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Gerstner waves and their generalizations in hydrodynamics and geophysics

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Abstract. To mark 220 years since the appearance of Gerstner's paper that proposed an exact solution to the hydrodynamic equations, an overview of exact solutions for water waves is given, each of which is a generalization of the Gerstner wave. Additional factors are coastal geometry, fluid rotation, varying pressure on the free surface, stratification, fluid compressibility, and background flows. Waves on a rotating Earth are studied in the *f*-plane approximation, and, in the near-equatorial region, also in the *β*-plane approximation. The flows are described in Lagrangian variables. For all waves in the absence of background flows, the trajectories of liquid particles are circles, as in the Gerstner wave (hence, their common name — Gerstner-like).

Keywords: Gerstner waves, Lagrangian coordinates, vorticity, Cauchy invariants, edge waves, Ptolemaic flows, rotating fluid, *f*-plane approximation, equatorially trapped waves

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On the 220th anniversary of the first exact solution in nonlinear wave theory

1. Introduction

The birth of the science studying nonlinear waves is traditionally associated with the first experiments by Scott Russel who, in the 1830–1840s, was the first to observe solitons propagating on the surface of a shallow channel [1]. In 1895, Korteweg and de Vries described this phenomenon mathematically based on an equation that came to be known later under their names [2]. However, the historically first analytical representation for nonlinear waves, describing a stationary wave with a trochoid profile on deep water, published by Franz Joseph Gerstner in 1802 [3,4], remained (and remains even now) out of the sight of many researchers. By virtue of some factors, which are discussed below, the Gerstner wave did not enjoy even 'one thousandth' of the attention paid to Korteweg–de Vries solitons. But the fact remains: first there was the Gerstner wave (Fig. 1).

The Korteweg–de Vries equation is derived from full equations of hydrodynamics (fluid dynamics) in the approximation of small nonlinearity and dispersion. Its soliton solution is a classical example of the nonlinear wave in which the effects of nonlinearity and dispersion equilibrate each other. On the other hand, the Gerstner wave is an exact solution of full equations of hydrodynamics, and it is to date the only example where these equations can be integrated for gravity waves on deep water. It would therefore be difficult to deny that, from a mathematical standpoint, the Gerstner wave is a much more significant achievement in the analytical theory of waves on water than shallow water solitons.

As is well known, more fame and much more attention than the Gerstner wave garnered was attracted by the weakly nonlinear Stokes wave [7]. This circumstance at first glance

Figure 1. Prague engineer and mechanic Franz Joseph Gerstner (1756–1831).

A short biographic reference. The name of Franz Joseph Gerstner was well known in the Czech lands and Austria. He was a professor at and the director of the Polytechnic Institute in Prague and the head of all hydraulic engineering. There was practically no engineering endeavor in the Czech lands that was undertaken without his involvement or advice [5]. The director of the Federal Institute of Technology in Zurich, Hans Straub (1895–1964), included F J Gerstner in the list of names of "great researchers and engineers" [6], not only as a scientist who proposed railway construction, but also as the creator of the trochoidal wave theory. As a matter of curiosity, his son (also a renowned engineer) — Franz Anton Gerstner — was the constructor of the first Russian railroad.

looks even more surprising, because the Stokes solution is written as a series in the small parameter of wave steepness. And yet such a representation for a periodic surface wave proved to be more practically important than the exact solution. The reason is the problem of realizability of the Gerstner wave. In contrast to the Stokes wave, the Gerstner wave is a vortical wave, and cannot occur in nature under the action of potential forces (Lagrange's theorem). Either the action of external nonpotential forces or special boundary conditions are needed to create Gerstner waves. For instance, H Lamb proposed that the Gerstner wave can be created from a shear flow with the same vorticity as in the wave [4]. In this case, the translational motion of fluid particles in the flow must be transformed into circular rotation (there is no drift flow in a Gerstner wave). The apparently special character of such a scenario did not favor the popularity of the Gerstner solution and its wide applications in practical computations. Nevertheless, based on the Gerstner formulas, A N Krylov developed the theory of ship pitching on a wavy sea [8] that "found wide applications in ship construction" [9]. Dubreil-Jacotin showed that Gerstner waves may exist in a fluid with arbitrary stratification [10].

The discovery of the mechanism of modulational instability for potential waves on water [11] seemed to finally turn the Gerstner theory into a hydrodynamical artifact — an elegant exact solution that is not realized in nature — but, unexpectedly, at about the same time, new examples of its applications started to emerge. Pollard modified the Gerstner solution for waves in a rotating fluid in the f-plane approximation [12]. Yih [13], Mollo-Christensen [14, 15], and, in a more complete form, Constantin [16] applied the Gerstner solution to describe edge waves propagating along a sloping coast (Refs [13–16] show that the results for a homogeneous fluid can be generalized to the case of stratified and rotating fluid). Furthermore, Mollo-Christensen gave a description of billows of Gerstner waves (clouds of trochoidal form) in a stratified atmosphere [17], and the authors of Refs [18, 19] found a cylindrical analog of Gerstner waves — epicycloidal waves propagating along the free surface of a cavity in a uniformly rotating fluid.

All these advances raised the status of the Gerstner solution to a substantial degree, but the question of its physical realization remained open. However, Monismith et al. [20] managed to solve this old problem by creating a Gerstner wave in laboratory conditions. The principle of the experiment was as follows. In a Gerstner wave, fluid particles move along circles, so there is no drift flow, which makes this wave different from the Stokes wave, whose propagation is accompanied by particle displacement (Stokes drift) to the wave propagation side. Thus, when generating a Gerstner wave, a flow was created in the direction against the wave. The absence of fluid particle drift in individual realizations pointed in favor of observing a Gerstner wave in experiment. The care with which the authors of Ref. [20] formulated their conclusions is worthy of special note. For their additional confirmation, they turned to analogous experiments carried out in other laboratories and showed that Gerstner waves were also observed earlier in three basins [21–23].

All the mentioned results were obtained in bounded channels with artificially (mechanically) generated waves. But, as noted by the authors of Ref. [20], similar observations (i.e., the absence of fluid particle drift) were also made for waves in the open ocean [24]. Thus, the existence of Gerstner waves was also confirmed in field conditions. Admittedly, for a wave steepness in excess of $1/3$, Gerstner waves are unstable against three-dimensional perturbations [25], but it has become possible to refer to them now as real physical waves. In turn, Weber pointed out that a drift flow may occur in a Gerstner wave, taking into account viscosity and surface films [26]. In Weber's opinion, wave motions of this type could plausibly have been observed by experimentalists since long ago (without realizing this and without attributing the observations to Gerstner waves) in laboratory basins.

During the last decade, the topic of Gerstner waves has got a second life, although it is difficult to say whether this is related to the experimental work [20] that could have inspired theoreticians. Numerous papers appeared generalizing Gerstner's solution to the case of nonconstant pressure on a free surface due to the action of wind, and taking into account Earth's rotation and stratification (a selection of such papers is given in the table in Section 2). Some of these papers used the term 'Gerstner-like' (or Gerstner type) waves, reflecting the link between the solutions obtained and the classical Gerstner wave, which, under some approximation, is their particular case.

The aim of present paper is to give a survey as complete as possible of works devoted to the Gerstner wave, its modifications, and its various generalizations. Not long ago, monographs focused on the Lagrangian description of fluid motion were published both in Russia and abroad [27, 28], but they barely touched this topic. In this respect, the present work is seen as rather important and timely. Besides, it seems appropriate to draw additional attention to Lagrangian fluid dynamics, the methods of which are used by a rather narrow circle of physicists. Finally, and perhaps most importantly, it makes great sense to compile at one time a full list of known exact solutions to this problem, especially because the year 2022 marks the 220th anniversary of the Gerstner paper.

2. Classical hydrodynamics

In the title of this paper, traditional (classical) hydrodynamics and geophysical fluid dynamics are separated as distinct components of general fluid mechanics with the aim of organizing the material and for the convenience of presentation. Geophysical fluid dynamics studies such motions of fluids when "an essential role is played by the *rotation* of the system as a whole and fluid *stratification*" (V M Kamenkovich, A S Monin, foreword to book [29]). Section 2 is devoted to traditional fluid dynamics, and Sections 3 and 4 deal with waves on the rotating Earth and in a stratified fluid, respectively.

2.1 Equations of hydrodynamics in Lagrangian variables

Traditionally, the motion of an ideal fluid is studied based on the following system of equations [4]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0\,, \qquad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{F}\,,$$

where ρ is the fluid density, $\mathbf{v}$ is its velocity, p is the pressure, and $\mathbf{F}$ is the external force per unit mass. The first of these equations is the continuity equation, and the second one is the Euler equation. All the introduced quantities are in the general case functions of three coordinates and time. In this description, called Eulerian, an observer is dealing with the velocity field at a given point.

There is an alternative way to describe fluid motion, called Lagrangian, where one follows the motion of fixed 'fluid particles' beginning from some time instant. The coordinates of such an individual particle, X, Y, Z, are considered to depend on three spatial coordinates, a, b, c, and time t,

$$X = X(a,b,c,t)\,, \quad Y = Y(a,b,c,t)\,, \quad Z = Z(a,b,c,t)\,. \tag{1}$$

These spatial variables can be taken as the initial coordinates of fluid particles X_0, Y_0, Z_0 at the time moment $t = 0$, i.e.,

$$a = X_0\,, \quad b = Y_0\,, \quad c = Z_0\,. \tag{2}$$

However, it is convenient to assume that each quantity a, b, c is some function of the coordinates of the initial fluid particle position and that they serve as labels for the individual fluid particle. Following the tradition, we call them Lagrangian coordinates.

The association of the variables a, b, c with the name of Lagrange is due to Dirichlet (1860), but, strictly speaking, it bears in itself an element of inaccuracy [27, 30]. In fact, Lagrange assumed that the fluid particle labels should satisfy condition (2). However, Euler, who, by the way, was ahead of Lagrange in formulating the method of describing fluid motion which came to be known later as Lagrangian, assumed the coordinates a, b, c to have a more general (accepted today) sense, calling them material variables. The accepted division of the methods describing fluid motion, as well as variables, into Eulerian and Lagrangian has a historical connotation, but in our opinion Dirichlet made a very wise decision in choosing the name for variables a, b, c.

