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Resonance scattering in a waveguide with identical thick perforated barriers

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ABSTRACT

We consider wave propagation across an infinite waveguide of an arbitrary bounded cross-section, whose interior is blocked by two identical thick perforated barriers with holes. When the holes are small, the waves over a broad range of frequencies are almost fully reflected. However, we show the existence of a resonance frequency at which the wave is almost fully transmitted, even for very small holes. Counter-intuitively, this resonance effect occurs for barriers of arbitrary thickness. We also discuss another asymptotic limit, in which the thickness of barriers grows to infinity but the fixed diameter of the holes can be large and even arbitrarily close to the diameter of the waveguide. The resonance scattering, which is known as tunneling effect in quantum mechanics, is demonstrated in a constructive way by rather elementary tools such as separation of variables and matching of the resulting series, in contrast to commonly used abstract methods such as searching for complex-valued poles of the scattering matrix or non-stationary scattering theory. In particular, we derived an explicit equation that determines the resonance frequency. The employed basic tools make the paper accessible to non-experts and educationally appealing.

1. Introduction

Since classic works by Rayleigh [1], scattering in waveguides is known to exhibit resonance features. The resonance character can be curvature-induced, be related to scattering near the waveguide cut-off frequency or to a resonance in the waveguide cross-section, or originate from obstacles forming a resonator that is weakly coupled to the waveguide [2–10]. In the latter case, the resonator can be either joint to the waveguide from outside, or made inside. The former setting is used in mufflers: incoming waves at frequencies far from the resonance one are almost fully transmitted; in contrast, the waves near the resonance frequency are almost fully reflected. In the second setting, when barriers with small holes are inserted inside the waveguide, the situation is different. If there is a single barrier with Dirichlet boundary condition, the incoming wave cannot "squeeze" through a small hole and is thus almost fully reflected. Intuitively, putting more barriers might seem to help further blocking the wave transmission. However, if there are two barriers, they can form a resonator,

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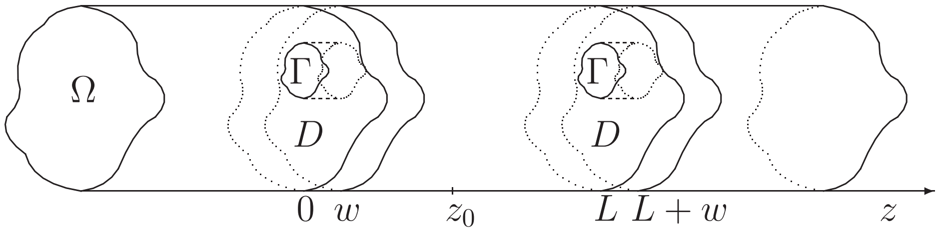

Fig. 1. Infinite cylinder $Q = \Omega \times \mathbb{R}$ of a bounded cross-section$\Omega \subset \mathbb{R}^2$, with two identical thick barriers ofcross-section D and thickness w, separated by distance $L - w$.Each barrier has a hole of cross-section $\Gamma = \Omega \backslash D$.

which is coupled to the waveguide, so that an incoming wave at the resonance frequency can be almost fully transmitted. This somewhat counter-intuitive effect in acoustics is known as tunneling effect in quantum mechanics [11].

In spite of a large amount of works on resonance scattering in physics literature, most of them were focused on approximate computations of the wave transmission coefficients (see, e.g., [12]). In turn, mathematical aspects of resonance scattering of the last type have been less studied (see [3,6,7] and references therein). One can also mention several works by Arsen'ev [13–15] who applied non-stationary scattering theory. While the geometric structure of the problem can be rather general, the derived results are typically formulated in the form of an alternative: either scattering is resonant at a given frequency, or this frequency corresponds to a trapping mode. Another technique of asymptotic expansions was applied by Sarafanov and co-workers [16,17] in order to analyze the limiting behavior in planar waveguides when the size of a hole in two barriers goes to zero.

In the present paper, we provide a much simpler analysis of the resonance scattering problem for the case of a waveguide of arbitrary constant bounded cross-section with two identical barriers that are perpendicular to the waveguide axis. This problem has two small parameters: the size of the hole and the difference between the wave frequency and the resonance frequency. Our goal is to reveal how these two parameters should be related to ensure wave transmission. In particular, we show that the width of barriers can be arbitrary large that may have interesting acoustic and electromagnetic applications (see Section 5). Moreover, we briefly discuss the limit of very trick barriers when the diameter of holes can be large and even comparable to the diameter of the waveguide. While former studies of resonance transmission commonly relied on the notion of resonances (i.e., complex-valued poles of the scattering matrix), we do not use this notion that facilitates all the proofs. In fact, our proofs are constructive and conceptually simple, even though some formulas are cumbersome. Showing a possibility of such mathematically simple proofs in resonance scattering problems is one of the educational goals of this paper.

