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High-dimensional Morse-Smale systems with king-saddles

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We introduce high-dimensional Morse-Smale systems with king-saddles. Every saddle of such system has a separatrix with heteroclinic intersections. We get the necessary and sufficient conditions of conjugacy of Morse-Smale diffeomorphisms with king-saddles. For every polar Morse-Smale system with two saddles on the n-sphere $\mathbb{S}^n$, one proves the existence of a king-saddle. This allows to get the necessary and sufficient conditions of conjugacy for such Morse-Smale diffeomorphisms on $\mathbb{S}^n$, $n \geq 4$. We give a simple sufficient condition of the existence of heteroclinic intersections for Morse-Smale systems on $\mathbb{S}^3$.

0. Introduction

In 1960, Smale [27] introduced the class of dynamical systems (flows and diffeomorphisms) which later were called Morse-Smale systems. By definition, the Morse-Smale systems are dynamical systems whose non-wandering sets consist of finite number of periodic hyperbolic orbits with transversal intersections of invariant manifolds of periodic hyperbolic orbits [27,28]. Morse-Smale systems splits into Morse-Smale diffeomorphisms (which are dynamical systems with discrete time) and Morse-Smale flows (which are dynamical systems with continuous time). Palis and Smale [23,24] proved that the Morse-Smale systems are structurally stable. Since a Morse-Smale system has zero topological entropy, one can say that the Morse-Smale systems are simplest structurally stable dynamical systems.

However, despite on a triviality of non-wandering set, the topological classification of Morse-Smale systems is a challenge problem. There are complete classifications of Morse-Smale systems on low-dimensional manifolds, and Morse-Smale diffeomorphisms on 3-manifolds, and some special classes on high-dimensional manifolds (see the surveys [9,18] concerning the topological classification of Morse-Smale systems). Andronov

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and Ponryagin [1] described Morse-Smale flows on compact plane domain. Maier [17] completely classified Morse-Smale diffeomorphisms of circle. Peixoto [25] got the complete topological classification of Morse-Smale flows without periodic trajectories on closed surfaces. Recently, Bonatti, Grines, and Pochinka [4] constructed a complete topological invariant for Morse-Smale diffeomorphisms of closed 3-manifolds. Let us mention the paper [3] where the authors of the paper classified 3-dimensional Morse-Smale diffeomorphisms whose saddles have the same Morse index and form a heteroclinic chain. As to high-dimensional Morse-Smale diffeomorphisms, there are few results. Grines, Gurevich, and Medvedev [6,7] classified high-dimensional Morse-Smale diffeomorphisms without heteroclinic intersections of separatrices of saddle periodic points (see also [8]).

In contract with [6,7], we consider high-dimensional Morse-Smale systems which have necessarily heteroclinic intersections, so-called systems with king-saddles. Roughly speaking, a king saddle has a separatrix which intersect separatrices of every other saddles (see exact definition below). We get the necessary and sufficient conditions of conjugacy of Morse-Smale diffeomorphisms with king-saddles. For polar Morse-Smale systems with two saddles on n-sphere, one proves the existence of king-saddles. This allows to get the necessary and sufficient conditions of conjugacy for such Morse-Smale systems. Let us formulate the main results (basic definitions of Dynamical Systems see in [2,26,28]).

Later on, M^n is a closed smooth and connected n-manifold, $n \geq 3$ or $n \geq 4$. A non-wandering set of Morse-Smale system $\mathcal{S}$ is denoted by $NW(\mathcal{S})$. Mainly, we recall some definitions only for diffeomorphisms, as this definitions for flows are similar.

Let $f : M^n \to M^n$ be a diffeomorphism of M^n, and p a periodic point of period $k \in \mathbb{N}$. The stable manifold $W^s(p)$ is defined to be the set of points $x \in M^n$ such that $\varrho(f^{kj}(x); p) \to 0$ as $j \to \infty$ where ϱ is a metric on M^n. The unstable manifold $W^u(p)$ is the stable manifold of p for the diffeomorphism f^{-1}. Stable and unstable manifolds are called invariant manifolds. It is well known that if p is hyperbolic, then every invariant manifold is an immersed submanifold homeomorphic to Euclidean space. Moreover, $W^s(p)$ and $W^u(p)$ are intersected transversally at p, and $\dim W^s(p) + \dim W^u(p) = n$.

A periodic orbit p is called a *sink periodic point* (resp., *source periodic point*) if $\dim W^s(p) = n$ and $\dim W^u(p) = 0$ (resp., $\dim W^s(p) = 0$ and $\dim W^u(p) = n$). A periodic point σ is called a *saddle periodic point* if $1 \leq \dim W^u(\sigma) \leq n-1$ (automatically, $1 \leq \dim W^s(\sigma) \leq n-1$). A sink (resp., source, saddle) periodic point p is called a *sink* (resp., *source, saddle*) provided p is a fixed point.

Let p, q be saddle periodic points such that $W^u(p) \cap W^s(q) \neq \emptyset$. Then the intersection $W^u(p) \cap W^s(q)$ is called *heteroclinic*. Note that according to the definition of Morse-Smale diffeomorphisms, the invariant manifolds $W^u(p)$, $W^s(q)$ are intersected transversally.

