# On values of $\mathfrak{s l}_{3}$ weight system on chord diagrams whose intersection graph is complete bipartite 

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#### Abstract

Each knot invariant can be extended to singular knots according to the skein rule. A Vassiliev invariant of order at most $n$ is defined as a knot invariant that vanishes identically on knots with more than $n$ double points. A chord diagram encodes the order of double points along a singular knot. A Vassiliev invariant of order $n$ gives rise to a function on chord diagrams with $n$ chords. Such a function should satisfy some conditions in order to come from a Vassiliev invariant. A weight system is a function on chord diagrams that satisfies so-called 4 -term relations. Given a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate invariant bilinear form, one can construct a weight system with values in the center of the universal enveloping algebra $U(\mathfrak{g})$. In this paper, we calculate $\mathfrak{s l}_{3}$ weight system for chord diagram whose intersection graph is complete bipartite graph $K_{2, n}$.


## 1 Introduction

Finite order knot invariants, which were introduced in [11] by V. Vassiliev near 1990, may be expressed in terms of weight systems, that is, functions on chord diagrams satisfying the so-called Vassiliev 4 -term relations. In paper [7], M. Kontsevich proved that over a field of characteristic zero every weight system corresponds to some finite order invariant. There are multiple approaches to constructing weight systems. In particular, D. Bar-Natan and M. Kontsevtch suggested a construction of a weight system coming from a finite dimensional Lie algebra endowed with an invariant nondegenerate bilinear form. The $\mathfrak{s l}_{2}$ Lie algebra weight system is the simplest case. Its values lie in the center of the universal enveloping algebra of the $\mathfrak{s l}_{2}$ Lie algebra, which, in turn, is isomorphic to the ring of polynomials in one variable (the Casimir element). The $\mathfrak{s l}_{2}$ weight system was studied in many papers. Despite the fact that this weight system can be defined easily, it is difficult to compute its value on a chord diagram using the definition because it is necessary to work with elements of a noncommutative algebra in order to do this. The Chmutov-Varchenko recurrence relations [3] simplify these computations significantly. A theorem by S. Chmutov and S. Lando [4] states that the value of the $\mathfrak{s l}_{2}$ weight system on a chord diagram
depends only on the intersection graph of this chord diagram, i.e. if two chord diagrams have isomorphic intersection graphs, the values of the weight system on these chord diagrams coincide.

On the other side, we don't have such good properties for the next reasonable case, namely, for the $\mathfrak{s l}_{3}$ weight system. The $\mathfrak{s l}_{3}$ Lie algebra weight system takes values in the center of the universal enveloping algebra of the $\mathfrak{s l}_{3}$ Lie algebra which is isomorphic to the ring of polynomials in TWO variables (the Casimir elements of degrees 2 and 3). For the $\mathfrak{s l}_{3}$ weight system, we do not have a result similar to the Chmutov-Varchenko recurrence relations for $\mathfrak{s l}_{2}$ weight system which could help us to compute its value. The Chmutov-Lando theorem also fails on $\mathfrak{s l}_{3}$ weight system, which means there are two different chord diagrams with different values in $\mathfrak{s l}_{3}$ weight system such that they have isomorphic intersection graphs.

Our main results concern explicit values of the $\mathfrak{s l}_{3}$ weight system on chord diagrams whose intersection graph is complete bipartite, with the size of one part equal to 2. In our computations, we use the results from [12]. Up to now, the explicit values of the $\mathfrak{s l}_{3}$ weight system has been known only in few examples and simple series. Our results imply a nontrivial conclusion that for the chord diagrams whose intersection graph is the complete bipartite graph $K_{2, n}$, the value of the $\mathfrak{s l}_{3}$ weight system depends on the second Casimir only.

A key role in our study is played by the Hopf algebra structure on the space of chord diagrams modulo 4 -term relations introduced by Kontsevich. Chord diagrams whose intersection graph is complete bipartite generate a Hopf subalgebra in this Hopf algebra. By analyzing the structure of this Hopf subalgebra, P. Filippova managed in $[5,6]$ to deduce the values of the $\mathfrak{s l}_{2}$ weight system on projections of the chord diagrams whose intersection graph is complete bipartite to the subspace of primitives. By combining our computations with her results, we obtain explicit expressions for the values on primitives of the $\mathfrak{s l}_{3}$ weight system.

The paper is organized as follows. In Sec. 2, we give definitions of the Hopf algebras of chord diagrams and Lie algebra weight system. In Sec. 3, we show the way to calculate $\mathfrak{s l}_{3}$ weight system and give some results on small chord diagrams. In Sec. 4, we give the definitions of the Hopf algebras of Jacobi diagrams and calculate some results on Jacobi diagrams. In Sec. 5, we prove the major theorem which is the formula for the values of the $\mathfrak{s l}_{3}$ weight system on chord diagrams whose intersection graph is complete bipartite and their projections to primitives.

In our presentation, we follow the approach of [9] to chord diagrams and weight systems, see also [2].

## 2 Hopf algebras of chord diagrams and Lie algebra weight systems

In this section we define the Hopf algebra of chord diagrams modulo 4-term relations.
Definition 2.1 (chord diagram) A chord diagram $D$ of order $n$ (or degree $n$ ) is an oriented circle (sometimes termed Wilson loop) with a distinguished set of $n$ disjoint pairs of distinct points, considered up to orientation preserving diffeomorphisms of the circle. We denote the set of chords of a chord diagram $D$ by [ $D$ ].