2. Formulation and main result

Let us consider scattering in a waveguide $Q_0 \subset \mathbb{R}^{d+1}$ of a bounded cross-section $\Omega \subset \mathbb{R}^d$, which contains two identical barriers of thickness w separated by distance $L - w$: $D \times (0, w)$ and $D \times (L, L+w)$, with $D \subset \Omega$ (Fig. 1):

$$Q_0 = (\Omega \times \mathbb{R}) \backslash \Big((D \times (0, w)) \cup (D \times (L, L+w)) \Big). \tag{1}$$

We study wave propagation through the waveguide Q_0 when the barriers are closing, i.e., the opening part of the barriers, $\Gamma = \Omega \backslash D$, is vanishing. As a similar problem for infinitely thin barriers was studied in [18], the main focus and novelty of the present paper is a finite arbitrary thickness w of barriers. The other limiting situation when $w \to \infty$ for a fixed and possibly large Γ will be discussed in Section 5.

We consider the stationary wave equation

$$\Delta u + k^2 u = 0 \qquad \text{in } Q_0, \tag{2}$$

with Dirichlet boundary condition on the waveguide walls and on the barriers,

$$u|_{\partial Q_0} = 0, \tag{3}$$

and standard radiation conditions

$$u(\boldsymbol{x}, z) = e^{i\gamma_1 z}\psi_1(\boldsymbol{x}) + r_1 e^{-i\gamma_1 z}\psi_1(\boldsymbol{x}) + \sum_{n=2}^{\infty} r_n e^{\gamma_n z}\psi_n(\boldsymbol{x}) \qquad (z < 0), \tag{4a}$$

$$u(\boldsymbol{x}, z) = t_1 e^{i\gamma_1 z}\psi_1(\boldsymbol{x}) + \sum_{n=2}^{\infty} t_n e^{-\gamma_n z}\psi_n(\boldsymbol{x}) \qquad (z > L+w), \tag{4b}$$

where r_n and t_n are unknown reflection and transmission coefficients, points in Q_0 are written as $(\boldsymbol{x}, z)$ (with $\boldsymbol{x} \in \Omega$ being the transverse coordinate and $z \in \mathbb{R}$ the longitudinal coordinate along the waveguide axis), $\psi_n(\boldsymbol{x})$ and λ_n are the $L_2(\Omega)$-normalized eigenfunctions and eigenvalues of the Laplace operator in the cross-section Ω:

$$-\Delta \psi_n = \lambda_n \psi_n, \qquad \psi_n|_{\partial\Omega} = 0 \quad (n = 1, 2, 3, \ldots), \tag{5}$$

and

$$\gamma_1 = \sqrt{k^2 - \lambda_1}, \quad \gamma_n = \sqrt{\lambda_n - k^2} \qquad (n \geq 2). \tag{6}$$

The reflection coefficients can be expressed by using the orthogonality of eigenfunctions $\{\psi_n\}$:

$$1 + r_1 = \left(u|_{z=0}, \psi_1\right)_{L_2(\Omega)}, \qquad r_n = \left(u|_{z=0}, \psi_n\right)_{L_2(\Omega)}. \tag{7}$$

In this paper, we prove the following result.

Theorem 1. *Let $\delta = \mathrm{diam}\{\Gamma\}$ be the diameter of the opening part Γ of the barriers. For any fixed wavelength k between $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$, we show that*

$$\lim_{\delta \to 0} r_1 = -1, \tag{8}$$

i.e., the wave is fully reflecting in the limit of closed barriers. In turn, for any non-empty Γ with $\delta > 0$ small enough, there exists a resonance wavelength k_D at which

$$r_1 \approx 0, \tag{9}$$

i.e., the wave is almost fully propagating across 2 almost closed barriers. In other words, for any $\varepsilon > 0$ there exists $\delta' > 0$ such that for any Γ with $\mathrm{diam}\{\Gamma\} < \delta'$, there exists k_D such that $|r_1| < \varepsilon$.

Moreover, as our proof is constructive, we will derive an equation, from which the resonance wavelength k_D can be found.

3. Derivation

We consider weak solutions of Eq. (2) from $H^{1,loc}(Q_0)$, i.e., the restriction of the solution to any finite subdomain Q' of Q_0 should belong to $H^1(Q')$. Moreover, the series determining the solution should converge in $L_2(\Omega)$. Under standard conditions on the boundary ∂Q, these solutions are smooth up to regular parts of the boundary.

3.1. Reduction to two single-barrier problems

First, we show that the original problem can be reduced to two problems in a half cylinder with a single barrier:

$$Q = (\Omega \times (-\infty, z_0)) \backslash (D \times (0, w)), \tag{10}$$

where $z_0 = (w + L)/2$.