To simplify the exposition, we'll assume that all periodic points of diffeomorphisms are fixed points (for flows, it holds automatically). This assumption holds for some iteration of diffeomorphism. Following [19,21], denote by $MS^{diff}(M^n; a, b, c)$ (resp., $MS^{flow}(M^n; a, b, c)$) the set of Morse-Smale diffeomorphisms $f : M^n \to M^n$ (resp., flows f^t) such that the non-wandering set $NW(f)$ (resp., $NW(f^t)$) consists of a sinks, b sources, and c saddles. Recall that due to Smale [27], $a \geq 1$ and $b \geq 1$. Set $MS(M^n; a, b, c) = MS^{diff}(M^n; a, b, c) \cup MS^{flow}(M^n; a, b, c)$.

A saddle σ_0 of $f \in MS^{diff}(M^n; a, b, c)$ is called an *u-king-saddle* provided $W^u(\sigma_0) \cap W^s(\sigma) \neq \emptyset$ for every another saddle σ of f. Similarly, σ_0 is called an *s-king-saddle* provided $W^s(\sigma_0) \cap W^u(\sigma) \neq \emptyset$ for every another saddle σ of f.

First, we show that any Morse-Smale system from the set $MS(\mathbb{S}^n; 1, 1, 2)$ has king-saddles. Let us recall that $MS(\mathbb{S}^n; 1, 1, 2)$ consists of Morse-Smale systems on the n-sphere $\mathbb{S}^n$ whose non-wandering set consists of a sink, a source, and two saddles. In particular, every system from $MS(\mathbb{S}^n; 1, 1, 2)$ is a polar gradient-like system. Recall that the Morse index of fixed point σ equals the dimension of the unstable manifold $W^u(\sigma)$ of σ.

Theorem 1. *Given any $n \geq 3$, $MS(\mathbb{S}^n; 1, 1, 2) \neq \emptyset$, and saddles of any system from the set $MS(\mathbb{S}^n; 1, 1, 2)$ has different Morse indexes. Moreover, any Morse-Smale system from the set $MS(\mathbb{S}^n; 1, 1, 2)$ has both u- and s-king-saddles. To be precise, the saddle with biggest Morse index is an u-king saddle, and the saddle with smallest Morse index is an s-king saddle.*

To prove this result, we construct the special decompositions of the n-sphere $\mathbb{S}^n$. To be precise, the following technical statement takes place. Below, $\mathbb{D}^k$ is the unit k-dimensional disk.

Proposition 1. *The following decompositions of $\mathbb{S}^n$, $n \geq 3$, hold:*

$$\mathbb{S}^n = \left(\mathbb{S}^1 \times \mathbb{D}^{n-1}\right) \bigcup_h \left(\mathbb{D}^2 \times \mathbb{S}^{n-2}\right) = \left(\mathbb{S}^2 \times \mathbb{D}^{n-2}\right) \bigcup_h \left(\mathbb{D}^3 \times \mathbb{S}^{n-3}\right) = \cdots =$$
$$= \left(\mathbb{S}^k \times \mathbb{D}^{n-k}\right) \bigcup_h \left(\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}\right) = \cdots = \left(\mathbb{S}^{n-2} \times \mathbb{D}^2\right) \bigcup_h \left(\mathbb{D}^{n-1} \times \mathbb{S}^1\right)$$

where $h : \partial\left(\mathbb{S}^k \times \mathbb{D}^{n-k}\right) = \mathbb{S}^k \times \mathbb{S}^{n-k-1} \to \mathbb{S}^k \times \mathbb{S}^{n-k-1} = \partial\left(\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}\right)$ is the natural identification provided $n \geq 4$, and h is a special identification provided $n = 3$.

For $n = 4$, the decomposition $\mathbb{S}^4 = \left(\mathbb{S}^2 \times \mathbb{D}^2\right) \cup_h \left(\mathbb{S}^1 \times \mathbb{D}^3\right)$ is the consequence of the more general result by Laudenbach and Poenaru [14].

Now, we are going to give a necessary and sufficient condition of conjugacy of Morse-Smale diffeomorphisms containing king-saddles. Recall that diffeomorphisms $f_1, f_2 : M^n \to M^n$ are called *conjugate*, if there is a homeomorphism $h : M^n \to M^n$ such that $h \circ f_1 = f_2 \circ h$.

In [20], the authors introduced the notation of equivalent embedding as follows. Let $M_1^k, M_2^k \subset M^n$ be topologically embedded k-manifolds, $1 \leq k \leq n-1$. We say that they *have the equivalent embedding* if there are neighborhoods $U(clos\, M_1^k)$, $U(clos\, M_2^k)$ of $clos\, M_1^k$, $clos\, M_2^k$ respectively and a homeomorphism $h : U(clos\, M_1^k) \to U(clos\, M_2^k)$ such that $h(M_1^k) = M_2^k$. Here, $clos\, N$ means the topological closure of N. This notation allows to classify Morse-Smale flows with non-wandering sets consisting of three equilibriums [20]. To be precise, it was proved that two such flows f_1^t, f_2^t are topologically equivalent if and only if the stable (or unstable) separatrices of saddles of f_1^t, f_2^t respectively have the equivalent embedding. Note that the notation of equivalent embedding goes back to a scheme introduced by Leontovich and Maier [15,16]. If one considers a conjugacy, we have to add conjugacy relations to the equivalent embedding. The modification of (global) conjugacy is a local conjugacy when the conjugacy holds in some neighborhoods of compact invariant sets. We introduce the intermediate notion, so-called a locally equivalent dynamical embedding (in short, dynamical embedding), as follows.

