

Lovász theta approach to eventown problem

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Abstract

We apply the bound on independence number via Lovász theta function to eventown problem and its generalizations over \mathbb{Z}_n .

Keywords: graph coloring, Lovász theta function, eventown problem.

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1 Introduction

The following bound was introduced by Lovász in his celebrated paper on Shannon capacity. In particular it implies famous Hoffman bound [8].

Theorem 1 (Lovász, [12]). *Let $G = (V, E)$ be a simple graph. Consider a symmetric real matrix A such that $A_{ij} = 1$ for every pair $\{i, j\} \notin E(G)$. Then*

$$\alpha(G) \leq \lambda_{max}(A),$$

where $\alpha(G)$ is the size of a maximal independent set of G and $\lambda_{max}(A)$ is the maximal eigenvalue of A .

The minimum of $\lambda_{max}(A)$ over the appropriate A is called *Lovász number* or *Lovász theta function* of a graph.

Also we need the following corollaries. Suppose that A and G satisfy the conditions of Theorem 1. Let c , λ_{max} and sp stand for the minimal entry, the maximal eigenvalue and the spectral radius of A , respectively.

Corollary 1. *Let I be a set with at most $\varepsilon|I|^2/2$ edges inside. Suppose that $(1 - c)\varepsilon < 1$. Then*

$$|I| \leq \frac{\lambda_{max}}{1 - (1 - c)\varepsilon}.$$

Corollary 2. *Let I and J be subsets of $V(G)$ with at most $\varepsilon|I| \cdot |J|$ edges between I and J (edges in $I \cap J$ are counted twice here). Suppose that $(1 - c)\varepsilon < 1$. Then*

$$|I| \cdot |J| \leq \left(\frac{sp}{1 - (1 - c)\varepsilon} \right)^2.$$

For the bounds on *disjoint* I and J one can make the class of appropriate matrices slightly wider, i.e. not demand $A_{ii} = 1$. Then one may combine the proof of Corollary 2 with Proposition 4.1 in the paper of Haemers [7].

Structure of the paper. Subsections 1.1 and 1.2 contain our results. Section 2 is devoted to proofs; we also provide the proof of Theorem 1 in the interest of completeness. We discuss relation to several subjects and propose some further questions in Section 3.

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1.1 A straightforward application to eventown problem

Let F be a family of subsets of $[n]$ is *eventown* if the intersection of any two members is even (in particular all sets have even size). Berlekamp [2] and Graver [6] independently proved F has at most $2^{\lfloor n/2 \rfloor}$ members, which is also best possible. The proof is very short up to general linear algebra. Note that every maximal eventown F is a linear subspace of \mathbb{F}_2^n ; otherwise one can replace F with $\text{span } F$. Since F lies in the orthogonal complement F^\perp and $\dim F + \dim F^\perp = n$, $|F|$ has the dimension at most $\lfloor n/2 \rfloor$.

Consider the following Hadamard matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Its spectrum is $\{\pm\sqrt{2}\}$. Then the spectrum of $M = A^{\otimes n}$ is $\{\pm 2^{\frac{n}{2}}\}$. Let us consider $G = (2^{[n]}, E)$ and $(X, Y) \in E$ iff $|X \cap Y|$ is odd. Then we identify elements of $2^{[n]}$ with $\{0, 1\}^n$ by usual way as well as indices for rows and columns of matrix M (where we mean that $A = (a_{ij})_{i,j \in \{0,1\}}$). Let $M = (m_{st})_{s,t \in \{0,1\}^n}$. Then for each $X, Y \in 2^{[n]}$ we have

$$m_{\chi(X), \chi(Y)} = \prod_{r=1}^n a_{(\chi(X))_r, (\chi(Y))_r} = (-1)^{|\{r \in \{1, \dots, n\} | a_{(\chi(X))_r, (\chi(Y))_r} = -1\}|} = (-1)^{|X \cap Y|}.$$

Thus we see that graph G and matrix M satisfy the conditions of Theorem 1.

Applying Theorem 1 one has $|F| \leq 2^{n/2}$. For even n we already get another proof of eventown theorem. For even n we should also recall that F is a linear subspace, so $|F| \leq 2^{\lfloor n/2 \rfloor}$.

