# Finite-dimensional reduction of systems of nonlinear diffusion equations 

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#### Abstract

We present a class of one-dimensional systems of nonlinear parabolic equations for which long-time phase dynamics can be described by an ODE with a Lipschitz vector field in $\mathbb{R}^{n}$. In the considered case of the Dirichlet boundary value problem sufficient conditions for a finite-dimensional reduction turn out to be much wider than the known conditions of this kind for a periodic situation.


Keywords: nonlinear parabolic equations, finite-dimensional dynamics on attractor; inertial manifold.

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## 1. Introduction

One of the main problems in the study of evolution equations is related to describing the final (at large time) behavior of their solutions. We consider systems of diffusion equations with Dirichlet boundary condition

$$
\begin{equation*}
\partial_{t} u=D \partial_{x x} u+f(x, u) \partial_{x} u+g(x, u), \quad u(0)=u(1)=0 \tag{1.1}
\end{equation*}
$$

on the closed interval $J=[0,1]$. Here $u=\left(u_{1}, \ldots, u_{m}\right), f$ and $g$ are sufficiently regular matrix and vector functions, respectively. We assume that the matrix $D$ of numerical coefficients is similar to a diagonal matrix with positive eigenvalues. In the case of $D=$ $\operatorname{diag}\left\{d_{1}, \ldots, d_{m}\right\}, d_{j}>0$, we deal with reaction-diffusion-convection equations. Under appropriate conditions on $f$ and $g$, system (1.1) induces a smooth dissipative semiflow $\left\{\Phi_{t}\right\}_{t \geq 0}$ in the phase space $X^{\alpha} \subset C^{1}\left(J, \mathbb{R}^{m}\right)$ with an appropriate $\alpha>0$, where $\left\{X^{\alpha}\right\}_{\alpha \geq 0}$ is the Hilbert semiscale [3] generated by the linear sectorial operator $u \rightarrow-D u_{x x}$ in $X=L^{2}\left(J, \mathbb{R}^{m}\right)$. In this situation, there exists a global attractor $[2,7,12]$ (in what follows, simply an attractor), i.e., a connected compact invariant set $\mathcal{A} \subset X^{\alpha}$ of a finite Hausdorff dimension uniformly attracting bounded subsets of $X^{\alpha}$ as $t \rightarrow+\infty$.

Our goal is to find conditions under which the dynamics on the attractor (final dynamics) of parabolic system (1.1) is finite-dimensional in the sense of [8]. This means that, for some ODE $\partial_{t} \xi=h(\xi)$ in $\mathbb{R}^{N}$ with Lipschitz vector field $h$, the resolving flow $\left\{\Theta_{t}\right\}$ and an invariant compact set $\mathcal{K} \subset \mathbb{R}^{N}$, the phase semiflows $\left\{\Phi_{t}\right\}_{t \geq 0}$ on $\mathcal{A}$ and $\left\{\Theta_{t}\right\}_{t \geq 0}$ on $\mathcal{K}$ are Lipschitz-conjugate. In this connection, we can speak [14] about the finite-dimensional reduction of evolution problem (1.1).

The main result in this paper (Theorem 4.3) ensures that the final phase dynamics of system (1.1) is finite-dimensional under the consistency condition

$$
\begin{equation*}
D f(x, u)=f(x, u) D, \quad(x, u) \in J \times \operatorname{co} \mathcal{A} \tag{1.2}
\end{equation*}
$$

where co $\mathcal{A}$ is the convex hull of $\mathcal{A}$.
It is well known [5] that, in the case of scalar diffusion $(D=d E$ with unit matrix $E)$ and $f=f(u), g=g(u)$, there exists an inertial manifold (IM), i.e., a finite-dimensional invariant $C^{1}$-surface in the phase space containing an attractor and exponentially attracting (with an asymptotic phase) all trajectories of the system as $t \rightarrow+\infty$. The presence of IM implies that the final dynamics is finite-dimensional, and an extensive literature deals with the existence of such manifolds (see, $[7,10,12,14]$ ). An original approach to these problems is presented in recent works of M. Anikushin (see [1] and the references therein).

In the periodic case ( $J$ is a circle of length 1 ), conditions ensuring that the final dynamics of systems (1.1) with $D=$ diag is finite-dimensional were obtained by the author in [11; p.13409]. Note that, in the class of periodic systems (1.1) with scalar diffusion, the first example of semilinear parabolic equation of mathematical physics that does not demonstrate such a dynamics was constructed in [6; Theorem 1.2].