(i) The first problem involves Dirichlet boundary condition on the cross-section at $z = z_0$:

$$\Delta u^D + k^2 u^D = 0 \quad \text{in } Q, \tag{11a}$$

$$u^D\big|_{\partial Q} = 0, \tag{11b}$$

$$u^D(\boldsymbol{x}, z) = e^{i\gamma_1 z}\psi_1(\boldsymbol{x}) + r_1^D e^{-i\gamma_1 z}\psi_1(\boldsymbol{x}) + \sum_{n=2}^{\infty} r_n^D e^{\gamma_n z}\psi_n(\boldsymbol{x}) \qquad (z < 0), \tag{11c}$$

$$u^D\big|_{z=z_0} = 0. \tag{11d}$$

(ii) The second problem involves Neumann boundary condition on the cross-section at $z = z_0$:

$$\Delta u^N + k^2 u^N = 0 \quad \text{in } Q, \tag{12a}$$

$$u^N\big|_{\partial Q} = 0, \tag{12b}$$

$$u^N(\boldsymbol{x}, z) = e^{i\gamma_1 z}\psi_1(\boldsymbol{x}) + r_1^N e^{-i\gamma_1 z}\psi_1(\boldsymbol{x}) + \sum_{n=2}^{\infty} r_n^N e^{\gamma_n z}\psi_n(\boldsymbol{x}) \qquad (z < 0), \tag{12c}$$

$$\left.\frac{\partial u^N}{\partial z}\right|_{z=z_0} = 0. \tag{12d}$$

From the solutions of these problems, we can construct the solution of the original scattering problem in Q_0. Indeed, let us extend the solution of u^D antisymmetrically and the solution u^N symmetrically onto Q_0:

$$u^D(\boldsymbol{x}, z) = -u^D(\boldsymbol{x}, 2z_0 - z) \quad (z > z_0), \tag{13a}$$

$$u^N(\boldsymbol{x}, z) = u^N(\boldsymbol{x}, 2z_0 - z) \quad (z > z_0). \tag{13b}$$

These extensions are the solutions of the Helmholtz equation (2) in Q_0, subject to Dirichlet conditions on ∂Q_0 (i.e., on the cylinder walls and on the barriers) and the following radiation conditions for $z > L + w$

$$u^D(\boldsymbol{x}, z) = -e^{i2\gamma_1 z_0} e^{-i\gamma_1 z}\psi_1(\boldsymbol{x}) - r_1^D e^{-i2\gamma_1 z_0} e^{i\gamma_1 z}\psi_1(\boldsymbol{x}) - \sum_{n=2} r_n^D e^{2\gamma_n z_0} e^{-\gamma_n z}\psi_n(\boldsymbol{x}), \tag{14a}$$

$$u^N(\boldsymbol{x}, z) = e^{i2\gamma_1 z_0} e^{-i\gamma_1 z}\psi_1(\boldsymbol{x}) + r_1^N e^{-i2\gamma_1 z_0} e^{i\gamma_1 z}\psi_1(\boldsymbol{x}) + \sum_{n=2} r_n^N e^{2\gamma_n z_0} e^{-\gamma_n z}\psi_n(\boldsymbol{x}). \tag{14b}$$

The half sum of u^D and u^N gives the solution of the original scattering problem, with

$$r_n = \frac{r_n^N + r_n^D}{2} \qquad (n \ge 1), \tag{15}$$

$$t_1 = \frac{r_1^N - r_1^D}{2} e^{-2i\gamma_1 z_0}, \qquad t_n = \frac{r_n^N - r_n^D}{2} e^{2\gamma_n z_0} \qquad (n \ge 2). \tag{16}$$

3.2. Dirichlet problem solution u^D

To find the solution u^D of the first problem in Q, let us also introduce the $L_2(\Gamma)$-normalized eigenfunctions and eigenvalues of the Laplace operator in the cross-section of the hole, $\Gamma = \Omega \backslash D$:

$$-\Delta\chi_n = \mu_n\chi_n, \qquad \chi_n|_{\partial\Gamma} = 0 \qquad (n = 1, 2, 3, \dots) \tag{17}$$

and set

$$\beta_n = \sqrt{\mu_n - k^2} \quad (n \ge 1). \tag{18}$$

The eigenfunctions χ_n form an orthonormal basis in $L_2(\Gamma)$. In the following, we consider that

$$\lambda_1 < k^2 < \min\{\mu_1, \lambda_2\} \tag{19}$$

so that the coefficients γ_n and β_n are real for all $n \ge 1$. Moreover, if the hole Γ is small, μ_1 is large so that k lies between $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$.

