# RIBBON DECOMPOSITION AND TWISTED HURWITZ NUMBERS 

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In memoriam S.M.Natanzon.


#### Abstract

Ribbon decomposition is a way to obtain a surface with boundary (compact, not necessarily oriented) from a collection of disks by joining them with narrow ribbons attached to segments of the boundary. Counting ribbon decompositions gives rise to a "twisted" version of the classical Hurwitz numbers (studied earlier in 7 in a different context) and of the cut-and-join equation. We also provide an algebraic description of these numbers and an explicit formula for them in terms of zonal polynomials.


## Introduction

A classical surgery in dimension 2 studies connected sums of spheres, that is, ways to obtain a compact surface from a collection of spheres by gluing cylinders to them. In this paper we apply similar technique to surfaces with boundary: they are obtained from a collection of disks by gluing rectangles ("ribbons") to their boundary. Like with the classical connected sum, to glue a ribbon one is to choose the orientation of the boundary at both points of gluing, so the ribbon glued may look twisted or not.

Representation of a surface with boundary as a union of disks with the ribbons attached will be called its ribbon decomposition. See Fig. 3 for examples: the upper picture is a ribbon decomposition of an annulus, the lower one, of a Moebius band.

Diagonals of ribbons form a graph embedded into the surface (a.k.a. fat graph, ribbon graph, combinatorial map, etc.), with all its vertices on the boundary. The edges adjacent to a given vertex are thus linearly ordered left to right (remember, an orientation of the boundary near every vertex is chosen); this ordering defines the embedding of the graph up to homotopy.

Fix a positive integer $m$ and a partition $\left(\lambda_{1} \geq \cdots \geq \lambda_{s}\right)$ of the number $n \stackrel{\text { def }}{=}$ $|\lambda| \stackrel{\text { def }}{=} \lambda_{1}+\cdots+\lambda_{s}$ into $s$ parts. The main object of study in this paper, the twisted Hurwitz numbers $h_{m, \lambda}^{\sim}$, have several definitions or models, as we call them. The first one, a topological model, uses ribbon decompositions. Denote by $\mathfrak{S}_{m, \lambda}$ the set of decompositions into $m$ ribbons of surfaces having boundary of $s$ components containing $\lambda_{1}, \ldots, \lambda_{s}$ vertices (endpoints of ribbon diagonals). Then the twisted

[^0]Hurwitz number is defined as

$$
\begin{equation*}
h_{m, \lambda}^{\sim} \stackrel{\text { def }}{=} \frac{1}{n!} \# \mathfrak{S}_{m, \lambda} \tag{0.1}
\end{equation*}
$$

Another model for $h_{m, \lambda}^{\sim}$ is algebraic. Consider a fixed-point-free involution

$$
\begin{equation*}
\tau=(1, n+1)(2, n+2) \ldots(n, 2 n) \tag{0.2}
\end{equation*}
$$

in the symmetric group $S_{2 n}$. Its centralizer is isomorphic to the reflection group of type $B_{n}$. Let $\sigma_{1}, \ldots, \sigma_{m} \in S_{2 n}$ be transpositions. A simple analysis (see Section 2 below) shows that the permutation

$$
\begin{equation*}
u \stackrel{\text { def }}{=} \sigma_{1} \ldots \sigma_{m} \tau \sigma_{m} \ldots \sigma_{1} \tau \in S_{2 n} \tag{0.3}
\end{equation*}
$$

is decomposed into independent cycles as $u=c_{1} c_{1}^{\prime} \ldots c_{s} c_{s}^{\prime}$ where $c_{i}^{\prime}=\tau c_{i} \tau$ for every $i=1, \ldots, s$. Denote by $\mathfrak{H}_{m, \lambda}$ the set of sequences $\left(\sigma_{1}, \ldots, \sigma_{m}\right)$ of $m$ transpositions such that the cycles $c_{1}, \ldots, c_{s}$ of the permutation $u$ of 0.3 have lengths $\lambda_{1}, \ldots, \lambda_{s}$. We prove (Theorem 2.4 that

$$
\begin{equation*}
h_{m, \lambda}^{\sim}=\frac{1}{n!} \# \mathfrak{H}_{m, \lambda} . \tag{0.4}
\end{equation*}
$$

The third model for $h_{m, \lambda}^{\sim}$ is algebro-geometric and is due to G. Chapuy and M. Dołęga [7], who generalized the classical notion of a branched covering to the non-orientable case. Let $N$ denote a closed surface (compact 2-manifold without boundary, not necessarily orientable), and $p: \widehat{N} \rightarrow N$, its orientation cover. Denote by $\overline{\mathbb{H}} \stackrel{\text { def }}{=} \mathbb{C} P^{1} /(z \sim \bar{z})=\mathbb{H} \cup\{\infty\}$ where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane; its closure $\overline{\bar{H}}$ is homeomorphic to a disk. Let $\pi: \mathbb{C} P^{1} \rightarrow \overline{\mathbb{H}}$ be the quotient map. A continuous map $f: N \rightarrow \overline{\bar{H}}$ is called [7] a twisted branched covering if there exists a branched covering $\widehat{f}: \widehat{N} \rightarrow \mathbb{C} P^{1}$ (in the classical sense, a holomorphic map) such that $\pi \circ \widehat{f}=f \circ p$, and all the critical values of $\widehat{f}$ are real. These requirements imply in particular that the ramification profile of any critical value $c \in \mathbb{R} P^{1} \subset \mathbb{C} P^{1}$ of $\widehat{f}$ has every part repeated twice: $\left(\lambda_{1}, \lambda_{1}, \ldots, \lambda_{s}, \lambda_{s}\right)$, and $\operatorname{deg} \widehat{f}=2 n$ is even. In this case we say that the ramification profile of the critical value $\pi(c) \in \partial \overline{\mathbb{H}}$ of the map $f: N \rightarrow \overline{\mathbb{H}}$ is $\lambda=\left(\lambda_{1}, \ldots, \lambda_{s}\right)$.

Twisted branched coverings are split into equivalence classes via right-left equivalence. Denote by $\mathfrak{D}_{m, \lambda}$ the set of equivalence classes of twisted branched coverings having $m$ critical values with the ramification profiles $2^{1} 1^{n-2}$ and one critical value $\infty$ with the ramification profile $\lambda$. Then

$$
\begin{equation*}
h_{m, \lambda}^{\sim}=\frac{1}{n!} \# \mathfrak{D}_{m, \lambda} . \tag{0.5}
\end{equation*}
$$

Note that we prove equations (0.4 and 0.5 differently. To prove (0.4 we establish a direct one-to-one correspondence $\Xi$ between the sets $\mathfrak{S}_{m, \lambda}$ and $\mathfrak{H}_{m, \lambda}$. To prove 0.5 we show (Theorems 2.10 and 2.12 that the generating function of the twisted Hurwitz numbers satisfies a PDE of parabolic type called twisted cut-and-join equation - just like standard Hurwitz numbers, whose generating function satisfies the "classical" cut-and-join [4]. Cardinalities of the sets $\mathfrak{D}_{m, \lambda}$ are shown in [7] to satisfy the same equation with the same initial data, so 0.5 follows. Finding a direct one-to-one correspondence between the sets $\mathfrak{D}_{m, \lambda}$ and $\mathfrak{S}_{m, \lambda}\left(\right.$ or $\left.\mathfrak{H}_{m, \lambda}\right)$ is a challenging topic of future research.

The paper contains three main sections in accordance with the three models described. In the first, "topological" section we study ribbon decompositions of
surfaces with boundary (rigged with marked points) and the graphs (with numbered vertices and edges) formed by the diagonals of ribbons. The graphs appear to be 1-skeleta of the surface, and the surface can be retracted to them (Theorem 1.9); also, the graphs behave nicely under the orientation cover of the surface (Theorems 1.11 and 1.13 .

Graph embeddings into oriented surfaces were studied earlier in a number of works (see [1] for the general theory without boundary, [5] for the disk and [6 for arbitrary surfaces and embeddings with a connected complement); they are in one-to-one correspondence with sequences of transpositions in the symmetric group. The cyclic structure of the product of the transpositions describes faces of the graph (i.e. connected components of its complement). The quantity of graphs with given faces is called a (classical) Hurwitz number and has been studied intensively during the last decades - the research involving dozens of authors and hundreds of works; its thorough review is far outside the scope of this paper. The algebraic model for twisted Hurwitz numbers, studied in Section 2 is a generalization of this correspondence. The section also contains an explicit formula for the cut-and-join equation (Theorem 2.12 ) and for the generating function of the twisted Hurwitz numbers (Theorem 2.15).

In the last section we study the notion of the branched covering defined in [7] and show that they form an algebro-geometric model for twisted Hurwitz numbers.

Acknowledgements. The research was partially funded by the HSE University Basic Research Program and by the International Laboratory of Cluster Geometry NRU HSE (RF Government grant, ag. No. 075-15-2021-608 dated 08.06.2021). The research of the first-named author was also supported by the Simons Foundation IUM grant 2021.

The authors are grateful to G. Chapuy and M. Dołęga who explained them a broader perspective beyond the phenomena studied and also helped much with the combinatorics of the Laplace-Beltrami equation.

We dedicate this article to the memory of our colleague Sergey Natanzon who fell victim of the COVID-19 pandemic. The subject of our research, to which Prof. Natanzon was always attentive, matches some of his favourite scientific topics - Hurwitz numbers and manifolds with boundary.

1. Surgery: a topological model for twisted Hurwitz numbers

### 1.1. General definitions.

Definition 1.1. Decorated-boundary surface (DBS) is a triple $\left(M,\left(a_{1}, \ldots, a_{n}\right)\right.$, $\left.\left(o_{1}, \ldots, o_{n}\right)\right)$ where $M$ is a compact surface (2-manifold) with boundary, $a_{1}, \ldots, a_{n} \in$ $\partial M$ are marked points and every $o_{i}$ is a local orientation of $\partial M$ (hence, of $M$ itself, too) in the vicinity of the point $a_{i}$, such that

- every connected component of $M$ has nonempty boundary, and
- every connected component of $\partial M$ contains at least one point $a_{i}$.

The DBS $M$ and $M^{\prime}$ with the same number $n$ of marked points are called equivalent if there exists a homeomorphism $h: M \rightarrow M^{\prime}$ such that $h\left(a_{i}\right)=a_{i}^{\prime}$ and $h_{*}\left(o_{i}\right)=o_{i}^{\prime}$ for all $i=1, \ldots, n$. The set of equivalence classes of DBS with $n$ marked points will be denoted $\mathcal{D \mathcal { B }}{ }_{n}$.

Pick marked points $a_{i}, a_{j} \in \partial M$, and let $\varepsilon_{i}, \varepsilon_{j} \in\{+,-\}$. Consider points $a_{i}^{\prime}, a_{j}^{\prime} \in$ $\partial M$ lying near $a_{i}, a_{j}$ and such that the boundary segment $a_{i} a_{i}^{\prime}$ is directed along the
orientation $o_{i}$ if $\varepsilon_{i}=+$ and opposite to it if $\varepsilon_{i}=-$; the same for $j$. Now take a long narrow rectangle ("a ribbon" henceforth) and glue its short sides to $\partial M$ as shown in Fig. 1. The result of gluing is homeomorphic to a surface $M^{\prime}$ with the boundary $\partial M^{\prime} \ni a_{1}, \ldots, a_{n}$. The boundary of $M^{\prime}$ near $a_{i}$ and $a_{j}$ contains a segment of $\partial M$ (the "old" part) and a segment of a long side of the ribbon glued (the "new" part); define local orientations $o_{i}^{\prime}, o_{j}^{\prime}$ of $\partial M^{\prime}$ near $a_{i}, a_{j}$ so that the orientations of the "old" parts would be preserved (see bold curved arrows in Fig. 11; for $k \neq i, j$ take $o_{k}^{\prime}=o_{k}$ by definition. Now $\left(M^{\prime},\left(a_{1}, \ldots, a_{n}\right),\left(o_{1}^{\prime}, \ldots, o_{n}^{\prime}\right)\right)$ is a DBS, so we defined a mapping $G[i, j]^{\varepsilon_{i}, \varepsilon_{j}}: \mathcal{D B S}_{n} \rightarrow \mathcal{D B S}_{n}$ called ribbon gluing. The ribbon gluing $G[i, j]^{\varepsilon_{i}, \varepsilon_{j}}$ will be called twisted if $\varepsilon_{i} \neq \varepsilon_{j}$, and non-twisted otherwise; compare the left and the right picture in Fig. 1.