Let $f_1, f_2 : M^n \to M^n$ be homeomorphisms of closed topological n-manifold M^n, $n \geq 2$, and N_1, N_2 invariant sets of f_1, f_2 respectively i.e. $f_i(N_i) = N_i$, $i = 1, 2$. We say that the sets N_1, N_2 *have the same dynamical embedding* if there are neighborhoods δ_1, δ_2 of $clos\ N_1$, $clos\ N_2$ respectively and a homeomorphism $h_0 : \delta_1 \cup f_1(\delta_1) \to M^n$ such that

$$h_0(\delta_1) = \delta_2, \qquad h_0(clos\ N_1) = clos\ N_2, \qquad h_0 \circ f_1|_{\delta_1} = f_2 \circ h_0|_{\delta_1} \tag{1}$$

Now we are ready to formulate the necessary and sufficient condition of conjugacy for Morse-Smale diffeomorphisms with king-saddles.

Theorem 2. *Let $f_i : M^n \to M^n$ be a Morse-Smale diffeomorphism of closed n-manifold M^n, $n \geq 3$, and σ_i an u(s)-king-saddle of f_i, $i = 1, 2$. Then f_1 and f_2 are topologically conjugate if and only if the invariant manifolds $W^{u(s)}(\sigma_1)$, $W^{u(s)}(\sigma_2)$ have the same dynamical embedding.*

Theorems 1 and 2 allows to get the necessary and sufficient conditions of conjugacy for Morse-Smale diffeomorphisms $MS^{diff}(\mathbb{S}^n; 1, 1, 2)$.

Theorem 3. *Let $f_i \in MS^{diff}(\mathbb{S}^n; 1, 1, 2)$ be a Morse-Smale diffeomorphism with saddles $\sigma_1^{(i)}$, $\sigma_2^{(i)}$, $i = 1, 2$, $n \geq 4$. Suppose that the Morse index of $\sigma_1^{(i)}$ is more than the Morse index of $\sigma_2^{(i)}$. Then f_1, f_2 are topologically conjugate if and only if one of the following conditions holds:*

- *the unstable manifolds $W^u(\sigma_1^{(1)})$, $W^u(\sigma_1^{(2)})$ have the same dynamical embedding;*
- *the stable manifolds $W^s(\sigma_2^{(1)})$, $W^s(\sigma_2^{(2)})$ have the same dynamical embedding.*

The following result gives the sufficient conditions of the existence of heteroclinic intersections for Morse-Smale systems on $\mathbb{S}^3$.

Theorem 4. *Let $f : \mathbb{S}^3 \to \mathbb{S}^3$ be a Morse-Smale diffeomorphism such that the number of sources equals the number of saddles with the Morse index two. Then f has heteroclinic intersections.*

For the number of sources (and the number of saddles with Morse index two) equals 1, this result was proved in [11].

Corollary 1. *Let f^t be a gradient-like Morse-Smale flow on $\mathbb{S}^3$. If the number of sources equals the number of saddles with the Morse index two, then f^t has heteroclinic intersections*

This corollary is useful for the problem on the existence of separators in magnetic fields in electrically conducting fluids, see [10].

The structure of the paper is the following. In Section 1, we formulate the main definitions and give some previous results. In Section 2, one proves main results.

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1. Preliminaries

Let $f : M \to M$ be a diffeomorphism and $p \in M$ a fixed hyperbolic point of f. Suppose that the restriction $f|_{W^u(p)}$ preserves the orientation of $W^u(p)$. Then $Ind_p(f) = (-1)^{\dim W^u(p)}$ is a topological index of f at p. Later on, we always assume that the restriction of diffeomorphism on an unstable manifold of fixed point is a preserving orientation mapping.

Let $f_{*k} : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R})$ be an isomorphism induced by $f : M \to M$ in the homology group $H_k(M, \mathbb{R}) = H_k(M)$, and $tr\ (f_{*k})$ a trace of f_{*k}, $0 \leq k \leq \dim M$. Suppose that the set $Fix\ (f)$ of fixed points of f consists of hyperbolic. The following Lefschetz formula holds:

$$\sum_{k=0}^{\dim M} (-1)^k tr\ (f_{*k}) = \sum_{p \in Fix\ (f)} Ind_p(f). \tag{2}$$

For the reference, we formulate the following result proved in [11] (see also [5] and [21], Proposition 1).

Proposition 2. *Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism, and $W^\tau(\sigma)$ an invariant manifold of dimension $2 \leq d \leq n-1$ of a saddle σ where $\tau \in \{s, u\}$. Suppose that $W^\tau(\sigma)$ has no heteroclinic* *intersections with other separatrices. Then $Sep^\tau(\sigma) = W^\tau(\sigma) \setminus \{\sigma\}$ belongs to unstable (if $\tau = s$) or stable (if $\tau = u$) manifold of some node (source or sink, respectively) periodic point, say N, and the topological closure of $Sep^\tau(\sigma)$ is a topologically embedded d-sphere that equals $W^\tau(\sigma) \cup \{N\} = S^d_\tau$. Moreover, S^d_τ is a locally flat embedded sphere provided $d \neq n-2$.*

Proof of Proposition 1. Let $\mathbb{R}^{n+1}$ be Euclidean space endowed with coordinates $(x_1, \dots, x_{n+1})$, and the n-sphere $\mathbb{S}^n$ defined by the equality

$$x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1, \quad n \geq 3.$$

Given any fixed number $1 \leq k \leq n-1$, $\mathbb{R}^{n+1} = \mathbb{R}^{k+1} \bigoplus \mathbb{R}^{n-k}$ where $\mathbb{R}^{k+1}$ is defined by $x_{k+2} = 0, \dots, x_{n+1} = 0$ and $\mathbb{R}^{n-k}$ is defined by $x_1 = 0, \dots, x_{k+1} = 0$. Clearly, the intersection $\mathbb{S}^n \cap \mathbb{R}^{k+1}$ is the k-sphere $\mathbb{S}^k_1 \subset \mathbb{S}^n$ defined by the equality

$$\mathbb{S}^k_1: \ x_1^2 + \cdots + x_{k+1}^2 = 1, \ x_{k+2} = 0, \dots, x_{n+1} = 0.$$