The vector space $\mathcal{A}$ spanned by chord diagrams over complex field $\mathbb{C}$ is graded,

$$
\mathcal{A}=\mathcal{A}_{0} \oplus \mathcal{A}_{1} \oplus \mathcal{A}_{2} \oplus \mathcal{A}_{3} \oplus \ldots
$$

Each component $\mathcal{A}_{n}$ is spanned by diagrams of the same order $n$.
Definition 2.2 (4-term elements) A 4-term (or $4 T$ ) element is the alternating sum of the following quadruples of diagrams:


Here all the four chord diagrams contain, in addition to the two depicted chords, one and the same set of other chords. For any vector space $V$, a function $f \in \operatorname{hom}_{\text {linear }}(\mathcal{A}, V)$ that vanishes on all 4 -term elements is called a weight system.

Now we define the Hopf algebra structure on $\mathcal{A} /\langle 4 T\rangle:=\mathcal{A}^{f r}$.
Definition 2.3 The product of two chord diagrams $D_{1}$ and $D_{2}$ is defined by cutting and gluing the two circles as shown


Modulo 4-term relationship, the product is well-defined.
Definition 2.4 The coproduct in the algebra $\mathcal{A}^{f r}$

$$
\delta: \mathcal{A}_{n}^{f r} \rightarrow \bigoplus_{k+l=n} \mathcal{A}_{k}^{f r} \otimes \mathcal{A}_{l}^{f r}
$$

is defined as follows. For a diagram $D \in \mathcal{A}_{n}^{f r}$ we put

$$
\delta(D):=\sum_{J \subseteq[D]} D_{J} \otimes D_{\bar{J}}
$$

The summation is taken over all subsets $J$ of the set of chords of $D$. Here $D_{J}$ is the diagram consisting of the chords that belong to $J$ and $\bar{J}=[D] \backslash J$ is the complementary subset of chords. To the entire space $\mathcal{A}^{f r}$, the operator $\delta$ is extended by linearity.

Claim 2.5 The vector space $A^{f r}$ endowed with the above product and coproduct is a commutative, cocommutative and connected bialgebra.

Definition 2.6 An element $p$ of a bialgebra is called primitive if $\delta(p)=1 \otimes p+$ $p \otimes 1$.

It is easy to show that primitive elements form a vector subspace $P\left(\mathcal{A}^{f r}\right)$ in the bialgebra $\mathcal{A}^{f r}$. Since any homogeneous component of a primitive element is primitive, such a vector subspace of a graded bialgebra is also graded $P_{n}=$ $P\left(\mathcal{A}^{f r}\right) \cap \mathcal{A}_{n}^{f r}$. An element of $\mathcal{A}_{n}$ is decomposable if it can be represented as a product of elements of order smaller than $n$.

Theorem $2.7([8,10])$ The projection $\pi(D)$ of a graph $D$ to the subspace of primitive elements whose kernel is the subspace spanned by decomposable elements in the Hopf algebra $D$ is given by the formula

$$
\begin{aligned}
\pi(D) & =D-1!\sum_{\substack{\left[D_{1}\right] \cup\left[D_{2}\right]=[D]}} D_{1} \cdot D_{2}+2!\sum_{\substack{\left[D_{1}\right] \cup\left[D_{2}\right] \cup\left[D_{3}\right]=[D]}} D_{1} \cdot D_{2} \cdot D_{3} \ldots \\
& =D-\sum_{i=2}^{|[D]|}(-1)^{i}(i-1)!\sum_{\substack{i \\
\bigcup_{j=1}^{i}\left[D_{j}\right]=[D] \\
\left[D_{j}\right] \neq \emptyset}}^{i} \prod_{j=1} D_{j}
\end{aligned}
$$

For example
Example 2.8 The element

is a primitive element, which is the projection of the argument in the left-hand side to the subspace of primitives.

Given a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate invariant bilinear form, one can construct a weight system with values in the center of its universal enveloping algebra $U(\mathfrak{g})$. These construction is due to M. Kontsevich [7] and D. Bar-Natan [1].

Definition 2.9 (Universal Lie algebra weight systems) Kontsevich's construction proceeds as follows. Let $\mathfrak{g}$ be a metrized Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a Lie algebra with an ad-invariant non-degenerate bilinear form $\langle\cdot, \cdot\rangle$. Let $m$ denote the dimension of $\mathfrak{g}$. Choose a basis $e_{1}, \ldots, e_{m}$ of $\mathfrak{g}$ and let $e_{1}^{*}, \ldots, e_{m}^{*}$ be the dual basis with respect to the form $\langle\cdot, \cdot\rangle$.

Given a chord diagram $D$ with $n$ chords, we first choose a base point on the circle, away from the ends of the chords of $D$. This gives a linear order on the endpoints of the chords, increasing in the positive direction of the Wilson loop. Assign to each chord $a$ an index, that is, an integer-valued variable, $i_{a}$. The values of $i_{a}$ will range from 1 to $m$, the dimension of the Lie algebra. Mark the first endpoint of the chord with the symbol $e_{i_{a}}$ and the second endpoint with $e_{i_{a}}^{*}$.