The continuity equation in the Lagrangian variables is written in a compact form with the help of the Jacobi matrix (the Jacobian)

$$\hat{R} = \begin{pmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{pmatrix}, \tag{3}$$

with elements being the derivatives of current particle coordinates over the Lagrangian variables. Matrix $\hat{R}$ describes the change in an infinitesimal fluid element $\mathrm{d}\mathbf{R}\{\mathrm{d}X, \mathrm{d}Y, \mathrm{d}Z\}$ that corresponds to the increment $\mathrm{d}\mathbf{a}\{\mathrm{d}a, \mathrm{d}b, \mathrm{d}c\}$ in the Lagrangian coordinates,

$$\mathrm{d}\mathbf{R} = \hat{R}\,\mathrm{d}\mathbf{a}\,. \tag{4}$$

The continuity equation in this notation becomes [27, 28]

$$\rho \det \hat{R} = \rho_0 \det \hat{R}_0\,, \quad \rho_0 = \rho\big|_{t=0}\,, \quad \hat{R}_0 = \hat{R}\big|_{t=0}\,, \tag{5}$$

where $\rho(a,b,c,t)$ is the density. For an incompressible fluid, the determinant of matrix $\hat{R}$ does not depend on time. If conditions (2) are enforced, matrix $\hat{R}_0$ is a unity one.

In order to obtain the motion equation in the Lagrangian form, we rewrite the Euler equation in the following way:

$$\mathbf{R}_{tt} = -\frac{1}{\rho}\,\nabla p + \mathbf{F}\,, \tag{6}$$

where $\mathbf{R} = \mathbf{R}\{X, Y, Z\}$. Equation (6) expresses Newton's second law for an individual particle. In order to exclude the differentiation over unknown functions X, Y, Z on its right-hand side, we make dot products of (6) with vectors $\mathbf{R}_a$, $\mathbf{R}_b$, and $\mathbf{R}_c$. The result is

$$\begin{aligned} \mathbf{R}_{tt}\mathbf{R}_a &= -\frac{1}{\rho}\,p_a + \mathbf{F}\mathbf{R}_a\,, \\ \mathbf{R}_{tt}\mathbf{R}_b &= -\frac{1}{\rho}\,p_b + \mathbf{F}\mathbf{R}_b\,, \\ \mathbf{R}_{tt}\mathbf{R}_c &= -\frac{1}{\rho}\,p_c + \mathbf{F}\mathbf{R}_c\,. \end{aligned} \tag{7}$$

With the help of the Jacobian matrix, these equations are written as [31]

$$\hat{R}^{\mathrm{T}}(\mathbf{R}_{tt} - \mathbf{F}) = -\frac{1}{\rho}\,\nabla_{\mathbf{a}}\,p\,. \tag{8}$$

The upper index T implies a transpose operation, and on the right-hand side the notation of gradient with respect to the Lagrangian variables $\mathbf{a}\{a, b, c\}$ is used.

2.2 Gerstner wave

Let us consider plane motions of an ideal incompressible fluid with a free surface in the field of Earth's gravity. Let the axis X be directed horizontally to the right, and the axis Y be directed vertically upward, and let a be the horizontal Lagrangian coordinate and b, the vertical one (fluid fills the half plane $b \leqslant 0$). The fluid is assumed to be infinitely deep. The equations of fluid dynamics in the Lagrangian form (5), (7) can then be written as

$$\frac{\partial}{\partial t}\frac{\partial(X, Y)}{\partial(a, b)} = \frac{\partial D_0}{\partial t} = 0\,, \tag{9}$$

$$X_{tt}X_a + Y_{tt}Y_a + gY_a = -\frac{1}{\rho}\,p_a\,, \tag{10}$$

$$X_{tt}X_b + Y_{tt}Y_b + gY_b = -\frac{1}{\rho}\,p_b\,, \tag{11}$$

where we took into account that the acceleration **g** due to gravity is directed against Y and the force is written as $\mathbf{F} = -\mathbf{g}$. The fluid is assumed to be homogeneous. Excluding pressure by cross-differentiation equations (10) and (11), one gets the condition that vorticity Ω is preserved along trajectories:

$$\frac{\partial}{\partial t}(X_{ta}X_b + Y_{ta}Y_b - X_{tb}X_a - Y_{tb}Y_a) = \frac{\partial(\Omega D_0)}{\partial t} = D_0\frac{\partial\Omega}{\partial t} = 0\,, \tag{12}$$

where

$$\Omega = D_0^{-1}\left[\frac{\partial(X_t, X)}{\partial(a,b)} + \frac{\partial(Y_t, Y)}{\partial(a,b)}\right].$$

To describe waves on a fluid surface, one should find solutions of equations (9) and (12), and also satisfy the conditions that pressure is constant on the free surface,

$$p\big|_{b=0} = p_0 = \text{const}\,, \tag{13}$$

and that vertical velocity oscillations decay with depth, $Y_t|_{b\to-\infty} = 0$.

Gerstner proposed an exact solution to this problem [3, 32]:

$$X = a - A\exp(kb)\sin(ka - \omega t)\,,$$
$$Y = b + A\exp(kb)\cos(ka - \omega t)\,, \quad b \leqslant 0\,, \tag{14}$$

where A is the wave amplitude, k is the wave number, and ω is the wave frequency. Just as in linear potential waves, the wave number and frequency are linked by the relationship

$$\omega^2 = gk\,, \tag{15}$$

which is a consequence of condition (13). At any time instant, the free surface presents a trochoid — a curve drawn by a point on a circle of radius A that rolls without slip along the horizontal line $Y = -A$. The trochoid moves with the speed $U = \omega k^{-1}$ to the right preserving its form, which is why Gerstner waves are also called trochoidal. Only solutions with $A \leqslant k^{-1}$ make physical sense; otherwise, the profile crosses itself. If $A = k^{-1}$, the crests in the profile become acute (with the angle equal to zero), such a limiting trochoid is called a cycloid (Fig. 2).

In the 1860s, solution (14) was rediscovered by three authors simultaneously: Froude [33], Rankine [34], and Reech [35]. For more than half a century, Gerstner's classical result stayed unnoticed. And this circumstance characterizes Gerstner as an outstanding scientist who was ahead of his time.

The coordinates of trajectory of an individual particle satisfy the relationship (see (14))

$$(X-a)^2 + (Y-b)^2 = A^2\exp(2kb)\,,$$

whence it follows that, in a fixed reference frame, each particle moves along circles with radius $A\exp(kb)$ (there is no particle drift in Gerstner waves). The initial position of fluid particles in Gerstner's solution does not coincide with Lagrangian coordinates, namely

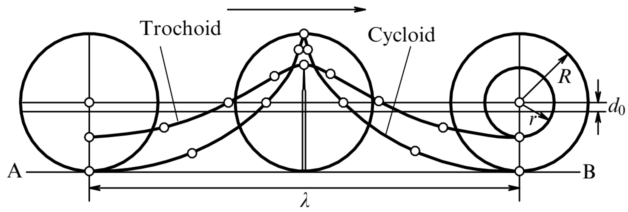

Figure 2. Cycloid ($R = k^{-1} = (2\pi)^{-1}\lambda$) and trochoid ($r < R$). Middle level of the cycloid is located lower than that of the trochoid by $d_0 = \pi(R^2 - r^2)\lambda^{-1}$. As applied to a Gerstner wave (14), this implies that, the larger the amplitude A, the lower the equilibrium level of fluid [32].

$$X_0 = a - A\exp(kb)\sin(ka)\,, \quad Y_0 = b + A\exp(kb)\cos(ka)\,.$$

One can, certainly, take X_0, Y_0 as the new Lagrangian variables, but in this case it would be impossible to propose explicit expressions for X and Y as functions of X_0 and Y_0.

The vorticity in the Gerstner wave is expressed as

$$\Omega_* = \frac{2k^3A^2U\exp(2kb)}{1 - k^2A^2\exp(2kb)}\,. \tag{16}$$

It is computed by using expressions (12) and (14). In the case of small wave steepness, when $\varepsilon = kA \ll 1$, the vorticity is given by the expression

$$\Omega_* = 2kU\exp(2kb)\varepsilon^2\left[1 + \exp(2kb)\varepsilon^2\right] + O(\varepsilon^6)\,. \tag{17}$$

From (17), it follows that, in the linear approximation, the Gerstner wave is irrotational and coincides with the linear Stokes wave. In the quadratic approximation, the vorticity in the Gerstner wave is $2kU\exp(2kb)\varepsilon^2$. It is equal in absolute value but opposite in sign to the vorticity of the Stokes drift induced by the potential Stokes wave. As demonstrated in Ref. [36], in the quadratic approximation, the following statement is valid:

$$\text{Stokes wave} = \text{Gerstner wave} + \text{Stokes drift}\,.$$

The Stokes wave is potential, and its vorticity is identically zero in all orders of perturbation theory. The Gerstner wave, as follows from formula (17), acquires additional vorticity in each even order of perturbation theory. Moreover, the profiles of both waves coincide up to the third order of perturbation theory [37] (see also [9]). Only in the fourth order of perturbation theory is a difference observed between the wave profiles.

Given solution (14), the pressure can be found using Eqns (10) and (11) [30] as

$$\frac{p - p_0}{\rho} = -gb - \frac{\omega^2A^2}{2}\left[1 - \exp(2kb)\right]. \tag{18}$$

The pressure in the fluid depends only on the coordinate b. As demonstrated by A S Monin [38], the only type of stationary waves in which pressure depends only on the vertical Lagrangian coordinate is trochoidal Gerstner waves. It is a remarkable fact that solution (14) remains valid in a stratified fluid with density $\rho(b)$ [8]. This happens because both the density and the pressure are functions of a single coordinate b, and hence a function of each other. In a barotropic fluid, the condition that vorticity (12) be preserved is the same as in a homogeneous fluid.