The inverse operation is called ribbon removal. To define it, take $\varepsilon \in\{+,-\}$ and fix a smooth simple (i.e. non-selfintersecting) curve $\gamma$ on $M$ joining $a_{i}$ and $a_{j}$ and transversal to $\partial M$ in its endpoints. Take now a point $a_{j}^{\prime} \in \partial M$ near $a_{i}$ and $a_{i}^{\prime} \in \partial M$ near $a_{j}$ (NB the subscripts!) such that the segment $a_{i} a_{j}^{\prime} \subset \partial M$ is directed according to the orientation $o_{i}$ if $\varepsilon=+$ and opposite to it if $\varepsilon=-$, and consider a "rectangle" $\Pi$ like in Fig. 1 . Then $M^{\prime} \stackrel{\text { def }}{=} M \backslash \operatorname{int}(\Pi)$ is homeomorphic to a surface with the boundary $\partial M^{\prime} \ni a_{1}, \ldots, a_{n}$. A local orientation $o_{i}^{\prime}$ of $\partial M^{\prime}$ near $a_{i}$ is defined by the same rule as for the ribbon gluing: $o_{i}$ and $o_{i}^{\prime}$ coincide on the intersection $\partial M^{\prime} \cap \partial M$ near $a_{i}$; the same for $o_{j}^{\prime}$, and also $o_{k}^{\prime} \stackrel{\text { def }}{=} o_{k}$ for all $k \neq i, j$. Now $\left(M^{\prime},\left(a_{1}, \ldots, a_{n}\right),\left(o_{1}^{\prime}, \ldots, o_{n}^{\prime}\right)\right)$ is a DBS obtained from the original DBS by the operation $R[\gamma]^{\varepsilon}$ of ribbon removal.
Remark 1.2. Local orientations $o_{i}$ and $o_{j}$ of $\partial M$ define orientations of the normal bundle to $\gamma$; we call $\gamma$ non-twisting if the orientations are the same, and twisting otherwise. Obviously, the segment $a_{j} a_{j}^{\prime}$ is directed along the orientation $o_{j}$ if $\varepsilon=+$ and $\gamma$ is non-twisting or $\varepsilon=-$ and $\gamma$ is twisting; otherwise $a_{j} a_{j}^{\prime}$ is directed opposite to $o_{j}$.

The operation $R[\gamma]^{\varepsilon}$ is a sort of inverse to ribbon gluing due to the following obvious statement:
Proposition 1.3. (1) Let $i, j \in\{1, \ldots, n\}, \varepsilon_{i}, \varepsilon_{j} \in\{+,-\}$ and $\gamma$ be a diagonal of the ribbon joining $a_{i}$ and $a_{j}$. Then $R[\gamma]^{\varepsilon_{i}} G[i, j]^{\varepsilon_{i}, \varepsilon_{j}}=\operatorname{id}_{\mathcal{D B S}_{n}}$.
(2) Let $\gamma$ be a simple smooth curve on $M$ joining $a_{i}$ and $a_{j}$ and transversal to the boundary, and $\varepsilon_{i} \in\{+,-\}$. Let $\varepsilon_{j} \in\{+,-\}$ be defined as $\varepsilon_{j}=\varepsilon_{i}$ if the curve $\gamma$ is non-twisting and $\varepsilon_{j}=-\varepsilon_{i}$ otherwise. Then $G[i, j]^{\varepsilon_{i}, \varepsilon_{j}} R[\gamma]^{\varepsilon_{i}}=$ $\mathrm{id}_{\mathcal{D B S}_{n}}$.

Remark 1.4. Gluing a ribbon decreases the Euler characteristics of the surface by 1 and removal, increases it by 1 .
1.2. Ribbon decompositions. By Definition 1.1 every connected component of a DBS contains a marked point. $M \in \mathcal{D B S}_{n}$ is called stable if every its connected component either contains at least two marked points or is a disk (with one marked point only).

Denote by $E_{n} \in \mathcal{D B S} \mathcal{S}_{n}$ a union of $n$ disks with one marked point on the boundary of each.

Proposition 1.5. $M \in \mathcal{D B S}_{n}$ is stable if and only if it can be obtained by gluing several ribbons to $E_{n}$. If $M$ is stable then its Euler characteristics $\chi(M)$ does not exceed $n$, and the number of ribbons is equal to $n-\chi(M)$.


Figure 1. Gluing ribbons

Proof. If a surface with a ribbon glued has a component with only one marked point then the gluing left this component intact. So, gluing a ribbon to a stable DBS keeps its stability, which proves the 'only if' part of the proposition ( $E_{n}$ is stable by definition).

To prove the 'if' part we will need a lemma:
Lemma 1.6. Let $n \geq 2$. Then for any $M \in \mathcal{D B S}_{n}$ which is connected and stable but is not a disk there exists a simple smooth nonseparating curve $\gamma$ joining two marked points.
"Nonseparating" here means that the complement of $\gamma$ is connected, too.
Proof of the lemma. $M$ contains at least two marked points. If $\partial M$ is not connected then take two marked points on different components of $\partial M$ and join them with a simple smooth curve $\gamma$; such curve is always nonseparating.

Let now $\partial M$ be connected. Then $M$ is a connected sum of a disk with several handles and/or Moebius bands. Let $S^{1} \subset M$ be a circle separating the disk from a handle or from a Moebius band, and let $p, q \in S^{1}$ be two points. There exists a nonseparating curve $\delta$ inside the handle or the Moebius band joining $p$ and $q$. Now pick a curve $\gamma_{1}$ joining $p$ with one marked point and $\gamma_{2}$ joining $q$ with another one. Then the union $\gamma \stackrel{\text { def }}{=} \gamma_{1} \cup \delta \cup \gamma_{2}$ is nonseparating as required.

Corollary 1.7. If $M \in \mathcal{D B S}_{n}$ is stable and $M \neq E_{n}$ then there exists a curve $\gamma$ on $M$ such that $M^{\prime} \stackrel{\text { def }}{=} R[\gamma]^{\varepsilon}(M)$ is stable (regardless of $\varepsilon$ ).

Proof of the corollary. A stable DBS different from $E_{n}$ contains a component with two or more marked points. If this component is a disk then take for $\gamma$ any simple curve joining these points. If it is not a disk then take for $\gamma$ the nonseparating curve of Lemma 1.6 .

The proposition is now proved using induction on the Euler characteristic of $M$. Every component of $M$ is a manifold with nonempty bounbdary, so the 2-nd Betti number of $M$ is zero and $\chi(M)=h_{0}(M)-h_{1}(M) \leq h_{0}(M) \leq n$; the equality is possible only if $M=E_{n}$. Let now $\chi(M)=n-m, m>0$. Use Corollary 1.7 to obtain a curve $\gamma$ in $M$ such that $M^{\prime}=R[\gamma]^{+}(M)$ is stable; by Remark 1.4 one has $\chi\left(M^{\prime}\right)=n-m+1$, so by the induction hypothesis $M^{\prime}$ can be obtained from $E_{n}$ by gluing $m-1$ ribbons. By assertion 2 of Proposition 1.3 there exist $i, j$ and $\varepsilon$ such that $M=G[i, j]^{+, \varepsilon}\left(M^{\prime}\right)$ - so, $M$ can be obtained by gluing $m$ ribbons.

Let now, again, $M \in \mathcal{D B S}_{n}$ be obtained by gluing of $m$ ribbons to $E_{n}$ :

$$
\begin{equation*}
M=G\left[i_{m}, j_{m}\right]^{\varepsilon_{m}, \delta_{m}^{\prime}} \ldots G\left[i_{1}, j_{1}\right]^{\varepsilon_{1}, \delta_{1}} E_{n} \tag{1.1}
\end{equation*}
$$

(that's what we will be calling a ribbon decomposition of $M$ ). For every ribbon, draw a diagonal joining its vertices $a_{i_{k}}$ and $a_{j_{k}}$, and assign the number $k$ to it. The union of the diagonals is a graph $\Gamma \subset M$ with $m$ numbered edges $r_{1}, \ldots, r_{m}$ and the marked points $a_{1}, \ldots, a_{n}$ as vertices; we call it a diagonal graph of the ribbon decomposition.

Let $a_{i}$ be a marked point of $M, \Gamma \subset M$ be a diagonal graph of a ribbon decomposition, and let $\ell_{1}, \ldots, \ell_{k}$ be the numbers of the edges of $\Gamma$ having $a_{i}$ as an endpoint, listed left to right according to the orientation $o_{i}$; denote $\mathcal{P}\left(a_{i}\right) \stackrel{\text { def }}{=}\left(\ell_{1}, \ldots, \ell_{k}\right)$.

Theorem 1.8. The diagonal graph $\Gamma$ has the following properties:
(1) (embedding) $\Gamma$ is embedded: its edges do not intersect one another or the boundary of $M$ except at endpoints.
(2) (anti-unimodality) For every vertex $a_{i}$ the sequence $\mathcal{P}\left(a_{i}\right)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ is anti-unimodal: there exists $p \leq k$ such that $\ell_{1}>\cdots>\ell_{p}<\cdots<\ell_{k}$.
(3) (twisting rule) In the notation of the above call the edges $\ell_{1}, \ldots, \ell_{p}$ negative at the endpoint $a_{i}$, and edges $\ell_{p}, \ldots, \ell_{k}$, positive (note that $\ell_{p}$ is both). Then any twisting edge of $\Gamma$ is positive at one of its endpoints and negative at the other one, and any non-twisting edge is either positive at both endpoints or negative at them.
(4) (retraction) The graph $\Gamma$ is a homotopy retract of the surface $M$.

Proof. Induction by the number $m$ of ribbons; the base $m=0$ is obvious. For any $m>0$, let $M=G\left[i_{m}, j_{m}\right]^{\varepsilon_{m} \delta_{m}} M^{\prime}$, and $\Gamma^{\prime} \subset M^{\prime} \subset M$ be the union of all the edges of $\Gamma$ except $m$.

Assertion 11 the internal points of the edge $m$ lie in the interior of the ribbon $r_{m}=M \backslash M^{\prime}$ and thus belong neither to $\Gamma^{\prime}$ nor to $\partial M$.

Assertion 22 after gluing the ribbon $r_{m}$ to $M^{\prime}$, the edge $m$ is either the leftmost or the rightmost of all the edges ending at $a_{i_{m}}$. Thus, if $\mathcal{P}\left(a_{i_{m}}\right)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ then either $\ell_{1}=m$ and $\ell_{2}, \ldots, \ell_{k}$ is anti-unimodal by the induction hypothesis, or $\ell_{k}=m$ and $\ell_{1}, \ldots, \ell_{k-1}$ is anti-unimodal. In both cases $\ell_{1}, \ldots, \ell_{k}$ is anti-uninmodal.

Assertion 3 is true for the edges of $\Gamma^{\prime} \subset M^{\prime}$ by the induction hypothesis. Apparently, this is preserved after the ribbon $r_{m}$ is glued. The edge $m$ is the diagonal of $r_{m}$; the long sides of $r_{m}$ lie in $\partial M$, and therefore the edge $m$ is adjacent to $\partial M$ at both its endpoints, from the right for one of them and from the left for the other. This proves assertion 3 for the edge $m$, too.

To facilitate induction for assertion 4, we make it a bit stronger: fix, for every $i$, a small segment $e_{i} \subset \partial M, a_{i} \in e_{i}$, and prove that there exists a homotopy retraction $f: M \rightarrow \Gamma$ such that $f(x)=a_{i}$ for all $x \in e_{i}$.


Figure 2. The open cover of $M$

By the induction hypothesis, such $f$ exists for $M^{\prime}$ and $\Gamma^{\prime}$. W.l.o.g. the ribbon $r_{m}$ containing $a_{i}$ and $a_{j}$ is glued to $M^{\prime}$ so that its short sides lie within the segments $e_{i}$ and $e_{j}$. Thus, the induction step is reduced to the following obvious statement: there exists a homotopy retraction of a rectangle $\Pi$ onto its diagonal $[a b]$ sending short sides and small neighbourhoods of the points $a, b \in \partial \Pi$ to the points $a$ and $b$.