We can consider a general solution of the Helmholtz equation in the domain $\Gamma \times (0, w)$

$$u^D(\boldsymbol{x}, z) = \sum_{n=1}^{\infty}\Big(e_{1n}\sinh(\beta_n(z - w)) + e_{2n}\sinh(\beta_n z)\Big)\chi_n(\boldsymbol{x}), \tag{20}$$

where the cofficients e_{1n} and e_{2n} can be expressed as

$$e_{1n} = -\frac{(u_0, \chi_n)_{L_2(\Gamma)}}{\sinh(\beta_n w)}, \qquad e_{2n} = \frac{(u_1, \chi_n)_{L_2(\Gamma)}}{\sinh(\beta_n w)}, \tag{21}$$

where

$$u_0 = u^D|_{z=0}, \qquad u_1 = u^D|_{z=w}. \tag{22}$$

Similarly, in the domain $\Omega \times (w, z_0)$, we have

$$u^D(\boldsymbol{x}, z) = e_1\sin(\gamma_1(z - z_0))\psi_1(\boldsymbol{x}) + \sum_{n=2}^{\infty} e_n\sinh(\gamma_n(z - z_0))\psi_n(\boldsymbol{x}), \tag{23}$$

where the coefficients e_n can be expressed as

$$e_1 = -\frac{(u_1, \psi_1)_{L_2(\Gamma)}}{\sin(\gamma_1\ell)}, \qquad e_n = -\frac{(u_1, \psi_n)_{L_2(\Gamma)}}{\sinh(\gamma_n\ell)}, \tag{24}$$

and

$$\ell = z_0 - w = \frac{L-w}{2}\,. \tag{25}$$

Remark. At this moment, we do not discuss the convergence of the series. Moreover, we will differentiate formally the series without studying the validity of this operation up to the introduction of Eqs. (27). These formal steps are just needed as a background to establish these equations. The solution of these equations will solve the problem (3). In fact, if Eqs. (27) have a solution in the functional space W (introduced below in Eq. (30)), it can be extended to the whole waveguide Q with the aid of Eqs. (11c, 20, 23). Indeed, these series determine u^D in the whole domain Q as an element of $H^{1,loc}(Q)$ that satisfies the Helmholtz equation, boundary and radiation conditions. The series determining the radiation condition converges in L_2 at any cross-section because $u \in H^{1,loc}(Q)$.

Using formal representations

$$\left.\frac{\partial u^D}{\partial z}\right|_{z=0-0} = -i\gamma_1 (u_0, \psi_1)\psi_1 + \sum_{n=2}^{\infty} \gamma_n (u_0, \psi_n)\psi_n + 2i\gamma_1\psi_1, \tag{26a}$$

$$\left.\frac{\partial u^D}{\partial z}\right|_{z=0+0} = -\sum_{n=1}^{\infty} \beta_n \Big(\mathrm{ctanh}(\beta_n w)(u_0, \chi_n) - \frac{1}{\sinh(\beta_n w)}(u_1, \chi_n)\Big)\chi_n, \tag{26b}$$

$$\left.\frac{\partial u^D}{\partial z}\right|_{z=w-0} = -\sum_{n=1}^{\infty} \beta_n \Big(\frac{1}{\sinh(\beta_n w)}(u_0, \chi_n) - \mathrm{ctanh}(\beta_n w)(u_1, \chi_n)\Big)\chi_n, \tag{26c}$$

$$\left.\frac{\partial u^D}{\partial z}\right|_{z=w+0} = -\gamma_1 \mathrm{ctan}(\gamma_1 \ell)(u_1, \psi_1)\psi_1 - \sum_{n=2}^{\infty} \gamma_n \mathrm{ctanh}(\gamma_n \ell)(u_1, \psi_n)\psi_n, \tag{26d}$$

and imposing the continuity of $\frac{\partial u}{\partial z}$ at $z=0$ and $z=w$, we obtain two functional equations on u_0 and u_1:

$$-i\gamma_1 (u_0, \psi_1)\psi_1 + A_0 u_0 + C u_1 = -2i\gamma_1\psi_1, \tag{27a}$$

$$B u_0 + \gamma_1 \mathrm{ctan}(\gamma_1\ell)(u_1, \psi_1)\psi_1 + A_1 u_1 = 0, \tag{27b}$$

where the operators A_0, A_1, B and C are defined as

$$A_0 f = \sum_{n=2}^{\infty} \gamma_n \big(f, \psi_n\big)_{L_2(\Gamma)} \psi_n + \sum_{n=1}^{\infty} \hat{\beta}_n \big(f, \chi_n\big)_{L_2(\Gamma)} \chi_n, \tag{28a}$$

$$A_1 f = \sum_{n=2}^{\infty} \gamma_n \mathrm{ctanh}(\gamma_n \ell) \big(f, \psi_n\big)_{L_2(\Gamma)} \psi_n + \sum_{n=1}^{\infty} \hat{\beta}_n \big(f, \chi_n\big)_{L_2(\Gamma)} \chi_n, \tag{28b}$$

$$B f = -\sum_{n=1}^{\infty} \frac{\beta_n}{\sinh(\beta_n w)} \big(f, \chi_n\big)_{L_2(\Gamma)} \chi_n, \tag{28c}$$

$$C f = -\sum_{n=1}^{\infty} \frac{\beta_n}{\sinh(\beta_n w)} \big(f, \chi_n\big)_{L_2(\Gamma)} \chi_n, \tag{28d}$$

and

$$\hat{\beta}_n = \beta_n \mathrm{ctanh}(\beta_n w). \tag{29}$$

We understand Eqs. (27) and the operators A_0, A_1, B, C as follows. Let us consider two Hilbert spaces