The treatise by Lamb [4] remained for a long time the only source abroad presenting the Gerstner wave theory. The discussion of wave properties there was rather concise, in contrast to that in Soviet textbook [32]. This triggered the appearance of articles [39, 40] and a chapter in monograph [41] dedicated directly to the Gerstner wave already in our time. An original feature of Refs [39–41] is the proof (in different ways) that transformation (14) is a diffeomorphism mapping the plane of material (Lagrangian) labels on the flow domain.

2.3 Edge waves along a sloping coast

Edge waves is the name for waves propagating along the coast. These waves attain maximum amplitude at the boundary with the land and rapidly decay offshore. All their energy is concentrated in a narrow coastal zone and barely leaks into the open ocean, so that one says that there is 'trapping' of wave energy, and also refers to these waves as trapped waves. Their study, as well as that of usual surface waves, was initiated by Stokes [42]. At present, such studies already form a separate branch of the theory of water waves (see, e.g., Refs [43, 44]), but the only exact solution in nonlinear wave theory is nevertheless related to the Gerstner approach [13–16] once again. We present this solution based on Ref. [16].

Let water be bounded by a sloping bottom that is at an angle α to the horizontal plane ($0 < \alpha < \pi/2$). We will consider edge waves propagating along the coast. Select the X-axis parallel to the coast, the Y-axis along the sloping bottom, and the Z-axis in the perpendicular direction (Fig. 3). The land-water boundary satisfies the conditions $-\infty < X < \infty$, $Y = b_0$, $Z = 0$, where b_0 is a constant ($b_0 \leqslant 0$). In this frame of reference, a unit mass of fluid will be affected by the force

$$\mathbf{f} = (0, -g\sin\alpha, -g\cos\alpha)\,. \tag{19}$$

If waves are absent, the fluid occupies the domain between the bottom plane ($Z = 0$) and the plane $Z = (b_0 - Y)\tan\alpha$. Let this fluid domain coincide with the set of Lagrangian labels, and let the variables a, b, c be counted along the axes X, Y, Z, respectively. The equation for the free surface in the Lagrangian frame takes the form

$$c = (b_0 - b)\tan\alpha\,, \quad b \leqslant b_0\,. \tag{20}$$

The problem geometry suggests that the motion in which fluid particles move in planes parallel to the bottom be considered. In this case, the Z component of velocity is absent and the impermeability condition along the Z-axis is ensured automatically. By analogy with expressions (14), we write the expression for such a two-dimensional flow as [16]

$$X = a - \frac{1}{k}\exp\left[k(b-c)\right]\sin\left(ka + \sqrt{gk\sin\alpha}\,t\right),$$
$$Y = b - c + \frac{1}{k}\exp\left[k(b-c)\right]\cos\left(ka + \sqrt{gk\sin\alpha}\,t\right), \tag{21}$$
$$Z = c + c\tan\alpha + \frac{\tan\alpha}{2k}\exp(2kb_0)\left\{1 - \exp\left[-2kc(1+\cot\alpha)\right]\right\}.$$

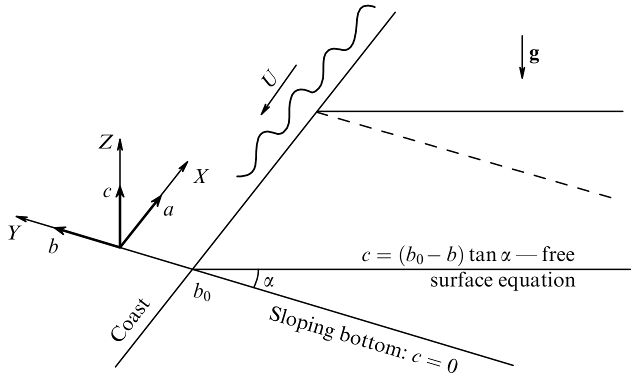

Figure 3. The problem geometry.

The first two equations of system (21) are similar to the Gerstner formulas, only the roles of the coordinate b and acceleration due to gravity is played by $b - c$ and $g\sin\alpha$. The wave frequency is $\sqrt{gk\sin\alpha}$ (naturally, the edge Stokes wave has the same frequency). The + sign in the arguments of trigonometric functions implies that the wave is propagating toward negative X (its speed is $U = \sqrt{g\sin\alpha/k}$) (see Fig. 3). The wave profile at the coast ($Z = c = 0$) corresponds to a trochoid without cusps ($b_0 < 0$) or a cycloid with cusps pointing upward ($b_0 = 0$).

The relationship for Z in (21) ensures that the pressure is constant on the free surface. Inserting condition (20) into equations (21), we obtain a parametric representation for the free surface. From this representation, it can be easily seen that the wave amplitude exponentially decays in the offshore direction ($b \to -\infty$), i.e., the wave is trapped. The vorticity in the wave has only the Z component, and its magnitude follows from formula (16) by replacing $A \to k^{-1}\exp(-kc)$ and adding a minus after the equal sign (the wave propagates to the left).

The instability of three-dimensional edge waves (21) was explored by Ionescu-Kruse [45] with the help of the Leblanc shortwave perturbation method [25]. It is proven that waves with a steepness larger than $(7/18)\sin\alpha$ are unstable.

2.4 Gouyon waves

Gerstner waves are vortical. However, they have a rather special form of vorticity (16). If expression (16) is considered in a weakly nonlinear limit as an expansion in the small wave steepness parameter, it contains only terms even in powers of ε (see relationship (17)). The multiplier with the powers of the parameter are the well-defined functions of coordinate b. However, in the Lagrangian description of stationary flow, the isolines of the streamfunction and the coordinate b coincide, and vorticity can be an arbitrary function of this coordinate. For this reason, a natural extension of the Gerstner wave is a periodic wave with a more general distribution of vorticity:

$$\Omega_*(b) = \sum_{n=1}^{\infty} \varepsilon_n \Omega_{*n}(b)\,. \tag{22}$$

The question of the description of a periodic wave with a general distribution of vorticity was studied for the first time by Dubreil-Jacotin [46], but Gouyon [47] did it in a more general form (see also Ref. [9]). Both researchers used Eulerian coordinates: in formula (22), the variable b should be replaced by the streamfunction. Gouyon found an explicit expression for the correction to the linear wave propagation speed, which is proportional to the steepness parameter; this is why it seems appropriate to refer to stationary waves with vorticity (22) as Gouyon waves.

In Ref. [48], which took into account not only Ω_{*1} but also Ω_{*2}, a quadratic correction to the wave propagation speed was found. Lagrangian variables were used for computations. Later, Gouyon's result was generalized to a spatial case [49]. The three-dimensional character of wave perturbations (the appearance of transverse modulation on the profile of a weakly nonlinear Gouyon wave) is the reason for the existence of a near-surface layer inside which the vorticity is an oscillating function.

In the case $\Omega_{*1} = 0$, $\Omega_{*2}(a, b) \neq 0$, in Lagrangian coordinates, a nonlinear Schrödinger equation (NSE) is derived describing the evolution of the complex amplitude of modulated wave $A(a, t)$ [50]:

$$\mathrm{i}\,\frac{\partial A}{\partial a} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t^2} - k\big(k^2|A|^2 + \gamma(a)\big)A = 0\,,$$
$$\gamma(a) = \frac{4k^2}{\omega}\int_{-\infty}^{0}\exp(2kb)\left(\int_{-\infty}^{b}\Omega_2(a, b')\,\mathrm{d}b'\right)\mathrm{d}b\,.$$

In the NSE, the coefficient by the nonlinear term depends on the vorticity distribution. For Gouyon waves, the magnitude of γ does not depend on the horizontal Lagrangian coordinate a. In the quadratic approximation, the weakly nonlinear Gerstner wave is a particular case of the Gouyon wave. Its vorticity is such (see (17)) that the coefficient by the nonlinearity in the NSE becomes zero [50, 51]. The effect of modulation instability is therefore absent for a Gerstner wave.

This is a rigorous result, but it could be foreseen by taking into account the dispersion relation for the Gerstner wave (15). Since it misses the amplitude, by virtue of the Lighthill criterion [52] the wave does not satisfy the conditions for modulational instability.

2.5 Epicycloid waves on the surface of a rotating cavity

There is a cylindrical analog of plain Gerstner waves. Consider wave motions in a fluid that partly fills a cylinder (a centrifuge) that is set in fast rotation with frequency ω around a horizontal axis. Under the action of centrifugal force, the fluid turns out to be pressed against the cylinder wall and rotates together with it around the central air core. The oscillations of the free surface occurring in this case are called centrifugal waves [53]. Their description in the linear approximation is well known [53, 54]. However, if the external radius of the centrifuge is considered infinite (much larger than the radius of air core R), the problem can be solved exactly [18, 19]. We will follow Ref. [19] in presenting the solution method.

Let the fluid rotation axis coincide with the direction of the Z-axis. Consider fluid motion in the XY plane. We introduce complex coordinates of fluid particle trajectory

$$W = X + \mathrm{i}Y\,, \quad \bar{W} = X - \mathrm{i}Y$$

and complex Lagrangian coordinates

$$\chi = a + \mathrm{i}b\,, \quad \bar{\chi} = a - \mathrm{i}b\,.$$

In this case, the system of equations of plane hydrodynamics (9), (12) can be written as the condition that two Jacobians be time independent [55, 56],

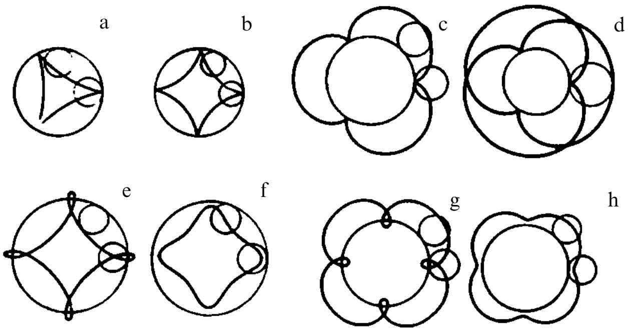

Figure 4. (a, b) Hypocycloids. (c, d) Epicycloids. (e) Elongated hypocycloid. (f) Shortened hypocycloid. (g) Elongated epicycloid. (h) Shortened epicycloid.

$$\frac{\partial}{\partial t}\frac{\partial(W, \bar{W})}{\partial(\chi, \bar{\chi})} = \frac{\partial D_0}{\partial t} = 0\,, \quad \frac{\partial}{\partial t}\frac{\partial(W_t, \bar{W})}{\partial(\chi, \bar{\chi})} = 0\,. \tag{23}$$

Through direct substitution, one can see that the expression

$$W = G(\chi)\exp(\mathrm{i}\lambda t) + F(\bar{\chi})\exp(\mathrm{i}\mu t)\,, \tag{24}$$

where G and F are analytical functions and λ and μ are real numbers, is an exact solution of system (23). The functions G and F are to a substantial degree arbitrary, since the only limitation on their choice is that the Jacobian D_0 does not become zero in the flow domain.