Let now $M \in \mathcal{D B S}_{n}$ and let $\Gamma \subset M$ be an embedded loopless graph with the vertices at the marked points of $M$ and the edges numbered $1, \ldots, m$. We call $\Gamma$ properly embedded if it satisfies all the assertions of Theorem 1.8 embedding, anti-unimodality, twisting rule and retraction. Connected components of the complement $M \backslash \partial M \backslash \Gamma$ will be called faces; connected components of $\partial M \backslash\left\{a_{1}, \ldots, a_{n}\right\}$, external edges, and connected components of $\Gamma \backslash\left\{a_{1}, \ldots, a_{n}\right\}$, internal edges.

Theorem 1.9. Vertices, edges and faces of a properly embedded graph $\Gamma$ form a cell decomposition of $M$ (as 0-cells, 1-cells and 2-cells, respectively) such that every face is adjacent to exactly one external edge. The total number of faces is $n$.

Proof. Let $\Gamma$ have $k$ faces $f_{1}, \ldots, f_{k}$. Cover $M$ with the open subsets shown in Fig. 2 .

Sets $e_{i}$ (neighbourhoods of internal edges) are homeomorphic to disks, $b_{i}$ (neighbourhoods of external edges) and $v_{i}$ (neighbourhoods of vertices), to half-disks; topology of faces $f_{i}$ is yet to be described. Connected components of all the nonempty intersections of the sets (including faces) are homeomorphic to disks or half-disks, too.

The nonempty intersections are:

- $2 m$ connected components of $f_{i} \cap e_{j}$ for all $i, j$;
- $n$ components of $f_{i} \cap b_{j}$ for all $i, j$;
- If $\delta_{j}$ is the valency of the $j$-th vertex of the graph, then there are $\delta_{j}+1$ components of $f_{i} \cap v_{j}$ for all $i$. The total number of components in $f_{i} \cap v_{j}$ is thus $\sum_{j}\left(\delta_{j}+1\right)=2 m+n$;
- $2 m$ components of $e_{i} \cap v_{j}$, for all $i, j$;
- $2 n$ components of $b_{i} \cap v_{j}$, for all $i, j$;
- $4 m$ components of $f_{i} \cap e_{j} \cap v_{k}$, for all $i, j, k$;
- $2 n$ components of $f_{i} \cap b_{j} \cap v_{k}$.

Thus the Euler characteristics of $M$ is

$$
\begin{aligned}
\chi(M) & =\sum_{i=1}^{k} \chi\left(f_{i}\right)+m+n+n-2 m-n-(2 m+n)-2 m-2 n+4 m+2 n \\
& =\sum_{i=1}^{k} \chi\left(f_{i}\right)-m
\end{aligned}
$$

On the other hand, $\Gamma$ is a retract of $M$, so $\chi(M)=\chi(\Gamma)=n-m$, hence $\sum_{i=1}^{k} \chi\left(f_{i}\right)=n$. Faces are connected open 2-manifolds, so $\chi\left(f_{i}\right) \leq 1$ for every $i$, and therefore $n \leq k$.

Closure of a face is a compact manifold with boundary, so it cannot retract to its boundary. It means that the boundary of any face is not a subset of the graph and must contain an external edge. The total number of external edges is $n$, and an external edge belongs to the boundary of one face only. This implies $n \geq k$ and therefore $n=k$ and $\chi\left(f_{i}\right)=1$ for every $i=1, \ldots, k$.

So each $f_{i}$ is a disk. Its closure contains one external edge and $k_{i}$ internal ones, as well as vertices, so it is an image of the map $Q_{i}$ from some $\left(k_{i}+1\right)$-gon to $M$. Every $Q_{i}$ sends sides of the polygon to the edges and vertices to vertices, so collectively the $Q_{i}, i=1, \ldots, k$, are characteristic maps of a cell decomposition.

Theorem 1.9 allows to prove the inverse of Theorem 1.8
Theorem 1.10. Let $M \in \mathcal{D B S}_{n}$ be stable and $\Gamma \subset M$ be a properly embedded graph. Then $\Gamma$ is the diagonal graph of a ribbon decomposition of $M$.

Proof. Induction by the the number $m$ of edges of $\Gamma$. The base: $m=0$ means that $\Gamma$ consists of $n$ isolated vertices. Since $M$ is a retract of $\Gamma$, one has $M=E_{n}$.

Let $m>0$. The edge $e_{m}$ of $\Gamma$ joins the vertices $a_{i}$ and $a_{j}$ (necessarily different) and separates faces $f_{p}$ and $f_{q}$ (which may be the same). By the anti-unimodality, $e_{m}$ is adjacent to $\partial M$ at both $a_{i}$ and $a_{j}$. Using Theorem 1.9 , consider a characteristic $\operatorname{map} Q_{p}$ of the cell $f_{p}$. It maps the side $v_{0} v_{1}$ of the polygon to the external edge of $f_{p}$ and the adjacent side $v_{1} v_{2}$, to $e_{m}$. Let $v^{\prime} \in v_{0} v_{1}$ be a point near the vertex $v_{1}, a_{i}^{\prime} \stackrel{\text { def }}{=} Q_{p}\left(v^{\prime}\right) \in \partial M$; consider the image $T_{p} \stackrel{\text { def }}{=} Q_{p}\left(v^{\prime} v_{1} v_{2}\right) \subset M$ of the triangle $v^{\prime} v_{1} v_{2}$. Then the union of $T_{p}$ and a similar triangle $T_{q} \subset f_{q}$ is a ribbon $H$ having $e_{m}$ as its diagonal.

Let $\Gamma^{\prime}$ be the graph $\Gamma$ with the edge $e_{m}$ removed. Take $\varepsilon=+$ if $\partial M$ near $a_{i}$ is oriented towards $a_{i}^{\prime}$, and $\varepsilon=-$ otherwise. Then $\Gamma^{\prime}$ is embedded into $M^{\prime} \stackrel{\text { def }}{=}$ $R\left[e_{m}\right]^{\varepsilon}(M)$; an immediate check shows that the embedding is proper, so $\Gamma^{\prime}$ is the diagonal graph of a ribbon decomposition of $M^{\prime}$ by the induction hypothesis. By Proposition $1.3 M$ can be obtained by gluing the ribbon $H$ to $M^{\prime}$, which finishes the induction.
1.3. Oriented case and the orientation cover. A DBS $M$ is called oriented if all the local orientations $o_{i}$ are consistent with a global orientation of the surface $M$. For an oriented $M$ the numbers of marked points read off the components of $\partial M$ according to the orientation form a cyclic decomposition of some permutation $\sigma \in S_{n}$ called the boundary permutation of $M$ (here and below we denote by $S_{n}$ the permutation group). In other words, for any $k=1, \ldots, n$ the marked point adjacent to $a_{k}$ in the positive direction of $\partial M$ is $a_{\sigma(k)}$.

It is easy to see that if $M$ is oriented and the gluing $G[i, j]^{\varepsilon_{i}, \varepsilon_{j}}$ is non-twisted $\left(\varepsilon_{i}=\varepsilon_{j}\right)$ then $G[i, j]^{\varepsilon_{i}, \varepsilon_{j}}(M) \in \mathcal{D B S}_{n}$ is oriented, too. A ribbon decomposition

$$
\begin{equation*}
M=G\left[i_{m}, j_{m}\right]^{++} \ldots G\left[i_{1}, j_{1}\right]^{++} E_{n} \tag{1.2}
\end{equation*}
$$

is called oriented; existence of such means, by obvious induction, that the surface $M$ is oriented.

Theorem 1.11. The diagonal graph $\Gamma$ of the oriented ribbon decomposition 1.2 has the following properties (in addition to those granted by Theorem 1.8):
(1) (vertex monotonicity) For every vertex $a_{i}$ of $\Gamma$ the sequence $\mathcal{P}\left(a_{i}\right)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ is increasing: $\ell_{1}<\cdots<\ell_{k}$.
(2) (face monotonicity) For every face $f_{i}$ of $\Gamma$ the numbers $\ell_{1}, \ldots, \ell_{p}$ of the internal edges adjacent to it are increasing if the count starts at the (only) external edge of $f_{i}$ and goes counterclockwise.
(3) (face separation) Every internal edge of $\Gamma$ separates two different faces.
(4) (boundary permutation) Let $a_{i_{k}}$ and $a_{j_{k}}$ be endpoints of the edge $e_{k}$ of $\Gamma, k=$ $1, \ldots, m$. Then the boundary permutation of $M$ is equal to $\left(i_{m} j_{m}\right) \ldots\left(i_{1} j_{1}\right) \in$ $S_{n}$.
Proof. Vertex monotonicity is a particular case of anti-unimodality of Theorem 1.8
If $\ell_{j}$ and $\ell_{j+1}$ are two internal edges on the boundary of $f_{i}$ sharing an endpoint $a$ then the orientation of the boundary near $a$ is consistent with the counterclockwise orientation of $f_{i}$. Then the vertex monotonicity implies $\ell_{j}<\ell_{j+1}$, which proves face monotonicity. The face monotonicity implies, in its turn, the face separation: as one moves around a face, the numbers of the internal edges seen are increasing and therefore cannot repeat.

Let $a_{k}, a_{s} \in \partial M$ be neighbouring vertices, that is, the endpoints of an external edge. By Theorem 1.9 and the face monotonicity, this is the sole external edge of a face $f$, its remaining sides being internal edges numbered $\ell_{1}<\cdots<\ell_{p}$, as one moves from $a_{k}$ to $a_{s}$. Consider an action of $S_{n}$ on the vertices of $M \in \mathcal{D B S}_{n}$ by permuting their numbers; in particular, the transposition $\left(i_{t} j_{t}\right)$ exchanges the numbers of the vertices joined by the $t$-th edge of the diagonal graph, leaving the other vertices intact. So, the transposition $\left(i_{\ell_{1}} j_{\ell_{1}}\right)$ moves $a_{k}$ to its neighbour at the face $f$; then the transposition $\left(i_{\ell_{2}}, j_{\ell_{2}}\right)$ (where $\ell_{2}>\ell_{1}$, so it is applied after the first one) moves it to the next vertex of the same face, etc.; eventually, $\sigma=\left(i_{m} j_{m}\right) \ldots\left(i_{1} j_{1}\right)$ moves $a_{k}$ to $a_{s}=a_{\sigma(k)}$.

Every manifold $M$ (possibly with boundary) has the orientation cover, uniquely defined up to an obvious isomorphism: it is an oriented manifold $\widehat{M}$ of the same dimension together with a fixed-point-free smooth involution $\mathcal{T}: \widehat{M} \rightarrow \widehat{M}$ reversing the orientation and such that $M$ is diffeomorphic to its orbit space.

The quotient map $\widehat{M} \rightarrow \widehat{M} / T=M$ is a 2 -sheeted covering, trivial iff $M$ is oriented. For 2-manifolds with boundary there is

Lemma 1.12. The orientation covering is trivial over the boundary of a 2-manifold.
Proof. The boundary $\partial M$ and its cover $\partial \widehat{M}$ are unions of circles. If the covering is nontrivial over the boundary then there is a component $C \subset \partial M$ covered by a $\mathcal{T}$-invariant circle $C^{\prime} \subset \partial \widehat{M}$.

A continuous map $A: S^{1} \rightarrow S^{1}$ has at least $|\operatorname{deg} A-1|$ fixed points, so the fixed-point-free map $\mathcal{T}: C^{\prime} \rightarrow C^{\prime}$ has degree 1 and therefore, being a covering, preserves orientation. Since $C^{\prime} \subset \partial \widehat{M}$, it means that $\mathcal{T}: \widehat{M} \rightarrow \widehat{M}$ also preserves local orientation at every point $a \in C^{\prime}$. But $\mathcal{T}$ is orientation-reversing everywhere - a contradiction.

