of l_i homeomorphic to $\mathbb{P}_0$ (so, we can assume $T(l_i) = \mathbb{P}_0$) such that $T(l_i) \backslash \{l_i\} = \mathbb{P}_0 \backslash \left(\mathbb{S}^1 \times \{0\}\right)$. By definition, $p|_{T(l_i)\backslash\{l_i\}}$ is isomorphic to p_0, $i = 1, \dots, k$.

Rational foliations on two-torus. Let $\mathcal{F}$ be a foliation without singularities on two-torus $\mathbb{T}^2$. The foliation $\mathcal{F}$ is called *rational* if every leaf of $\mathcal{F}$ is an embedded circle. Obviously, all leaves define the same nontrivial element of the fundamental group $\pi_1(\mathbb{T}^2)$. Now, suppose that $\mathbb{T}^2$ is the boundary of the canonical solid torus $\mathbb{P}_0$. In this case, any leaf of $\mathcal{F}$ is called a *meridian* provided the leaf is a homotopy trivial curve in $\mathbb{P}_0$. Clearly, a meridian is the boundary of some embedded two-disk in $\mathbb{P}_0$. A simple closed curve $l \subset \mathbb{T}^2$ is called a *parallel* if l transversally intersects some meridian at a unique point. Obviously, a parallel defines a nontrivial element of $\pi_1(\mathbb{T}^2)$.

Note that the fibres $p_0^{-1}(\cdot)$ of the fibre bundle p_0 form a foliation denoted by $\mathcal{F}$. The leaves of $\mathcal{F}$ are annuluses that are transversal to boundary $\mathbb{T}^2 = \partial\mathbb{P}_0$. The intersections of this leaves with $\mathbb{T}^2$ produce the rational foliation denoted by F_0. Thus, F_0 is a foliation generated by parallels on the torus $\mathbb{T}^2$.

2. Previous results

For a codimension one basic set Ω of A-diffeomorphism, it was was proved independently by Plykin [34] and Williams [46] that Ω is either attractor or repeller. This result holds even for endomorphisms [17]. For A-flows, the proof is similar. So, we give only the scheme of the proof in the particular case when the dimension of basic set equals two.

Lemma 1. *Let Λ be a two-dimensional basic set of A-flow f^t on a closed three-manifold. Then Λ is either attractor or repeller. Moreover, Λ is an attractor if and only if $W^u(x) \subset \Lambda$ for all points $x \in \Lambda$. Similarly, Λ is a repeller if and only if $W^s(x) \subset \Lambda$ for all points $x \in \Lambda$.*

Scheme of the proof. Take $x \in \Lambda$ and put by definition,

$$\widehat{W}^{uu}(x) = W^{uu}(x) \cap \Lambda, \widehat{W}^{uu}_{\epsilon}(x) = W^{uu}_{\epsilon}(x) \cap \Lambda, \widehat{W}^{ss}(x)$$
$$= W^{ss}(x) \cap \Lambda, \widehat{W}^{ss}_{\epsilon}(x) = W^{ss}_{\epsilon}(x) \cap \Lambda.$$

A relative neighbourhood $\widehat{V}$ of $x \in \Lambda$ in Λ is homeomorphic to the product $\widehat{W}^{uu}_{\epsilon}(x) \times \widehat{W}^{ss}_{\epsilon}(x) \times \mathbb{R}$. Since $\dim \Lambda = 2$, $\dim\, \widehat{W}^{uu}(x) + \dim\, \widehat{W}^{ss}(x) = 1$. It follows from the existence of hyperbolic structure that $\dim\,(W^{uu}(x) \times W^{ss}(x) \times \mathbb{R}) = 3$. Hence,

$$2 = \dim\, W^{uu}(x) + \dim\, W^{ss}(x) \geqslant \dim\, \widehat{W}^{uu}(x) + \dim\, \widehat{W}^{ss}(x) = 1.$$

Therefore, either $\dim\, W^{uu}(x) = \dim\, \widehat{W}^{uu}(x)$ or $\dim\, W^{ss}(x) = \dim\, \widehat{W}^{ss}(x)$. Suppose for definiteness, $\dim\, W^{uu}(x) = \dim\, \widehat{W}^{uu}(x)$. Note that $\dim\, W^{uu}(x) = \dim\, \widehat{W}^{uu}(x) = 1$, otherwise Λ is a trivial basic set. Due to [23], the equality $\dim\, \widehat{W}^{uu}(x) = 1$ implies the existence of an interior point, say y_0, in $\widehat{W}^{uu}(x)$. Hence, there is $\delta > 0$ such that $W^{uu}_{\delta}(y_0) \subset \Lambda$. As a consequence, $W^{uu}(y_0) \subset \Lambda$ and $W^{u}(y_0) \subset \Lambda$. Due to a local product structure on a basic set, $W^{u}(x) \subset \Lambda$ for all points $x \in \Lambda$. Hence, Λ is an attractor. Similarly, if $\dim\, W^{ss}(x) = \dim\, \widehat{W}^{ss}(x)$ then Λ is a repeller. □

Lemma 2. *Let Λ_a be a nontrivial attractor of A-flow f^t on a closed three-manifold. Then Λ_a is expanding if and only if Λ_a is two-dimensional.*

Proof. Suppose Λ_a is a two-dimensional attractor. By lemma 1, $W^u(x) \subset \Lambda_a$ for any $x \in \Lambda_a$. Since Λ_a is nontrivial, $\dim W^{uu}(x) \geqslant 1$. It follows that $\dim W^u(x) = 2$ for any $x \in \Lambda_a$. Hence, Λ_a is expanding.