A particle in flows (24) moves along a circle with radius $|F|$ whose center, in turn, rotates along a circle with radius $|G|$. If the ratio of frequencies μ and λ is positive, the particle trajectory will be an epicycloid, and if it is negative, it will be a hypocycloid (Fig. 4); the number of petals in the curves depends on the frequency ratio. Such orbits were followed by planets in the Ptolemaic picture of the world, which is why this type of flows is called Ptolemaic [55, 56].

Gerstner waves (14) in complex variables are written as

$$W = \chi + \mathrm{i}A\exp\big[\mathrm{i}(k\bar{\chi} - \omega t)\big]\,, \quad \mathrm{Im}\,\chi \leqslant 0\,. \tag{25}$$

They belong to the set of Ptolemaic motions (24), but, since frequency λ for them is equal to zero, fluid particles simply move over circles.

If we assume that waves on the surface of a rotating cavity also belong to the class of Ptolemaic flows, then the functions G and F should be defined by boundary conditions. Since at infinity the fluid rotates as a whole, in formula (24) we should set

$$G(\chi) = R\exp(\mathrm{i}k\chi) = v\,, \quad \lambda = \omega\,, \tag{26}$$

and assume that $|F| \to 0$ at $|v| \to \infty$; in the plane of Lagrangian variable χ, the domain corresponds to a half-band in the lower half-plane $\mathrm{Im}\,\chi \leqslant 0$, $0 \leqslant \mathrm{Re}\,\chi \leqslant 2\pi$ or the outer part of a circle of radius R in the plane of complex variable v.

The function F will be found from the condition that pressure be constant on the free surface of a cavity, which corresponds to the equality $|v| = R$. Taking into account relationships (10) and (11), the expression for pressure can be written as (we neglect the action of the gravity force)

$$\frac{p}{\rho} = \frac{1}{2}\,\omega^2|v|^2 + \frac{1}{2}\,\mu^2|F|^2 + \operatorname{Re}\int\left(\omega^2 v\bar{F}' + \mu^2\bar{F}\right)\exp\left[\mathrm{i}(\lambda-\mu)t\right]\mathrm{d}v\,.$$

In order that the pressure be constant on the free surface, it is necessary that the coefficient with the temporal multiplier become zero. This happens when

$$F(\bar{v}) = A\bar{v}^{-q^2}\,, \quad q = \frac{\mu}{\omega}\,, \tag{27}$$

where A is a constant. The final expression for W will be obtained by inserting equalities (26), (27) into expression (24):

$$W = v\exp(\mathrm{i}\omega t) + A\bar{v}^{-q^2}\exp(\mathrm{i}q\omega t)\,, \tag{28}$$

from where it can be easily concluded that the trajectories of fluid particles are shortened epicycloids ($q > 0$) and hypocycloids ($q < 0$), and profiles of propagating waves are epicycloids with the number of petals $q^2 - 1$, and for the profile to be closed, q^2 should be an integer. For the constant that defines the amplitude of waves, there is an upper limit R^{q^2+1}/q^2, when cusps are forming on a free surface (for larger values of A, the profile has loops (Fig. 4a, c, e, g), which cannot be realized physically).

Epicycloidal waves are vortical. The vorticity in them is written as

$$\Omega_* = \frac{2\omega\left(1 - q^5A^2|v|^{-2(q^2+1)}\right)}{1 - q^4A^2|v|^{-2(q^2+1)}}\,,$$

whence it can be seen that the magnitude of vorticity depends on the sign of q, among other factors.

Let us find the angular velocity ω_0 of rotation of a stationary wave profile. The rotation of fluid as a whole with this frequency is characterized by a common multiplier $\exp(\mathrm{i}\omega t)$ in the expression for W; therefore, in the reference frame where the profile is at rest, solution (28) will take the form

$$W = v\exp\left[\mathrm{i}(\omega-\omega_0)t\right] + A\bar{v}^{-q^2}\exp\left[\mathrm{i}(q\omega-\omega_0)t\right].$$

In this frame, the particle trajectories coincide with the profile shape; hence, the equality

$$q\omega - \omega_0 = q^2(\omega - \omega_0)$$

is valid, from which we find

$$\omega_0 = \frac{q\omega}{q+1}\,.$$

In the reference frame rotating with angular velocity ω, the frequency of profile rotation is $(q+1)^{-1}\omega$, so that waves that correspond to negative q move in the rotation direction, and those that correspond to positive values move against it.

We note that Inogamov [18] arrived at an analogous result, also using complex Lagrangian coordinates. He, however, did not introduce the concept of Ptolemaic flows, having guessed the solution in the form (28) [18], and did not consider possible generalizations.

The waves just studied have an interesting parallel in magnetohydrodynamics. If the centrifugal force pushes the rotating particles to the periphery, the Lorentz force in a magnetic field, by contrast, keeps charged particles in a rotating core. Reference [57] solves two problems: (1) on azimuthal waves in a column of uniformly charged electron gas placed in a homogeneous longitudinal magnetic field, taking into account the effect of finite temperature; (2) in an equilibrium form of a plasma cylinder in an analogous external field. In both cases, the boundary of the transverse section is a hypocycloid. For azimuthal waves, the spectrum of eigenfrequencies is found. The frequency, as in all Gerstner-type waves, does not depend on the amplitude.

2.6 Generalized Gerstner waves for variable free surface pressure

Traditionally in the theory of water waves, it is required that pressure be constant on a free surface. However, this condition can be violated in the presence of wind. Then, the effect of this violation can be modeled through the formation of inhomogeneous and nonstationary pressure distribution on the free surface, and one can study the influence of given boundary conditions on the wave evolution.

Let us consider the generalizations of Gerstner waves of this kind. Assume that the flow domain in the Lagrangian variables occupies the lower half-plane, and the fluid motion is described by the expression

$$W = G(\chi) + F(\bar{\chi})\exp(-\mathrm{i}\omega t)\,. \tag{29}$$

This motion belongs to the family of Ptolemaic flows, but function G can in this case differ from the linear one, and function F can differ from the exponential one (see (25)). In expression (29), function G defines the level with respect to which the particles on the free surface rotate, and the module of function F defines the radius of their circular rotation (the wave amplitude). The particles are at rest in deep regions, so the following condition should be obeyed:

$$|F| \to 0 \quad \text{at} \quad b \to -\infty\,.$$

Since function F is an analytical one, it reaches its maximum on the free surface. Hence, it follows that the free surface particles will oscillate with the largest amplitude.

Wave solution (29) corresponds to the following pressure distribution:

$$\frac{p-p_0}{\rho} = -g\,\operatorname{Im}\left[G + F\exp(-\mathrm{i}\omega t)\right] + \frac{1}{2}\,\omega^2|F|^2 + \operatorname{Re}\left[\exp(\mathrm{i}\omega t)\int\omega^2 G'\bar{F}\,\mathrm{d}\chi\right],$$

where p_0 is a constant. In the general case, the pressure varies periodically with time and is nonuniform along the free surface $\operatorname{Im}\chi = 0$. In essence, we have a whole class of exact solutions which describe complex free surface dynamics for inhomogeneous and harmonically varying pressure along it. The vorticity of waves (29) is given by the equality

$$\Omega_* = \frac{2\omega|F'|^2}{|G'|^2 - |F'|^2}\,.$$

Different examples of generalized Gerstner waves (29) are studied in a series of papers [58–62]. Their details are summarized in the Table. The Ptolemaic solutions allow a broad class of nonstationary phenomena to be analyzed on a model level. We look at two of them.

Table. Examples of generalized Gerstner waves (α and β are constants that are different in each example).

Wave model	$G(\chi)$	$F(\bar{\chi})$	Reference
Oscillating standing soliton	χ	$\dfrac{\beta}{(\bar{\chi}+\mathrm{i})^n}\,,\quad \beta>0\,,\quad n\geqslant 2$	[58]
Oscillating soliton in the background of a Gerstner wave	χ	$\mathrm{i}A\exp(\mathrm{i}k\bar{\chi})+\dfrac{\beta}{(\chi+\mathrm{i})^n}$	[58]
Breather overturning on calm water	$\chi-\dfrac{\mathrm{i}\beta}{(\chi-\mathrm{i})^2}$	$\dfrac{\mathrm{i}\beta}{(\bar{\chi}+\mathrm{i})^2}$	[59]
Nonstationary Gerstner waves	$\chi+\dfrac{\beta}{\chi-\mathrm{i}\alpha}$	$\mathrm{i}A\exp(\mathrm{i}k\bar{\chi})$	[60]
Rogue wave inside a packet of Gerstner waves	$\chi+\dfrac{\mathrm{i}}{k}\ln\left(1+P\left(\dfrac{\chi}{\alpha}\right)\right)$, $P\left(\dfrac{\chi}{\alpha}\right)=\dfrac{\mathrm{i}\beta}{\mathrm{i}\alpha-\chi}$	$\mathrm{i}A\overline{\left(1+P\left(\dfrac{\chi}{\alpha}\right)\right)}\exp(\mathrm{i}k\bar{\chi})$	[61]
Rogue wave in the background of a Gerstner wave	$\chi-\dfrac{\mathrm{i}\beta}{(\chi-\mathrm{i}\alpha)^2}$	$-\mathrm{i}A\exp(\mathrm{i}k\bar{\chi})+\dfrac{\mathrm{i}\beta}{(\bar{\chi}+\mathrm{i}\alpha)^2}$	[62]

2.6.1 Breather overturning. Consider a Ptolemaic flow of the following form [59]:

$$W=\chi-\frac{\mathrm{i}\beta}{(\chi-\mathrm{i})^2}+\frac{\mathrm{i}\beta}{(\bar{\chi}+\mathrm{i})^2}\exp(-\mathrm{i}\omega t)\,,\quad \operatorname{Im}\chi<0\,. \qquad (30)$$

In expression (30), the quantities W, χ, β are assumed to be dimensionless. The transition to dimensional variables is carried out by the transformations $W\to\alpha_* W$, $\chi\to\alpha_*\chi$, $\beta\to\alpha_*^3\beta$, where α_* is some scale with the dimension of length. The dynamics of a free fluid surface for flow (30) are presented in Fig. 5. Solution (30) evolves differently, depending on the magnitude of β [59]. We begin the analysis from $t_0=\pi/\omega$. At this time moment, the shape of the free surface is symmetric with respect to the vertical axis passing through the point of maximum deflection (Fig. 5a) with a height of 2β (or $2\alpha_*\beta$ in the dimensional form). For negative β, the free surface has a trough. When $\beta>0$, the profile has a crest.