Let a fixed-point-free involution $\tau \in S_{2 n}$ be defined by 0.2 . The notion of an orientation cover can be extended to decorated-boundary surfaces as follows: $\widehat{M} \in \mathcal{D} \mathcal{B S}_{2 n}$ with the marked points $b_{1}, \ldots, b_{2 n}$ is called the orientation cover of $M \in \mathcal{D B S} S_{n}$ with the marked points $a_{1}, \ldots, a_{n}$ if $\widehat{M}$ is oriented and there exists a fixed-point-free smooth involution $\mathcal{T}: \widehat{M} \rightarrow \widehat{M}$ reversing the orientation and such that $\mathcal{T}\left(b_{k}\right)=b_{\tau(k)}$ for all $k=1, \ldots, 2 n$, and also there exists a diffeomorphism $p: \widehat{M} / \mathcal{T} \rightarrow M$ between the orbit space and $M$ such that $p\left(\left\{b_{k}, b_{\tau(k)}\right\}\right)=a_{k}$ for all $k=1, \ldots, n$.

For $M \in \mathcal{D B S}_{n}$ the surface $\widehat{M}$ and involution $\mathcal{T}: \widehat{M} \rightarrow \widehat{M}$ are uniquely defined; the marked points are $p^{-1}\left(a_{1}\right) \stackrel{\text { def }}{=}\left\{b_{i}, b_{i+n}\right\} \subset \partial \widehat{M}$. The numbering of the two points $b_{i}$ and $b_{i+n}$ depends on the local orientation $o_{i}$ of $\partial M$ at $a_{i}$ and is fixed by the following rule: the mapping $p: \partial \widehat{M} \rightarrow \partial M$ preserves the orientation at $b_{i}$ and reverses it at $b_{i+n}, i=1, \ldots, n$. Thus, for every $M \in \mathcal{D B} \mathcal{S}_{n}$ an orientation cover $\widehat{M} \in \mathcal{D B S}_{2 n}$ is unique.

Let $1 \leq i \leq n$ and $\varepsilon \in\{+,-\}$. Denote $i^{\varepsilon}= \begin{cases}i, & \varepsilon=+, \\ \tau(i), & \varepsilon=-.\end{cases}$
Theorem 1.13. Let $M$ be a DBS of equation 1.1. Then its orientation cover is

$$
\begin{equation*}
\widehat{M}=G\left[i_{m}^{\varepsilon_{m}} j_{m}^{\delta_{m}}\right]^{++} \ldots G\left[i_{1}^{\varepsilon_{1}} j_{1}^{\delta_{1}}\right]^{++} G\left[i_{1}^{-\varepsilon_{1}} j_{1}^{-\delta_{1}}\right]^{++} \ldots G\left[i_{m}^{-\varepsilon_{m}} j_{m}^{-\delta_{m}}\right]^{++} E_{n} \tag{1.3}
\end{equation*}
$$

The involution $\mathcal{T}: \widehat{M} \rightarrow \widehat{M}$ maps the ribbon $r_{\ell}$ to the ribbon $r_{2 m+1-\ell}$ for all $\ell=1, \ldots, 2 m$.

See Figure 3 for an example. The Roman numerals there mean faces, Arabic numbers, vertices, and the circled numbers mark diagonals of the ribbons.

Proof. Let $a_{i}$ be a marked point of $M$ with $\mathcal{P}\left(a_{i}\right)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ where $\ell_{1}>\cdots>$ $\ell_{p}<\cdots<\ell_{k}$, and let $b_{i}, b_{\tau(i)} \in \widehat{M}$ be its preimages. Use the induction on $m$ to prove the theorem showing simultaneously that $\mathcal{P}\left(b_{i}\right)=\left(m+1-\ell_{1}, \ldots, m+\right.$ $\left.1-\ell_{p}, m+\ell_{p+1}, \ldots, m+\ell_{k}\right)$ and $\mathcal{P}\left(b_{\tau(i)}\right)=\left(m+1-\ell_{k}, \ldots, m+1-\ell_{p+1}, \ell_{p}+\right.$ $\left.m, \ldots, \ell_{1}+m\right)$.

The base $m=0$ is obvious. For $m>0$ let $M=G[i, j]^{\varepsilon \delta} M^{\prime}$ where $i, j, \varepsilon, \delta$ mean $i_{m}, j_{m}, \varepsilon_{m}, \delta_{m}$, respectively. If $\mathcal{P}_{M^{\prime}}\left(a_{i}\right)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ where $\ell_{1}>\cdots>\ell_{p}<\cdots<$ $\ell_{k}$ then $\mathcal{P}_{M}\left(a_{i}\right)=\left(\ell_{1}, \ldots, \ell_{k}, m\right)$ if $\varepsilon=+$ and $\mathcal{P}_{M}\left(a_{i}\right)=\left(m, \ell_{1}, \ldots, \ell_{k}\right)$ if $\varepsilon=-$; the same for $a_{j}$ (depending on $\delta$ instead of $\varepsilon$ ).

Denote by $\widehat{M}^{\prime}$ the orientation cover of $M^{\prime}$ and define $\widehat{M}$ by $(1.3)$. By the induction hypothesis $\widehat{M}^{\prime}$ is a subset of $\widehat{M}$ (a union of all the ribbons except $r_{1}$ and $r_{2 m}$ ). Extend $\mathcal{T}: \widehat{M^{\prime}} \rightarrow \widehat{M^{\prime}}$ to the involution $\widehat{M} \rightarrow \widehat{M}$ sending $r_{1}$ to $r_{2 m}$ and vice versa;


Figure 3. Covering of the Moebius band $M=G[1,2]^{++} G[2,3]^{++} G[1,3]^{+-} E_{3}$ by an annulus
also extend the homeomorphism $\rho: \widehat{M}^{\prime} / \mathcal{T} \rightarrow M^{\prime}$ to a map $\widehat{M} / \mathcal{T} \rightarrow M$ sending $r_{1}$ and $r_{2 m}$ to the $m$-th ribbon of $M$. To complete the proof we are to check that the extended $\mathcal{T}$ and $\rho$ are continuous on the boundary of the ribbons $r_{1}$ and $r_{2 m}$.

By the induction hypothesis, $\mathcal{P}_{\widehat{M}^{\prime}}\left(b_{i}\right)=\left(m-\ell_{1}, \ldots, m-\ell_{p}, \ell_{p+1}+m-1, \ldots, \ell_{k}+\right.$ $m-1)$ and $\mathcal{P}_{\widehat{M}^{\prime}}\left(b_{\tau(i)}\right)=\left(m-\ell_{k}, \ldots, m-\ell_{p+1}, \ell_{p}+m-1, \ldots, \ell_{1}+m-1\right)$. So, if $\varepsilon=+$ then $\mathcal{P}_{\widehat{M}}\left(b_{i}\right)=\left(m+1-\ell_{1}, \ldots, m+1-\ell_{p}, \ell_{p+1}+m, \ldots, \ell_{k}+m, 2 m\right)$ and $\mathcal{P}_{\widehat{M}^{\prime}}\left(b_{\tau(i)}\right)=\left(1, m+1-\ell_{k}, \ldots, m+1-\ell_{p+1}, \ell_{p}+m, \ldots, \ell_{1}+m\right)$, and if $\varepsilon=-$ then $\mathcal{P}_{\widehat{M}}\left(b_{i}\right)=\left(1, m+1-\ell_{1}, \ldots, m+1-\ell_{p}, \ell_{p+1}+m, \ldots, \ell_{k}+m\right)$ and $\mathcal{P}_{\widehat{M}^{\prime}}\left(b_{\tau(i)}\right)=\left(m+1-\ell_{k}, \ldots, m+1-\ell_{p+1}, \ell_{p}+m, \ldots, \ell_{1}+m, 2 m\right)$; the same for $b_{j}$ and $b_{\tau(j)}$, with $\delta$ instead of $\varepsilon$.

Thus, if $\varepsilon=+$ then the ribbon $r_{2 m}$ is adjacent to $r_{\ell_{k}+m}$ and the ribbon $r_{1}$, to $r_{m+1-\ell_{k}}$; the $m$-th ribbon of $M=G[i, j]^{\varepsilon \delta} M^{\prime}$ is adjacent to the ribbon numbered $\ell_{k}$. By the induction hypothesis, $\mathcal{T}$ exchanges $r_{\ell_{k}+m}$ and $r_{m+1-\ell_{k}}$, so the extensions of $\mathcal{T}$ and $\rho$ are continuous on the "long" sides of $r_{2 m}$ and $r_{1}$ containing $b_{i}$ and $b_{\tau(i)}$,
respectively. The proof in the case $\varepsilon=-$ is the same. A similar analysis of $\mathcal{P}\left(b_{j}\right)$ and $\mathcal{P}\left(b_{\tau(j)}\right)$ for $\delta=+$ and $\delta=-$ shows that $\mathcal{T}$ and $\rho$ are continuous on the other sides of $r_{2 m}$ and $r_{1}$, too.

## 2. Algebraic model and twisted cut-And-Join equation

2.1. Algebraic preliminaries and twisted Hurwitz numbers. Recall [2] that the reflection group $B_{n}$ is the semidirect product $S_{n} \ltimes(\mathbb{Z} / 2 \mathbb{Z})^{n}$ where $S_{n}$ acts on $(\mathbb{Z} / 2 \mathbb{Z})^{n}$ by permuting the factors; it is generated by reflections $s_{i j} \stackrel{\text { def }}{=}(i j) \ltimes \mathbb{1}$ and $l_{i} \stackrel{\text { def }}{=} \mathrm{id} \ltimes \mathbf{1}_{i}$ where $\mathbf{1} \stackrel{\text { def }}{=}(1, \ldots, 1)$ and $\mathbf{1}_{i} \stackrel{\text { def }}{=}(1, \ldots,-1, \ldots, 1)(-1$ at the $i$ th place). The group $B_{n}$ can be embedded into $S_{2 n}$ as a centralizer $C(\tau)$ where $\tau$, as above, is defined by 0.2 ; the isomorphism $A: C(\tau) \rightarrow B_{n}$ is given by $A(\sigma)=\lambda_{\sigma} \ltimes\left(\varepsilon_{\sigma}^{(1)}, \ldots, \varepsilon_{\sigma}^{(n)}\right)$ where

$$
\lambda_{\sigma}(i)=\left\{\begin{array}{ll}
\sigma(i), & \sigma(i) \leq n, \\
\sigma(i)-n, & \sigma(i) \geq n+1
\end{array} \quad \text { and } \quad \varepsilon_{\sigma}^{(i)}= \begin{cases}1, & \sigma(i) \leq n \\
-1, & \sigma(i) \geq n+1\end{cases}\right.
$$

Let $C^{\sim}(\tau) \stackrel{\text { def }}{=}\left\{\sigma \in S_{2 n} \mid \tau \sigma=\sigma^{-1} \tau\right\}$ (a "twisted centralizer" of $\tau$ ).
Lemma 2.1. Let $\sigma=c_{1} \ldots c_{m} \in C^{\sim}(\tau)$ where $c_{1}, \ldots, c_{m}$ are independent cycles. Then for every $i$

- either there exists $j \neq i$ such that $c_{i}=\left(u_{1} \ldots u_{k}\right)$ and $c_{j}=\left(\tau\left(u_{k}\right) \ldots \tau\left(u_{1}\right)\right)$;
- or $c_{i}$ has even length $2 k$ and looks like $c_{i}=\left(u_{1} \ldots u_{k} \tau\left(u_{k}\right) \ldots \tau\left(u_{1}\right)\right)$.

In the first case we say that the cycles $c_{i}$ and $c_{j}$ are $\tau$-symmetric, and in the second case the cycle $c_{i}$ is $\tau$-self-symmetric.
Proof. Let $c_{i}=\left(u_{1}^{(i)} \ldots u_{k_{i}}^{(i)}\right)$ for all $i=1, \ldots, m$. Then $\tau \sigma \tau^{-1}=c_{1}^{\prime} \ldots c_{m}^{\prime}$ where $c_{i}^{\prime}=\left(\tau\left(u_{1}^{(i)}\right) \ldots \tau\left(u_{k_{i}}^{(i)}\right)\right)$. On the other side, $\sigma^{-1}=c_{1}^{\prime \prime} \ldots c_{m}^{\prime \prime}$ where $c_{i}^{\prime \prime}=\left(u_{k_{i}}^{(i)} \ldots u_{1}^{(i)}\right)$. Once a cycle decomposition is unique, every $c_{i}^{\prime \prime}$ must be equal to some $c_{j}^{\prime}$. If $j \neq i$ then $c_{i}$ and $c_{j}$ are $\tau$-symmetric, and if $j=i$ then $c_{i}$ is $\tau$-self-symmetric.