Let $op(F)$ denote the number of distinct pairs $f_1, f_2 \in F$ for which $|f_1 \cap f_2|$ is odd. O'Neill [13] showed that for $1 \leq s \leq 2^{\lfloor n/2 \rfloor} - 2^{\lfloor n/4 \rfloor}$ there is a family F with $|F| = 2^{\lfloor n/2 \rfloor} + s$ and $op(F) = s \cdot 2^{\lfloor n/2 \rfloor - 1}$. Also he conjectured that this example is tight and proved the conjecture for $s = 1, 2$. The application of Corollary 1 gives twice weaker bound for even n (and much weaker bound for odd n).

Theorem 2. *Let $|F| = 2^{n/2} + s$ for some integer s . Then*

$$op(F) \geq s \cdot 2^{\lfloor \frac{n}{2} \rfloor - 2}.$$

1.2 An application to k -town problem

Let F be a family of vectors from $\{0, \dots, k-1\}^n$, such that $(f_1, f_2) = 0 \pmod{k}$ for any $f_1, f_2 \in F$ (in particular for $f_1 = f_2$); such F is further called a *k -town family*.

First, for prime k the classical argument gives the tight upper bound $k^{\lfloor n/2 \rfloor}$.

If k is square free we can obtain the same bound from the prime k case in the following way. Let us see that the case $k = pq$ is a formal consequence of the cases $k = p$, $k = q$ if p, q are coprime. Indeed let $F \subset (\mathbb{Z}/pq\mathbb{Z})^n$ such that $(f_1, f_2) = 0$ for each $f_1, f_2 \in F$. Then by assumption the number of residues modulo p for elements F does not exceed $p^{\frac{n}{2}}$ and number of residues modulo q for elements F does not exceed $q^{\frac{n}{2}}$. Then by the Chinese remainder theorem $|F| \leq p^{\frac{n}{2}} q^{\frac{n}{2}}$.

The observations above should be folklore, meanwhile we do not know how to prove the first inequality in the following theorem without spectral graph theory for an arbitrary k .

Theorem 3. *If F is a k -town family then*

$$|F| \leq k^{\frac{n}{2}}.$$

Moreover suppose that k is prime, and $(f_1, f_2) \neq 0 \pmod{k}$ for at most $\varepsilon|F|^2$ pairs $f_1, f_2 \in F$. If $\varepsilon < \frac{k-1}{k}$ then

$$|F| \leq \frac{k^{\frac{n}{2}}}{1 - \frac{k}{k-1}\varepsilon}.$$

Obtaining an example of k -town F with $|F| = k^{\frac{n}{2}}$ in the case of prime k and even n is equivalent to finding a set of $\frac{n}{2}$ pairwise (and-self) orthogonal linear independent vectors in $(\mathbb{Z}/k\mathbb{Z})^n$. There are a lots of such sets; for example for a prime $k = 4t+1$ one can consider vectors of the form $v_j = e_{2j-1} + \varepsilon^t e_{2j}$ for $1 \leq j \leq \frac{n}{2}$ (here $\{e_j\}$ — is the standard basis in $(\mathbb{Z}/k\mathbb{Z})^n$, ε is a primitive root in $\mathbb{Z}/k\mathbb{Z}$).

For a general prime k one can choose v_1, v_2, \dots inductively and almost arbitrarily such that $v_j \in \langle v_1, \dots, v_{j-1} \rangle^\perp \setminus \langle v_1, \dots, v_{j-1} \rangle$ and $(v_j, v_j) = 0$. This can be done: indeed, for $j < \frac{n}{2} - 2$ we can choose arbitrary $4 < \dim \langle v_1, \dots, v_{j-1} \rangle^\perp - \dim \langle v_1, \dots, v_{j-1} \rangle$ linearly independent vectors and find v_j in their span (as any quadratic form with ≥ 3 variables over finite field has an isotropic vector). When $j = \lfloor \frac{n}{2} \rfloor - 2$, we can choose 4-dimensional $V \subset (\langle v_1, \dots, v_{j-1} \rangle^\perp \setminus \langle v_1, \dots, v_{j-1} \rangle) \cup 0$ as well, and it is also well-known that there exists $v_{\frac{n}{2}-1}, v_{\frac{n}{2}} \in V$ such that $(v_{\frac{n}{2}-1}, v_{\frac{n}{2}-1}) = (v_{\frac{n}{2}-1}, v_{\frac{n}{2}}) = (v_{\frac{n}{2}}, v_{\frac{n}{2}}) = 0$.