Now suppose that Λ_a is an expanding attractor. Then $\dim \Lambda_a = \dim W^u(x)$ for any $x \in \Lambda_a$. If we assume that $\dim W^u(x) = 1$ then $\dim W^{uu}(x) = 0$. This implies that Λ_a is trivial since $\dim W^s(x) = 3$. Hence, $\dim W^u(x) = 2$. This follows that Λ_a is two-dimensional. □

For references, we formulate the key statement proved by Bowen [8]. The topological closure of set N will be denoted by $\mathrm{clos}(N)$.

Proposition 1. *Let f^t be an A-flow and Ω a nontrivial basic set of f^t. Then Ω is non-mixing if and only if the restriction $f^t|_{\Omega}$ is a dynamical τ-time suspension over some A^*-homeomorphism $\varphi^*_x : clos(W^{uu}(x)) \to clos(W^{uu}(x))$ for some $\tau > 0$ and any $x \in \Omega$. Moreover, if* $\mathrm{clos}(W^{uu}(x)) \cap \mathrm{clos}(W^{uu}(y)) \neq \emptyset$ *for $x, y \in \Omega$ then* $\mathrm{clos}(W^{uu}(x)) = \mathrm{clos}(W^{uu}(y))$.

Note that the property of Anosov flow to be mixing is closely related with the notion of C-density introduced by Anosov [3] and Bowen [8]. Anosov [2] (with some assumptions on measure) and Plante [32] (without assuming on measure) proved that a transitive Anosov flow is C-dense if and only if the flow is mixing. Thus, a transitive Anosov flow is not C-dense if and only if the flow is non-mixing.

The crucial technical statement for the proof of theorem 1 is the following assertion.

Lemma 3. *Let f^t be an A-flow on an orientable closed three-manifold M^3 such that the non-wandering set* $\mathrm{NW}(f^t)$ *contains a two-dimensional non-mixing attractor Λ_a. Then there is a neighbourhood $U(\Lambda_a)$ of Λ_a such that*

- *$U(\Lambda_a) \subset W^s(\Lambda_a)$ is an attracting domain of Λ_a;*
- *The boundary $\partial U(\Lambda_a)$ is transversal to f^t and consists of finitely many components $T^2_1, \ldots, T^2_k$ where each T^2_i is homeomorphic to the two-torus $\mathbb{T}^2$;*
- *The flow f^t in $U(\Lambda_a)$ has a global section.*

Proof. Recall that due to Bowen's brilliant result [8] (see proposition 1), the restriction $f^t|_{\Lambda_a}$ of f^t on Λ_a is a dynamical τ-time suspension over some A^*-homeomorphism $\varphi_* : \Pi_0 \to \Pi_0$ where Π_0 is the topological closure of $W^{uu}(x_0)$, $x_0 \in \Lambda_a$. Thus, $\varphi_* = f_\tau|_{\Pi_0}$ is the τ-time shift along the trajectories of the flow f^t, and $f_\tau^m(\Pi_0) = \varphi_*^m(\Pi_0) = \Pi_0$ for any $m \in \mathbb{Z}$. Moreover, $\Lambda_a = \cup_{0 \leqslant t < \tau} f_t(\Pi_0)$ and $f_{t_1}(\Pi_0) \cap f_{t_2}(\Pi_0) = \emptyset$ provided $t_1 \neq t_2, t_1, t_2 \in [0;\ \tau)$. In addition, the t-time shift $f_t|_{\Pi_0} : \Pi_0 \to f_t(\Pi_0)$ is a homeomorphism for any $t \in [0;\ \tau)$. Taking a circle S^1 as $[0;\ \tau]/0 \simeq \tau$ one gets the fibre bundle

$$p_a : \Lambda_a \to S^1 = [0;\tau]/0 \simeq \tau \text{ where } p_a(x) = t \text{ provided } x \in f_t(\Pi_0), \tag{1}$$

with the fibre Π_0. Hence, there is a minimal period for periodic trajectories of Λ_a. Choose the point $x_0 \in \Lambda_a$ belonging to a periodic trajectory $l(x_0)$ with the minimal period $\tau_0 = k\tau > 0, k \in \mathbb{N}$.