$$W_i = \left\{ v \in L_2(\Gamma) \ : \ \sum_{n=2}^{\infty} \gamma_n^{(i)} \big(v, \psi_n\big)^2_{L_2(\Gamma)} + \sum_{n=1}^{\infty} \hat{\beta}_n \big(v, \chi_n\big)^2_{L_2(\Gamma)} < \infty \right\}, \quad i = 0, 1, \tag{30}$$

with the inner products

$$(f, g)_{W_i} = \sum_{n=2}^{\infty} \gamma_n^{(i)} (f, \psi_n)_{L_2(\Gamma)} \, (g, \psi_n)_{L_2(\Gamma)} + \sum_{n=1}^{\infty} \hat{\beta}_n (f, \chi_n)_{L_2(\Gamma)} \, (g, \chi_n)_{L_2(\Gamma)}, \tag{31}$$

where $\gamma_n^{(0)} = \gamma_n$ and $\gamma_n^{(1)} = \gamma_n \mathrm{ctanh}(\gamma_n \ell)$. Since $\mathrm{ctanh}(\gamma_n \ell)$ rapidly tends to 1 as $n \to \infty$, these functional spaces are equivalent, i.e., any function belonging to W_0, also belongs to W_1, and vice-versa. For this reason, we do not distinguish W_0 and

W_1 and denote either of them as W. It is easy to see that these functional spaces are also equivalent to $H^{\frac{1}{2}}(\Gamma)$ but this equivalence is not needed in the following. Note also that for any $v \in W$,

$$||v||_W \ge c_0\, ||v||_{L_2(\Gamma)}, \tag{32}$$

with a strictly positive constant c_0.

The operators B and C are bounded in $L_2(\Gamma)$ and their norms are small if the diameter of the opening Γ is small, see Section 4. The operators A_0 and A_1 in Eqs. (28a, 28b) can also be rigorously defined; however, for our purposes, it is sufficient to understand these operators in terms of the associated quadratic forms, i.e. by setting

$$\big(A_i f, g\big)_{L_2(\Gamma)} = \big(f, g\big)_W, \quad i = 0, 1. \tag{33}$$

As the operators A_0 and A_1 are positive definite (in the sense of positive definite quadratic forms determined by A_0 and A_1), their inverses A_0^{-1} and A_1^{-1} are well defined (see discussion in Section 4).

Applying A_0^{-1} and A_1^{-1} to Eq. (27a) and Eq. (27b) respectively, we rewrite them in a matrix operator form

$$\underbrace{\begin{pmatrix} I & A_0^{-1}C \\ A_1^{-1}B & I \end{pmatrix}}_{=M} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \begin{pmatrix} -i\gamma_1 (u_0, \psi_1) A_0^{-1}\psi_1 \\ \gamma_1 \mathrm{ctan}(\gamma_1 \ell)(u_1, \psi_1) A_1^{-1}\psi_1 \end{pmatrix} = \begin{pmatrix} -2i\gamma_1 A_0^{-1}\psi_1 \\ 0 \end{pmatrix}, \tag{34}$$

where I is the identity operator. We multiply Eq. (34) by the operator inverse to the operator M,

$$M^{-1} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \begin{pmatrix} I & -A_0^{-1}C \\ -A_1^{-1}B & I \end{pmatrix},$$

where

$$R_1 = (I - A_0^{-1}CA_1^{-1}B)^{-1}, \qquad R_2 = (I - A_1^{-1}BA_0^{-1}C)^{-1}. \tag{35}$$

We obtain the functional equations

$$u_0 - i\gamma_1 (u_0, \psi_1) R_1 A_0^{-1}\psi_1 - \gamma_1 \mathrm{ctan}(\gamma_1 \ell)(u_1, \psi_1) R_1 A_0^{-1} C A_1^{-1}\psi_1 = -2i\gamma_1 R_1 A_0^{-1}\psi_1, \tag{36a}$$

$$u_1 + i\gamma_1 (u_0, \psi_1) R_2 A_1^{-1} B A_0^{-1}\psi_1 + \gamma_1 \mathrm{ctan}(\gamma_1 \ell)(u_1, \psi_1) R_2 A_1^{-1}\psi_1 = 2i\gamma_1 R_2 A_1^{-1} B A_0^{-1}\psi_1. \tag{36b}$$