Theorem 2.2. There exists a one-to-one correspondence between the following three sets:
(1) The quotient (the set of left cosets) $S_{2 n} / B_{n}$ where we assume $B_{n}=C(\tau)$;
(2) The set $B_{n}^{\sim}$ of permutations $\sigma \in C^{\sim}(\tau)$ such that their cycle decomposition contains no $\tau$-self-symmetric cycles.
(3) The set of fixed-point-free involutions $\lambda \in S_{2 n}$.

The size of each set is $(2 n-1)!!=1 \times 3 \times \cdots \times(2 n-1)$.
Proof. To prove the theorem we will construct injective maps $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$.
$1 \rightarrow 2$ let $\sigma \in S_{2 n}$ be an element of the coset $\lambda \in S_{2 n} / B_{n}$; take $Q(\lambda) \stackrel{\text { def }}{=}$ $[\sigma, \tau] \stackrel{\text { def }}{=} \sigma \tau \sigma^{-1} \tau$. Since $\tau$ is an involution, one has $\tau Q(\lambda) \tau=\tau \sigma \tau \sigma^{-1}=Q(\lambda)^{-1}$, so $Q(\lambda) \in C^{\sim}(\tau)$. If $\sigma^{\prime} \in \lambda$ is another element of the coset then $\sigma^{\prime}=\sigma \rho$ where $\rho \tau=\tau \rho$ and therefore $\left[\sigma^{\prime}, \tau\right]=\sigma \rho \tau \rho^{-1} \sigma^{-1} \tau=\sigma \tau \rho \rho^{-1} \sigma^{-1} \tau=Q(\lambda)$, so the map $Q: S_{2 n} / B_{n} \rightarrow C^{\sim}(\tau)$ is well-defined. If $Q(\lambda)=Q\left(\lambda^{\prime}\right)$ where $\lambda, \lambda^{\prime} \in S_{2 n} / B_{n}$ are represented by $\sigma$ and $\sigma^{\prime}$, respectively, then $\sigma \tau \sigma^{-1} \tau=\sigma^{\prime} \tau\left(\sigma^{\prime}\right)^{-1} \tau$, which is equivalent to $\left(\sigma^{\prime}\right)^{-1} \sigma \tau=\tau\left(\sigma^{\prime}\right)^{-1} \sigma$. So $\left(\sigma^{\prime}\right)^{-1} \sigma \in B_{n}$, and $\lambda=\lambda^{\prime}$, so $Q$ is injective.

Prove that actually $Q(\lambda) \in B_{n}^{\sim} \subset C^{\sim}(\tau)$. Suppose it is not the case, that is, $Q(\lambda)$ contains a $\tau$-self-symmetric cycle $c=\left(u_{1} \ldots u_{k} \tau\left(u_{k}\right) \ldots \tau\left(u_{1}\right)\right)$. It implies
that $\tau Q(\lambda)$ has a fixed point $u=u_{k}$. On the other hand, $\tau Q(\lambda)=(\tau \sigma) \tau(\tau \sigma)^{-1}$ is conjugate to $\tau$ and is therefore a product of $n$ independent transpositions having no fixed points - a contradiction.

2 2 $\rightarrow 3$ the condition $\tau \sigma \tau^{-1}=\sigma^{-1}$ is equivalent to $(\sigma \tau)^{2}=\mathrm{id}$. If $\sigma=c_{1} c_{2} \ldots c_{k}$ then $\sigma \tau$ sends every element of the cycle $c_{i}$ to an element of its $\tau$-symmetric cycle $c_{j}$. So if $j \neq i$ for all $i$ then the involution $\sigma \tau$ has no fixed points. The map $\sigma \mapsto \sigma \tau$ is obviously injective.
$3 \rightarrow 1$ if $\psi$ is a fixed-point-free involution then its cycle decomposition is a product of $n$ independent transpositions, and therefore $\psi$ belongs to the same conjugacy class in $S_{2 n}$ as $\tau: \psi=\sigma \tau \sigma^{-1}$ for some $\sigma \in S_{2 n}$. Denote by $R(\psi) \in S_{2 n} / B_{n}$ the left coset containing $\sigma$. The equality $\sigma_{1} \tau \sigma_{1}^{-1}=\sigma_{2} \tau \sigma_{2}^{-1}$ is equivalent to $\left(\sigma_{1} \sigma_{2}^{-1}\right) \tau=\tau\left(\sigma_{1} \sigma_{2}^{-1}\right)$, that is, $\sigma_{1} \sigma_{2}^{-1} \in B_{n}$. So the left cosets containing $\sigma_{1}$ and $\sigma_{2}$ are the same and $R(\psi) \in S_{2 n} / B_{n}$ is well-defined. If $R\left(\psi_{1}\right)=R\left(\psi_{2}\right)$ where $\psi_{i}=\sigma_{i} \tau \sigma_{i}^{-1}, i=1,2$, then $\sigma_{1} \sigma_{2}^{-1} \in B_{n}$ and therefore $\psi_{1}=\psi_{2}$; thus, $R$ is an injective map.

Fix a partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{s}\right),|\lambda|=n$, and denote by $B_{\lambda}^{\sim} \subset B_{n}^{\sim}$ the set of permutations whose decomposition into independent cycles consists of $s$ pairs of $\tau$-symmetric cycles of lengths $\lambda_{1}, \ldots, \lambda_{s}$. Apparently, $B_{n}^{\sim}=\bigsqcup_{|\lambda|=n} B_{\lambda}^{\sim}$.

Proposition 2.3. $B_{\lambda}^{\sim}$ is a $B_{n}$-conjugacy class in $S_{2 n}$.
Proof. Let $\sigma=c_{1} c_{1}^{\prime} \ldots c_{s} c_{s}^{\prime} \in B_{n}^{\sim}$ be the cycle decomposition where $c_{i}$ and $c_{i}^{\prime}$ are $\tau$-symmetric for all $i$ : $c_{i}^{\prime}=\tau c_{i}^{-1} \tau$. Let $x \in B_{n}$, that is, $x \tau=\tau x$. Then $x \sigma x^{-1}=$ $x c_{1} x^{-1} \cdot x \tau c_{1}^{-1} \tau x^{-1} \cdots \cdots x c_{s} x^{-1} \cdot x \tau c_{s}^{-1} \tau x^{-1}$. The permutations $\tilde{c}_{i} \stackrel{\text { def }}{=} x c_{i} x^{-1}$ and $\tilde{c}_{i}^{\prime}=x c_{i}^{\prime} x^{-1}$ are cycles of length $\lambda_{i}$, and they are $\tau$-symmetric: $\tau \tilde{c}_{i} \tau=\tau x c_{i} x^{-1} \tau=$ $x \tau c_{i} \tau x^{-1}=x\left(c_{i}^{\prime}\right)^{-1} x^{-1}=\left(\tilde{c}_{i}^{\prime}\right)^{-1}$. Thus, $x \sigma x^{-1} \in B_{\lambda}^{\sim}$.

On the other side, let $\tilde{\sigma}=\tilde{c}_{1} \tilde{c}_{1}^{\prime} \ldots \tilde{c}_{s} \tilde{c}_{s}^{\prime} \in B_{\lambda}^{\sim}$. Let $\tilde{c}_{i}=\left(v_{1}^{(i)} \ldots v_{\lambda_{i}}^{(i)}\right)$, so $\tilde{c}_{i}^{\prime}=\left(\tau\left(v_{\lambda_{i}}^{(i)}\right) \ldots \tau\left(v_{1}^{(i)}\right)\right)$. Define an element $x \in S_{2 n}$ such that $x\left(u_{s}^{(i)}\right)=v_{s}^{(i)}$ and $x\left(\tau\left(u_{s}^{(i)}\right)\right)=\tau\left(v_{s}^{(i)}\right)$. Then $x \sigma x^{-1}=\tilde{\sigma}$ and $x \tau=\tau x$ (that is, $x \in B_{n}$ ).

Now denote by $\mathfrak{S}_{m, \lambda}$ the set of decompositions into $m$ ribbons of the surfaces $M \in \mathcal{D B S}_{n}$ such that $\partial M$ has $s$ components containing $\lambda_{1}, \ldots, \lambda_{s}$ marked points.

Let $\mathcal{G} \in \mathfrak{S}_{m, \lambda}$ be a ribbon decomposition of $M \in \mathcal{D} \mathcal{B S}{ }_{n}$. Denote by $\widehat{M} \in \mathcal{D B S}_{2 n}$ the orientation cover of $M$ with a ribbon decomposition given by (1.3). Now define $\Xi(\mathcal{G}) \stackrel{\text { def }}{=}\left(\sigma_{1}, \ldots, \sigma_{m}\right)$ where each $\sigma_{k} \stackrel{\text { def }}{=}\left(i_{k}^{\varepsilon_{k}}, j_{k}^{\delta_{k}}\right) \in S_{2 n}$ is a transposition; here we are using the notation of Theorem 1.13. Denote

$$
\begin{aligned}
\mathfrak{H}_{m, \lambda} & \stackrel{\text { def }}{=}\left\{\left(\sigma_{1}, \ldots, \sigma_{m}\right) \mid \forall s=1, \ldots, m \sigma_{s}=\left(i_{s} j_{s}\right), j_{s} \neq \tau\left(i_{s}\right)\right. \\
& \left.\sigma_{1} \sigma_{2} \ldots \sigma_{m}\left(\tau \sigma_{m} \tau\right) \ldots\left(\tau \sigma_{1} \tau\right) \in B_{\lambda}^{\sim}\right\}
\end{aligned}
$$

Theorem 2.4. For any $\lambda$ and $m$ the map $\Xi$ is a one-to-one correspondence between $\mathfrak{S}_{m, \lambda}$ and $\mathfrak{H}_{m, \lambda}$.
Proof. Let $\mathcal{G} \in \mathfrak{S}_{m, \lambda}$ be a ribbon decomposition of $M \in \mathcal{D B S}_{n}$. By Theorem 1.13 the diagonal of the $k$-th ribbon in the ribbon decomposition (1.3) of $\widehat{M}$ joins the marked points numbered $i_{k}^{\varepsilon_{k}}$ and $j_{k}^{\delta_{k}}$ if $1 \leq k \leq m$ and the points numbered $i_{k-m}^{-\varepsilon_{k-m}}$ and $j_{k-m}^{-\delta_{k-m}}$ if $m+1 \leq k \leq 2 m$. Then by Property 4 of Theorem 1.11 the boundary
permutation of $\widehat{M}$ is

$$
\sigma=\sigma_{1} \sigma_{2} \ldots \sigma_{m}\left(\tau \sigma_{m} \tau\right) \ldots\left(\tau \sigma_{1} \tau\right)
$$

it has the cyclic type $\left(\lambda_{1}, \lambda_{1}, \ldots, \lambda_{s}, \lambda_{s}\right)$ by the definition of $\mathfrak{S}_{m, \lambda}$. On the other hand, $\tau \sigma \tau=\left(\tau \sigma_{1} \tau\right) \ldots\left(\tau \sigma_{m} \tau\right) \sigma_{m} \ldots \sigma_{1}=\sigma^{-1}$ because $\sigma_{k}$ and $\tau \sigma_{k} \tau$ are involutions for all $k$. Thus, $\sigma \in B_{n}^{\sim}$, hence $\sigma \in B_{\lambda}^{\sim}$, and so $\Xi(\mathcal{G}) \in \mathfrak{H}_{m, \lambda}$.

The map $\Xi$ is obviously one-to-one.
Corollary 2.5. The boundary permutation of the orientation cover $\widehat{M} \in \mathcal{D B S}_{2 n}$ of any $M \in \mathcal{D B S}_{n}$ belongs to $B_{n}^{\sim}$.