Let us show that $W^{ss}(x_0) \cap \Lambda_a \subset \Pi_0$. Suppose the contrary. Then there is a point $y_0 \in W^{ss}(x_0) \cap \Lambda_a$ such that $y_0 \notin \Pi_0$. First, we consider the case $y_0 \in W^u(x_0)$. We see that $W^u(x_0) = \cup_{z \in l(x_0)} W^{uu}(z) \ni y_0$. Then there exists a unique point $y_1 \in l(x_0) \cap W^{uu}(y_0)$. It follows from $y_1 \in l(x_0)$ that $f_{\tau_0}(y_1) = y_1$. Denote by Π_1 the topological closure of $W^{uu}(y_1)$. According to proposition 1, $f_{\tau_0}^m(\Pi_1) = \Pi_1$, $m \in \mathbb{Z}$. We have to prove that $\Pi_1 = \Pi_0$. Since $y_0 \in W^{ss}(x_0)$, $f_{\tau_0}^m(y_0) \to f_{\tau_0}^m(x_0) = x_0$ as $m \to \infty$. At the same time, $f_{\tau_0}^m(y_0) \in f_{\tau_0}^m(\Pi_1) = \Pi_1$ because of $W^{uu}(y_0) = W^{uu}(y_1)$. Therefore, Π_0 is intersected with Π_1. According to proposition 1, $\Pi_1 = \Pi_0$. Hence, $y_0 \in \Pi_0$. This contradiction concludes the proof in the case $y_0 \in W^u(x_0)$. Now, consider the case $y_0 \notin W^u(x_0)$. Since $\Lambda_a = \cup_{t \geqslant 0} f_t(\Pi_0)$, there is $t_0 \geqslant 0$ such that $f_{-t_0}(y_0) \in \Pi_0$. It follows from $\Pi_0 = \mathrm{clos}(W^{uu}(x_0))$ that the point $f_{-t_0}(y_0)$ is approximated by the point of $W^{uu}(x_0)$. The continuous dependence of unstable manifolds implies the existence of the sequence $y_k \in W^u(x_0)$ such that $y_k \to y_0$ as $k \to \infty$. The first case above implies that $y_k \in \Pi_0$. Since the set Π_0 is closed, $y_0 \in \Pi_0$.

Thus, $W^{ss}(x_0) \cap \Lambda_a \subset \Pi_0$. The continuous dependence of strongly stable manifolds implies that $W^{ss}(x) \cap \Lambda_a \subset \Pi_0$ for every point $x \in W^{uu}(x_0)$, and hence, for any $x \in \Pi_0$. By construction, $W^{uu}(x_0)$ is dense in Π_0. It follows from lemma 1 that $W^{uu}(x_0) \subset \Pi_0 \subset \Lambda_a$. Again, the continuous dependence of strongly stable and unstable manifolds implies that for all $x, y \in \Pi_0$ one holds $W^{uu}(x) \cap W^{ss}(y) \subset \Pi_0$.

Set $S_0 = \bigcup_{x \in \Pi_0} W^{ss}(x)$. Note that Π_0 endowed with the hyperbolic structure induced from Λ_a. It follows from the continuous dependence of strongly stable manifolds that S_0 is a topological (noncompact) surface. It follows from proposition 1 that S_0 is a global section for the flow $f^t|_{W^s(\Lambda_a)}$. Moreover, since $f_\tau(\Pi_0) = \Pi_0$ and strongly (unstable and stable) manifolds are invariant under t-time shifts, $f_\tau(S_0) = S_0$. This means that $\varphi = f_\tau|_{S_0} : S_0 \to S_0$ is a dynamical suspension which is a continuation of φ_*. Due to [21], $W^s(\Lambda_a) = \cup_{x \in \Lambda_a} W^{ss}(x)$. Since $\Lambda_a = \cup_{0 \leqslant t < \tau} f_t(\Pi_0)$ and $f_{t_1}(\Pi_0) \cap f_{t_2}(\Pi_0) = \emptyset$ provided $t_1 \neq t_2$, $t_1, t_2 \in [0;\ \tau)$, we see that $W^s(\Lambda_a) = \cup_{0 \leqslant t < \tau} f_t(S_0)$ and $f_{t_1}(S_0) \cap f_{t_2}(S_0) = \emptyset$ provided $t_1 \neq t_2$, $t_1, t_2 \in [0;\ \tau)$. It follows from (1) that there is the fibre bundle

$$P_W : W^s(\Lambda_a) \to S^1 = [0;\tau]/0 \simeq \tau \text{ where } P_W(x) = t \text{ provided } x \in f_t(S_0), \tag{2}$$

with the fibre S_0. The fibres of the bundle (2) form the foliation denoted by F_W.

Since Λ_a is a two-dimensional basic set of f^t and $\Lambda_a = \cup_{0 \leqslant t \leqslant \tau} f_t(\Pi_0)$, $\Lambda_a \cap S_0 = \Pi_0$ and Π_0 is a one-dimensional closed transitive invariant set of $\varphi = f_\tau|_{S_0} : S_0 \to S_0$. Since Λ_a is an expanding attractor, Π_0 is an attracting set consisting of unstable manifolds of points $x \in \Pi_0$ under φ. The local product structure on Λ_a induces the local product structure on Π_0 under φ. This allows us to construct the special bunches of Π_0 similarly bunches of one-dimensional expanding attractors of surface A-diffeomorphisms. To be precise, let $\mathcal{B}$ be a bunch of Λ_a consisting of the boundary unstable manifolds $W^u(p_1), \ldots, W^u(p_r)$ where $O(p_i)$ is a boundary periodic trajectory, $i = 1, \ldots, r$. Let $[x_1, y_1]^{ss}, \ldots, [x_{r+1}, y_{r+1}]^{ss}$ be the connecting arcs of $\mathcal{B}$. Here, we suppose that $x_{r+1} = x_1$, $y_{r+1} = y_1$ provided $r = 1$. Due to the above construction, one can assume that the union $[x_1, y_1]^{ss} \cup [y_1, x_2]^{uu} \cup \ldots \cup [y_r, x_{r+1}]^{uu} \cup [x_{r+1}, y_{r+1}]^{ss}$ is a closed simple curve denoted by $c(\mathcal{B})$ belongs to S_0 where $p_{i+1} \in [y_i, x_{i+1}]^{uu}$, $i = 0, \ldots, r-1$.