Now, write the product of all the $e_{i_{a}}$ and all the $e_{i_{a}}^{*}$, in the order in which they appear on the Wilson loop of $D$, and take the sum of the $m^{n}$ elements of the universal enveloping algebra $U(\mathfrak{g})$ obtained by substituting all possible values of the indices $i_{a}$ into this product. Denote by $\phi_{\mathfrak{g}}(D)$ the resulting element of $U(\mathfrak{g})$.

Claim 2.10 The above construction has the following properties:

1. the element $\phi_{\mathfrak{g}}(D)$ does not depend on the choice of the base point on the diagram;
2. it does not depend on the choice of the basis $e_{i}$ of the Lie algebra;
3. it belongs to the ad-invariant subspace

$$
U(\mathfrak{g})^{\mathfrak{g}}=\{x \in U(\mathfrak{g}) \mid x y=y x \text { for all } y \in \mathfrak{g}\}=Z U(\mathfrak{g})
$$

4. This map from chord diagrams to $Z U(\mathfrak{g})$ satisfies the 4-term relations. Therefore, it extends to a homomorphism of commutative algebras $\mathcal{A}^{\text {fr }} \rightarrow$ $Z U(\mathfrak{g})$.

Remark If $D$ is a chord diagram with $n$ chords, then

$$
\phi_{\mathfrak{g}}(D)=c^{n}+\{\text { terms of degree less than } 2 n \text { in } U(g)\}
$$

where $c=e_{1} \otimes e_{1}^{*}+\cdots+e_{m} \otimes e_{m}^{*} \in U(\mathfrak{g})$ is the quadratic Casimir element. Indeed, we can permute the endpoints of chords on the circle without changing the highest term of $\phi_{\mathfrak{g}}(D)$ since all the additional summands arising as commutators have degrees smaller than $2 n$. Therefore, the highest degree term of $\phi_{\mathfrak{g}}(D)$ does not depend on $D$ with a given number $n$ of chords. Finally, if $D$ is a diagram with $n$ isolated chords, that is, the $n$th power of the diagram with one chord, then $\phi_{\mathfrak{g}}(D)=c^{n}$.

## 3 The $\mathfrak{s l}_{3}$ weight system

In this section, we concentrate on the weight system associated to the Lie algebra $\mathfrak{S H}_{3}$.

Definition 3.1 (Weight systems associated with representations) A linear representation $T: \mathfrak{g} \rightarrow \operatorname{End}(V)$ extends to a homomorphism of associative
algebras $U(T): U(\mathfrak{g}) \rightarrow \operatorname{End}(V)$. The composition of following three maps (with the last map being the trace)

$$
\mathcal{A} \xrightarrow{\phi_{\mathfrak{g}}} U(\mathfrak{g}) \xrightarrow{U(T)} \operatorname{End}(V) \xrightarrow{T r} \mathbb{C}
$$

by definition gives the weight system associated with the representation

$$
\phi_{\mathfrak{g}}^{T}=\operatorname{Tr} \circ U(T) \circ \phi_{\mathfrak{g}}
$$

Consider the standard representation of the Lie algebra $\mathfrak{s l}_{3}$ as the space of $3 \times 3$ matrices with zero trace. It is an eight-dimensional Lie algebra spanned by the matrices

$$
\begin{aligned}
& E_{1}=\left(\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad E_{2}=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right), \quad E_{3}=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad H_{1}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right), \\
& F_{1}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad F_{2}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{array}\right), \quad F_{3}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right), \quad H_{2}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right)
\end{aligned}
$$

with the commutators

$$
\begin{array}{llll}
{\left[E_{i}, F_{j}\right]=\delta_{i j} H_{i},} & {\left[H_{i}, H_{j}\right]=0,} & {\left[H_{i}, E_{i}\right]=2 E_{i},} & {\left[H_{i}, F_{i}\right]=-2 F_{i},} \\
{\left[H_{1}, E_{2}\right]=-E_{2},} & {\left[H_{2}, E_{1}\right]=-E_{1},} & {\left[H_{2}, E_{3}\right]=E_{3},} & {\left[H_{1}, E_{3}\right]=E_{3},} \\
{\left[H_{1}, F_{2}\right]=F_{2},} & {\left[H_{2}, F_{1}\right]=F_{1},} & {\left[H_{2}, F_{3}\right]=-F_{3},} & {\left[H_{1}, F_{3}\right]=-F_{3}}
\end{array}
$$

We shall use the symmetric bilinear form $\langle x, y\rangle=\operatorname{Tr}(x y)$ :

$$
\begin{array}{llll}
\left\langle E_{i}, E_{j}\right\rangle=0, & \left\langle F_{i}, F_{j}\right\rangle=0, & \left\langle H_{i}, E_{j}\right\rangle=0, & \left\langle H_{i}, F_{j}\right\rangle=0 \\
\left\langle E_{i}, F_{j}\right\rangle=\delta_{i j}, & \left\langle H_{i}, H_{i}\right\rangle=2, & \left\langle H_{1}, H_{2}\right\rangle=-1 &
\end{array}
$$