In $\mathbb{R}^{k+1}$, this sphere bounds the closed $(k+1)$-disk denoted by $\mathbb{D}^{k+1}_1$. Similarly, the intersection $\mathbb{S}^n \cap \mathbb{R}^{n-k}$ is the $(n-k-1)$-sphere $\mathbb{S}^{n-k-1}_2 \subset \mathbb{S}^n$ defined by the equality

$$\mathbb{S}^{n-k-1}_2: \ x_1 = 0, \dots, x_{k+1} = 0, \ x_{k+2}^2 + \cdots + x_{n+1}^2 = 1.$$

In $\mathbb{R}^{n-k}$, the sphere $\mathbb{S}^{n-k-1}_2$ bounds closed $(n-k)$-disk denoted by $\mathbb{D}^{n-k}_2$.

Let M^n_1 be the n-manifold defined as follows

$$M^n_1: \ x_1^2 + \cdots + x_{k+1}^2 = 1, \ x_{k+2}^2 + \cdots + x_{n+1}^2 \leq 1.$$

Clearly, $M^n_1 = \mathbb{S}^k_1 \times \mathbb{D}^{n-k}_2$. Similarly, set

$$M^n_2: \ x_1^2 + \cdots + x_{k+1}^2 \leq 1, \ x_{k+2}^2 + \cdots + x_{n+1}^2 = 1.$$

Clearly, $M^n_2 = \mathbb{D}^{k+1}_1 \times \mathbb{S}^{n-k-1}_2$. The manifolds M^n_1, M^n_2 have the common boundary $\partial M^n_1 = \partial M^n_2$ that equals the intersection

$$M^n_1 \cap M^n_2 = \partial M^n_1 = \partial M^n_2 = \mathbb{S}^k_1 \times \mathbb{S}^{n-k-1}_2: \ x_1^2 + \cdots + x_{k+1}^2 = 1, \ x_{k+2}^2 + \cdots + x_{n+1}^2 = 1.$$

Let us prove that the union $M^n_1 \cup M^n_2$ homeomorphic to $\mathbb{S}^n$. Indeed, for any point $p_0 = (x_{1,0}, \dots, x_{n+1,0}) \in \mathbb{S}^n$, we denote by l_0 the ray starting at the origin $(0, \dots, 0) \in \mathbb{R}^{n+1}$ and passing through p_0. Let us show that l_0 intersects $M^n_1 \cup M^n_2$ at a unique point. If the parameter $t \in [1; \infty)$ increases, then the point $p_t = (\sqrt{t}x_{1,0}, \dots, \sqrt{t}x_{n+1,0})$ moves from p_0 along l_0 to infinity. The terms

$$\begin{gathered} w_t = (tx_{1,0})^2 + \cdots + (tx_{n+1,0})^2 = t^2(x_1^2 + \cdots + x_{n+1}^2) = \\ = (tx_{1,0})^2 + \cdots + (tx_{k+1,0})^2 + (tx_{k+2,0})^2 + \cdots + (tx_{n+1,0})^2, \end{gathered}$$

increases boundless beginning with 1. This follows that there is $t_0 \in [1; \infty)$ such that $w_t > 1$. In addition, one of the following possibility holds

a) $(tx_{1,0})^2 + \cdots + (tx_{k+1,0})^2 < 1, \ (tx_{k+2,0})^2 = \cdots + (tx_{n+1,0})^2 = 1$;
b) $(tx_{1,0})^2 + \cdots + (tx_{k+1,0})^2 = 1, \ (tx_{k+2,0})^2 = \cdots + (tx_{n+1,0})^2 < 1$;
c) $(tx_{1,0})^2 + \cdots + (tx_{k+1,0})^2 = 1, \ (tx_{k+2,0})^2 = \cdots + (tx_{n+1,0})^2 = 1$.

In the case (a), the intersection $l_0 \cap M^n_1 \cup M^n_2$ consists of a unique point p_{t_0} belonging to the interior of the manifold M^n_2. In the case (b), the intersection $l_0 \cap M^n_1 \cup M^n_2$ consists of a unique point p_{t_0} belonging to the interior of the manifold M_1^n. At last in the case (c), the intersection $l_0 \cap M_1^n \cup M_2^n$ consists of a unique point p_{t_0} belonging to the intersection of the boundaries of the manifolds M_1^n, M_2^n. Thus, the corresponding $p_0 \longrightarrow p_{t_0}$ defines the mapping $\vartheta : \mathbb{S}^n \to M_1^n \cup M_2^n$. By similar way, one can show that there is the inverse mapping $\vartheta^{-1} : M_1^n \cup M_2^n \to \mathbb{S}^n$. We see that the union $M_1^n \cup M_2^n$ is a piecewise linear manifold. Since the mapping $\vartheta : \mathbb{S}^n \to M_1^n \cup M_2^n$ is a projection along rays, ϑ is a homeomorphism. This completes the proof. $\square$

Let us introduce invariant sets that determine dynamics of Morse-Smale diffeomorphism. Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism whose non-wandering set $NW(f)$ consists of the set $\alpha(f)$ of sources, the set $\omega(f)$ of sinks, and the set $\sigma(f)$ of saddles. Set

$$A(f) = \omega(f) \bigcup_{s_i \in \sigma(f)} W^u(s_i), \quad R(f) = \alpha(f) \bigcup_{s_i \in \sigma(f)} W^s(s_i).$$

We see that $A(f)$ (respectively, $R(f)$) is the union of $\omega(f)$ (resp., $\alpha(f)$) and unstable (resp., stable) manifolds of saddles from the set $\sigma(f)$. The following statement proved in [22] gives the necessary and sufficient conditions of conjugacy of Morse-Smale diffeomorphisms.