We cover $c(\mathcal{B})$ by segments of strongly stable manifolds through $[y_1, x_2]^{uu} \cup \ldots \cup [y_r, x_{r+1}]^{uu}$ to get an annulus A with the middle circle $c(\mathcal{B})$. Since $c(\mathcal{B}) \subset S_0$, $A \subset S_0$. Note that according proposition 1, the trajectories $O(p_i)$, $i = 1, \ldots, r$, have the same period denoted by $T(\mathcal{B}) > 0$. The set $C_{\mathcal{B}} = \bigcup_{0 \leqslant t \leqslant T(\mathcal{B})} f_t\,(c(\mathcal{B}))$ is a cylinder which intersects A through finitely many circles (simple closed curves). Since $c(\mathcal{B})$ belongs to S_0, $C_{\mathcal{B}} \backslash \{c(\mathcal{B})\}$ intersects the leaves of F_W through the circles each isotopic to $c(\mathcal{B})$ on the cylinder $C_{\mathcal{B}}$. Therefore, these circles form the rational foliation consisting of nontrivial loops.

Let us move $c(\mathcal{B})$ to a closed simple curve $\tilde{c} \subset A$ such that $\tilde{c}$ has no intersection with Λ_a and $\tilde{c}$ cuts transversally $W^{ss}_{\emptyset}(p_i)$, $i = 1, \ldots, r$, in the annulus A. In addition, one can assume that $\tilde{c} \subset S_0$. Slightly deforming $\tilde{c} \subset A$, one gets the cylinder $\tilde{C} = \bigcup_{0 \leqslant t \leqslant T(\mathcal{B})} f_t\,(\tilde{c}))$ such that $\tilde{C}$ is transversal to $f^{\,t}$ and the intersection $A \cap \tilde{C}$ consists of finitely many disjoint simple curves. A standard procedure allows to construct a closed surface $T^2(\mathcal{B})$ for the bunch $\mathcal{B}$ (this procedure is similar to the construction of closed transversal for a flow starting with a closed curve that consists of an arc of trajectory and a transversal segment [4]). Moreover, it is possible to make the deformation of $\tilde{C}$ along the fibre of the bundle (2) so that the intersections $T^2(\mathcal{B}) \cap P_W^{-1}(x)$, $x \in S^1 = [0;\ \tau]/0 \simeq \tau$, produce the rational foliation denoted by $\widetilde{F}_W$. Due to the Euler–Poincaré formula (see for example [33], ch 3), the Euler characteristic of $T^2(\mathcal{B})$ equals zero. Hence, $T^2(\mathcal{B})$ is either a torus or Klein bottle. But the possibility to get a Klein bottle instead of the torus $T^2(\mathcal{B})$ fails because of orientability of M^3. Continuing by similar way for every bunch of Λ_a, one gets the desired components $T_1^2, \ldots, T_k^2$ of the boundary $\partial U(\Lambda_a)$ where each T_i^2 homeomorphic to the two-torus $\mathbb{T}^2$. $\square$

In the case when an A-flow is Anosov flow, lemma 3 agrees with the results by Brunella [9] who proved that the basic sets of nontransitive Anosov flow are separated by torii.

3. Proofs of main results

Proof of theorem 1. According to lemma 3, there is a neighbourhood $U(\Lambda_a) \subset W^s(\Lambda_a)$ of Λ_a such that $U(\Lambda_a)$ is an attracting domain of Λ_a, and the boundary $\partial U(\Lambda_a)$ is transversal to $f^{\,t}$ and consists of finitely many components $T_1^2, \ldots, T_k^2$ where each T_i^2 homeomorphic to the two-torus $\mathbb{T}^2$. The transversality of each T_i^2 to the trajectories of $f^{\,t}$ and the inclusion $U(\Lambda_a) \subset W^s(\Lambda_a)$ imply that every positive semi-trajectory starting at a point of T_i^2 belongs to $U(\Lambda_a)$ and never intersects again $\cup_{j=1} T_j^2$. This follows that the union $T_a = \cup_{j=1} T_j^2$ divides $W^s(\Lambda_a)$ into two domains $U(\Lambda_a)$ and $U_{\text{out}} = W^s(\Lambda_a) \backslash U(\Lambda_a)$. Clearly, every negative semi-trajectory starting at a point of T_a belongs to U_{out} and never intersects T_a again. Taking in mind the continuous dependence of trajectories on initial conditions, one gets that U_{out} is homeomorphic (in the interior topology) to $\left(\cup_{i=1}^k T_i^2\right) \times (-\infty; 0)$ that is the disjoint union $\cup_{i=1}^k \left(T_i^2 \times (-\infty; 0)\right) = U_{\text{out}}$. To construct a compactification of $W^s(\Lambda_a)$ it is enough to get a compactification for every $T_i^2 \times (-\infty; 0)$ by a circle.

Take the canonical solid torus $\mathbb{P}_0 = \mathbb{D}^2 \times \mathbb{S}^1$. There is a vector field $\vec{v}$ on $\mathbb{P}_0$ such that $\vec{v}$ is directed transversally on the boundary $\partial\mathbb{P}_0$ outside of $\mathbb{P}_0$ and $\vec{v}$ has a unique periodic trajectory $l_0 = \{0\} \times \mathbb{S}^1$ that is a repeller of $\vec{v}$. Take a homeomorphism $\vartheta_i : \partial\mathbb{P}_0 \to T_i^2$, $i = 1, \ldots, k$. Using ϑ_i and the negative semi-trajectories of $\vec{v}$ starting at $\partial\mathbb{P}_0$, one can construct a homeomorphism $\widetilde{\vartheta}_i$ between $\mathbb{P}_0 \backslash \{l_0\}$ and $T_i^2 \times (-\infty; 0) = T_i^3$ for every $i = 1, \ldots, k$.