Figure 5 plots the evolution of a breather during one period of oscillations. The perturbation of the free surface changes its form but does not move as a single whole. For this reason, we can call it a breather. The breather profile varies with time. Two qualitatively different regimes are possible: (a) for $\beta=-0.25$ or $\beta=0.5$, the profile has no inflection points; (b) for $\beta=0.85$, the profile has inflection points, and breather overturning is observed.

We are interested in the profile with $\beta=0.85$ plotted by the solid line in Fig. 5. The steepness of its forward front increases with time, and at $t_1=4.25/\omega$ (Fig. 5b) an inflection point appears for the first time. This point is characterized by the vertical tangent to the profile shown by the dashed line. Further, the profile has two inflection points until the time instant $t_3=5.9/\omega$, when they coalesce. The inflection point disappears afterwards. At the moment $t=2\pi/\omega$, the surface becomes flat.

During the following half-cycle, all evolution stages are symmetrically repeated (Fig. 5e–g), and the inflection points form on the left breather slope. An inflection point appears at $t_4=6.7/\omega$ when the steepness is rather small. Observing such a situation in natural conditions is hardly possible. However, the breather dynamics during the first half-cycle resemble very much the breakup of oceanic waves. Closer to the time moment when a vertical tangent appears in the profile, one needs to account for viscosity, which will destroy the solution considered at some moment $t_*>t_1$. Thus, expression (30)

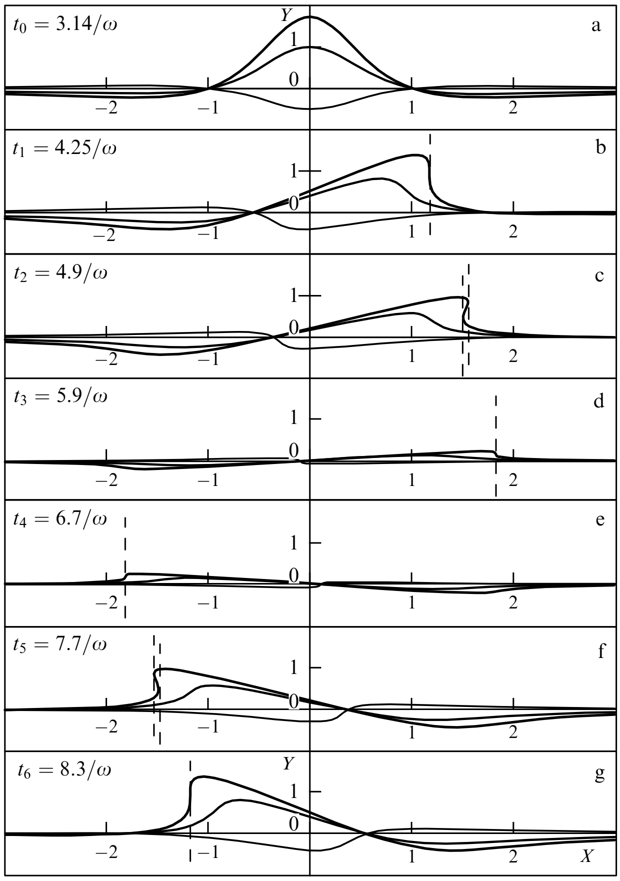

Figure 5. Evolution of a breather profile for various values of β. Solid black line corresponds to $\beta=0.85$, thin black line, to $\beta=0.5$, and light line, to $\beta=-0.25$. X and Y coordinates are normalized with α_*.

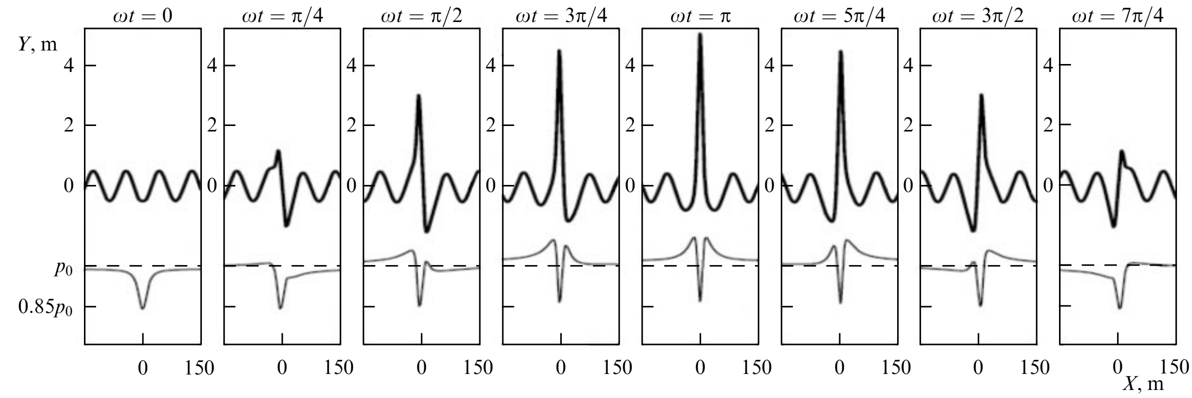

Figure 6. Formation of a rogue wave in the background of a Gerstner wave.

describes wave overturning in the interval (t_0, t_*). For example, one can choose $t_* = t_2$.

We stress that expression (30) corresponds to inhomogeneous and nonstationary pressure distribution on the free surface. The pressure oscillates in anti-phase with profile oscillations.

2.6.2 Rogue wave in the Gerstner wave background. Figure 6 depicts the dynamics of a wave surface for expression (29) with functions G and F that correspond to the last row in table [62]. Numerical computations were carried out for the case $A = 0.5$ m, $k = 0.074$ m^{-1}, $\alpha = 12$ m, $\beta = 328$ m^3, $\omega = \sqrt{gk} = 0.85$ s^{-1}, $\lambda = 84.9$ m. At the initial moment $(t = 0)$, the shape of the free surface (the upper curve) coincides exactly with the Gerstner wave profile. Later on, a peak starts to grow on the profile, reaching a maximum at the moment $t = \pi/\omega$, and then decreasing and disappearing near the end of the period. The largest peak height, $h = 2\beta/\alpha^2 + A \approx 5.1$ m, is eight times larger than the amplitude of Gerstner wave A. This is why the peak formation can be considered the birth of a rogue wave (see Ref. [63] for details of the rogue wave phenomenon). The reason is the pressure applied at the surface. The lowest curve in Fig. 6 shows the deviation of free surface pressure from atmospheric pressure p_0. At each free surface point, the pressure varies with time, but its negative jump in the region of the wave peak is about 100 mm Hg.

3. Exact solutions for waves taking into account Earth's rotation

We select the reference frame on the rotating Earth as shown in Fig. 7. Its origin is at latitude Φ, the X-axis is directed eastward, the Y-axis is directed northward, and the Z-axis is directed vertically upward. In this reference frame, the vector of Earth's rotation $\mathbf{\Omega}$ lies in the plane YZ. In the rotating reference frame, each particle will be affected by the Coriolis force and centrifugal force in addition to the gravity force, and the equation of motion takes the following form [29]:

$$\mathbf{R}_{tt} + 2\mathbf{\Omega} \times \mathbf{R}_t = -\frac{1}{\rho}\,\nabla p + \nabla\Phi - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R})\,, \qquad (31)$$

where $\Phi = -gZ$ is the geopotential, the Coriolis acceleration is on the left-hand side, and the centrifugal acceleration is on the right-hand side, but with a minus sign. The centrifugal force has a gradient character, and equation (31) can be

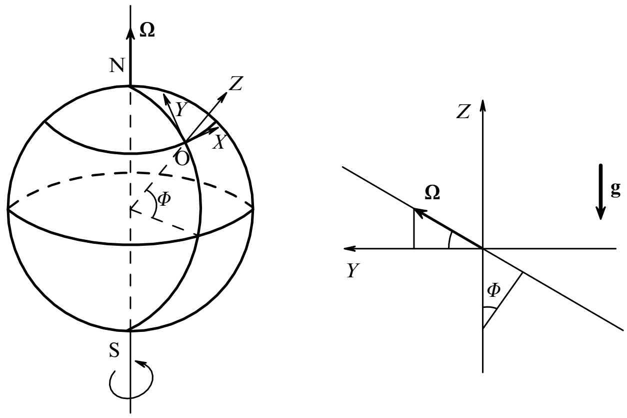

Figure 7. Coordinate system on Earth's surface.

rewritten as

$$\mathbf{R}_{tt} + 2\mathbf{\Omega} \times \mathbf{R}_t = -\frac{1}{\rho}\,\nabla H\,,$$
$$H = \frac{p}{\rho} - \Phi + \Phi_{\mathrm{c}}\,, \quad \Phi_{\mathrm{c}} = -\frac{1}{2}(\mathbf{\Omega} \times \mathbf{R})^2\,, \qquad (32)$$

where Φ_{c} is the potential of centrifugal forces.