Proposition 3. *Two Morse-Smale diffeomorphisms f_1, $f_2 : M^n \to M^n$ of closed n-manifold M^n, $n \geq 2$, are conjugate if and only if one of the following conditions holds:*

- *the sets $A(f_1)$, $A(f_2)$ have the same dynamical embedding;*
- *the sets $R(f_1)$, $R(f_2)$ have the same dynamical embedding.*

2. Proofs of main results

Proof of Theorem 1. It is enough to construct a polar Morse-Smale flow $\in MS^{flow}(\mathbb{S}^n; 1, 1, 2)$, since a t-time shift along trajectories gives a Morse-Smale diffeomorphism $\in MS^{diff}(\mathbb{S}^n; 1, 1, 2)$.

Due to Proposition 1, there is the decomposition $\mathbb{S}^n = \left(\mathbb{S}^k \times \mathbb{D}^{n-k}\right) \bigcup \left(\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}\right)$ of $\mathbb{S}^n$. Let $\vec{v}_{1,k}$ be a vector field on the k-sphere $\mathbb{S}_1^k = \mathbb{S}^n \cap \mathbb{R}^{k+1}$ such that the non-wandering set of $\vec{v}_{1,k}$ consists of the source at the point $(1, 0, \ldots, 0)$ and the sink at the point $(-1, 0, \ldots, 0)$. Such vector field induces a flow sometime called a sink-source flow.

Denote by $A_1(x_1, \ldots, x_{k+1})$, …, $A_{k+1}(x_1, \ldots, x_{k+1})$ the coordinates of the vector field $\vec{v}_{1,k}$, i.e.

$$\vec{v}_{1,k}(x_1, \ldots, x_{k+1}) = \left(A_1(x_1, \ldots, x_{k+1}), \ldots, A_{k+1}(x_1, \ldots, x_{k+1})\right).$$

Let

$$\vec{\boldsymbol{V}}_1 = (A_1(x_1, \ldots, x_{k+1}), \ldots, A_{k+1}(x_1, \ldots, x_{k+1}), x_{k+2}, \ldots, x_{n+1})$$

be the vector field on $M_1^n = \mathbb{S}_1^k \times \mathbb{D}_2^{n-k}$. Since the vector field

$$\vec{v}_{k+1,n+1} = (0, \ldots, 0, x_{k+2}, \ldots, x_{n+1})$$

has the hyperbolic source $(0, \ldots, 0)$ on the disk $\mathbb{D}_2^{n-k}$, the vector field $\vec{\boldsymbol{V}}_1$ has the hyperbolic source $\alpha = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ and the hyperbolic saddle $\sigma_1 = (-1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. By construction, the vector field $\vec{\boldsymbol{V}}_1$ has the repelling set $\mathbb{S}_1^k$. Therefore,

$$W^s(\sigma_1) = \mathbb{S}_1^k \setminus \{\alpha\} = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \cdots + x_{k+1}^2 = 1,\ x_{k+2} = 0, \ldots, x_{n+1} = 0,\ x_1 \neq 1\},$$

$$W^u(\sigma_1) \cap M_1^n \subset \{(x_1, \dots, x_{n+1}) : x_1 = -1, x_2 = 0, \dots, x_{k+1} = 0\}.$$

We see that σ_1 is a saddle of the type $(n-k, k)$. Since the vector field $\vec{v}_{k+1,n+1}$ on $\partial\mathbb{D}_2^{n-k}$ is directed outside of $\mathbb{D}_2^{n-k}$ and transversal to $\partial\mathbb{D}_2^{n-k}$, the vector field $\vec{\boldsymbol{V}}_1$ on $\partial M_1^n = \mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1}$ is directed outside of $M_1^n = \mathbb{S}_1^k \times \mathbb{D}_2^{n-k}$ and is transversal to the boundary $\partial M_1^n = \mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1}$.

Similarly, one constructs the vector field $\vec{\boldsymbol{V}}_2$ on $M_2^n = \mathbb{D}_1^{k+1} \times \mathbb{S}_2^{n-k-1}$ as follows

$$\vec{\boldsymbol{V}}_2 = (-x_1, \dots, -x_{k+1}, B_{k+2}(x_{k+2}, \dots, x_{n+1}), \dots, B_{n+1}(x_{k+2}, \dots, x_{n+1}))$$

where the vector field

$$\vec{w}_{2,k+1}(x_{k+2}, \dots, x_{n+1}) = (B_{k+2}(x_{k+2}, \dots, x_{n+1}), \dots, B_{n+1}(x_{k+2}, \dots, x_{n+1}))$$