Let us introduce a topology on the union $T_i^3 \cup l_i$ where $l_i = l_0$. We assume that the set T_i^3 is endowed with the initial topology. Take a point $x \in l_i = l_0$, and let $U(x)$ be a neighbourhood of $x \in l_0 \subset \mathbb{P}_0$ in the solid torus $\mathbb{P}_0$. Then $U(x)\backslash\{l_0\}$ is an open set in $\mathbb{P}_0$. Hence, $\widetilde{\vartheta}_i\left(U(x)\backslash\{l_0\}\right) = \widetilde{U}(x)$ is an open set in T_i^3. Put the union $\widetilde{U}(x) \cup (U(x) \cap l_0)$ to be a neighbourhood of x in $T_i^3 \cup l_i$. It is easy to see that such neighbourhoods introduce the topology on $T_i^3 \cup l_i$. This gives the compactification of T_i^3 by the closed curve l_i for every $i = 1, \ldots, k$. As a consequence, one gets the compactification $W^s(\Lambda_a)\cup_{i=1}^k l_i$ denoted by $M(\Lambda_a)_{\vartheta_1,\ldots,\vartheta_k}$. Below, we describe the homeomorphisms $\vartheta_1, \ldots, \vartheta_k$ in details to get the final compactification $M(\Lambda_a)$. It is easy to check that $M(\Lambda_a)_{\vartheta_1,\ldots,\vartheta_k}$ is a closed topological manifold. Since any topological three-manifold admits a unique structure of smooth manifold, $M(\Lambda_a, \vartheta_1, \ldots, \vartheta_k)$ is endowed with the structure of smooth manifold which is the extension of the smooth structure on $W^s(\Lambda_a)$.

Let P_i^3 be a copy of $\mathbb{P}_0$, and $\vec{v}_i = \vec{v}$ the vector field with closed curve $l_i = l_0$ in P_i^3, $i = 1, \ldots, k$. By construction, $M(\Lambda_a, \vartheta_1, \ldots, \vartheta_k) = U(\Lambda_a)\cup_{\vartheta_1} P_1^3 \cup \ldots \cup_{\vartheta_k} P_k^3$. Slightly deforming the vector fields $\vec{v}_i$, $i = 1, \ldots, k$, one can assume that this fields and the restriction $f^t|_{U(\Lambda_a)}$ form the smooth flow $\tilde{f}^t$ that is the extension of $f^{\,t}$ to $M(\Lambda_a, \vartheta_1, \ldots, \vartheta_k)$. Clearly, $NW(\tilde{f}^t) = \Lambda_a\cup_{i=1}^k l_i$. Since l_i are repelling periodic trajectories of $\vec{v}_i$, $i = 1, \ldots, k$, the trajectories $l_1, \ldots, l_k$ are repelling isolated periodic trajectories of $\tilde{f}^t$.

We keep the notation of the proof of lemma 3. Take the foliation F_W generated by the fibres of the bundle P_W, see (2). By construction, given any T_i^2, the intersections of the leaves with T_i^2 form a rational foliation $F(T_i^2)$ such that each leaf of $F(T_i^2)$ belongs to a leaf of F_W. Therefore, P_W induces the fibre bundle $P_W|_{U(\Lambda_a)} : U(\Lambda_a) \to S^1 = [0; \tau]/0 \simeq \tau$ such that the restriction $F_W|_{U(\Lambda_a)}$ of F_W on $U(\Lambda_a)$ is a foliation whose leaves are the fibres of the bundle $P_W|_{U(\Lambda_a)}$.

Recall that $\mathbb{P}_0 = \mathbb{S}^1 \times \mathbb{D}^2$ is the canonical solid torus, and $p_0 : \mathbb{S}^1 \times \left(\mathbb{D}^2\backslash\{0\}\right) \to S^1$ is the fibre bundle where $p_0(\vartheta, z) = \frac{z}{|z|}$, $\vartheta \in \mathbb{S}^1 = \partial\mathbb{D}^2$, $z \in \mathbb{D}^2\backslash\{0\}$, $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. The fibres $p_0^{-1}(\cdot)$ form a foliation denoted by $\mathcal{F}$. The leaves of $\mathcal{F}$ are annuluses transversal to boundary $\mathbb{T}^2 = \partial\mathbb{P}_0$. The intersections of this leaves with $\mathbb{T}^2$ produce the rational foliation F_0 generated by parallels on the torus $\mathbb{T}^2 = \partial\mathbb{P}_0$. We know that rational foliations are topologically equivalent [4, 30]. Hence there are the mapping $\vartheta_i : \partial\mathbb{P}_0 \to T_i^2$ taking the leaves of the foliation F_0 to the leaves of $F(T_i^2)$. This gives the continuation of the fibre bundle $P_W|_{U(\Lambda_a)}$ to T_i^3, $i = 1, \ldots, k$. It follows that the collection $\{l_1, \ldots, l_k\}$ is a fibred link in $M(\Lambda_a)$.