In some cases were k is non-prime we can obtain examples of different nature. For example, when $k = m^2$ for some integer m one can consider the set of vectors of the form (mx_1, \dots, mx_n) . This example shows that in the case of k being a perfect square the first inequality of Theorem 3 is also tight for an odd n .

Suppose that we are interested in the scalar product t instead of 0. Then the statement of Theorem 3 can be slightly improved.

Corollary 3. Let F be a family of vectors from $\{0, \dots, k-1\}^n$, and $(f_1, f_2) = t \pmod{k}$ for every $f_1, f_2 \in F$. Then

$$|F| \leq c(t, k) \cdot k^{\frac{n}{2}}$$

for some $\frac{1}{\sqrt{2}} < c(t, k) \leq 1$. Moreover if $\frac{k}{\gcd(k, t)}$ tends to infinity $c(t, k)$ tends to $\frac{1}{\sqrt{2}}$ (gcd stands for the greatest common divisor).

Assume also that k is prime and $(f_1, f_2) \neq t \pmod{k}$ for at most $\varepsilon|F|^2$ pairs $f_1, f_2 \in F$ for some $t \neq 0$. If $\varepsilon < \frac{k-1}{k}$ then

$$|F| \leq c(k) \frac{k^{\frac{n}{2}}}{1 - \frac{k}{k-1}\varepsilon},$$

where $c(k) < 1$ and $c(k) \rightarrow \frac{2\sqrt{2}}{\pi}$ with $k \rightarrow \infty$.

2 Proofs

Proof of Theorem 1. Let I be an independent set, denote by χ_I its characteristic vector. Then

$$(A\chi_I, \chi_I) = |I|^2.$$

From the other hand, one has

$$(A\chi_I, \chi_I) = \sum c_i^2 \lambda_i \leq \left(\sum c_i^2 \right) \lambda_{max} = |I| \cdot \lambda_{max},$$

where $\chi_I = \sum c_i v_i$ is the decomposition of χ_I via orthonormal eigenbasis $\{v_i\}$ of A . (We use that symmetric matrix has real spectrum and the length of χ_I is the same in the standard basis and in $\{v_i\}$, i.e. $\sum c_i^2 = |I|$) \square

Proof of Corollary 1. Denote by χ_I the characteristic vector of I . Then

$$(A\chi_I, \chi_I) \geq |I|^2 \cdot (1 - (1-c)\varepsilon).$$

From the other hand, one has

$$(A\chi_I, \chi_I) = \sum c_i^2 \lambda_i \leq \left(\sum c_i^2 \right) \lambda_{max} = |I| \cdot \lambda_{max},$$

where $\chi_I = \sum c_i v_i$ is the decomposition of χ_I via orthonormal eigenbasis $\{v_i\}$ of A . \square

Proof of Corollary 2. Denote by χ_I and χ_J the characteristic vectors of I and J respectively. Then

$$(A\chi_I, \chi_J) \geq |I| \cdot |J| \cdot (1 - (1-c)\varepsilon).$$

From the other hand, one has

$$(A\chi_I, \chi_J) = \sum c_i d_i \lambda_i \leq \left(\sum |c_i| \cdot |d_i| \right) \text{sp} \leq \sqrt{\left(\sum c_i^2 \right) \left(\sum d_i^2 \right)} \cdot \text{sp} = \sqrt{|I| \cdot |J|} \cdot \text{sp},$$

where $\chi_I = \sum c_i v_i$ and $\chi_J = \sum d_i v_i$ are the decompositions of χ_I and χ_J via orthonormal eigenbasis $\{v_i\}$ of A . \square

Proof of Theorem 3. Consider the following $k \times k$ matrix:

$$a_{jl} = \phi^{jl},$$

where ϕ is a primitive k -th root of unity and $0 \leq j, l \leq k-1$ and the matrix $M = A^{\otimes n}$. Then we have $A^4 = k^2 E$, therefore $|\lambda| = (k^{\frac{1}{2}})^n = k^{\frac{n}{2}}$ for each eigenvalue λ of M , i.e., $\text{sp}(M) = k^{\frac{n}{2}}$.