Corollary 2.6. If the twisted Hurwitz number $h_{m, \lambda}^{\sim}$ is defined by equation 0.1) then equality (0.4) takes place.

Example 2.7. For $m=1$, any $n=|\lambda|$ and any transposition $\sigma \neq(i, i+n)$ the permutation $\mu=\sigma \tau \sigma \tau$ belongs to $B_{n, 2^{1} 1^{n-2}}$. Now $\# \mathfrak{H}_{1,2^{11^{n-2}}}$ is the total number of all transpositions $\sigma \in S_{2 n}$ except $(i, i+n)$, which is $\frac{1}{2}(2 n)(2 n-1)-n=2 n(n-1)$. So, $h_{1,2^{1} 1^{n-2}}=\frac{2 n(n-1)}{n!}=\frac{2}{(n-2)!}$ and $h_{1, \lambda}=0$ for all other $\lambda$.

Let $m=2, n=|\lambda|=2$; here $\tau=(13)(24) \in S_{4}$. The set $B_{2}^{\sim} \subset B_{2}=C(\tau) \subset S_{4}$ is a union of two conjugacy classes, $B_{[2]}^{\sim}=\{(12)(34),(14)(23)\}$ and $B_{[1,1]}^{\sim}=\{e\}$.

Consider the permutation $\mu \stackrel{\text { def }}{=} \sigma_{1} \sigma_{2} \tau \sigma_{2} \sigma_{1} \tau$ where $\sigma_{1}, \sigma_{2} \in\{(12),(14),(23),(34)\}$; totally, there are 16 of them. It is easy to see that $\mu=e \in B_{[1,1]}^{\sim}$ if and only if $\sigma_{2}=\sigma_{1}$ or $\sigma_{2}=\tau \sigma_{1} \tau$; the remaining 8 pairs of transpositions $\left(\sigma_{1}, \sigma_{2}\right)$ give $\mu \in B_{[2]}^{\sim}$. This gives $h_{2,[1,1]}^{\sim}=h_{2,[2]}^{\sim}=\frac{8}{2!}=4$.

For $n=3, m=2$ the calculations (in $S_{6}$ ) are similar but more cumbersome, giving eventually $h_{2,[1,1,1]}^{\sim}=h_{2,[2,1]}^{\sim}=4$ and $h_{2,[3]}^{\sim}=16$.
2.2. Twisted cut-and-join operator. Now denote

$$
\begin{equation*}
\mathcal{C}_{\lambda}^{\sim} \stackrel{\text { def }}{=} \sum_{\sigma \in B_{\lambda}^{\sim}} \sigma \in \mathbb{C}\left[B_{n}^{\sim}\right] . \tag{2.1}
\end{equation*}
$$

(a conjugacy class sum). Also, call the set

$$
\mathcal{Z}\left(B_{n}^{\sim}\right) \xlongequal{\text { def }}\left\{y \in \mathbb{C}\left[B_{n}^{\sim}\right] \mid x y x^{-1}=y \forall x \in B_{n}\right\}
$$

a twisted center of $B_{n}$. It is clear that $\mathcal{C}_{\lambda}^{\sim}$ belong to $\mathcal{Z}\left(B_{n}^{\sim}\right)$ and form a basis in it.
Let $\mathbb{C}[p]$ be a space of polynomials of the countable set of variables $p=\left(p_{1}, p_{2}, \ldots\right)$. Assume $\operatorname{deg} p_{k}=k$ for all $k$ and denote by $\mathbb{C}[p]_{n}$ the space of homogeneous polynomials of degree $n$. A linear map $\Psi: \mathcal{Z}\left(B_{n}^{\sim}\right) \rightarrow \mathbb{C}[p]_{n}$ defined by

$$
\begin{equation*}
\Psi\left(\mathcal{C}_{\lambda}^{\sim}\right)=p_{\lambda} \stackrel{\text { def }}{=} p_{\lambda_{1}} \ldots p_{\lambda_{s}} \tag{2.2}
\end{equation*}
$$

is obviously an isomorphism of vector spaces.
Define an operator $\mathfrak{C} \mathfrak{J}^{\sim}: \mathcal{Z}\left(B_{n}^{\sim}\right) \rightarrow \mathcal{Z}\left(B_{n}^{\sim}\right)$ by

$$
\mathfrak{C}^{\sim}(\sigma)=\sum_{\substack{1 \leq i<j \leq 2 n \\ j \neq \tau(i)}}(i j) \sigma(\tau(i) \tau(j))
$$

Definition 2.8. The twisted cut-and-join operator is a linear map $\mathcal{C} \mathcal{J}^{\sim}: \mathbb{C}[p]_{n} \rightarrow$ $\mathbb{C}[p]_{n}$ making the following diagram commutative:


Let $\lambda, \mu$ be partitions such that $|\lambda|=|\mu|=n$. Take an element $\sigma \in B_{\lambda}^{\sim}$ and consider a set

$$
S(\sigma ; \mu) \stackrel{\text { def }}{=}\left\{(i, j) \mid \leq i, j \leq 2 n, j \neq i, \tau(i),(i j) \sigma_{*}(\tau(i) \tau(j)) \in B_{\mu}^{\sim}\right\}
$$

Proposition 2.3 implies that for every $x \in B_{n}$ and $\sigma \in B_{\lambda}^{\sim}$ the map $(i, j) \mapsto$ $(x(i), x(j))$ is a bijection between $S\left(x \sigma x^{-1}, \mu\right)$ and $S(\sigma, \mu)$. So, the size of the set $S(\sigma, \mu)$ for $\sigma \in B_{\lambda}^{\sim}$ depends on $\lambda$ and $\mu$ only.

We will be using "physical" notation for matrix elements of a linear operator: $\mathfrak{C} \mathfrak{J}^{\sim}\left(\mathcal{C}_{\lambda}^{\sim}\right)=\sum_{\mu}\langle\lambda| \mathfrak{C} \mathfrak{J}^{\sim}|\mu\rangle \mathcal{C}_{\mu}^{\sim}$

Theorem 2.9. $\langle\lambda| \mathfrak{C J}^{\sim}|\mu\rangle=\frac{1}{2} \# S(\sigma, \mu)$ for any $\sigma \in B_{\lambda}^{\sim}$.
Proof. By definition,

$$
\begin{equation*}
\mathfrak{C} \mathfrak{J}^{\sim}\left(\mathcal{C}_{\lambda}^{\sim}\right)=\sum_{\sigma \in B_{\lambda}^{\sim}} \mathfrak{C} \mathfrak{J}^{\sim}(\sigma)=\sum_{\sigma \in B_{\lambda}^{\sim}} \sum_{\substack{1 \leq i<j \leq 2 n \\ j \neq \tau(i)}}(i j) \sigma(\tau(i) \tau(j)) \tag{2.4}
\end{equation*}
$$

As it was noted above, 2.4 is a sum of identical summands, so

$$
\mathfrak{C} \mathcal{J}^{\sim}\left(\mathcal{C}_{\lambda}^{\sim}\right)=\# B_{\lambda}^{\sim} \sum_{\mu} \sum_{\substack{1 \leq i<j \leq 2 n \\ j \neq \tau(i) \\(i j) \sigma(\tau(i) \tau(j)) \in B_{\mu}^{\sim}}}(i j) \sigma(\tau(i) \tau(j))
$$

for any fixed $\sigma \in B_{\lambda}^{\sim}$, and therefore

$$
\begin{aligned}
\mathfrak{C} \mathfrak{J}^{\sim}\left(\mathcal{C}_{\lambda}^{\sim}\right)= & \sum_{\mu} \sum_{\substack{1 \leq i<j \leq 2 n \\
j \neq \tau(i) \\
(i j) \sigma(\tau(i) \tau(j)) \in B_{\mu}^{\sim}}} \sum_{\tau \in B_{\mu}} \tau \\
= & \frac{1}{2} \sum_{\mu} \#\left\{(i, j) \mid j \neq i, \tau(i),(i j) \sigma(\tau(i) \tau(j)) \in B_{\mu}^{\sim}\right\} \mathcal{C}_{\mu}^{\sim}
\end{aligned}
$$

Consider the generating function $\mathcal{H}^{\sim}(\beta, p)$ of the twisted Hurwitz numbers defined as follows:

$$
\begin{equation*}
\mathcal{H}^{\sim}(\beta, p)=\sum_{m \geq 0} \sum_{\lambda} \frac{h_{m, \lambda}^{\sim}}{m!} p_{\lambda_{1}} p_{\lambda_{2}} \ldots p_{\lambda_{s}} \beta^{m} \tag{2.5}
\end{equation*}
$$

Theorem 2.10. $\mathcal{H}^{\sim}$ satisfies the cut-and-join equation $\frac{\partial \mathcal{H}^{\sim}}{\partial \beta}=\mathcal{C} \mathcal{J}^{\sim}\left(\mathcal{H}^{\sim}\right)$.
Proof. Fix a positive integer $n$ and denote by $\mathcal{H}_{n}^{\sim}$ a degree $n$ homogeneous component of $\mathcal{H}^{\sim}$. The twisted cut-and-join operator preserves the degree, so $\mathcal{H}^{\sim}$ satisfies the cut-and-join equation if and only if $\mathcal{H}_{n}^{\sim}$ does (for each $n$ ).

Let

$$
\mathcal{G}_{n} \stackrel{\text { def }}{=} \sum_{m \geq 0} \sum_{\lambda:|\lambda|=n} \frac{n!h_{m, \lambda}^{\sim}}{m!} \mathcal{C}_{\lambda}^{\sim} \beta^{m} \in \mathbb{C}\left[S_{2 n}\right]
$$

where $\mathcal{C}_{\lambda}^{\sim}$ is defined by 2.1). An elementary combinatorial reasoning gives

$$
\mathcal{G}_{n}=\sum_{m \geq 0} \frac{\beta^{m}}{m!}\left(\mathfrak{C} \mathcal{J}^{\sim}\right)^{m}\left(e_{2 n}\right)
$$

where $e_{2 n} \in S_{2 n}$ is the unit element. Clearly $\mathfrak{C} \mathcal{J}^{\sim}\left(\mathcal{G}_{n}\right)=\sum_{m \geq 0} \frac{\beta^{m}}{m!}\left(\mathfrak{C} \mathfrak{J}^{\sim}\right)^{m+1}\left(e_{2 n}\right)=$ $\sum_{m \geq 1} \frac{\beta^{m-1}}{(m-1)!}\left(\mathfrak{C} \mathfrak{J}^{\sim}\right)^{m}\left(e_{2 n}\right)=\frac{\partial \mathcal{G}_{n}}{\partial \beta}$. Applying $\Psi$ one obtains $\Psi \mathfrak{C} \mathfrak{J}^{\sim}\left(\mathcal{G}_{n}\right)=\Psi\left(\frac{\partial \mathcal{G}_{n}}{\partial \beta}\right)=$ $\frac{\partial}{\partial \beta} \Psi\left(\mathcal{G}_{n}\right)$. By 2.2$), \Psi\left(\mathcal{G}_{n}\right)=\mathcal{H}_{n}^{\sim}$, hence $\frac{\partial}{\partial \beta} \Psi\left(\mathcal{G}_{n}\right)=\frac{\partial \mathcal{H}_{n}^{\sim}}{\partial \beta}$. By the definition of the twisted cut-and-join operator, $\Psi \mathfrak{C} \mathcal{J}^{\sim}\left(\mathcal{G}_{n}\right)=\mathcal{C} \mathcal{J}^{\sim}\left(\Psi\left(\mathcal{G}_{n}\right)\right)=\mathcal{C} \mathcal{J}^{\sim}\left(\mathcal{H}_{n}^{\sim}\right)$, and the equality $\frac{\partial \mathcal{H}_{n}^{\sim}}{\partial \beta}=\mathcal{C} \mathcal{J}^{\sim}\left(\mathcal{H}_{n}^{\sim}\right)$ follows.
Corollary 2.11. $\mathcal{H}^{\sim}(\beta, p)=\exp \left(\beta \mathcal{C} \mathcal{J}^{\sim}\right) \exp \left(p_{1}\right)$.
Proof. It follows from (0.1) that $h_{0, \lambda}=\frac{1}{n!}$ if $\lambda=1^{n}$ and $h_{0, \lambda}=0$ otherwise. Thus, $\mathcal{H}^{\sim}(0, p)=\exp \left(p_{1}\right)$, and the formula follows from Theorem 2.10 .
2.3. Explicit formulas. In this section we prove explicit formulas for the cut-andjoin operator (Theorem 2.12) and for twisted Hurwitz numbers (Theorem 2.15).