One can easily check that it is ad-invariant and non-degenerate. The corresponding dual basis is

$$
H_{1}^{*}=\frac{2}{3} H_{1}+\frac{1}{3} H_{2}, H_{2}^{*}=\frac{1}{3} H_{1}+\frac{2}{3} H_{2}, E_{i}^{*}=F_{i}, F_{i}^{*}=E_{i}
$$

and, hence

$$
\begin{aligned}
& \phi_{\mathfrak{s l}_{3}}(\bigcirc)=c_{2}=\sum_{i} e_{i} e_{i}^{*}=\frac{2}{3} H_{1}^{2}+\frac{2}{3} H_{2}^{2}+\frac{1}{3}\left(H_{1} H_{2}+H_{2} H_{1}\right)+\sum_{i=1}^{3}\left(E_{i} F_{i}+F_{i} E_{i}\right) \\
& \phi_{\mathfrak{s l}_{3}}^{S t}(\oslash)=\operatorname{Tr}\left(\frac{8}{3} \times i d_{3}\right)=8
\end{aligned}
$$

In addition,

$$
\phi_{\mathfrak{s l}_{3}}^{S t}(\bigotimes)=\operatorname{Tr}\left(\sum_{i} e_{i} e_{j} e_{i}^{*} e_{J}^{*}\right)=\operatorname{Tr}\left(-\frac{8}{9} \times i d_{3}\right)=-\frac{8}{3}
$$

Indeed,

$$
\phi_{\mathfrak{s l}_{3}}(\bigotimes)=\left(c_{2}-\lambda\right) c_{2}
$$

and we have $\left(\frac{8}{3}-\lambda\right) \frac{8}{3}=-\frac{8}{9}$, then $\lambda=3$. So $\phi_{\mathfrak{s l}_{3}}(\bigotimes)=\left(c_{2}-3\right) c_{2}$.
Because of the remark in the end of the previous section, the degree of a Lie algebra weight system of chord diagram with $n$ chords is at most $2 n$. By computing the results for sufficiently many irreducible representations, we can reconstruct the value of the universal weight system. Here are some results for the Lie algebra $\mathfrak{s l}_{3}$ in small orders.

|  | intersection graph | $\mathfrak{s l}_{3}$ weight system | $\mathfrak{s l}_{3}$ weight system of the projection in primitive space |
| :---: | :---: | :---: | :---: |
| $\bigcirc$ | $K_{1}$ | $c_{2}$ | $c_{2}$ |
| $\otimes$ | $K_{2}$ | $c_{2}\left(c_{2}-3\right)$ | $-3 c_{2}$ |
| $\square$ | $K_{1,2}$ | $c_{2}\left(c_{2}-3\right)^{2}$ | $9 c_{2}$ |
| $\otimes$ | $K_{3}$ | $c_{2}\left(c_{2}-3\right)\left(c_{2}-6\right)$ | $18 c_{2}$ |
| \# | $K_{2,2}$ | $c_{2}\left(c_{2}^{3}-12 c_{2}^{2}+63 c_{2}-99\right)$ | $c_{2}\left(9 c_{2}-99\right)$ |
| $\pm$ | $K_{4} \backslash\{e\}$ | $c_{2}\left(c_{2}^{3}-15 c_{2}^{2}+81 c_{2}-126\right)$ | $c_{2}\left(9 c_{2}-126\right)$ |
| * | $K_{4}$ | $c_{2}\left(c_{2}^{3}-18 c_{2}^{2}+108 c_{2}-180\right)$ | $c_{2}\left(9 c_{2}-180\right)$ |
| 4 | $K_{2,3}$ | $c_{2}\left(c_{2}^{4}-18 c_{2}^{3}+162 c_{2}^{2}-648 c_{2}+873\right)$ | $-c_{2}\left(135 c_{2}-873\right)$ |
| * | $K_{5}$ | $c_{2}\left(c_{2}^{4}-30 c_{2}^{3}+360 c_{2}^{2}-1764 c_{2}+2664\right)$ | $-c_{2}\left(324 c_{2}-2664\right)$ |
| \% | $K_{2,4}$ | $\begin{array}{r} c_{2}\left(c_{2}^{5}-24 c_{2}^{4}+306 c_{2}^{3}-\right. \\ \left.1998 c_{2}^{2}+6273 c_{2}-7227\right) \end{array}$ | $c_{2}\left(1485 c_{2}-7227\right)$ |
| * | $K_{3,3}$ | $\begin{array}{r} \frac{3}{4} c_{3}^{2}+c_{2}\left(c_{2}^{5}-27 c_{2}^{4}+\right. \\ \left.405 c_{2}^{3}-3339 c_{2}^{2}+13014 c_{2}-17595\right) \\ \hline \end{array}$ | $\begin{array}{r} \frac{3}{4} c_{3}^{2}-c_{2}\left(99 c_{2}^{2}-\right. \\ \left.4374 c_{2}+17595\right) \\ \hline \end{array}$ |
| * | $K_{6}$ | $\begin{array}{r} c_{3}^{2}+c_{2}\left(c_{2}^{5}-45 c_{2}^{4}+\right. \\ \left.900 c_{2}^{3}-8826 c_{2}^{2}+38196 c_{2}-54288\right) \\ \hline \end{array}$ | $\begin{array}{r} c_{3}^{2}-c_{2}\left(132 c_{2}^{2}-\right. \\ \left.10872 c_{2}+54288\right) \end{array}$ |