Multiplying each of these equations by ψ_1 and integrating over the hole Γ, we get two linear equations which can be written in a matrix form as

$$\begin{pmatrix} 1 + a & b\,\mathrm{ctan}(\gamma_1 \ell) \\ c & 1 + d\,\mathrm{ctan}(\gamma_1 \ell) \end{pmatrix} \begin{pmatrix} (u_0, \psi_1) \\ (u_1, \psi_1) \end{pmatrix} = 2 \begin{pmatrix} a \\ c \end{pmatrix}, \tag{37}$$

where

$$a = -i\gamma_1 (R_1 A_0^{-1}\psi_1, \psi_1)_{L_2(\Gamma)}, \tag{38a}$$

$$b = -\gamma_1 (R_1 A_0^{-1} C A_1^{-1}\psi_1, \psi_1)_{L_2(\Gamma)}, \tag{38b}$$

$$c = i\gamma_1 (R_2 A_1^{-1} B A_0^{-1}\psi_1, \psi_1)_{L_2(\Gamma)}, \tag{38c}$$

$$d = \gamma_1 (R_2 A_1^{-1}\psi_1, \psi_1)_{L_2(\Gamma)}. \tag{38d}$$

Since $(u_i, \psi_1)_{L_2(\Omega)} = (u_i, \psi_1)_{L_2(\Gamma)}$ for both $i = 0, 1$ given that $(u_0)|_D = (u_1)|_D = 0$ according the boundary condition (3), we did not specify the functional space for these two scalar products. Inverting the 2×2 matrix in Eq. (37), one finds (u_0, ψ_1) and (u_1, ψ_1).

Taking the limit $z \to 0$ in the radiation condition (11c), multiplying it by ψ_1 and integrating over Ω, the reflection coefficient r_1^D can be expressed as

$$r_1^D = (u_0, \psi_1)_{L_2(\Omega)} - 1 = \frac{a - 1 + (ad - bc - d)\mathrm{ctan}(\gamma_1 \ell)}{a + 1 + (ad - bc + d)\mathrm{ctan}(\gamma_1 \ell)}. \tag{39}$$

This is the main technical result of this paper that determines resonance scattering properties.

3.3. Neumann problem solution u^N

As the analysis for the Neumann problem (12) is almost identical, we just sketch the major steps and modifications. The ansatz (20) for u^N on $\Gamma \times (0, w)$ is the same, while Eq. (23) is modified to account for the Neumann condition at $z = z_0$:

$$u^N(\boldsymbol{x}, z) = e_1 \cos(\gamma_1(z - z_0))\psi_1(\boldsymbol{x}) + \sum_{n=2}^{\infty} e_n \cosh(\gamma_n(z - z_0))\psi_n(\boldsymbol{x}), \tag{40}$$

where the coefficients e_n are

$$e_1 = \frac{(u_1, \psi_1)_{L_2(\Gamma)}}{\cos(\gamma_1 \ell)}, \qquad e_n = \frac{(u_1, \psi_n)_{L_2(\Gamma)}}{\cosh(\gamma_n \ell)}. \tag{41}$$

The continuity conditions at $z = 0$ imply the same functional relation (27a), with the same operators A_0 and C. In turn, the ansatz (40) modifies the relation (27b) representing the continuity conditions at $z = w$ as

$$Bu_0 - \gamma_1 \tan(\gamma_1 \ell)(u_1, \psi_1) + \bar{A}_1 u_1 = 0, \tag{42}$$

where the operator B remains unchanged, while A_1 is replaced by

$$\bar{A}_1 f = \sum_{n=2}^{\infty} \gamma_n \tanh(\gamma_n \ell)\big(f, \psi_n\big)_{L_2(\Gamma)} \psi_n + \sum_{n=1}^{\infty} \hat{\beta}_n \big(f, \chi_n\big)_{L_2(\Gamma)} \chi_n. \tag{43}$$

As both $\tanh(\gamma_n \ell)$ and $\mathrm{ctanh}(\gamma_n \ell)$ rapidly tend to 1 as $n \to \infty$, the former operator A_1 from Eq. (28b) and the new operator $\bar{A}_1$ are close and have similar properties. Repeating the analysis of Section 3.2, we finally get

$$r_1^N = \big(u^N|_{z=0}, \psi_1\big)_{L_2(\Omega)} - 1 = \frac{\bar{a} - 1 - (\bar{a}\bar{d} - \bar{b}\bar{c} - \bar{d}) \tan(\gamma_1 \ell)}{\bar{a} + 1 - (\bar{a}\bar{d} - \bar{b}\bar{c} + \bar{d}) \tan(\gamma_1 \ell)}, \tag{44}$$

where the coefficients $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are still given by Eqs. (38), in which A_1 is replaced by $\bar{A}_1$.

3.4. Resonance transmission

It is important to emphasize that the reflection coefficient r_1^D in Eq. (39) depends on $\mathrm{ctan}(\gamma_1 \ell)$ and on the coefficients a, b, c, d. Here, $\mathrm{ctan}(\gamma_1 \ell)$ is determined by the wavelength k, the resonator half-length ℓ, and the shape of the cross-section Ω, but does not depend on the hole Γ. In turn, the coefficients a, b, c, d depend on the hole diameter δ. As shown in Section 4 below, the coefficients a, b, c, d vanish as the diameter δ of the hole Γ goes to 0. As a consequence, for a fixed wavelength k, we obtain in the limit of the vanishing hole:

$$r_1^D \to -1 \quad (\delta \to 0). \tag{45}$$

Similarly, one deduces from the analysis of Eq. (44):

$$r_1^N \to -1 \quad (\delta \to 0). \tag{46}$$

Substituting these expressions into Eq. (15), we get

$$r_1 \to -1 \quad (\delta \to 0), \tag{47}$$

i.e., the wave is fully reflected in the limit of two closed barriers, as intuitively expected.