Theorem 2.12. The twisted cut-an-join operator is given by

$$
\begin{align*}
\mathcal{C} \mathcal{J}^{\sim} & =\sum_{i, j \geq 1}(i+j) p_{i} p_{j} \frac{\partial}{\partial p_{i+j}}+2 i j p_{i+j} \frac{\partial^{2}}{\partial p_{i} \partial p_{j}}+\sum_{k \geq 1} k(k-1) p_{k} \frac{\partial}{\partial p_{k}} \\
& =\sum_{i, j \geq 1}(i+j)\left(p_{i} p_{j}+p_{i+j}\right) \frac{\partial}{\partial p_{i+j}}+2 i j p_{i+j} \frac{\partial^{2}}{\partial p_{i} \partial p_{j}} \tag{2.6}
\end{align*}
$$

(The two formulas are equivalent because there are $k-1$ pairs $(i, j)$ such that $i, j \geq 1$ and $i+j=k$.)

To prove Theorem 2.12 we calculate explicitly the matrix elements $\langle\lambda| \mathfrak{G}^{\sim}|\mu\rangle$ for all possible $\lambda, \mu$.

Let $\sigma \in S_{n}$ and $1 \leq i<j \leq n$. The cycle structure of the product $\sigma^{\prime}=(i j) \sigma$ depends on the cycle structure of $\sigma$ and on $i$ and $j$ as follows: if $i$ and $j$ belong to the same cycle $\left(x_{1}, \ldots, x_{\ell}\right)$ of $\sigma$ (where $i=x_{1}, j=x_{m}$ ), then $\sigma^{\prime}$ contains two cycles $\left(x_{1}, \ldots, x_{m-1}\right)$ and $\left(x_{m}, \ldots, x_{\ell}\right)$ instead ("a cut"). If $i$ and $j$ are in different cycles $\left(x_{1}, \ldots, x_{m}\right)$ and $\left(y_{1}, \ldots, y_{k}\right)$ (where $i=x_{1}$ and $\left.j=y_{1}\right)$ then $\sigma^{\prime}$ contains the cycle $\left(x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{k}\right)$ instead ("a join").

Let now $\sigma \in B_{\lambda}^{\sim} \subset B_{n}^{\sim}$ where $\lambda=1^{a_{1}} 2^{a_{2}} \ldots n^{a_{n}}$ (in other words, the element $\sigma \in S_{2 n}$ contains $a_{s}$ pairs of $\tau$-symmetric cycles of length $s$ for $s=1, \ldots, n$ ). Let $1 \leq i<j \leq 2 n, j \neq \tau(i)$ and $\sigma^{\prime} \stackrel{\text { def }}{=}(i j) \sigma(\tau(i) \tau(j)) \in B_{\mu}^{\sim}$. The cyclic structure of $\sigma^{\prime}$ depends on the positions of $i, j, \tau(i), \tau(j)$ and on the cycles of $\sigma$; there are three possible cases shown in Fig. 4 .
Case 1. Here $\mu$ is obtained from $\lambda$ by a cut:

$$
\begin{array}{ll}
\mu=1^{a_{1}} \ldots m^{a_{m}+1} \ldots k^{a_{k}+1} \ldots \ell^{a_{\ell}-1} \ldots n^{a_{n}}, & m+k=\ell, m<k \\
\text { or } & \\
\mu=1^{a_{1}} \ldots m^{a_{m}+2} \ldots \ell^{a_{\ell}-1} \ldots n^{a_{n}}, & m=\ell / 2
\end{array}
$$



Figure 4. Terms of $\mathfrak{C} \mathfrak{I}^{\sim}$

For a fixed $\sigma \in B_{\lambda}^{\sim}$ look for $i, j$ such that $\sigma^{\prime} \stackrel{\text { def }}{=}(i j) \sigma(\tau(i) \tau(j)) \in B_{\mu}^{\sim}$. The element $\sigma$ contains $2 a_{\ell}$ cycles of length $\ell$, so there are $2 \ell a_{\ell}$ possible positions for $i$. In $\sigma^{\prime}$ the elements $i$ and $j$ are in different cycles; if $m<k$ then $m$ may be the length of either. So if $m<k$ then $j$ should be in the same cycle in $\sigma$ as $i$, and the distance between them is either $m$ or $k$. So there are two possible positions for $j$ once $i$ is chosen, and $\langle\mu| \mathfrak{G J}^{\sim}|\lambda\rangle=\frac{1}{2} \# S(\sigma, \mu)=2 \ell a_{\ell}$. If $m=k=\ell / 2$ then the position for $j$ is unique and $\langle\mu| \mathfrak{C} \mathfrak{J}^{\sim}|\lambda\rangle=\ell a_{\ell}$.
Case 2. Here $\mu$ is obtained from $\lambda$ by a join:

$$
\begin{aligned}
& \mu=1^{a_{1}} \ldots m^{a_{m}-1} \ldots k^{a_{k}-1} \ldots \ell^{a_{\ell}+1} \ldots n^{a_{n}}, \quad m+k=\ell, m<k \\
& \text { or } \\
& \mu=1^{a_{1}} \ldots m^{a_{m}-2} \ldots \ell^{a_{\ell}+1} \ldots n^{a_{n}}, \quad m=\ell / 2
\end{aligned}
$$

If $m<k$ then $i$ may belong to the cycle of either length. If $i$ belongs to the cycle of length $m$ then there are $2 m a_{m}$ possible positions for it (cf. Case 1) and $2 k a_{k}$ positions for $j$; vice versa if $i$ belongs to the cycle of length $k$. The matrix element is then $\langle\mu| \mathfrak{C J}^{\sim}|\lambda\rangle=4 m k a_{m} a_{k}$. If $m=k=\ell / 2$ then $i$ and $j$ belong to cycles of the same length $m$; the cycle containing $j$ contains neither $i$ nor $\tau(i)$. Hence there are $4 a_{m}\left(a_{m}-1\right)$ possibilities for choosing a pair of cycles to contain $i$ and $j$, and $m^{2}$ possible positions for $i$ and $j$ in them. Therefore $\langle\mu| \mathfrak{C} \mathfrak{J}^{\sim}|\lambda\rangle=2 m^{2} a_{m}\left(a_{m}-1\right)$.

Case 3. Here $\mu=\lambda$. Like in the previous cases we have $2 \ell a_{\ell}$ possible positions for $i$ and $\ell-1$ positions for $j \neq \tau(i)$ (in the cycle $\tau$-symmetric to the one containing $i$ ) once $i$ is fixed. Thus, $\langle\mu| \mathfrak{G}^{\sim}|\lambda\rangle=\sum_{\ell} 2 \ell(\ell-1) a_{\ell}$.
Proof of Theorem 2.12. It follows from Theorem 2.9 and Definition 2.8 that $\mathcal{C} \mathcal{J}^{\sim} p_{\lambda}=$ $\sum_{\mu}\langle\lambda| \mathfrak{G} \mathfrak{J}^{\sim}|\mu\rangle p_{\mu}$.

For a given $\lambda$ there are three types of $\mu$ such that $\langle\lambda| \mathfrak{C J}^{\sim}|\mu\rangle \neq 0$ listed above. Hence $\mathcal{C} \mathcal{J}^{\sim}$ is a sum of three terms.

Suppose $\mu$ is like in Case 1 with $m<k$. The monomial $p_{\lambda}$ contains $p_{m}^{a_{m}} p_{k}^{a_{k}} p_{\ell}^{a_{\ell}}$ and the monomial $p_{\mu}$ contains $p_{m}^{a_{m}+1} p_{k}^{a_{k}+1} p_{\ell}^{a_{\ell}-1}$; the other factors are the same. So the term in 2.6 acting on $p_{\lambda}$ and giving $p_{\mu}$ is $2 \ell p_{m} p_{k} \frac{\partial}{\partial p_{\ell}} p_{\lambda}=2 \ell a_{\ell} p_{\mu}=\langle\mu| \mathfrak{C} \mathcal{J}^{\sim}|\lambda\rangle p_{\mu}$ (actually there are two equal terms in the sum: $i=m, j=k$ or vice versa, hence the factor 2 ).

If $\mu$ is like in Case 1 with $m=\ell / 2$ then $p_{\lambda}$ contains $p_{\ell / 2}^{a_{\ell / 2}} p_{\ell}^{a_{\ell}}$ and $\mu$ contains $p_{\ell / 2}^{a_{\ell / 2}+2} p_{\ell}^{a_{\ell}-1}$. So the only term in 2.6) acting on $p_{\lambda}$ and giving $p_{\mu}$ is $\ell p_{\ell / 2}^{2} \frac{\partial}{\partial p_{\ell}} p_{\lambda}=$ $\ell a_{\ell} p_{\mu}=\langle\mu| \mathfrak{C J}^{\sim}|\lambda\rangle p_{\mu}$.

The calculations for the two remaining cases are similar.
By Theorem 2.12, $\mathcal{C J}^{\sim}=\mathcal{C} \mathcal{J}_{0}+\mathcal{R}$ where

$$
\mathcal{C} \mathcal{J}_{0}=\sum_{i, j \geq 1}(i+j) p_{i} p_{j} \frac{\partial}{\partial p_{i+j}}+i j p_{i+j} \frac{\partial^{2}}{\partial p_{i} \partial p_{j}}
$$

is the classical cut-and-join, and

$$
\mathcal{R}=\sum_{i, j \geq 1} p_{i+j}\left((i+j) \frac{\partial}{\partial p_{i+j}}+i j \frac{\partial^{2}}{\partial p_{i} \partial p_{j}}\right) .
$$

A one-parametric family

$$
\Delta_{\alpha} \stackrel{\text { def }}{=} \mathcal{C} \mathcal{J}_{0}+(\alpha-1) \mathcal{R}=\sum_{i, j \geq 1}(i+j)\left(p_{i} p_{j}+(\alpha-1) p_{i+j}\right) \frac{\partial}{\partial p_{i+j}}+\alpha i j p_{i+j} \frac{\partial^{2}}{\partial p_{i} \partial p_{j}}
$$

is called [3] the Laplace-Beltrami operator; in particular, $\Delta_{1}=\mathcal{C} \mathcal{J}_{0}$ is the classical cut-and-join and $\Delta_{2}=\mathcal{C} \mathcal{J}^{\sim}$, the twisted cut-and-join. By the classical results of [3, p. 376 and after], the eigenvalues (and eigenvectors) of $\Delta_{\alpha}$ are indexed by partitions. The eigenvalue corresponding to $\lambda=\left(\lambda_{1} \geq \cdots \geq \lambda_{s}\right)$ is equal to

$$
e(\lambda, \alpha)=\sum_{i=1}^{s} \lambda_{i}\left(\alpha \lambda_{i}+2-2 i-\alpha\right)
$$

The corresponding eigenvector is a polynomial $J_{\lambda}^{(\alpha)}(p)$ of degree $|\lambda| \stackrel{\text { def }}{=} \lambda_{1}+\cdots+\lambda_{s}$ called Jack polynomial; it is normalized so that the coefficient at $p_{1}^{n}$ in it is 1 . Polynomials $Z_{\lambda} \stackrel{\text { def }}{=} J_{\lambda}^{(2)}$ are called zonal.
Theorem 2.13 ([3]).

$$
\sum_{\lambda} \frac{J_{\lambda}^{(\alpha)}(p) J_{\lambda}^{(\alpha)}(q)}{H_{\lambda}(\alpha) H_{\lambda}^{\prime}(\alpha)}=\exp \left(\sum_{k \geq 1} \frac{p_{k} q_{k}}{k \alpha}\right)
$$

where $H_{\lambda}(\alpha) \stackrel{\text { def }}{=} \prod_{(i, j) \in Y(\lambda)}(\alpha a(i, j)+\ell(i, j)+1)$ and $H_{\lambda}^{\prime}(\alpha) \stackrel{\text { def }}{=} \prod_{(i, j) \in Y(\lambda)}(\alpha a(i, j)+$ $\ell(i, j)+\alpha)$. Here $Y(\lambda)$ is the Young diagram of the partition $\lambda$, and $a(i, j)$ and $\ell(i, j)$ are the arm and the leg, respectively, of the cell $(i, j) \in Y(\lambda)$.