Let us see that A (and, as a consequence, M) has an eigenbasis in \mathbb{R}^n . Let $\{e_i\}$ be a standard basis. Then moving to the basis $\{e_1, \frac{1}{\sqrt{2}}(e_j \pm e_{k+2-j})\}$ or to $\{e_1, e_{\frac{k}{2}}, \frac{1}{\sqrt{2}}(e_j \pm e_{k+2-j})\}$ for even k (note that it is a unitary transformation) we obtain a block matrix with two blocks of the form A_1, iA_2 , where A_1, A_2 are the real symmetric matrices (they have the sizes $\frac{k+1}{2}$ и $\frac{k-1}{2}$ respectively for odd k and $\frac{k}{2} + 1, \frac{k}{2} + 1$ for even k). As a matter of fact, for each $2 \leq j \leq k$ we have

$$A(e_j + e_{k+2-j}) = \sum_{l=1}^k \phi^{(j-1)(l-1)} e_l + \sum_{l=1}^k \phi^{(k+1-j)(l-1)} e_l = 2 + \sum_{l=2}^k (\phi^{(j-1)(l-1)} + \overline{\phi^{(j-1)(l-1)}}) e_l = 2 + \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} 2 \cdot \text{Re}(\phi^{(j-1)(l-1)}) (e_l + e_{k+2-l})$$

and for each $2 \leq j \leq k, j \neq \frac{k}{2}$

$$A(e_j - e_{k+2-j}) = \sum_{l=1}^k \phi^{(j-1)(l-1)} e_l - \sum_{l=1}^k \phi^{(k+1-j)(l-1)} e_l = \sum_{l=2}^k (\phi^{(j-1)(l-1)} - \overline{\phi^{(j-1)(l-1)}}) e_l = i \cdot \sum_{l=2}^{\lfloor \frac{k}{2} \rfloor} 2 \cdot \text{Im}(\phi^{(j-1)(l-1)}) (e_l - e_{k+2-l}).$$

Thus we see that this change of the base (over \mathbb{R}) leads to real symmetric matrix and to pure imaginary symmetric matrix, which both have real eigenbasis, therefore A has real eigenbasis as well. Hence $N := \operatorname{Re} M$ has the same real eigenbasis with M . Obviously, all eigenvalues of N lie in $\{\pm k^{n/2}, 0\}$.

Let us consider $G = ((\mathbb{Z}/k\mathbb{Z})^n, E)$, where $(X, Y) = ((x_1, \dots, x_n), (y_1, \dots, y_n)) \in E$ iff $\sum_{r=1}^n x_r y_r \neq 0$. Then we also identify $(\mathbb{Z}/k\mathbb{Z})^n$ as indices for rows and columns of matrix M (taking all indices in A modulo k). Let $M = (m_{st})_{s,t \in \{0, \dots, k-1\}^n}$. Now for each $X, Y \in ((\mathbb{Z}/k\mathbb{Z})^n, (X, Y) \notin E$ we have

$$m_{X,Y} = \prod_{r=1}^n a_{x_r, y_r} = \phi^{\sum_{r=1}^n x_r y_r} = \phi^0 = 1.$$

Thus graph G and matrix M (and also N) satisfy the conditions of Theorem 1, so the first statement of the theorem is proved.

Now let k be a prime number. An immediate application of Corollary 1 gives

$$|F| \leq \frac{k^{\frac{n}{2}}}{1 - t\varepsilon},$$

where $t = 1 - \cos\left(\left\lfloor \frac{k}{2} \right\rfloor \frac{2\pi}{k}\right)$, which tends to 2 with $k \rightarrow \infty$. So we modify the proof of Corollary 1 in the following way.