Forming a dot product of equation (31) and $\mathbf{R}_{a_i}$, we obtain motion equations in the Lagrangian coordinates:

$$\mathbf{R}_{tt}\mathbf{R}_{a_i} + 2(\mathbf{\Omega} \times \mathbf{R}_t)\mathbf{R}_{a_i} = -H_{a_i}\,,$$
$$i = 1, 2, 3\,, \quad \{a_i\} = \{a, b, c\}\,. \qquad (33)$$

Together with the continuity equation (5), three equations (33) make a system of equations of an ideal incompressible fluid in Lagrangian variables in a rotating reference frame.

In Sections 3.1 and 3.2, we will study various wave motions when:

(a) the projections of Earth's angular velocity can be considered nonvarying in the entire flow domain: the Coriolis parameters $f = 2\Omega_Z = 2\Omega\sin\Phi$ and $\tilde{f} = 2\Omega_Y = 2\Omega\cos\Phi$ are assumed to be constant (the f-plane approximation);

(b) near-equatorial flows are in the band of low latitudes $\Phi \sim Y/R_*$, R_* is Earth's radius, the Coriolis parameters are $f = \beta Y$, $\beta = 2\Omega/R$ and $\tilde{f} = 2\Omega$ (the β-plane approximation).

The representation of vector $\mathbf{\Omega}$ in each case will be different, but a general result can be formulated for these cases [64, 65]. We eliminate the gradient term from equations (33) by taking cross derivatives and, after intermediate computations, obtain the equation

$$\frac{\partial}{\partial t}\left[\mathbf{R}_{ta_j}\mathbf{R}_{a_i} - \mathbf{R}_{ta_i}\mathbf{R}_{a_j} + 2(\mathbf{\Omega}\times\mathbf{R}_{a_j})\mathbf{R}_{a_i}\right] = 0\,, \qquad i \neq j\,,$$

which is equivalent to the preservation condition for three invariants, S_1, S_2, S_3,

$$\mathbf{R}_{tb}\mathbf{R}_c - \mathbf{R}_{tc}\mathbf{R}_b + 2(\mathbf{\Omega}\times\mathbf{R}_b)\mathbf{R}_c = S_1(a,b,c)\,, \tag{34}$$

$$\mathbf{R}_{tc}\mathbf{R}_a - \mathbf{R}_{ta}\mathbf{R}_c + 2(\mathbf{\Omega}\times\mathbf{R}_c)\mathbf{R}_a = S_2(a,b,c)\,, \tag{35}$$

$$\mathbf{R}_{ta}\mathbf{R}_b - \mathbf{R}_{tb}\mathbf{R}_a + 2(\mathbf{\Omega}\times\mathbf{R}_a)\mathbf{R}_b = S_3(a,b,c)\,, \tag{36}$$

which are functions of only Lagrangian coordinates. Equations (34)–(36) are the consequence of motion equations. Together with continuity equation (5), they constitute a system of fluid dynamical equations of an ideal incompressible fluid in a rotating reference frame.

If $\mathbf{\Omega} = 0$, equations (34)–(36) take the following form:

$$\mathbf{R}_{tb}\mathbf{R}_c - \mathbf{R}_{tc}\mathbf{R}_b = S_{10}(a,b,c)\,, \tag{37}$$

$$\mathbf{R}_{tc}\mathbf{R}_a - \mathbf{R}_{ta}\mathbf{R}_c = S_{20}(a,b,c)\,, \tag{38}$$

$$\mathbf{R}_{ta}\mathbf{R}_b - \mathbf{R}_{tb}\mathbf{R}_a = S_{30}(a,b,c)\,, \tag{39}$$

where the index 0 denotes the motion in a nonrotating reference frame. Expressions (37)–(39) were first written by Cauchy (1815), and Lamb gives them in his book [4]. The functions S_{10}, S_{20}, S_{30}, called the Cauchy invariants [27, 28, 66–71], are equal to circulations around three infinitesimal closed contours with planes perpendicular to the coordinate axes [4]. Formulas (37)–(39) offer generalizations of the Cauchy invariants for motion in a rotating reference frame.

3.1 Gerstner waves in a rotating fluid

For the f-plane approximation (f is the constant Coriolis parameter), Pollard gave the following exact solution of equations of fluid dynamics [12]:

$$\begin{cases} X = a - \dfrac{Am}{k}\exp(mc)\sin\left[k(a-Ut)\right], \\ Y = b + f\,\dfrac{Am}{k^2U}\exp(mc)\cos\left[k(a-Ut)\right], \\ Z = c + A\exp(mc)\cos\left[k(a-Ut)\right], \end{cases} \tag{40}$$

where A and m are positive constants, k and U are, respectively, the wave number and phase velocity of the wave. Inserting (40) into continuity equation (5), we obtain

$$\det\hat{R} = 1 - m^2A^2\exp(2mc)\,.$$

The flow domain is given by the condition $c \leqslant c_0 < 0$, and, to preserve the one-to-one character of mapping (40) (the determinant should not turn to zero), it is required that the inequality $A \leqslant 1/[m\exp(mc)]$ be valid, which ensures that there are no self-crossings in the wave profile (in a Gerstner wave, the role of parameter m is played by the wave number).

Inserting (40) into expressions (34)–(36), on the one hand, we check that they are correct, and on the other hand, we compute the values of generalized Cauchy invariants

$$S_1 = 0\,,$$
$$S_2 = m(k^2 - m^2)UA^2\exp(2mc) + \tilde{f}\left[1 - m^2A^2\exp(2mc)\right],$$
$$S_3 = f\,.$$

Equation (35) also defines the parameter m as

$$m^2 = \frac{k^4U^2}{k^2U^2 - f^2}\,. \tag{41}$$

Thus, in solution (40), there remains the only free parameter A defining the wave amplitude.

Wave oscillations of fluid particles decay exponentially with the depth, ensuring bottom impermeability ($c = -\infty$). In order to find the pressure, we should insert expressions (40) into equations (33) and neglect the centrifugal force. The expression for the pressure takes the form

$$p - p_0 = \rho\,\frac{mgA^2}{2}\left[\exp(2mc) - \exp(2mc_0)\right] - \rho g(c - c_0)\,. \tag{42}$$

Just as for a Gerstner wave, the pressure depends only on the vertical Lagrangian coordinate. When deriving expression (42), the wave dispersion relation is found by requiring that the pressure at the free surface be time independent:

$$U^2(k^2U^2 - f^2) = (g - \tilde{f}U)^2\,.$$

If rotation is absent (the Coriolis parameters are zero), the latter expression coincides with that for Gerstner waves. The wave travels from the west to the east, and its crests are parallel to the Y-axis.

From relationships (40) and (41), it follows that fluid particles move along circles:

$$(X-a)^2 + (Y-b)^2 + (Z-c)^2 = \frac{m^2A^2}{k^2}\exp(2mc)\,.$$

The center of each such circle is located at the point (a,b,c), which does not coincide with the initial particle position, and the rotation radius is $mA\exp(mc)/k$. Comparing the last two expressions of solution (40), we conclude that the motion looks similar in all planes that are parallel to the plane

$$Y - f\,\frac{m}{k^2U}\,Z - b + f\,\frac{m}{k^2U}\,c = 0\,,$$

which makes an angle $\arctan[fm/(k^2U)]$ with the Z-axis. The trajectories of circular motion lie in these planes. At the equator, $f = 0$, $m = k$, and the Pollard solution transforms into the Gerstner solution (see (14): the role of b is now played by coordinate c). At the equator, the particles oscillate in the plane XZ; for $f \neq 0$, the plane of their oscillations is inclined in each hemisphere to the respective pole. However, as concluded by Pollard himself [12], the magnitude of this angle is extremely small.

The stability of Pollard waves was studied in a short-wave limit. When some slope threshold is surpassed, these waves become unstable against perturbations that are perpendicular to the propagation direction (west–east) [69].

Constantin and Monismith analyzed the propagation of Pollard waves in the background of homogeneous zonal flow U_0. If we add the term $-U_0t$ in the first of equations (40), they will continue to be an exact solution to the equations of rotating fluid. But the dispersion relation will be modified. Exploring it, the authors of Ref. [72] pointed out that two types of waves are possible. The first one, described by Pollard, presents a slightly modified Gerstner wave in which there is no particle drift. The second type (an inertial Gerstner wave), characterized by a slower propagation speed, is 'attached' to the mean flow and cannot occur in its absence.

The effect of Earth's rotation can also be taken into account for edge Gerstner waves [15]. The form of the exact solution resembles expressions (21), but the scales of exponential decay along the vertical and horizontal directions are now different. The papers by Mollo-Christensen [14, 15] are written in an exclusively concise way. Their ideas are discussed in a more exhaustive and complete way in Ref. [73] for the near-equatorial domain and in Refs [74, 75] for an arbitrary latitude.

3.2 Waves in a near-equatorial domain

Close to the equator, in the β-plane approximation, equations (34)–(36) take the form [65]

$$\begin{aligned}
&\frac{\partial(X_t,X)}{\partial(b,c)}+\frac{\partial(Y_t,Y)}{\partial(b,c)}+\frac{\partial(Z_t,Z)}{\partial(b,c)}\\
&\qquad+2\Omega\,\frac{\partial(Z,X)}{\partial(b,c)}+\beta Y\,\frac{\partial(X,Y)}{\partial(b,c)}=S_1(a,b,c)\,,\\
&\frac{\partial(X_t,X)}{\partial(c,a)}+\frac{\partial(Y_t,Y)}{\partial(c,a)}+\frac{\partial(Z_t,Z)}{\partial(c,a)}\\
&\qquad+2\Omega\,\frac{\partial(Z,X)}{\partial(c,a)}+\beta Y\,\frac{\partial(X,Y)}{\partial(c,a)}=S_2(a,b,c)\,,\\
&\frac{\partial(X_t,X)}{\partial(a,b)}+\frac{\partial(Y_t,Y)}{\partial(a,b)}+\frac{\partial(Z_t,Z)}{\partial(a,b)}\\
&\qquad+2\Omega\,\frac{\partial(Z,X)}{\partial(a,b)}+\beta Y\,\frac{\partial(X,Y)}{\partial(a,b)}=S_3(a,b,c)\,.
\end{aligned} \tag{43}$$

The f-plane approximation at the equator follow from these equations if one takes $\beta = 0$ (in this case, $f = 0$, $\tilde{f} = 2\Omega$).