Substituting $q_{1}=\alpha, q_{2}=q_{3}=\cdots=0$ and taking into account the normalization of the Jack polynomials one obtains
Corollary 2.14. $\sum_{\lambda} \frac{\alpha^{|\lambda|} J_{\lambda}^{(\alpha)}(p)}{H_{\lambda}(\alpha) H_{\lambda}^{\prime}(\alpha)}=\exp \left(p_{1}\right)$.
Taking now $\alpha=2$ and substituting the formula of Corollary 2.14 into Corollary 2.11, one obtains

## Theorem 2.15.

$$
\mathcal{H}^{\sim}(\beta, p)=\sum_{\lambda} \exp \left(2 \beta \sum_{i} \lambda_{i}\left(\lambda_{i}-i\right)\right) \frac{2^{|\lambda|} Z_{\lambda}(p)}{H_{\lambda}(2) H_{\lambda}^{\prime}(2)}
$$

This is a twisted analog of the formula expressing the usual Hurwitz numbers via Schur polynomials, see [4].

Example 2.16. The zonal polynomials $Z_{\lambda}$ for small $\lambda$ are:
$Z_{[1]}=p_{1}$, with $H_{[1]}(2) H_{[1]}^{\prime}(2)=2$
$Z_{[1,1]}=p_{1}^{2}-p_{2}$ with $H_{[1,1]}(2) H_{[1,1]}^{\prime}(2)=12$
$Z_{[2]}=p_{1}^{2}+2 p_{2}$ with $H_{[2]}(2) H_{[2]}^{\prime}(2)=24$
$Z_{[1,1,1]}=p_{1}^{3}-3 p_{2} p_{1}+2 p_{3}$ with $H_{[1,1,1]}(2) H_{[1,1,1]}^{\prime}(2)=144$
$Z_{[2,1]}=p_{1}^{3}+p_{2} p_{1}-2 p_{3}$ with with $H_{[2,1]}(2) H_{[2,1]}^{\prime}(2)=80$
$Z_{[3]}=p_{1}^{3}+6 p_{2} p_{1}+8 p_{3}$ with $H_{[3]}(2) H_{[3]}^{\prime}(2)=720$
This gives us the first few terms in the expansion of $\mathcal{H}^{\sim}(\beta, p)$ :

$$
\begin{aligned}
\mathcal{H}^{\sim}(\beta, p)= & p_{1}+\frac{p_{1}^{2}}{6}\left(2 e^{-2 \beta}+e^{4 \beta}\right)+\frac{p_{2}}{3}\left(-e^{-2 \beta}+e^{4 \beta}\right)+\frac{p_{1}^{3}}{90}\left(9 e^{2 \beta}+e^{12 \beta}+5 e^{-6 \beta}\right) \\
+ & \frac{p_{2} p_{1}}{30}\left(2 e^{12 \beta}+3 e^{2 \beta}-5 e^{-6 \beta}\right)+\frac{p_{3}}{45}\left(4 e^{12 \beta}-9 e^{2 \beta}+5 e^{-6 \beta}\right)+\ldots \\
= & \left(p_{1}+\frac{p_{1}^{2}}{2}+\frac{p_{1}^{3}}{6}+\ldots\right)+\beta\left(2 p_{1}+2 p_{1} p_{2}+\ldots\right) \\
& \quad+\beta^{2}\left(2 p_{1}^{2}+2 p_{2}+2 p_{1}^{3}+2 p_{1} p_{2}+8 p_{3}+\ldots\right)+\ldots
\end{aligned}
$$

they agree with 2.5 and Example 2.7

## 3. Algebro-geometric model: Twisted branched coverings

A classical notion of the branched covering was extended to the non-orientable case by G. Chapuy and M. Dołęga in [7]. Let $N$ denote a closed surface (compact 2-manifold without boundary, not necessarily orientable), and p: $\widehat{N} \rightarrow N$, its orientation cover. As above, denote by $\mathcal{T}: \widehat{N} \rightarrow \widehat{N}$ an orientation-reversing involution without fixed points such that $p \circ \mathcal{T}=p$. Also denote by $\mathcal{J}: \mathbb{C} P^{1} \rightarrow \mathbb{C} P^{1}$ the complex conjugation, and let $\overline{\mathbb{H}} \stackrel{\text { def }}{=} \mathbb{C} P^{1} /(z \sim \mathcal{J}(z))=\mathbb{H} \cup\{\infty\}$ where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane; $\overline{\mathbb{H}}$ is homeomorphic to a disk. Denote by $\pi: \mathbb{C} P^{1} \rightarrow \overline{\mathbb{H}}$ the quotient map.
Definition $3.1([7])$. A continuous map $f: N \rightarrow \overline{\mathbb{H}}$ is called a twisted branched covering if there exists a branched covering $\widehat{f}: \widehat{N} \rightarrow \mathbb{C} P^{1}$ such that
(1) $\pi \circ \widehat{f}=f \circ p$, and
(2) all the critical values of $\widehat{f}$ are real.

Property $\sqrt{11}$ is equivalent to saying that $\widehat{f}$ is a real map with respect to $\mathcal{T}$, that is, $\widehat{f} \circ \mathcal{T}=\mathcal{J} \circ \widehat{f}$. The involution $\mathcal{T}$ has no fixed points, so the critical points of $\widehat{f}$ come in pairs $(a, \mathcal{T}(a))$, the ramification profile of every critical value $c \in \mathbb{R} P^{1} \subset \mathbb{C} P^{1}$ of $\widehat{f}$ has every part repeated twice: $\left(\lambda_{1}, \lambda_{1}, \ldots, \lambda_{s}, \lambda_{s}\right)$, and $\operatorname{deg} \widehat{f}=2 n$ is even. In this case we say that the ramification profile of the critical value $\pi(c) \in \partial \overline{\mathbb{H}}$ of the $\operatorname{map} f: N \rightarrow \overline{\mathbb{H}}$ is $\lambda=\left(\lambda_{1}, \ldots, \lambda_{s}\right)$.

The twisted branched covering $f$ is called simple if all its critical values, except possibly $\infty \in \overline{\mathbb{H}}$, have the ramification profile $2^{1} 1^{n-2}$. (Equivalently, each critical value of $\widehat{f}$ has 2 simple critical points and $2 n-4$ regular points as preimages.) Let $u \in \partial \overline{\mathbb{H}}$ be a regular (not critical) value of $f$; then the preimage $f^{-1}(u) \subset N$ consists of $n$ points. Fix a bijection $\nu: f^{-1}(u) \rightarrow\{1, \ldots, n\}$ (a labeling); then the triple $(f, u, \nu)$ is called a labeled simple twisted branched covering.

Labeled simple twisted branched coverings are split into equivalence classes via right-left equivalence: $\left(f_{1}, u_{1}, \nu_{1}\right) \sim\left(f_{2}, u_{2}, \nu_{2}\right)$ if there exist orientation-preserving diffeomorphisms $D_{1}: \widehat{N} \rightarrow \widehat{N}$ and $D_{2}: \mathbb{C} P^{1} \rightarrow \mathbb{C} P^{1}$ such that

- $\left(f_{1}\right.$ transforms to $\left.f_{2}\right) \widehat{f}_{1} \circ D_{1}=D_{2} \circ \widehat{f}_{2}$,
- $\left(D_{1}\right.$ and $D_{2}$ are equivariant) $\mathcal{T} \circ D_{1}=D_{1} \circ \mathcal{T}$ and $D_{2} \circ \mathcal{J}=\mathcal{J} \circ D_{2}$,
- $\left(D_{1}, D_{2}\right.$ preserve labeling) $D_{2}\left(\pi^{-1}\left(u_{1}\right)\right)=\pi^{-1}\left(u_{2}\right)$ and $\nu_{2} \circ D_{1}=\nu_{1}$.

For an integer $m \geq 0$ and a partition $\lambda$ denote by $\mathfrak{D}_{m, \lambda}$ the set of equivalence classes of labeled simple twisted branched coverings having $m$ simple critical values and such that the ramification profile of $\infty$ is $\lambda$.

Theorem 3.2. $\# \mathfrak{D}_{m, \lambda}=\# \mathfrak{S}_{m, \lambda}=\# \mathfrak{H}_{m, \lambda}=n!h_{m, \lambda}^{\sim}$.
Proof. The generating function $\mathcal{D}(\beta, p) \stackrel{\text { def }}{=}=\sum_{m \geq 0} \sum_{\lambda} \frac{\# \mathfrak{D}_{m, \lambda}}{n!m!} p_{\lambda_{1}} p_{\lambda_{2}} \ldots p_{\lambda_{s}} \beta^{m}$ is shown in [7. Theorem 6.5 for $b=1$ ] to satisfy the twisted cut-and-join equation $\frac{\partial \mathcal{D}}{\partial \beta}=\mathcal{C} \mathcal{J}^{\sim}(\mathcal{D})$ where $\mathcal{C} \mathcal{J}^{\sim}$ is given by equation 2.6).

Let $m=0$, so the branched covering $\widehat{f} \in \mathfrak{D}_{0, \lambda}$ is unramified except possibly over $\infty$. Denote by $N_{0} \subset \widehat{N}$ any connected component of $\widehat{N}$, by $n_{0}$, the degree of $\left.\widehat{f}\right|_{N_{0}}$, and $\left.k \stackrel{\text { def }}{=} \# \widehat{f}\right|_{N_{0}} ^{-1}(\infty)$. Then the Euler characteristic $\chi\left(\mathbb{C} P^{1} \backslash\{\infty\}\right)=1$ and therefore $\chi\left(N_{0} \backslash \widehat{f}^{-1}(\infty)\right)=\chi\left(N_{0}\right)-k=n_{0}$. The set $N_{0}$ is a smooth compact 2-manifold, so $2 \geq \chi\left(N_{0}\right)=n_{0}+k$, implying $n_{0}=k=1$. It means that $\widehat{f}$ is unramified over $\infty$, too, so $\lambda=1^{n}$ and $\widehat{f}$ is a collection of $n$ orientation-preserving diffeomorphisms of spheres. Obviously, $\widehat{f}$ is unique up to the right-left equivalence described above. Thus, $\# \mathfrak{D}_{0,1^{n}}=1$ and $\# \mathfrak{D}_{0, \lambda}=0$ for other $\lambda$. $\operatorname{So}, \mathcal{D}(0, p)=\exp \left(p_{1}\right)$, and Corollary 2.11 implies that $\mathcal{D}(\beta, p) \equiv \mathcal{H}^{\sim}(\beta, p)$ proving the theorem.

Remark. Note that unlike Theorem 2.4 we do not know any "natural" one-to-one map between the sets $\mathfrak{D}_{m, \lambda}$ and $\mathfrak{S}_{m, \lambda}\left(\right.$ or $\left.\mathfrak{H}_{m, \lambda}^{\sim}\right)$. Finding one is a challenging topic of future research.

In [7], a one-parametric generalization of Hurwitz numbers is defined by counting twisted branched coverings with parameter-dependent weights. The parameter value $b=0$ gives classical Hurwitz numbers, and $b=1$, twisted Hurwitz numbers. A natural one-to-one correspondence between $\mathfrak{D}_{m, \lambda}$ and $\mathfrak{S}_{m, \lambda}$ would allow to transfer these weights to ribbon decompositions and to define parametric Hurwitz numbers using them.

Note also that in [7], more general two-part Hurwitz numbers were studied; currently we do not know other models for them.

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[^0]:    2010 Mathematics Subject Classification. Primary 57N05, secondary 05C30.
    Key words and phrases. Surface with boundary, Hurwitz number, Jack polynomial.
    The research was partially funded by the HSE University Basic Research Program and by the International Laboratory of Cluster Geometry NRU HSE (RF Government grant, ag. No. 075-15-2021-608 dated 08.06.2021). The research of the first-named author was also supported by the Simons Foundation IUM grant 2021.