For every root ϕ the matrix $N = N(\phi)$ satisfies

$$(N(\phi)\chi_F, \chi_F) = \sum_{i,j \in F} N_{ij}(\phi)$$

and

$$(N(\phi)\chi_F, \chi_F) \leq |F| \cdot k^{\frac{n}{2}}.$$

Summing up these inequalities for all k -th roots except 1 one has

$$(k-1) \cdot |F|^2 \cdot (1 - \varepsilon) - \varepsilon \cdot |F|^2 \leq (k-1) \cdot |F| \cdot k^{\frac{n}{2}} \quad (1)$$

since for every i, j that corresponds to sets with nonzero scalar product

$$\sum_{\phi \neq 1} N_{ij}(\phi) = -1,$$

and for i, j that corresponds to sets with zero scalar product

$$\sum_{\phi \neq 1} N_{ij}(\phi) = k - 1.$$

Rewriting (1) finishes the proof. □

Proof of Corollary 3. Fix a primitive k -th root of unity ϕ and consider the same matrices A and M as in the previous proof. Let us consider $G = ((\mathbb{Z}/k\mathbb{Z})^n, E)$, where $(X, Y) = ((x_1, \dots, x_n), (y_1, \dots, y_n)) \in E$ iff $\sum_{r=1}^n x_r y_r \neq t$. Then G and $N := \operatorname{Re}(\phi^t M)$ satisfy the conditions of Theorem 1. Note that N and M shares a real eigenbasis, so all eigenvalues of N lie in $\operatorname{Re}\{\pm \phi^t k^{n/2}, \pm \phi^t k^{n/2} i\}$. Hence the spectral radius of N lies between $\frac{1}{\sqrt{2}} k^{n/2}$ and $k^{n/2}$.

Lets, check the second part of first proposition. Note that for any coprime x, y we have $\frac{xy}{\gcd(xy, t)} = \frac{x}{\gcd(x, t)} \cdot \frac{y}{\gcd(y, t)}$ and $c(t, xy) \leq c(t, x)c(t, y)$ – this follows from remarks before Theorem 3. Therefore we need only treat the case $k = p^s$ for prime p .

Note that any upper bound for $|F|$ with some $t = t_0$ is also an upper bound for $t = t_0 r^2, r \in (\mathbb{Z}/k\mathbb{Z})^*$. Indeed, if $(f_1, f_2) = t_0 r^2$ for each $f_1, f_2 \in F$, then $(\frac{1}{r} f_1, \frac{1}{r} f_2) = t_0$ and we can apply a bound for $t = t_0$. Now we are to show that we can choose $r = r(k, t)$ such that $|\operatorname{Re}(\phi^{tr^2})|$ tends to $\frac{1}{\sqrt{2}}$ if $\frac{k}{\gcd(k, t)}$ tends to infinity – in this case $|\operatorname{Re}(\phi^{tr^2} i^l)|$ tends to $\frac{1}{\sqrt{2}}$ as well.

To obtain this, it's sufficient to choose r such that $|\frac{tr^2}{k/8}|$ tends to 1 (with suitable choice of representative of tr^2 modulo k).

Any $t \in \mathbb{Z}/p^s\mathbb{Z}$ we can consider as $t = p^m t', r \in \mathbb{Z}/p^{s-m}\mathbb{Z}$ and replacing (t, p^s) with (r, p^{s-m}) (this doesn't change both ratios in question) we can assume that $t \in (\mathbb{Z}/p^s\mathbb{Z})^*$ (and $\frac{k}{\gcd(k, t)} = k$).

Let $p > 2$. Let's prove that we can choose l such that $|l| < k^{\frac{1}{2}}$ and $t = lw^2$ modulo k for some w . Suppose that t is non-square modulo k (else we can take $l = 1$). If $p = 2$, then all non-squares are known to be of the form $3w^2, 5w^2$ or $7w^2$ for some w , so we can take $l \in \{1, 3, 5, 7\}$ (when $s > 5$). If $p > 2$ we can take l_1, l_2 with $|l_i| < \sqrt{p}$ such that $l_1 = l_2 t$ modulo p by Thues lemma. Then for some i l_i is a non-square modulo m and therefore modulo $k = p^s$. As t is non square as well we can write $t = l_i w^2$ and take $l = l_i$.