3.2.1 The *f*-plane approximation. Consider two-dimensional near-equatorial flows in planes parallel to the plane XZ, i.e., $Y = b$. Continuity equation (5) takes the form

$$\frac{\partial(X,Z)}{\partial(a,c)}=S_0(a,c)\,, \tag{44}$$

where S_0 is a time-independent function, and from the equations for the Cauchy invariants it follows that

$$\frac{\partial(X_t,X)}{\partial(a,c)}+\frac{\partial(Z_t,Z)}{\partial(a,c)}=-S_2(a,c)-2\Omega S_0(a,c)\,. \tag{45}$$

Systems of equations (44) and (45) are equivalent to systems (9) and (12), with the only difference being that the role of coordinate b is now played by the variable c. Relationships (44) and (45) are the system of equations of two-dimensional fluid dynamics, so that all exact solutions for plane waves in a nonrotating fluid will also be valid in the f-plane approximation. However, the expressions for wave vorticity and dispersion relations will change. Following similar reasoning, Hsu described the Gerstner wave in the near-equatorial latitude band in the f-approximation [76], and Kluczek studied the effect of a homogeneous flow on this wave [77]. The steepness threshold after which the Gerstner wave becomes unstable is found in Ref. [78].

Solution of equations (44), (45) in the form $X = X(a,c,t)$, $Z = Z(a,c,t)$ can be generalized by assuming

$$Y=b+\sigma(a,c)t\,,$$

which corresponds to adding a meridional flow with the profile $\sigma(a,c)$ to the known two-dimensional flow. This changes the vorticity of the net flow and the invariants S_1 and S_3, which now become $S_1 = -\sigma'_c$ and $S_3 = \sigma'_a$. Henry studied the effect of such flows on a Gerstner wave [79], Kluczek, on a Gerstner wave in a flow [77], and Abrashkin, on Ptolemaic flows in the near-equatorial band [64].

This repository of exact solutions should be enlarged with a description of the Gerstner wave generated by a traveling harmonic pressure wave [64]. The pressure p on the profile of the Gerstner wave is expressed as (see (14), $b \to c$)

$$\begin{aligned}
\frac{p-p_0}{\rho}&=-gc+\frac{\omega(\omega+2\Omega)}{2}\,A^2\exp(2kc)\\
&\quad+\left[\omega(\omega+2\Omega)k^{-1}-g\right]A\cos(ka-\omega t)\,.
\end{aligned} \tag{46}$$

Traditionally, the boundary condition for the free surface for water waves is that the pressure be constant ($c = 0$). Then, equating the factor by the cosine in equation (46) to zero, the wave dispersion relation is derived [76]. However, one may assume that a pressure distribution in the form of a harmonic traveling wave is maintained at the free surface because of wind,

$$p^*=p_1+p_2A\cos(ka-\omega t)\,, \tag{47}$$

where p_1 and p_2 are constants that satisfy the following relationships:

$$\begin{aligned}
p_1&=p_0+\frac{\omega(\omega+2\Omega)}{2}\,\rho A^2\,,\\
p_2&=\rho\left[\omega(\omega+2\Omega)k^{-1}-g\right]A\,.
\end{aligned} \tag{48}$$

When conditions (48) are observed, one may say that Gerstner solution (14) corresponds to stationary trochoidal waves on a fluid surface maintained by an external pressure (47). For known ω and k, the second relationship specifies the wave amplitude A, and the first one, the quantity p_0. The free surface elevation is expressed through the formula $Y = A\cos(ka - \omega t)$; thus, in the case of positive p_2, the pressure varies in phase with the profile; otherwise, in antiphase. The case $p_2 = 0$ corresponds to a Gerstner wave with constant pressure on the profile.

Solving the quadratic equation in (48) with respect to ω, one finds the dispersion relation for the waves:

$$\omega=\pm\sqrt{\Omega^2+\left(g+\frac{p_2}{\rho A}\right)k}-\Omega\,. \tag{49}$$

Let us assume that $(g + p_2/\rho A) > 0$. Taking the $+$ sign, we obtain a wave propagating to the east, while the minus sign corresponds to westward propagation. It is interesting to compare dispersion relation (49) with that for equatorial waves on a homogeneous flow [77]. The quantity $p_2/(2\rho A\Omega)$ is analogous to the flow velocity. Thus, wind either accelerates or decelerates waves, depending on the sign of p_2.

The solution considered here is interesting from the standpoint of both creating Gerstner waves in the laboratory and natural conditions. As hinted at by our analysis, wind can be one of its generation mechanisms.

3.2.2 Trapped waves (*β*-plane approximation). For the case when the Coriolis parameter varies linearly with latitude (this approximation is called the β-plane), Constantin found an exact solution of system (43):

$$\begin{cases} X = a - \dfrac{1}{k} \exp\{k[c - h(b)]\} \sin[k(a - Ut)], \\ Y = b, \\ Z = c + \dfrac{1}{k} \exp\{k[c - h(b)]\} \cos[k(a - Ut)], \end{cases} \tag{50}$$

where $h(b) = \beta b^2/[2(kU + 2\Omega)]$, and the wave phase velocity is expressed as

$$U = \frac{\sqrt{\Omega^2 + kg} - \Omega}{k}.$$

The Lagrangian variables a, b vary in the limits $(-\infty, \infty)$ and c lies in the limits $(-\infty, c_0)$, where $c_0 < 0$. Relationships (50) describe equatorial surface waves traveling at speed U eastward [80]. They are spatially periodic waves with amplitude decaying exponentially in the meridional direction, which is why they are called trapped. For $h = 0$, expressions (50) reduce to the Gerstner solution. The additional exponentially decaying multiplier by the amplitude is the highlight of this exact solution.

The expressions for the generalized Cauchy invariants in waves (50) take the following form [65]:

$$S_1 = 0, \quad S_2 = 2\Omega - 2(kU + \Omega)\exp(2\zeta),$$
$$S_3 = \beta b\left[1 - \frac{2(kU + \Omega)}{kU + 2\Omega}\exp(2\zeta)\right], \quad \zeta = k[c - h(b)]. \tag{51}$$

The zonal component of vector $\mathbf{S}\{S_1, S_2, S_3\}$ equals zero. The vorticity $\boldsymbol{\omega}$ for waves (50) is expressed as follows [80]:

$$\boldsymbol{\omega} = S_0^{-1}\{-bkU^2g^{-1}\beta\exp\zeta\sin\theta,\ -2kU\exp(2\zeta),$$
$$bkU^2g^{-1}\beta[\exp\zeta\cos\theta - \exp(2\zeta)]\}, \tag{52}$$
$$S_0^{-1} = 1 - \exp(2\zeta), \quad \theta = k(a - Ut).$$

All three vorticity components differ from zero, and the zonal and vertical components depend on time. A comparison of formulas (51) and (52) gives a clear indication of the difference between the vectors of vorticity and Lagrangian invariants.

The pressure in a fluid is written as

$$p - p_0 = \rho g\left(\frac{\exp(2\zeta)}{2k} - c\right) - \rho g\left(\frac{\exp(2kc_0)}{2k} - c_0\right). \tag{53}$$

The shape of the free surface at the equator where $b = 0$ is obtained from (50) by setting $c = c_0$. The trapped equatorial waves can be qualitatively compared with edge waves along a vertical coastal wall at the equator.

Trapped equatorial waves, just like their simpler analogs — Gerstner waves and Gerstner-type edge waves — are linearly unstable when their steepness exceeds some threshold [81].

Expressions (50) are a unique example of an exact solution of fluid dynamics equations in the equatorial domain. They can be generalized to the case of uniform zonal near-surface flow [82–84], to account for centrifugal forces [85], to correct the gravity force in the framework of the standard β-plane model [86], and for edge waves by a sloping bottom lying parallel to the equator [87]. References [88, 89] use relationships (50) to describe waves propagating in the background of zonal flow at an arbitrary latitude. They use a nontraditional β-plane approximation when the parameter f varies with latitude and the parameter $\tilde{f}$ is assumed to be constant. This approximation for the spatial problem is rather rough, and the solutions obtained for waves cannot be called exact.

4. Waves in a stratified fluid

A vast bibliography devoted to Gerstner-type waves in a rotating fluid is chiefly based on three solutions for a homogeneous fluid that describe:

— Gerstner waves [3];
— waves over a sloping bottom [13–16];
— trapped near-equatorial waves [80].

Namely these will also remain in our focus in this section.

4.1 Continuous stratification

Assume that some exact solution $X(a, b, c, t)$, $Y(a, b, c, t)$, $Z(a, b, c, t)$ is known for Gerstner-type waves in a homogeneous fluid with density ρ^0. One of characteristic features of such waves is that their pressure depends only on two Lagrangian variables: $p^0 = p^0(\varphi(b, c))$, where $\varphi(b, c)$ is a function constant on the free surface (the superscript indicates that it is pressure in a homogeneous fluid).

Assume that the expressions for X, Y, Z are the same in a stratified fluid, but now the density ρ_s is some function of $\varphi(b, c)$. This implies that $\rho_s = \rho_s(p^0)$. Since the density of a stratified fluid does not depend on time, continuity equation (5) will be satisfied. In order to also keep motion equations (7) valid, we need to set

$$\frac{1}{\rho^0}\frac{\partial p^0}{\partial a_i} = \frac{1}{\rho_s(p^0)}\frac{\partial p_s}{\partial a_i}, \quad a_i = \{a, b, c\},$$

where p_s is the pressure in the stratified fluid. As a consequence of the written condition,

$$p_s = \int \frac{\rho_s(p^0)}{\rho^0}\, dp^0.$$

Pressure p_s is a function of pressure in a homogeneous fluid. By choosing the constant of integration, we satisfy the condition that pressure be constant on the free surface.