## 2. Preliminaries

In what follows, if necessary, we will use the technique developed in [11]. All preliminary constructions in Sections 2 and 3 are carried out for the case $D=$ diag. Let us write system (1.1) in the form of a semilinear parabolic equation (SPE)

$$
\begin{equation*}
\partial_{t} u=-A u+F(u) \tag{2.1}
\end{equation*}
$$

in the real Hilbert space $X=L^{2}\left(J, \mathbb{R}^{m}\right)$ equipped with the norm $\|\cdot\|$. Here we have $A: u \rightarrow-D u_{x x}$ with Dirichlet boundary condition and the nonlinearity $F: u \rightarrow$
$f(x, u) \partial_{x} u+g(u)$. For the linear positive definite operator $A$, we put $X^{\alpha}=\mathcal{D}\left(A^{\alpha}\right)$ with $\alpha \geq 0$ and $X_{0}=X$. Then $\|u\|_{\alpha}=\left\|A^{\alpha} u\right\|$. Note that the function $F$ is of class $W^{2}\left(X^{\alpha}, X\right)$ if

$$
\begin{equation*}
F \in C^{2}\left(X^{\alpha}, X\right) \bigcap \operatorname{Lip}\left(X^{\alpha}, X\right) \quad \text { and } \quad\|F(u)\| \leq M, \quad u \in X^{\alpha} \tag{2.2}
\end{equation*}
$$

for some $\alpha \in[0,1)$. In this case, $\operatorname{SPE}(2.1)$ generates [3] a smooth compact resolving semiflow $\left\{\Phi_{t}\right\}_{t \geq 0}$ in the phase space $X^{\alpha}$. Assumption (2.2) implies [10; Lemma 1.1] the $X^{\alpha}$-dissipativity of (2.1):

$$
\limsup _{t \rightarrow+\infty}\left\|\Phi_{t} u\right\|_{\alpha} \leq r
$$

for some $r>0$ uniformly in $u \in$ balls in $X^{\alpha}$. Under such conditions, there exists [2,7,12] a compact attractor $\mathcal{A} \subset X^{\alpha}$ consisting of all bounded complete trajectories $\{u(t)\}_{t \in \mathbb{R}} \subset$ $X^{\alpha}$. In fact, $\mathcal{A} \subset X^{1}$ due to the smoothing action of the parabolic equation [3]. Simple argument [11; p.13410] shows that, in all constructions concerning SPE (2.1), one can replace the nonlinearity exponent $\alpha$ by any value $\alpha_{1} \in(\alpha, 1)$, and if condition (2.2) is satisfied in the pair of spaces $\left(X^{\theta}, X^{\theta+\alpha}\right)$ with $\theta>0$ instead of $\left(X, X^{\alpha}\right)$, then all the listed properties of the dynamics are preserved for the phase space $X^{\theta+\alpha}$. In what follows, functions $Y_{1} \rightarrow Y_{2}$ of class (2.2) will arise for some Banach spaces $Y_{1}, Y_{2}$.

As in [11], we will use sufficient conditions for the final dynamics to be finitedimensional [9]. Assume that $G(u)=F(u)-A u$ is the vector field (2.1), $\mathcal{N}=\mathcal{A} \times \mathcal{A}$, and $Y$ is a Banach space.

Definition 2.1 ([9]). A continuous field $\Pi: \mathcal{N} \rightarrow Y$ is said to be regular if, for any $u, v \in \mathcal{A}$, the function $\Pi\left(\Phi_{t} u, \Phi_{t} v\right):[0,+\infty) \rightarrow Y$ is of class $C^{1}$ with the derivative $\partial_{t} \Pi(u, v)$ uniformly bounded in $(u, v) \in \mathcal{N}$ at zero.