Here $c_{2}, c_{3}$ are the Casimir elements, of degree 2 and 3 , respectively, in

$$
\begin{aligned}
& Z U\left(\mathfrak{s l}_{3}\right), \\
& c_{2}= \sum_{i=1}^{8} e_{i} e_{i}^{*}, \quad\left\{e_{i}\right\}_{i \in[8]} \text { is a basis of } \mathfrak{s l}_{3} . \\
& c_{3}=-\frac{4}{3}\left(2 H_{1} H_{1} H_{1}-3 H_{1} H_{1}\left(H_{1}+H_{2}\right)-3 H_{1}\left(H_{1}+H_{2}\right)\left(H_{1}+H_{2}\right)+\right. \\
&+2\left(H_{1}+H_{2}\right)\left(H_{1}+H_{2}\right)\left(H_{1}+H_{2}\right)+9 E_{1} F_{1} H_{1}-18 E_{1} F_{1}\left(H_{1}+H_{2}\right)- \\
&-18 E_{3} F_{3} H_{1}+9 E_{3} F_{3}\left(H_{1}+H_{2}\right)+9 E_{2} F_{2} H_{1}+9 E_{2} F_{2}\left(H_{1}+H_{2}\right)- \\
&-27 E_{1} F_{3} E_{2}-27 F_{1} E_{3} F_{2}+18 H_{1}\left(H_{1}+H_{2}\right)-9\left(H_{1}+H_{2}\right)\left(H_{1}+H_{2}\right)- \\
&\left.-18 H_{1}+9\left(H_{1}+H_{2}\right)\right)
\end{aligned}
$$

The factor $-\frac{4}{3}$ in $c_{3}$ is due to the requirement that under the standard representation $c_{3}$ takes value $\frac{80}{3}$, which is the image of the Casimir element $\sum_{i, j, k} e_{i j} e_{j k} e_{k i}-$ $\frac{1}{3}\left(\sum_{i} e_{i i}\right)^{3}$ in $Z U\left(g l_{3}\right)$.

## 4 Jacobi diagrams and Lie algebra weight systems

When computing the values of the $\mathfrak{s l}_{3}$ weight system, we will require the results in [12] about recurrence relations for the values of this weight system on Jacobi diagrams. To this end, we recall the notion of closed Jacobi diagram. These diagrams provide a better understanding of the primitive space $P \mathcal{A}$, see, e.g. [2].

Definition 4.1 A closed Jacobi diagram (or, simply, a closed diagram) is a connected trivalent graph with a distinguished embedded oriented cycle, called Wilson loop, and a fixed cyclic order of half-edges at each vertex not on the Wilson loop. Half the number of the vertices of a closed diagram is called the degree, or order, of the diagram. This number is always an integer.

In the pictures below, we shall always draw the diagram inside its Wilson loop, which will be assumed to be oriented counterclockwise unless explicitly specified otherwise. Inner vertices will also be assumed to be oriented counterclockwise.

Chord diagrams are exactly those closed Jacobi diagrams all of whose vertices lie on the Wilson loop.

Definition 4.2 The vector space of closed diagrams $\mathcal{C}_{n}^{S T U}$ is the space spanned by all closed diagrams $\mathcal{C}_{n}$ of degree $n$ modulo the STU relations:


The three diagrams $S, T$ and $U$ must be identical outside the shown fragment. We write $\mathcal{C}^{S T U}$ for the direct sum of the spaces $\mathcal{C}_{n}^{S T U}$ for all $n \geq 0$.

Now we shall define a bialgebra structure in the space $\mathcal{C}^{S T U}$.
Definition 4.3 The product of two closed diagrams is defined in the same way as for chord diagrams: the two Wilson loops are cut at arbitrary places and then glued together into one loop, in agreement with the orientations:


Definition 4.4 The internal graph of a closed diagram is the graph obtained by erasing the Wilson loop. A closed diagram is said to be connected if its internal graph is connected. The connected components of a closed diagram are defined as the connected components of its internal graph.

In the sense of this definition, any chord diagram of order $n$ consists of $n$ connected components - the maximal possible number.

Now, the construction of the coproduct proceeds in the same way as for chord diagrams, the chords being replaced by the more general connected components.
Definition 4.5 Let $D$ be a closed diagram and $[D]$ the set of its connected components. For any subset $J \subseteq[D]$, denote by $D_{J}$ the closed diagram with only those components that belong to $J$ and by $D_{\bar{J}}$ the "complementary" diagram $(\bar{J}:=[D] \backslash J)$. We set

$$
\delta(D):=\sum_{J \subseteq[D]} D_{J} \otimes D_{\bar{J}}
$$

Now, for each $n=0,1,2, \ldots$, we have a natural inclusion $\lambda: \mathcal{A}_{n} \rightarrow \mathcal{C}_{n}$.
Claim 4.6[1] The inclusion $\lambda$ gives rise to an isomorphism of bialgebras $\lambda$ : $\mathcal{A}^{f r} \rightarrow \mathcal{C}^{S T U}$.

By definition, connected closed diagrams are primitive with respect to the coproduct $\delta$. It may sound surprising that the converse is also true:

Claim 4.7 [1] The primitive space $P$ of the bialgebra $\mathcal{C}^{S T U}$ coincides with the linear span of connected closed diagrams.

Since every closed diagram is a linear combination of chord diagrams, the weight system $\phi_{\mathfrak{g}}$ can be treated as a function on $\mathcal{C}^{S T U}$ with values in $U(\mathfrak{g})$.