At last, the unstable manifolds of the repelling periodic trajectories $l_1, \ldots, l_k$ are three-dimensional open submanifolds of M^3. Clearly, they intersect transversally the two-dimensional stable manifolds of the points of Λ_a. Since the non-wandering set $\mathrm{NW}(\tilde{f}^t) = \Lambda_a\cup_{i=1}^k l_i$ has a hyperbolic structure, $\tilde{f}^t$ is an A-flow satisfying a strong transversality condition. It follows from [20] that $\tilde{f}^t$ is a structurally stable flow. By construction, $\tilde{f}^t$ is a nonsingular flow. □

Proof of theorem 2. Since $L = \{l_1, \ldots, l_k\} \subset M^3$ is a fibred link, the manifold $M^3\backslash L$ can be considered as a mapping torus manifold $M^3\backslash L = \left(\operatorname{int} M^2 \times [0; 1]\right)/(x, 1) \sim (g(x), 0)$ where $g : \operatorname{int} M^2 \to \operatorname{int} M^2$ is a diffeomorphism of the interior of some compact surface M^2 with boundary components $C_1, \ldots, C_k$ corresponding to the knots $l_1, \ldots, l_k$. By definition, every leaf $p^{-1}(\cdot)$ of the fibre bundle $p : M^3\backslash L \to S^1$ is homeomorphic to $\operatorname{int} M^2$. In addition, the topological closure of all leaves $p^{-1}(x)$, $x \in S^1$, are compact surfaces (each homeomorphic to M^2) with the boundary components $l_1, \ldots, l_k$. Hence, we can extend g to the diffeomorphism $M^2 \to M^2$ denoted again by g such that $g|_{\cup_{i=1}^k C_i} = id$.

Let us show that slightly deforming g near ∂M^2, one can assume that every C_i is a repelling set of g. Let a_i be an annulus having the boundary component C_i. Then C_i is diffeomorphic to $C_i \times [0;\ 1]$ that admits a flow ϕ^t along the second factor such that the curves $C_i \times \{0\}$, $C_i \times \{0\}$ form the fixed point set of ϕ^t. Denote by ϕ_t the t-shift along the trajectories of ϕ^t. Since g is a diffeomorphism, $Dg > 0$. Hence, $g \circ \phi_t$ is a diffeomorphism with repelling set C_i for sufficiently large t. Thus, one can assume that $g \circ \phi_t$ has the repelling set $\cup_{i=1}^k C_i$. Clearly, $g \circ \phi_t$ is diffeotopic to g.

It follows from [41], theorem 2 (see also [31]) that $g \circ \phi_t$ can be approximated by an A-diffeomorphism g_0 homotopic to $g \circ \phi_t$ with the repelling set $\cup_{i=1}^k C_i$. Let us take a point x_* near $\cup_{i=1}^k C_i$ such that x_* and $g_0(x_*)$ belong to attracting region of g_0^{-1}. Then there is a path p_* connecting the points x_*, $g_0(x_*)$ such that p_* belongs to the wandering set of g_0. Slightly deforming g_0 in a small neighbourhood of p_* one can get an A-diffeomorphism denoted again by g_0 such that x_* becomes an attracting fixed point of g_0.

Take an attracting neighbourhood U of x_* homeomorphic to a disk. We know that there is an attracting neighbourhood U_P of the classical Plykin attractor Λ_P of diffeomorphism $g_P : \mathbb{S}^2 \to \mathbb{S}^2$ such that U_P is also homeomorphic to a disk [39]. One can change g_0 inside U replacing $g_0|_U$ by the mapping $g_P|_{U_P}$ with the Plykin attractor Λ_P so that the diffeomorphism g_* obtained is an A-diffeomorphism. By construction, g_* is homotopic to g_0, and g_* is an A-diffeomorphism with the attractor Λ_P.

Since g_* is homotopic to g, the mapping torus $\left(\mathrm{int}\, M^2 \times [0; 1]\right)/(x, 1) \sim (g_*(x), 0)$ is homeomorphic to $M^3 \backslash L$. Therefore, the dynamical suspension $\mathrm{sus}^t(g_*)$ of g_* is an A-flow on the manifold $M^3 \backslash L$. Since g_* coincide with $g \circ \phi_t$ outside of U, $\mathrm{sus}^t(g_*)$ can be extended to an A-flow $f^{\,t}$ on M^3 with the repelling periodic trajectories $l_1, \ldots, l_k$. By construction, the non-wandering set $\mathrm{NW}(f^{\,t})$ contains the non-mixing two-dimensional attractor Λ_a corresponding the Plykin attractor Λ_P. This completes the proof. □