has the hyperbolic source at the point $(0, \dots, 0, 1)$ and hyperbolic sink at the point $(0, \dots, 0, -1)$ on the $(n-k-1)$-sphere $\mathbb{S}_2^{n-k-1} \subset \mathbb{S}^n$. By construction, the vector field $\vec{\boldsymbol{V}}_2$ has the attracting set $\{0\} \times \mathbb{S}_2^{n-k-1} \subset M_2^n$. Therefore, the vector field $\vec{\boldsymbol{V}}_2$ has the hyperbolic saddle $\sigma_2 = (0, \dots, 0, 1)$ and the hyperbolic sink $\omega = (0, \dots, -1)$. Similarly, one can show that

$$W^u(\sigma_2) = \mathbb{S}_2^{n-k-1} \setminus \{\omega\} = \{(x_1, \dots, x_{n+1}) : x_1 = 0, \dots, x_{k+1} = 0, x_{k+2}^2 + \dots + x_{n+1}^2 = 1, x_{n+1} \neq -1\}$$
$$W^s(\sigma_2) \cap M_2^n \subset \{(x_1, \dots, x_{n+1}) : x_1^2 + \dots + x_{k+1}^2 \leq 1,\ x_{k+2} = 0, \dots, x_n = 0, x_{n+1} = 1\}.$$

We see that σ_2 is a saddle of the type $(n-k-1, k+1)$. Since the vector field $\vec{v}_{k+1,n+1}$ on $\partial\mathbb{D}_1^{k+1}$ is directed inside of $\mathbb{D}_1^{k+1}$ and transversal to $\partial\mathbb{D}_1^{k+1}$, the vector field $\vec{\boldsymbol{V}}_2$ on $\partial M_2^n = \mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1}$ is directed inside of $M_2^n = \mathbb{D}_1^{k+1} \times \mathbb{S}_2^{n-k-1}$ and transversal to the boundary $\partial M_2^n = \mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1}$. It follows from Proposition 1 that the vector fields $\vec{\boldsymbol{V}}_1$, $\vec{\boldsymbol{V}}_2$ form the piecewise-linear vector field $\vec{\boldsymbol{V}}$ on $M_1^n \cup M_2^n = \mathbb{S}^n$. Clearly, the integrable curves of $\vec{\boldsymbol{V}}$ satisfy the uniqueness condition (given any point, there is a unique integrable curve through the point).

By construction, the non-wandering set of $\vec{\boldsymbol{V}}$ consists of the hyperbolic source $\alpha = (1, 0, \dots, 0)$, the hyperbolic sink $\omega = (0, \dots, 0, -1)$, and the hyperbolic saddles $\sigma_1 = (-1, 0, \dots, 0)$, $\sigma_2 = (0, \dots, 0, 1)$. Note that the saddles have different topological indexes $ind(\sigma_1) = (-1)^{n-k}$ and $ind(\sigma_2) = (-1)^{n-k-1}$.

It is easy to see that $W^s(\sigma_1) \cap W^u(\sigma_2) = \emptyset$ while the intersection $W^u(\sigma_1) \cap W^s(\sigma_2)$ is non-empty and consists of the integrable curve passing through the point $(-1, 0, \dots, 0, 1)$.

Indeed, by construction, $W^u(\sigma_1) \cap M_1^n = \{\sigma_1\} \times \mathbb{D}_2^{n-k}$, $W^s(\sigma_2) \cap M_1^n = \mathbb{D}_1^{k+1} \times \{\sigma_2\}$. Hence, $W^u(\sigma_1) \cap \partial M_1^n = \{\sigma_1\} \times \mathbb{S}_2^{n-k-1}$, $W^s(\sigma_2) \cap \partial M_1^n = \mathbb{S}_1^k \times \{\sigma_2\}$. The spheres $\{\sigma_1\} \times \mathbb{S}_2^{n-k-1}$, $\mathbb{S}_1^k \times \{\sigma_2\}$ are intersected transversally at a unique point $\{\sigma_1\} \times \{\sigma_2\}$. This implies that the intersection $W^u(\sigma_1) \cap W^s(\sigma_2)$ is non-empty and consists of the integrable curve passing through the point $(-1, 0, \dots, 0, 1)$. To be precise, the heteroclinic intersection $W^u(\sigma_1) \cap W^s(\sigma_2)$ is the union of the segments $I_1 = \{(x_1, \dots, x_{n+1}) : x_1 = -1, x_2 = 0, \dots, x_n = 0, 0 < x_{n+1} \leq 1\}$, $I_1 = \{(x_1, \dots, x_{n+1}) : -1 \leq x_1 < 0, x_2 = 0, \dots, x_n = 0, x_{n+1} = 1\}$. Except the point $(-1, 0, \dots, 0, 1)$, the intersection $W^u(\sigma_1) \cap W^s(\sigma_2)$ is transversal because of the structure of the vector fields $\vec{\boldsymbol{V}}_1$, $\vec{\boldsymbol{V}}_2$.

A tubular neighborhood U of the common boundary $\mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1}$ of the manifolds M_1^n and M_2^n is homeomorphic to $\mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1} \times (-1; +1)$. One boundary component $C_1 = \mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1} \times \{-1\}$ belongs to M_1^n while another boundary component $C_2 = \mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1} \times \{+1\}$ belongs to M_2^n. One can take U such that the both components C_1, C_2 of the boundary ∂U are transversal to the vector field $\vec{\boldsymbol{V}}$. It allows to smooth out $\vec{\boldsymbol{V}}$ near $\mathbb{S}_1^k \times \mathbb{S}_2^{n-k-1}$ to get a smooth vector field equal to $\vec{\boldsymbol{V}}$ outside of U. This gives a polar Morse-Smale vector field with two saddles on $\mathbb{S}^n$. Hence, $MS(\mathbb{S}^n; 1, 1, 2) \neq \emptyset$.