Let us now consider the case of two *almost* closed barriers, i.e., δ is small but strictly positive. By continuity arguments, one can argue that r_1 remains close to -1 for most wavelengths, except for the resonance one. Indeed, for a fixed hole Γ, Eq. (39) implies

$$r_1^D = 1 \tag{48}$$

under the condition on the wavelength k:

$$1 + d\,\mathrm{ctan}(\gamma_1 \ell) = 0. \tag{49}$$

The wavelength k_D determined by this equation, is the resonance wavelength of the resonator with Dirichlet condition (11d), and it cannot be the resonance frequency of the same resonator with Neumann condition (12d). As a consequence, Eq. (46) is still applicable at k_D, and Eqs. (46,48) imply

$$r_1 \approx 0, \tag{50}$$

i.e., the wave almost fully propagates across 2 almost closed barriers. In other words, we have shown that for the hole diameter δ small enough, the waveguide is almost totally reflecting for most wavelengths, except for the resonance wavelength k_D, at which it is almost fully propagating.

4. Estimates for operators

4.1. Estimates for operators A_0^{-1} and A_1^{-1}

As the operators A_0 and A_1 are positive definite (in the sense of positive definite quadratic forms determined by A_0 and A_1), their inverses A_0^{-1} and A_1^{-1} are well defined and for all $f \in L_2(\Gamma)$:

$$||A_i^{-1} f||_{L_2(\Gamma)} \le c_1 ||f||_{L_2(\Gamma)} \qquad (i = 0, 1), \tag{51}$$

for some $c_1 > 0$, and

$$||A_i^{-1} f||_W \le c_2 ||f||_{L_2(\Gamma)} \qquad (i = 0, 1), \tag{52}$$

for some $c_2 > 0$.

Indeed, the inverse $A_0^{-1} f$ is defined in a weak sense as the solution of the equation

$$\sum_{n=2}^{\infty} \gamma_n \big(A_0^{-1} f, \psi_n\big)_{L_2(\Gamma)} \big(v, \psi_n\big)_{L_2(\Gamma)} + \sum_{n=1}^{\infty} \hat{\beta}_n \big(A_0^{-1} f, \chi_n\big)_{L_2(\Gamma)} \big(v, \chi_n\big)_{L_2(\Gamma)} = (f, v)_{L_2(\Gamma)}$$

for any $v \in W_0$. Substituting $v = A_0^{-1} f$ into this equation, we obtain

$$\begin{aligned} ||A_0^{-1} f||_W^2 &= \sum_{n=2}^{\infty} \gamma_n \big(A_0^{-1} f, \psi_n\big)_{L_2(\Gamma)} \big(A_0^{-1} f, \psi_n\big)_{L_2(\Gamma)} \\ &+ \sum_{n=1}^{\infty} \hat{\beta}_n \big(A_0^{-1} f, \chi_n\big)_{L_2(\Gamma)} \big(A_0^{-1} f, \chi_n\big)_{L_2(\Gamma)} \\ &= \big(f, A_0^{-1} f\big)_{L_2(\Gamma)}, \end{aligned}$$

from which

$$||A_0^{-1} f||_W^2 \le ||f||_{L_2(\Gamma)} \, ||A_0^{-1} f||_{L_2(\Gamma)} \le c_3 \, ||f||_{L_2(\Gamma)} \, ||A_0^{-1} f||_W,$$

with $c_3 > 0$, where we used (32). We conclude that

$$||A_0^{-1} f||_W \le c_3 \, ||f||_{L_2(\Gamma)}.$$

A similar bound can be obtained for A_1^{-1}.

4.2. Estimates for operators B and C

The estimates for operators B and C are much stronger. Indeed,

$$\begin{aligned} ||Bf||_{L_2(\Gamma)}^2 &= \left\| \sum_{n=1}^{\infty} \frac{\beta_n}{\sinh \beta_n} \big(f, \chi_n\big)_{L_2(\Gamma)} \chi_n \right\|_{L_2(\Gamma)}^2 \\ &\le 2 \sum_{n=1}^{\infty} \frac{\beta_n^2}{\sinh^2 \beta_n} \big(f, \chi_n\big)_{L_2(\Gamma)}^2 \underbrace{||\chi_n||_{L_2(\Gamma)}^2}_{=1} \le \frac{2\beta_1^2}{\sinh^2 \beta_1} \, ||f||_{L_2(\Gamma)}^2, \end{aligned}$$

given that β_n monotonously grow with n, whereas the function $z/\sinh(z)$ is monotonously decreasing. We get thus

$$||B||_{L_2} \le c_4 \frac{\beta_1}{\sinh \beta_1} \longrightarrow 0 \qquad \text{as } \delta \to 0,$$

with $c_4 > 0$. In fact, as the diameter $\delta = \mathrm{diam}\{\Gamma\}$ of the hole Γ vanishes, the eigenvalue μ_1 goes to infinity, implying very fast decay of $||B||_{L_2}$. The same analysis holds for $||C||_{L_2}$.