Proof of theorem 3. First, we construct an A-flow with two-dimensional attractor on an n-sphere S^n for any $n \geqslant 4$. Due to corollary 1, there is an A-flow $f^{\,t}$ on S^3 with a two-dimensional expanding attractor Λ. Take S^3 to be smoothly embedded in S^4 such that $S^4 \backslash S^3$ is the disjoint union of four-balls B_1^4, B_2^4. One can continue $f^{\,t}$ to S^4 to get an A-flow, say f_4^t, such that f_4^t has a unique source at each B_i^4, $i = 1, 2$, and S^3 is an attracting set for f_4^t. Indeed, let (x_1, x_2, x_3) be local coordinates in a neighbourhood of a point of S^3. Suppose the flow $f^{\,t}$ is defined by the system $\dot{x}_1 = p_1(x_1, x_2, x_3),\ \dot{x}_2 = p_2(x_1, x_2, x_3),\ \dot{x}_3 = p_3(x_1, x_2, x_3)$. One can introduce a coordinate x_4 such that the unequally $x_4 > 0$ corresponds points in B_1^4. Then the system $\dot{x}_1 = p_1, \dot{x}_2 = p_2, \dot{x}_3 = p_3, \dot{x}_4 = -x_4$ defines locally a flow which can be extended to desired flow f_4^t on B_1^4. Similarly, one can get f_4^t on B_2^4. Since $\Lambda \subset S^3$ is an attractor and S^3 is an attracting set, Λ is the two-dimensional attractor of f_4^t. Continuing by similar way, on can construct an A-flow, say f_n^t, on S^n with the two-dimensional expanding attractor Λ for any $n \geqslant 5$.

Now, let M^n be an arbitrary closed n-manifold, $n \geqslant 4$. Due to Smale [42], there is a gradient-like Morse–Smale flow g^t on M^n such that g^t has a sink s_0. By construction, f_n^t has an isolated source, say r_n. Take out neighbourhoods $U(s_0)$, $U(r_n)$ of s_0, r_n such that the boundaries $\partial U\,(s_0$, $\partial U(r_n)$ are transversal to g^t and f_n^t respectively. One can glue $M^n \backslash U(s_0)$ with $S^n \backslash U(r_n)$ under a diffeomorphism $\partial U(s_0) \to \partial U(r_n)$ to get a connected sum $M^n \sharp S^n$ homeomorphic to M^n. Then the flows g^t and f_n^t define the A flow desired on M^n. This completes the proof. □

Proof of theorem 4. Let $M'(\Lambda_a)$ be a casing with a fibred link $L' = \{l'_1, \ldots, l'_k\}$, and $\tilde{f}^t$ a nonsingular A-flow on $M'(\Lambda_a)$ whose non-wandering set consists of attractor Λ_a and repelling isolated periodic trajectories $l'_1, \ldots, l'_k$. Due to lemma 3, there is a neighbourhood $U(\Lambda_a)$ of Λ_a is an attracting domain of Λ_a such that the boundary $\partial U(\Lambda_a)$ is transversal to $\tilde{f}^t$ and consists of torii $T_1^2, \ldots, T_k^2$. Since every l_j, $j = 1, \ldots, k$, is an isolated repelling periodic trajectory, there is a neighbourhood $U(l_j)$ of l_j homeomorphic to a solid torus such that $U(l_j)$ belongs to the basin of l_j and the torus $\partial U(l_j)$ is transversal to $\tilde{f}^t$. Without loss of generality one can assume that all $U(l_j)$, $j = 1, \ldots, k$, do not intersect $U(\Lambda_a)$. Since $\tilde{f}^t$ is a flow of attractor–repeller type with no non-wandering points except $\Lambda_a \cup \{l'_1, \ldots, l'_k\}$, any positive semi-trajectory starting from $\partial U(l_j)$ intersects some torus, say $T^2_{j_1}$, at a unique point, and after that the semi-trajectory never leaves $U(\Lambda_a)$. A simply connectedness of $\partial U(l_j)$ implies that there is a homeomorphism $\theta_j : \partial U(l_j) \to T^2_{j_1}$ that is a forward Poincare mapping. Such homeomorphisms exist for every $j = 1, \ldots, k$. Hence, $M'(\Lambda_a)$ can be obtained up to homeomorphism by the procedure described in the proof of theorem 1.

Now, we keep the notation of the proof of theorem 1. We see that one remains a freedom to choose the mapping $\vartheta_i : \partial\mathbb{P}_0 \to T^2_i$, $i = 1, \ldots, k$ to get the casing $M(\Lambda_a)$. Any casing $M'(\Lambda_a) \in \mathbb{F}(\Lambda_a)$ is obtained by some mapping $\vartheta'_i : \partial\mathbb{P}_0 \to T^2_i$, $i = 1, \ldots, k$. Then the mapping $\vartheta'_i \circ \vartheta_i^{-1} : \partial\mathbb{P}_0 \to T^2_i$, $i = 1, \ldots, k$, induce a surgery of the link $L = \{l_1, \ldots, l_k\}$ to get $M'(\Lambda_a)$ from $M(\Lambda_a)$. This completes the proof. □

Remark. One can show that there are infinitely many casings $M(\Lambda_a)$ with $M(\Lambda_a)\backslash L(\Lambda_a)$ endowed with a hyperbolic structure. Indeed, let us fix some $M(\Lambda_a) \in \mathbb{F}(\Lambda_a)$. Denote by $M^{\text{int}}(\Lambda_a)$ the set $M(\Lambda_a)\backslash(\cup_{i=1}^k P^3_i)$. Slightly modifying the restriction $f^t|_{M^{\text{int}}(\Lambda_a)}$, one can get a flow f^t_1 with the global section $S_0 \cap M^{\text{int}}(\Lambda_a)$ such that the set of the circles $\tilde{c}$ becomes an invariant set under the τ-time shift $f^t_{1,\tau}$. Since the circles $\tilde{c}$ have no intersections with $\Lambda_a \cap S_0$, one can assume that the support of the modification has no intersection with Λ_a. Therefore, one can assume that the Poincare forward mapping $f^t_{1,\tau} : S_0 \cap M^{\text{int}}(\Lambda_a) \to S_0 \cap M^{\text{int}}(\Lambda_a)$ is an A^*-homeomorphism with the non-wandering set consisting of $\tilde{c}$ and one-dimensional basic set $\Lambda_a \cap S_0$. It follows from [14] that $f^t_{1,\tau}$ semi-conjugates to a pseudo-Anosov homeomorphism. Due to [44], $M^{\text{int}}(\Lambda_a) = M(\Lambda_a)\backslash(\cup_{i=1}^k P^3_i)$ is endowed with the structure of hyperbolic manifold.