The STU relation, which defines the algebra $\mathcal{C}$, gives us a hint how to do it. Namely, if we assign elements $e_{i}, e_{j}$ to the endpoints of chords of the T- and Udiagrams from the STU relations,

then it is natural to assign the commutator $\left[e_{i}, e_{j}\right]$ to the trivalent vertex on the Wilson loop of the S-diagram.

Generally, $\left[e_{i}, e_{j}\right]$ may not be a basis vector. A diagram with an endpoint marked by a linear combination of the basis vectors should be understood as the corresponding linear combination of diagrams marked by basis vectors. This understanding implies a useful

Lemma 4.8 The degree of the value of a Lie algebra weight system on a closed diagram $D$ is less or equal than the number of legs of $D$.

## 5 Values of the $\mathfrak{s l}_{3}$ weight system on certain Jacobi diagrams

Given a weight system $w$, we write $\bar{w}:=w \circ \pi$ for its composition with the projection to the subspace of primitives along the subspace of decomposable elements.

Here is an important lemma.
Lemma 5.1 (leaf lemma) [4] Let $w_{\mathfrak{g}}$ be the weight system associated to a metrized simple Lie algebra $\mathfrak{g}$ where the metric is proportional to the Killing form, with proportionality coefficient $\lambda$, $c$ the quadratic Casimir element in $U(\mathfrak{g})$. Then

$$
w_{\mathfrak{g}}(D)=\left(c-\frac{1}{2 \lambda}\right) w_{\mathfrak{g}}\left(D_{a}\right)
$$

for any Jacobi diagram $D$ and a leaf a (a chord intersecting a single leg) in it, $D_{a}$ being the Jacobi diagram $D$ with the chord a removed.


For small degree Jacobi diagrams, we can resolve all the vertices and get the linear combination of chord diagram. For example

$$
\square=\square-2 \square+\square
$$

Based on the above table, we have

$$
w_{\mathfrak{S r}_{3}}(\Xi)=c_{2}\left(c_{2}-3\right)^{2}-2 c_{2}^{2}\left(c_{2}-2\right)+c_{2}^{3}=9 c_{2}
$$

Here are the results of similar computations for some small Jacobi diagrams

| Jacobi diagram | $\mathfrak{s l}_{3}$ weight system |
| ---: | ---: |
| $\square$ | $9 c_{2}$ |
| $\square$ | $c_{2}\left(9 c_{2}+9\right)$ |
| $H$ | $c_{2}\left(27 c_{2}+63\right)$ |
|  | $c_{2}\left(189 c_{2}+117\right)$ |

Denote by $J_{i, j}$ the order $i+j+2$ Jacobi diagram with $i-1$ cells and $j$ chords crossing cells. $(i, j \geq 0)$


Specifically,

and $J_{0, j}$ is the chord diagram whose intersection graph is the complete bipartite graph $K_{2, j}$.

Our first main result is the following
Theorem 5.2 For any simple Lie algebra $\mathfrak{g}$ endowed with the scalar product proportional to the Killing form with proportionality coefficient $\lambda$, one has

$$
\bar{w}_{\mathfrak{g}}\left(J_{i, j}\right)=\bar{w}_{\mathfrak{g}}\left(J_{i-1, j+1}\right)+\frac{1}{\lambda} \bar{w}_{\mathfrak{g}}\left(J_{i-1, j}\right)
$$

Theorem 5.2 implies the following
Corollary 5.3 The following assertions are true:

1. the value $\bar{w}_{\mathfrak{g}}\left(J_{0, j}\right)$ has degree at most 4 ;
2. we have $\bar{w}_{\mathfrak{g}}\left(J_{i, 0}\right)=\sum_{k=0}^{i}\binom{i}{k} \lambda^{-k} \bar{w}_{\mathfrak{g}}\left(J_{0, i-k}\right)$;
3. we have $\sum_{n=0} \bar{w}_{\mathfrak{g}}\left(J_{n, 0}\right) \frac{x^{n}}{n!}=e^{\frac{x}{\lambda}} \sum_{n=0} \bar{w}_{\mathfrak{g}}\left(J_{0, n}\right) \frac{x^{n}}{n!}$.

## 6 Proof of Theorem 5.2

In order to prove Theorem 5.2, we need the leaf lemma after projection
Lemma 6.1 (leaf lemma after projection) For any simple Lie algebra $\mathfrak{g}$ endowed with the scalar product proportional to the Killing form with proportionality coefficient $\lambda$, one has

$$
\bar{w}_{\mathfrak{g}}(D)=-\frac{1}{2 \lambda} \bar{w}_{\mathfrak{g}}\left(D_{a}\right)
$$

for any Jacobi diagram $D$ and any leaf $a$ in it.


Proof. By the formula for the projection,

$$
\pi(D)=D-\sum_{i=2}^{|[D]|}(-1)^{i}(i-1)!\sum_{\substack{i \\ \bigcup_{j=1}\left[D_{j}\right]=[D] \\\left[D_{j}\right] \neq \emptyset}} \prod_{j=1}^{i} D_{j}
$$

where the sum is taken over all unordered splittings of the set $[D]$ of connected components of $D$ into nonempty subsets $\left[D_{j}\right]$ and $D_{j}$ is the sub-diagram induced by $\left[D_{j}\right]$. Now, after picking a chord $a$, we rewrite the previous formula in the form