Now, we have to prove that saddles of any system from the set $MS(\mathbb{S}^n; 1, 1, 2)$ have different Morse indexes. It is sufficient to prove this statement for the Morse-Smale diffeomorphisms $MS^{diff}(\mathbb{S}^n; 1, 1, 2)$, since the desired statement follows for $MS^{flow}(\mathbb{S}^n; 1, 1, 2)$ automatically.

Let $f \in MS^{diff}(\mathbb{S}^n; 1, 1, 2)$. We see that $H_0(\mathbb{S}^n) = H_n(\mathbb{S}^n) = 1$, $H_k(\mathbb{S}^n) = 0$, $1 \le k \le n-1$. It follows that for $\mathbb{S}^n$, the Lefschetz formula (2) gives the following equality

$$1 + (-1)^n = \sum_{p \in Fix\ (f)} Ind_p(f). \tag{3}$$

Denote by α the source and by ω the sink of f. Clearly, $Ind_\alpha(f) = (-1)^n$ and $Ind_\omega(f) = 1$. Therefore, the formula (3) becomes

$$\sum_{p \in Fix\ (f) \setminus (\alpha \cup \omega)} Ind_p(f) = 0. \tag{4}$$

Let σ_i be a saddle of f and μ_i the Morse index of σ_i, $i = 1, 2$. Then (4) implies the equality

$$(-1)^{\mu_1} + (-1)^{\mu_2} = 0. \tag{5}$$

As a consequence, $\mu_1 \ne \mu_2$. Thus, the saddles of $f \in MS^{diff}(\mathbb{S}^n; 1, 1, 2)$ have different Morse indexes. Later on for definiteness, we'll assume $\mu_1 < \mu_2$.

Let us prove that $f \in MS^{diff}(\mathbb{S}^n; 1, 1, 2)$ has both u- and s-king-saddles. First, we show that the following inclusions hold

$$W^u(\sigma_1) \setminus \sigma_1 \subset W^s(\omega), \quad W^s(\sigma_2) \setminus \sigma_2 \subset W^u(\alpha).$$

Indeed, according to [28], a supporting manifold is the union of stable (unstable) invariant manifolds of non-wandering points. Taking in mind $W^s(\alpha) = \alpha$ and $W^u(\omega) = \omega$, one gets

$$\mathbb{S}^n = W^s(\sigma_1) \cup W^s(\omega) \cup W^s(\sigma_2) \cup \{\alpha\} = W^u(\sigma_1) \cup W^u(\alpha) \cup W^u(\sigma_2) \cup \{\omega\}. \tag{6}$$

Recall that stable and unstable manifolds of Morse-Smale diffeomorphism intersect transversally (if an intersection is not empty). Suppose for a moment that $W^u(\sigma_1) \cap W^s(\sigma_2) \ne \emptyset$. Since $W^u(\sigma_1) = \mu_1$ and $\dim W^s(\sigma_2) = n - \mu_2$, $\mu_1 + (n - \mu_2) < n$. Hence, the intersection $W^u(\sigma_1) \cap W^s(\sigma_2)$ is not transversal. Therefore, $W^u(\sigma_1) \cap W^s(\sigma_2) = \emptyset$. Note that Morse-Smale diffeomorphisms have no homoclinic points. It follows from (6) that $W^u(\sigma_1) \setminus \sigma_1 \subset W^s(\omega)$. The second inclusion $W^s(\sigma_2) \setminus \sigma_2 \subset W^u(\alpha)$ is proved similarly.

Due to Proposition 2, the union $S^{\mu_1}_\omega \overset{\text{def}}{=} W^u(\sigma_1) \cup \{\omega\}$ is a topologically embedded μ_1-sphere and the union $S^{n-\mu_2}_\alpha \overset{\text{def}}{=} W^s(\sigma_2) \cup \{\alpha\}$ is a topologically embedded $(n - \mu_2)$-sphere.

To finish the proof we have to prove $W^s(\sigma_1) \cap W^u(\sigma_2) \ne \emptyset$. Suppose the contrary. It follows from (6) that $W^s(\sigma_1)$ has no heteroclinic intersections. By Proposition 2, the union $W^s(\sigma_1) \cap \{\alpha\}$ is a topologically embedded $(n - \mu_1)$-sphere denoted by $S^{n-\mu_1}_1$. Since f is a Morse-Smale diffeomorphism, the invariant manifolds $W^s(\sigma_1)$ and $W^u(\sigma_1)$ intersect transversally at unique point σ_1. Hence, the index of the intersection $S^{\mu_1}_\omega \cap S^{n-\mu_1}_1$ equals -1 or $+1$. This is impossible because of $H_1(\mathbb{S}^n, \mathbb{Z}) = \cdots = H_{n-1}(\mathbb{S}^n, \mathbb{Z}) = 0$. We see that $W^s(\sigma_1) \cap W^u(\sigma_2) \ne \emptyset$. Thus, the saddle with biggest Morse index is an u-king saddle, and the saddle with smallest Morse index is an s-king saddle. This completes the proof. $\square$

Remark. Note that a saddle of Morse-Smale diffeomorphism f can not be u- and s-king saddle simultaneously except it is a unique saddle. Indeed, suppose σ is an u- and s-king saddle, and there is another saddle σ_1. Then $W^s(\sigma) \cap W^u(\sigma_1) \ne \emptyset$ and $W^u(\sigma) \cap W^s(\sigma_1) \ne \emptyset$. Due to Lemma 7.2 [28] (see also Lemma 1.20 [13]), the diffeomorphism f must have infinitely many periodic points. This is impossible for a Morse-Smale diffeomorphism.