Through similar reasoning, it was found that Gerstner waves [10], edge waves over a sloping bottom (both classical [13, 90] and equatorial in the f-plane approximation [13, 91]), and also trapped near-equatorial waves [80], can exist in stratified fluids. The Gerstner solution remains valid in a stratified fluid with an arbitrary distribution $\rho_s(b)$ [8]. For all other solutions, the profile of stratification depends on some particular expressions already containing two coordinates b and c. This is a rather wide class of stratifications, but, understandably, not the most general.

One more, totally unanticipated, applicability domain for Gerstner-like solutions was mentioned by Godin [92, 93], who noted that, when the pressure, entropy, and density remain constant in a moving particle, the descriptions of incompressible motions of compressible fluid and flows of incompressible fluid are kinematically equivalent. This means that the motion of an incompressible medium for which the mentioned thermodynamical values in a fluid particle are invariants and satisfy the equation of state for a compressible medium also describes the motion of this medium. For Gerstner waves and their 'modifications' that were considered (an example of an edge wave is treated in detail in Refs [92, 93]), the pressure and density can be chosen such that their relation corresponds, for example, to an ideal or polytropic gas. Then, the wave Gerstner-like solution will also describe the motion of compressible fluid with the chosen equation of state.

4.2 Waves in layers with density discontinuities

Mollo-Christensen [94] drew attention to two original circumstances related to the Gerstner wave.

Consider a two-layer model of a fluid (Fig. 8). Let the densities of the lower and upper layers be constant and equal respectively to ρ_1 and ρ_* ($\rho_1 > \rho_*$). Let us assume that a Gerstner wave (14) is propagating in the lower layer in the background of a uniform flow U, and the upper layer hosts a uniform flow with the velocity equal to that of the Gerstner wave $U + \omega/k$. In this case, the velocity at the interface between the layers will be continuous, and the pressure on both sides will coincide if we assume that the wave frequency ω satisfies the condition

$$\omega^2 = \frac{\rho_1 - \rho_*}{\rho_1} gk \,,$$

which coincides with the dispersion relation for linear waves at the interface between two fluid layers. The solution constructed in this way describes an internal Gerstner wave on a homogeneous flow and can be generalized to the cases of arbitrarily stratified lower fluid and uniformly rotating fluid [94].

A particular case of the two-layer model by Mollo-Christensen when the upper fluid is at rest and $U = -\omega/k$ was studied in Ref. [95]. This condition corresponds to a stopped wave, i.e., to the motion of fluid particles with density ρ_1 by trochoidal trajectories in the negative direction of the X-axis (the author of Ref. [95] does not mentions this). The ideas of Mollo-Christensen were extended in a more complete form to zonally propagating waves in a near-equatorial domain. Using the f-plane approximation, a description of internal Gerstner waves [96], their transformation on the flow [97], and a meridional current together with it [98] was given.

The second finding of the author of Ref. [94] is even more elegant. Let us rewrite expressions (14), changing the sign with the sine and in exponent powers (the flow is studied in the XY-plane):

$$X = a + A \exp(-kb) \sin(ka - \omega t) \,,$$
$$Y = b + A \exp(-kb) \cos(ka - \omega t) \,. \qquad (54)$$

These relationships also satisfy the system of hydrodynamic equations (9) and (12), but, in order to constrain them to the waves with bounded amplitude, it should be assumed that $b > 0$. The wave amplitude will decrease with an increase in the vertical Lagrangian coordinate, which corresponds to a flipped-over trochoidal Gerstner wave, with its troughs looking now upward. We will refer to such waves as anti-Gerstner.

As a final result, Ref. [94] proposed a model of three-layer fluid flow (Fig. 9):

— in the lower fluid ($b = b_1 < 0$) with density ρ_1, a Gerstner wave with frequency ω_1 propagates in the direction X in the background of a uniform flow U_1;

— in the upper fluid ($b = b_2 > 0$) with density ρ_2, an anti-Gerstner wave with frequency ω_2 propagates along the X-axis in the background of a uniform flow U_2;

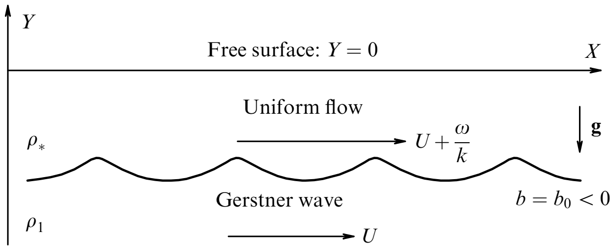

Figure 8. Two-layer model by Mollo-Christensen.

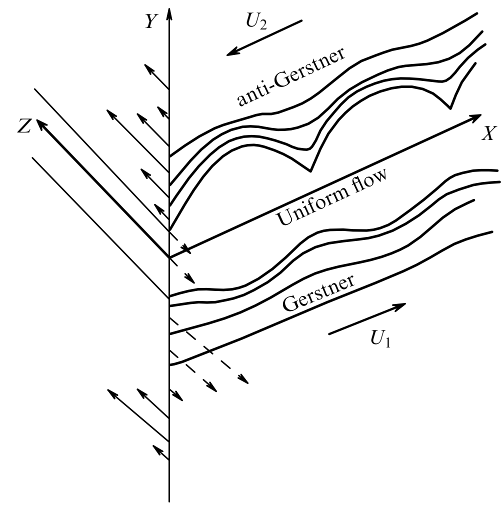

Figure 9. Three-layer model by Mollo-Christensen. Flow directions are shown for geostrophic billows [94]. Arrows along the Z-axis show the meridional flow with the velocity changing its direction.

— in the central part ($b_1 \leqslant b \leqslant b_2$), a fluid with density $\rho_*(\rho_1 > \rho_* > \rho_2)$, trapped by waves, flows homogeneously in the direction of the X-axis with the velocity

$$U_1 + \frac{\omega_1}{k} = U_2 + \frac{\omega_2}{k} \,,$$

where k is the same wavenumber for both waves. The expressions for frequencies are found from the condition of pressure continuity at the interfaces [94].

This flow scheme serves as a model for clouds trapped by waves and moving together with them. Mollo-Christensen introduced for them the term billows to designate big waves or rollers. Many billows do not contain condensed water and therefore are invisible, but their characteristic feature is precisely the finite magnitude of perturbation amplitude. Reference [94] deals with two cases: gravity billows in a nonrotating medium and geostrophic billows ($g = 0$, but the rotation of the atmosphere is accounted for). A combined case when both the action of gravity and the rotation of Earth are taken into account is obtained by merging these two solutions. As applied to the ocean, the idea of a three-layer model for the Gerstner–anti-Gerstner system was used in [99].

Multilayer fluid motions with trapped Gerstner-like waves in a near-equatorial domain were studied by Constantin [100, 101]. The role of Gerstner waves in these studies is played by near-equatorial waves [80], discovered by the same author. Reference [100] considers equatorially trapped waves in thermocline propagating eastward (symmetric with respect to the equator). Fluid above the thermocline is assumed to be at rest (an analog of the two-layer Mollo-Christensen model). Reference [101] studies an anti-Gerstner variant of waves when the amplitude of wave perturbations is decaying above the thermocline, and there is a uniform flow below it. The stability of these two-layer models is explored in Refs [102–104], and Refs [105, 106] deal with their generalizations for the case of additional background flows. Kluczek, based on the example of Ref. [101], constructed a solution for the internal Pollard wave [107, 108], which is an anti-Pollard wave in our terminology. However, it would be more precise to call all the waves considered in this article Gerstner-like.

Weber, studying the properties of weakly nonlinear waves at the interface between two fluids [109, 110], proposed that any wave for which there is no particle drift be called a Gerstner-like wave. All solutions analyzed or mentioned here obviously share this property. In this sense, apparently, it is appropriate to speak about a family of different waves that can be unified under the name of Gerstner and referred to as Gerstner-like.

5. Conclusions

This paper presents a complete set of exact solutions for waves in fluid that are related to the classical Gerstner solution. We should not fail to note that practically half of the studies listed in the reference list appeared during the last decade. One can say without exaggeration that the Gerstner solution for waves on deep water has acquired a 'second youth.' We specially mention the extensive contribution of the scientific school led by University of Vienna Professor Adrian Constantin in the development of this area. It is also appropriate to note that the name of Franz Joseph Gerstner enjoys extraordinary respect in Austria because of his contributions to education and engineering. However, he will always be remembered in the history of science as the author of the first exact solution in the nonlinear wave theory.

All families of Gerstner-like waves are vortical. This is why there is an unavoidable question on the mechanism of vorticity generation. In the framework of the model of ideal fluid, the vorticity can be naturally linked to shear flows (set initially), in the background of which the waves are generated. In particular, H Lamb explained the possibility of Gerstner wave formation namely in this way [4]. This mechanism operates when epicyclic waves are excited on a cavity surface in a rotating fluid (cylindrical Gerstner waves [18, 19]). Monismith et al. observed Gerstner waves by creating a background flow in the near-surface layer, which compensated the drift of fluid particles in the direction of wave propagation [20]. Gerstner waves were also observed in other basins [21–23] and the open ocean [24].

There might exist, however, another scenario for the formation of vortex surface waves related to the action of viscosity near the free surface. For example, in the approximation of viscous fluid, standing waves on the water surface in an oscillating container (Faraday ripples) are potential in the linear approximation. However, they become vortical owing to the action of viscosity in the quadratic approximation [111]. This result reminds us of an analogous property of the Gerstner wave which carries no vorticity in the linear approximation but acquires it in the next approximation. And there is one more interesting circumstance. The vorticity of capillary ripples measured in the experiment varied directly proportionally to the wave steepness squared [111]. For sufficiently small values of steepness, the vorticity in the Gerstner wave behaves similarly. We admit that the results for standing waves cannot be directly transferred to traveling waves; furthermore, in this case, the vorticity vector for ripples is directed vertically. And yet this example indicates that on a qualitative level a scenario with a viscous near-surface layer seems to be quite realistic. Furthermore, if there is an elastic film on the fluid surface, the effect of vertical vorticity generation is enhanced [112], indicating the possibility of more efficient Gerstner wave generation in fluids with an elastic film on their surface (see also Ref. [24]). The scenario with a viscous sub-layer also plays a defining role in problems of surface vortical wave excitation by winds. However, the generation mechanism for the Gerstner wave still remains open. This is a task for the future.

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