$$
\begin{aligned}
\pi(D)= & D-\sum_{i=2}^{|[D]|}(-1)^{i}(i-1)!\sum_{\substack{\bigcup_{\begin{subarray}{c}{i \\
j=1 \\
j \\
\left[D_{j}\right] \neq \emptyset ;\left\{D_{j}\right]=[D]} }}\{a\}=D_{1}}\end{subarray}}^{i} \prod_{j=2}^{i} D_{j}+\sum_{\substack{\bigcup_{j=1}^{i}\left[D_{j}\right]=[D] \\
\left[D_{j}\right] \neq \emptyset ;\{a, b\} \subseteq D_{1}}}^{i} D_{j} \\
& \left.+\sum_{\substack{\bigcup_{j=1}^{i}\left[D_{j}\right]=[D] \\
\left[D_{j}\right] \neq \emptyset ;\{a\} \subsetneq D_{1} ; b \notin D_{1}}}\left(\{a\} \sqcup D_{a, 1}\right) \prod_{j=2}^{i} D_{j}\right)
\end{aligned}
$$

Now, applying $w_{\mathfrak{g}}$ to both sides and using the Leaf lemma we obtain the required:

$$
\begin{aligned}
& \bar{w}_{\mathfrak{g}}(D)=\left(c-\frac{1}{2 \lambda}\right) w_{\mathfrak{g}}\left(D_{a}\right)-\sum_{i=2}^{|[D]|}(-1)^{i}(i-1)!\sum_{\substack{i-1 \\
\sum_{j=1}^{[ }\left[D_{j}\right]=[D] \\
\left[D_{j}\right] \neq \emptyset}} c w_{\mathfrak{g}}\left(\prod_{j=1}^{i-1} D_{j}\right) \\
& \left.+\sum_{\substack{i \\
\bigsqcup_{j=1}^{i}\left[D_{a, j}\right]=\left[D_{a}\right] \\
\left[D_{a, j}\right] \neq \emptyset}}\left(c-\frac{1}{2 \lambda}\right) w_{\mathfrak{g}}\left(\prod_{j=1}^{i} D_{a, j}\right)+\sum_{\substack{i \\
j=1 \\
\left[D_{a, j}\right] \neq \emptyset}}^{\substack{i}} \sum_{\substack{ \\
i}}(i-1) c w_{\mathfrak{g}}\left(\prod_{j=1}^{i} D_{a, j}\right)\right) \\
& =\left(c-\frac{1}{2 \lambda}\right) w_{\mathfrak{g}}\left(D_{a}\right)-c w_{\mathfrak{g}}\left(D_{a}\right) \\
& -\sum_{i=2}^{\left|\left[D_{a}\right]\right|}\left((-1)^{i+1} i!+(-1)^{i}(i-1)!(i-1)\right) \sum_{\substack{\bigsqcup_{j=1}^{i}\left[D_{a, j}\right]=\left[D_{a}\right] \\
\left[D_{a, j}\right] \neq \emptyset}} c w_{\mathfrak{g}}\left(\prod_{j=1}^{i} D_{a, j}\right) \\
& +\left(2 c-\frac{1}{2 \lambda}\right)\left(\bar{w}_{\mathfrak{g}}\left(D_{a}\right)-w_{\mathfrak{g}}\left(D_{a}\right)\right) \\
& =\left(c-\frac{1}{2 \lambda}\right) w_{\mathfrak{g}}\left(D_{a}\right)-c w_{\mathfrak{g}}\left(D_{a}\right)-c\left(\bar{w}_{\mathfrak{g}}\left(D_{a}\right)-w_{\mathfrak{g}}\left(D_{a}\right)\right) \\
& +\left(c-\frac{1}{2 \lambda}\right)\left(\bar{w}_{\mathfrak{g}}\left(D_{a}\right)-w_{\mathfrak{g}}\left(D_{a}\right)\right) \\
& =-\frac{1}{2 \lambda} \bar{w}_{\mathfrak{g}}\left(D_{a}\right)
\end{aligned}
$$

In order to do the induction step, let us resolve the extreme internal vertices in $J_{i, j}$ by means of the $S T U$ relations:


The diagram on the left is $J_{i, j}$. The first diagram on the right is $J_{i-1, j+1}$. The second and third diagrams on the right are $J_{i-1, j}$ with a leaf, respectively. The fourth diagram on the right is $J_{i-1, j}$ times an isolated chord.

Applying $\bar{w}_{\mathfrak{g}}$ to both sides of the equation and using the fact that a product of
two nontrivial Jacobi diagrams projects to 0 in primitives, we get Theorem 5.2:

$$
\begin{aligned}
\bar{w}_{\mathfrak{g}}\left(J_{i, j}\right) & =\bar{w}_{\mathfrak{g}}\left(J_{i-1, j+1}\right)+2 \times \frac{1}{2 \lambda} \bar{w}_{\mathfrak{g}}\left(J_{i-1, j}\right)+0 \\
& =\bar{w}_{\mathfrak{g}}\left(J_{i-1, j+1}\right)+\frac{1}{\lambda} \bar{w}_{\mathfrak{g}}\left(J_{i-1, j}\right)
\end{aligned}
$$

## 7 Values of the $\mathfrak{s l}_{3}$ weight system on the special family $J_{i, 0}$ of Jacobi diagrams

Let $W(\cdot)$ denote the weight system associated to $\mathfrak{s l}_{3}$ and the matrix trace as the invariant bilinear form. Below, we will make use of the following theorem about the values of this weight system on Jacobi diagrams.