Now take $x \in \mathbb{Z}$ such that $x^2 \in \left(\frac{k}{8|l|} - \left(\frac{k}{|l|}\right)^{\frac{1}{2}}, \frac{k}{8|l|} + \left(\frac{k}{|l|}\right)^{\frac{1}{2}}\right)$, then $||lx^2| - \frac{k}{8}| \leq (kl)^{\frac{1}{2}} < k^{\frac{3}{4}}$ and $|\frac{lx^2}{k}|$ tends to 1 when k tends to infinity, as desired (we put $r = \frac{x}{w}$ modulo k).

Now we prove the second part of Corollary. Repeat the proof of the corresponding part of Theorem 3. Then the right part of (1) is replaced with

$$\sum_{a=1}^{k-1} \text{sp Re}(\phi^{ta} M).$$

Note that for every a

$$\begin{aligned} \text{sp Re}(\phi^{ta} M) &= \max(\text{Re}(\phi^{ta}), \text{Re}(\phi^{ta} \cdot i), -\text{Re}(\phi^{ta}), -\text{Re}(\phi^{ta} \cdot i)) \cdot k^{n/2} = \\ &= \max\left(\left|\cos \frac{2\pi ta}{k}\right|, \left|\cos\left(\frac{2\pi ta}{k} + \frac{\pi}{2}\right)\right|, \left|\cos\left(\frac{2\pi ta}{k} + \pi\right)\right|, \left|\cos\left(\frac{2\pi ta}{k} + \frac{3\pi}{2}\right)\right|\right) k^{n/2}. \end{aligned}$$

After summation over $1 \leq a \leq k-1$ and taking $k^{n/2}$ out of the brackets and dividing by $k-1$, one has a Riemann sum for the following integral

$$\frac{\int_0^{\pi/4} \cos(x) dx}{\pi/4} = \frac{2\sqrt{2}}{\pi}.$$

□

3 Discussion

3.1 Related subjects

It turns out that Theorem 3 is closely related with several well-studied themes.

Erdős distance problem over a finite field. Let \mathbb{F}_q be a finite field. Hart and Iosevich [9] proved that if $t \neq 0$ and $E \subset \mathbb{F}_q^n$, then

$$|\{(x, y) \in E \times E : x \cdot y = t\}| = \frac{|E|^2}{q} + R(E),$$

where

$$|R(E)| \leq q^{\frac{n-1}{2}} |E|.$$

In the notation of Theorem 3 (for this part we assume k to be prime) it implies inequality

$$|F| \leq \frac{k^{\frac{n-1}{2}}}{\frac{k-1}{k} - \varepsilon}.$$

This bound is better than Theorem 3 for a large prime k , but it essentially requires the condition $t \neq 0$. The bound for $t = 0$ is obtained in [10] and is slightly weaker than Theorem 3.

Define the distance in an n -dimensional vector space over a finite field as

$$\text{dist}(\bar{x}, \bar{y}) = (x_1 - y_1)^2 + \dots + (x_n - y_n)^2,$$

i.e. we omit the square root in the standard formula. Let $\Delta(E)$ be the set of all distances between points in E . The Erdős distance problem is to determine the smallest possible cardinality of the set $\Delta(E)$ over all sets E of a given size.

In the case of prime $k > 2$ Theorem 3 gives the following elementary bound which is far from the Iosevich–Rudnev estimate [11]. Let E be a set with $|\Delta(E)| = s$. Without loss of generality $(0, \dots, 0) \in E$; otherwise one may shift E . Then E lies on s spheres centered at $(0, \dots, 0)$. By a pigeon-hole principle there is a sphere containing a subset $E' \subset E$ of size at least $(|E| - 1)/s$. Since k is odd and E' lie on a sphere, there are at most s different scalar products between vectors of E' . Let t be a most popular scalar product in $E' \times E'$. Then one may apply Theorem 3 with $\varepsilon = \frac{s-1}{s}$ and receive that

$$|E'| \leq \frac{k^{n/2}}{1 - \frac{k}{k-1} \frac{s-1}{s}} = \frac{s(k-1)k^{n/2}}{k-s}.$$

Hence

$$|E| \leq \frac{s^2(k-1)k^{n/2}}{k-s} + 1.$$

For instance, $s = k-1$ means that we evaluate the size of a set without a single distance r by $k^{\frac{n+6}{2}}$, meanwhile Iosevich–Rudnev estimate is $Ck^{\frac{n+1}{2}}$ with an absolute constant $C > 0$.