Proof of Theorem 2. Let $f_i : M^n \to M^n$ be a Morse-Smale diffeomorphism of closed n-manifold M^n, $n \geq 2$, $i = 1, 2$. We assume that all points of the non-wandering set $NW(f_i)$ are fixed. Suppose for definiteness that f_i has an u-king saddle σ_i, $i = 1, 2$. Obviously, if f_1 and f_2 are topologically conjugate then the invariant manifolds $W^u(\sigma_1)$, $W^u(\sigma_2)$ have the same dynamical embedding.

Let us show that if $W^u(\sigma_1)$, $W^u(\sigma_2)$ have the same dynamical embedding then f_1 and f_2 are topologically conjugate. Since σ_i is an u-king saddle, $W^u(\sigma_i) \cap W^s(\sigma) \neq \emptyset$ for every another saddle σ of f_i. Due to [27], $clos\ W^u(\sigma) \subset clos\ W^u(\sigma_i)$. It follows from [5], Proposition 2.1, that any sink belongs to the topological closure of unstable manifold of some saddle of f_i. As a consequence, $A(f_i) = clos\ W^u(\sigma_i)$. Therefore, the same dynamical embedding of $W^u(\sigma_1)$, $W^u(\sigma_2)$ implies the same dynamical embedding of $A(f_1)$, $A(f_2)$. Due to Proposition 3, the diffeomorphisms f_1, f_2 are topologically conjugate. This completes the proof. □

Proof of Theorem 3. Denote by $\mu_1^{(i)}$ and $\mu_2^{(i)}$ the Morse indexes the saddles $\sigma_1^{(i)}$, $\sigma_2^{(i)}$ respectively, $i = 1, 2$. Due to the conditions, $\mu_1^{(i)} > \mu_2^{(i)}$. According to Theorem 1, the saddle with biggest Morse index is an u-king saddle, and the saddle with smallest Morse index is an s-king saddle. Hence, $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ are u-king saddles of f_1 and f_2 respectively. Similarly, $\sigma_2^{(1)}$ and $\sigma_2^{(2)}$ are s-king saddles of f_1 and f_2 respectively. It follows from Theorem 2 that f_1, f_2 are topologically conjugate if and only if the unstable manifolds $W^u(\sigma_1^{(1)})$, $W^u(\sigma_1^{(2)})$ have the same dynamical embedding, or the stable manifolds $W^s(\sigma_2^{(1)})$, $W^s(\sigma_2^{(2)})$ have the same dynamical embedding. This completes the proof. □

Proof of Theorem 4. Let $\sigma_1, \ldots, \sigma_k$ be the saddles with the Morse index two. By the conditions, f has the same number of sources denoted by $\alpha_1, \ldots, \alpha_k$. Due to [5], the set $R = \cup_{i=1}^k \alpha_i \cup_{j=1}^k \sigma_j$ is a repelling connecter graph. Since the number of sources equals the number of saddles with the Morse index 2, R has a unique simple cycle denoted by C_R. This implies the existence of tubular neighborhood $U(R)$ of R belonging to the repelling domain of R such that $U(R)$ homeomorphic to a sold torus. Denote by $\sigma_{i_1}, \ldots, \sigma_{i_s}$ the saddles belonging to C_R. Taking $U(R)$ sufficiently small if necessary, one can assume that each unstable manifold $W^u(\sigma_j)$, $j = i_1, \ldots, i_s$, has the intersection with the torus $\partial U(R)$ being a meridian of $\partial U(R)$.

According the Lefshetz formula (see, for example [26]), the number of sinks equals the number of saddles with the Morse index one. Therefore, one can construct the attracting set $A = \cup_i \omega_i \cup_j \tilde{\sigma}_j$ with the tubular neighborhood $U(A)$ homeomorphic to a solid torus where ω_i are sinks and $\tilde{\sigma}_j$ are saddles with the Morse index one. Moreover, the stable manifolds of the saddles $\tilde{\sigma}_j$ forming a cycle in A has the intersection with the torus $\partial U(A)$ being a meridian of $\partial U(A)$. Due to [5], the pair A, R forms a global dynamic attractor-repeller of f. Hence, $f^m(\partial U(R)) \subset U(A)$ and $f^{-m}(\partial U(A)) \subset U(R)$ for some $m \in \mathbb{N}$. Clearly, $f^m(\partial U(R))$ does not bound a domain in $U(A) \setminus C_A$, and $f^{-m}(\partial U(A))$ does not bound a domain in $U(R) \setminus C_R$. It follows from [12], Theorem 3.3, that $f^m(\partial U(R))$ is parallel to $\partial U(A)$, and $f^{-m}(\partial U(A))$ is parallel to $\partial U(R)$. As a consequence, the sphere $\mathbb{S}^3$ can be represented as the union $U(A) \cap_\vartheta U(R)$ of the solid torii $U(A)$, $U(R)$ with the identification the their boundaries under the diffeomorphisms $\vartheta : \partial U(A) \to \partial U(R)$. Let μ be a standard meridian on $\partial U(A)$. Then $\vartheta(\mu)$ have to intersect a standard meridian on $\partial U(R)$. Therefore, there exists an unstable manifold $W^u(\sigma_{i_l})$ that intersects a stable manifold $W^s(\tilde{\sigma}_r)$ of some saddles σ_{i_l}, $\tilde{\sigma}_r$. Hence, f has heteroclinic intersections. □

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