Theorem 7.1 (Kenichi Kuga and Shutaro Yoshizumi [12]) For Jacobi diagrams that are different only in parts depicted as below, the following relations hold:

1. $W(\bigcap)=6 W(\mid)$,
2. $W($

3. $W(\square \square)=W(\square)+W( \rangle\langle+3 W(D()+$ $+3 W(>)+3 W(>)$.

In particular, for the Jacobi diagrams $J_{i, 0}$, the above formulas yield


We know


By Theorem 7.1(2), we have


By Theorem 7.1(1), we have


By Theorem 7.1(2) and the STU-relations, we have


Combining all these together, we get

$$
W\left(J_{i, 0}\right)=W\left(J_{i-1,0}\right)+6 W\left(J_{i-2,0}\right)+3 \times 6^{i-2} c_{2}^{2}
$$

Now, we have $W\left(J_{1,0}\right)=9 c_{2}$ and $W\left(J_{2,0}\right)=9 c_{2}\left(c_{2}+9\right)$. By induction, we get

$$
W\left(J_{i, 0}\right)=c_{2}\left(\left(\frac{6^{i}}{8}+\frac{27}{40}(-2)^{i}+\frac{1}{5} 3^{i}\right) c_{2}+\frac{9}{5}\left(3^{i}-(-2)^{i}\right)\right) \text { for }(i \geq 1)
$$

and $W\left(J_{0,0}\right):=0$, we have the exponential generating functions form

$$
\sum_{n=0} W\left(J_{n, 0}\right) \frac{x^{n}}{n!}=c_{2}\left(\left(\frac{1}{8} e^{6 x}+\frac{27}{40} e^{-2 x}+\frac{1}{5} e^{3 x}\right) c_{2}+\frac{9}{5}\left(e^{3 x}-e^{-2 x}\right)\right)-c_{2}^{2}
$$

By Corollary 5.3(3), we conclude that

$$
\begin{aligned}
& \sum_{n=0} \bar{w}_{\mathfrak{S I}_{3}}\left(K_{2, n}\right) \frac{x^{n}}{n!}=\sum_{n=0} \bar{w}_{\mathfrak{S I}_{3}}\left(J_{0, n}\right) \frac{x^{n}}{n!}=e^{-6 x} \sum_{n=0} \bar{w}_{\mathfrak{s l}_{3}}\left(J_{n, 0}\right) \frac{x^{n}}{n!} \\
& \sum_{n=0} \bar{w}_{\mathfrak{S l}_{3}}\left(K_{2, n}\right) \frac{x^{n}}{n!}=\frac{c_{2}}{40}\left(\left(27 c_{2}-72\right) e^{-8 x}+\left(8 c_{2}+72\right) e^{-3 x}-40 c_{2} e^{-6 x}+5 c_{2}\right)
\end{aligned}
$$

Now we can reconstruct the values of the $\mathfrak{s l}_{3}$ weight system on the chord diagrams $J_{2, n}$ by making use of a result in [5], which says

$$
\sum_{n=0} \pi\left(K_{2, n}\right) \frac{x^{n}}{n!}=e^{-K_{0,1} x} \sum_{n=0} K_{2, n} \frac{x^{n}}{n!}-\left(\sum_{n=0} \pi\left(K_{1, n}\right) \frac{x^{n}}{n!}\right)^{2}
$$

We get the values of $\mathfrak{s l}_{3}$ weight system on chord diagrams whose intersection graph is complete bipartite $K_{2, n}$.

Theorem 7.2 We have

$$
\begin{aligned}
\sum_{n=0} w_{\mathfrak{s l}_{3}}\left(K_{2, n}\right) \frac{x^{n}}{n!} & =\frac{c_{2}}{40}\left(\left(27 c_{2}-72\right) e^{\left(c_{2}-8\right) x}+\left(8 c_{2}+72\right) e^{\left(c_{2}-3\right) x}+5 c_{2} e^{c_{2} x}\right) \\
w_{\mathfrak{s l}_{3}}\left(K_{2, n}\right) & =\frac{c_{2}}{40}\left(\left(27 c_{2}-72\right)\left(c_{2}-8\right)^{n}+\left(8 c_{2}+72\right)\left(c_{2}-3\right)^{n}+5 c_{2}^{n+1}\right)
\end{aligned}
$$

Our computations show, in particular, that, for the chord diagrams whose intersection graph is $K_{2, n}$, their projection to the subspace of primitives can be represented as a linear combination of connected Jacobi diagrams with at most 4 legs. It has been conjectured earlier by S . Lando that the value of the weight system $\mathfrak{s l}_{2}$ on a projection to the subspace of primitives of a chord diagram is a polynomial in the quadratic Casimir element $c$ whose degree does not exceed half the length of the largest cycle in the intersection graph of the chord diagram. There is a lot of evidence supporting this conjecture, see, for example [5, 6]. The following more general conjecture may explain Lando's one, and is, probably, easier to prove.

Conjecture 7.3 (S. Lando, Z. Yang) Let $D$ be a chord diagram, and let $\ell$ be the length of the longest cycle in it. Then the projection $\pi(D)$ of $D$ to the subspace of primitives is a linear combination of connected Jacobi diagrams with at most $\ell$ legs. In particular, the value of a weight system associated to an arbitrary Lie algebra $\mathfrak{g}$ on this projection has degree at most $\ell$.

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