A bound via singular numbers. Let $G = (V, E)$ be a simple graph. Suppose that A is a complex matrix such that $A_{ij} = 1$ for every pair $\{i, j\} \notin E(G)$. Then

$$\alpha(G) \leq \sigma_{\max}(A),$$

where $\alpha(G)$ is the size of a maximal independent set of G and $\sigma_{\max}(A)$ is the maximal singular value of A (i.e. the square root of a maximal eigenvalue of a self-adjoint operator A^*A , where A^* denotes the adjoint of A). The proof immediately follows from the main theorem of [3].

For matrix A in the proof of Theorem 3 one has

$$A^*A = kE,$$

so every singular number of matrix A is equal to \sqrt{k} . Hence every singular number of $M = A^{\otimes n}$ is equal to $k^{n/2}$. This implies the first inequality of Theorem 3.

Hypergraph discrepancy and asymptotic precision of Theorem 2 when ε is close to $1/2$. A hypergraph is a pair (V, E) , where V is a finite set whose elements are called vertices and E is a family of subsets of V , called edges. A vertex 2-coloring of a hypergraph (V, E) is a map $\pi : V \rightarrow \{\pm 1\}$. A *discrepancy* of a coloring π is the largest value of $|\sum_{v \in e} \pi(v)|$ over $e \in E$.

Consider an explicit hypergraph $H = (V, E)$, where $V = [N] \times [N]$ and edges have form $I \times J$ for $I, J \subset [N]$. Note that every $\{\pm 1\}$ matrix of size N produces a 2-coloring of H . Let $N = 2^n$ and χ be the coloring from $A^{\otimes n}$ (recall that A is 2×2 Hadamard matrix). Let $I \times J$ be an edge of H providing the discrepancy disc of χ ; without loss of generality let disc be positive. Then

$$\text{disc} = (A^{\otimes n} \chi_I, \chi_J) = (1 - 2\varepsilon) \cdot |I| \cdot |J|,$$

where ε satisfies the conditions of Theorem 2. Theorem 2 implies

$$\text{disc}^2 = |I|^2 \cdot |J|^2 (1 - 2\varepsilon)^2 \leq 2^n |I| \cdot |J| \leq 2^{3n} = N^3. \quad (2)$$

Astashkin proved [1] that H has the discrepancy at least $cN^{3/2}$ for every coloring χ and some absolute constant $c > 0$, i.e. inequality (2) is precise up to an absolute constant. It means that Theorem 2 is precise up to an absolute constant in the case when $|I|$ and $|J|$ are close to N and $\frac{1}{2} - \varepsilon$ is of order $2^{-n/2}$.

3.2 Further questions

Generalization on k -eventown. We say that $F \subset 2^{[n]}$ is k -eventown if the size of the intersection of any (not necessarily different) $f_1, f_2 \in F$ is zero modulo k . The problem of determine the maximal size of k -eventown was studied by Frankl and Odlyzko [4]. They found a nice construction based on Hadamard matrices of k -eventowns of size at least $(ck)^{\lfloor n/(4k) \rfloor}$, where $c > 0$ is an absolute constant. In addition, they showed that any k -eventown has size at most $2^{O(\log k/k)n}$ as n tends to ∞ .

In particular, for $k = 3$ the best known lower and upper bounds are $24^{\lfloor n/12 \rfloor}$ and $2^{\lfloor n/2 \rfloor}$ respectively.

Generalization on t -wise k -eventown. We say that $F \subset 2^{[n]}$ is t -wise k -eventown if the size of the intersection of any different $f_1, \dots, f_t \in F$ is zero modulo k . Note that a 2-wise k -eventown is not the same as an k -eventown, since in the former we do not require that the sets themselves have size zero modulo k .

Sudakov and Vieira [14] show that a t -wise eventown has for $t \geq 3$ a unique extremal configuration and obtain a stability result for this problem. Gishboliner, Sudakov and Tomon [5] show that for every k there is $t = t(k)$ that the size of any t -wise k -eventown is bounded by

$$|F| \leq 2^{\lfloor n/k \rfloor} + \text{const}(k, t).$$

A generalization of the methods of the paper, if exists, will have a deal with some tensor analysis.

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