From the above estimates we deduce that

$$||A_0^{-1} C A_1^{-1} B f||_W \to 0 \qquad \text{as } \delta \to 0$$

so that the operator R_1 defined in Eq. (35), is bounded in W. The same is true for R_2.

From these estimates we finally obtain that

$$\big(R_1 A_0^{-1} \psi_1, \psi_1\big)_{L_2(\Gamma)} \le c_5 ||\psi_1||_{L_2(\Gamma)}^2 \to 0 \qquad \text{as } \delta \to 0, \tag{53}$$

implying that a, given by Eq. (38a), also vanishes as $\delta \to 0$. Similar estimates take place for b, c, and d.

5. Extensions and applications

In this paper, we studied wave propagation in an infinite waveguide of arbitrary but constant cross-section, which is the most common setting for acoustic and electromagnetic applications [19–22]. We have shown that the inclusion of two identical barriers with holes allows one to build a filter that blocks waves with a prescribed range of frequencies but transmits the wave at some resonance frequency from that range. Most surprisingly, such a resonance scattering can be achieved for barriers of arbitrary thickness, which is a new and rather counter-intuitive result. Moreover, we derived the exact explicit expression for the reflection coefficient that allows us to elucidate the mechanisms of the resonance transmission and to construct a practically relevant asymptotic description of acoustic and electromagnetic filters.

It is worth stressing that the diameter δ of the hole and the thickness w of barriers are two major parameters of the problem. We mainly emphasized on the geometric setting, in which the hole is small but the thickness of two barriers can be arbitrary. Such a narrow hole limit is convenient for applying various asymptotic tools developed for spectral and scattering problems [10,18,23–25]. Using these powerful tools, one can further generalize our results by considering, e.g., a junction of two waveguides of different cross-sections, holes of variable cross-section, or even of a waveguide of variable cross-section, for which the separation of variables is not applicable. However, industrial manufacturing of devices with small holes is sophisticated and thus expensive. For this reason, a common practical choice for the geometric setting is a waveguide with barriers that possess relatively large holes. For instance, one can use a coaxial line as a barrier in electromagnetic filters, in which case the barrier is small while the size of the hole is actually comparable to that of the waveguide. Keeping the requirement of the absence of propagating waves in these holes, one can achieve the resonance transmission by increasing the thickness of barriers. In mathematical terms, one should deal here with the other limit when the diameter of the hole δ is arbitrary but fixed while the thickness w of the barriers goes to infinity. Conventional narrow-hole asymptotic tools have to be re-adjusted to be able to deal with this limit. In turn, the mathematical analysis presented in this paper is still applicable to describe the resonance transmission in such devices. To illustrate this statement, we provide a sketch of the proof of the existence of a resonance frequency.

At any fixed wavelength k, let us analyze the limit of Eq. (44) as the barriers thickness w goes to infinity. Reproducing the arguments of Section 4, one can show that $||B|| \to 0$, $||C|| \to 0$, $||R_i|| \to 1$ in the limit $w \to \infty$. As a consequence, we get $\bar{b}\,\bar{c} \to 0$, while $\bar{a} \to a$ and thus Eq. (44) yields

$$\lim_{w\to\infty} r_1^N = \frac{a-1}{a+1}\,. \tag{54}$$

In other words, for a fixed k, the reflection coefficient for the Neumann problem is

$$r_1^N = \frac{a-1}{a+1} + \varepsilon(k), \tag{55}$$

where $\varepsilon(k)$ is arbitrarily small given that the barriers thickness w is large enough. Changing the wavelength k, one can make the reflection coefficient of the Dirichlet problem, r_1^D, to be arbitrarily close to $-r_1^N$. For this purpose, setting

$$r_1^D = -\frac{a-1}{a+1} \tag{56}$$

and using Eq. (39) for r_1^D, we get an equation on the resonance wavelength

$$1 + \left(d - \frac{abc}{a^2-1}\right)\mathrm{ctan}(\gamma_1 \ell) = 0, \tag{57}$$

which is an analog of our former Eq. (49). Due to the definition of a and the properties of the involved operators, a^2 remains separated from 1 in the limit $w \to \infty$ for the considered range of wavelengths. Since b and c are arbitrarily small in that range, the existence of a solution of Eq. (57) is equivalent to the existence of a solution of Eq. (49), as we proved in the paper. As a consequence, $r_1 = r_1^D + r_1^N = \varepsilon(k) \approx 0$, and the waveguide becomes fully transmitting at the wavelength determined by Eq. (57).

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