Proof of theorem 5. Let $\varphi : W^s(\Lambda_1) \to W^s(\Lambda_2)$ be the homeomorphism taking the trajectories of the flow $f^t_1|_{W^s(\Lambda_1)}$ to the trajectories of the flow $f^t_1|_{W^s(\Lambda_2)}$. Take a point $x_1 \in \Lambda_1$. Due to proposition 1, the restriction $f^t_1|_{W^s(\Lambda_1)}$ is the dynamical τ_1-time suspension over some A^*-homeomorphism $\psi_1 : \text{clos}(W^{uu}(x_1)) \to \text{clos}(W^{uu}(x_1))$ with some $\tau_1 > 0$. Similarly, the restriction $f^t_2|_{W^s(\Lambda_2)}$ is the dynamical τ_2-time suspension over some A^*-homeomorphism $\psi_2 : \text{clos}(W^{uu}(x_2)) \to \text{clos}(W^{uu}(x_2))$ with some $\tau_2 > 0$ where $x_2 = \varphi(x_1)$. Since φ is a conjugacy for any $t \in \mathbb{R}$, φ takes the strong unstable manifolds of f^t_1 to the strong unstable manifolds of f^t_2. Then $\varphi\,[W^{uu}(x_1)] = W^{uu}(\varphi(x_1)) = W^{uu}(x_2)$. As a consequence, $\varphi\,[\text{clos}\, W^{uu}(x_1)] = \text{clos}\, W^{uu}(x_2)$. Due to proposition 1, $\tau_1 = \tau_2$. It follows that φ conjugates ψ_1 and ψ_2. Hence, φ takes the foliation $F(T^2_{i,1})$ to the foliation $F(T^2_{i,2})$ for every $i = 1, \ldots, k$.

We keep the notation of the proof of theorem 1. Let $M(\Lambda_j)_{\vartheta_1^{(j)},\ldots,\vartheta_k^{(j)}}$ be the casing of Λ_j obtained by using homeomorphisms $\vartheta_i^{(j)} : \partial\mathbb{P}_0 \to T^2_{i,j}$, $i = 1, \ldots, k$, $j = 1, 2$ where $\mathbb{P}_0$ is the canonical solid torus. To continue φ to a homeomorphism $M(\Lambda_1)_{\vartheta_1^{(1)},\ldots,\vartheta_k^{(1)}} \to M(\Lambda_2)_{\vartheta_1^{(2)},\ldots,\vartheta_k^{(2)}}$ one needs to continue φ on $\mathbb{P}_0$. To do this, we need to change the homeomorphisms $\vartheta_i^{(2)}$ for every $i = 1, \ldots, k$. Denote by μ a meridian of $\mathbb{P}_0$. By construction, the curve $\vartheta_i^{(j)}(\mu)$ is a closed simple curve on $T^2_{i,j}$ transversally intersecting the leaves of the foliation $F(T^2_{i,j})$, $j = 1, 2$. Moreover, since μ intersects each parallel at a unique point, $\vartheta_i^{(j)}(\mu)$ intersects each leaf of the foliation $F(T^2_{i,j})$ at a unique point also. Since φ takes the foliation $F(T^2_{i,1})$ to the foliation $F(T^2_{i,2})$, $\varphi \circ \vartheta_i^{(1)}(\mu)$ is a closed simple curve on $T^2_{i,2}$ transversally intersecting each leaf of the foliation $F(T^2_{i,2})$ at a unique point, $i = 1, \ldots, k$. This follows that there is a Dehn twist $D : T^2_{i,2} \to T^2_{i,2}$ taking $\vartheta_i^{(2)}(\mu)$ to $\varphi \circ \vartheta_i^{(1)}(\mu)$ such that every leaf of $F(T^2_{i,2})$ is invariant under D. Set $\widehat{\vartheta}_i^{(2)} = D \circ \vartheta_i^{(2)} : \partial P^3_0 \to T^2_i$. Since D keeps the foliation $F(T^2_{i,2})$, $M(\Lambda_2)_{\widehat{\vartheta}_1^{(2)},\ldots,\widehat{\vartheta}_k^{(2)}}$ is a casing of Λ_2. By construction, φ takes $\vartheta_i^{(1)}(\mu)$ to a curve homotopy $\widehat{\vartheta}_2^{(2)}(\mu)$. It is well-known, that a homeomorphism of the boundary of solid torus can be extended to the whole solid torus provided the boundary homeomorphism keeps meridians. Moreover, one can get the extension keeping the central axis of the solid torus. Hence, φ can be extended to the homeomorphism $M(\Lambda_1) \to M(\Lambda_2)_{\widehat{\vartheta}_1^{(2)},\ldots,\widehat{\vartheta}_k^{(2)}}$ taking the fibred link $L(\Lambda_1)$ to $L(\Lambda_2)$. $\square$

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