The smoothness of the semiflow $\left\{\Phi_{t}\right\}$ and the invariance of the compact set $\mathcal{A} \subset X^{\alpha}$ ensure the regularity of the identity embedding $\mathcal{N} \rightarrow X^{\alpha} \times X^{\alpha}$, and hence, the regularity of each field $\Pi: \mathcal{N} \rightarrow Y$ that can be continued to a $C^{1}$-mapping into the $X^{\alpha} \times X^{\alpha}{ }_{-}$ neighborhood of the set $\mathcal{N}$. In this situation, we have $\partial_{t} \Pi(u, v)=\Pi^{\prime}(u, v)(G(u), G(v))$, where $(\cdot)^{\prime}$ is the Frechet differentiation. Under condition (2.2) on the nonlinearity $F$, the function $u \rightarrow G(u)$ on $\mathcal{A}$ is continuous and even Hölder [8] in the $X^{\alpha}$-metric. The regular fields $\mathcal{N} \rightarrow Y$ form a linear structure and even a multiplicative one if $Y$ is a

Banach algebra. In the latter case, if all elements $\Pi(u, v) \in Y$ are invertible, then the field $\Pi^{-1}$ is also regular.

We will start from the decomposition

$$
\begin{equation*}
G(u)-G(v)=\left(T_{0}(u, v)-T(u, v)\right)(u-v), \quad(u, v) \in \mathcal{N} \tag{2.3}
\end{equation*}
$$

where $T_{0} \in \mathcal{L}\left(X^{\alpha}\right)$ and $T \in \mathcal{L}\left(X^{1}, X\right)$ are unbounded linear operators in $X$ similar to positive definite ones. We let

$$
\Sigma_{T}=\bigcup_{u, v \in \mathcal{A}} \operatorname{spec} T(u, v)
$$

denote the total spectrum of the operators $T$.
We will need a particular case [9; Theorem 2.8] in the situation $\Sigma_{T} \subset \mathbb{R}^{+}$.
Theorem 2.2. Assume that $F \in W^{2}\left(X^{\alpha}, X\right)$ and

$$
\begin{equation*}
T(u, v)=S^{-1}(u, v) H(u, v) S(u, v) \tag{2.4}
\end{equation*}
$$

on $\mathcal{N}$, where the unbounded self-adjoint linear operators $H(u, v)$ are positive definite in $X$, the fields $S, S^{-1}: \mathcal{N} \rightarrow \mathcal{L}(X)$ and $T_{0}: \mathcal{N} \rightarrow \mathcal{L}\left(X^{\alpha}, X\right)$ are regular, and the field $T_{0}: \mathcal{N} \rightarrow \mathcal{L}\left(X^{\alpha}\right)$ is bounded. If in addition, the set $\mathbb{R}^{+} \backslash \Sigma_{T}$ contains intervals $\left(a_{k}-\xi_{k}, a_{k}+\xi_{k}\right)$ with $a_{k}>\xi_{k}>0$ such that

$$
\begin{equation*}
\xi_{k} \rightarrow \infty, \quad a_{k}^{\alpha / 2}=o\left(\xi_{k}\right) \tag{2.5}
\end{equation*}
$$

as $k \rightarrow+\infty$, then the final $X^{\alpha}$-dynamics of SPE (2.1) is finite-dimensional.
We further assume that the matrix function $f=f(x, u)$ and the vector functions $g=g(x, u)$ in (1.1) satisfy the regularity conditions:
(H) $\quad f, g$ of class $C^{\infty}$ on $J \times \mathbb{R}^{m}$ are finite in $u$ and $f(x, 0)=g(x, 0)=0$ for $x=0,1$.

We let $\mathcal{H}^{s}=\mathcal{H}^{s}(J)$ denote generalized Sobolev $L^{2}$-spaces (spaces of Bessel potentials $[3,13])$ of scalar functions on $J$ with arbitrary $s \geq 0$. If $s>1 / 2$, then $\mathcal{H}^{s} \subset C(J)$ and $\mathcal{H}^{s}$ is a Banach algebra [13; Sec. 2.8.3]. The differentiation operator $\partial_{x}$ belongs to $\mathcal{L}\left(\mathcal{H}^{s+1}, \mathcal{H}^{s}\right)$. In fact, the $X^{s}$ are closed subspaces (with equivalent norm) in the spaces $\mathcal{H}^{2 s}\left(J, \mathbb{R}^{m}\right)$ of vector functions, and $X^{s}=\mathcal{H}^{2 s}\left(J, \mathbb{R}^{m}\right)$ for $s \leq 1 / 4$. For $s>1 / 4$, the space $X^{s}$ consists of elements $u \in \mathcal{H}^{2 s}\left(J, \mathbb{R}^{m}\right)$ such that $u(0)=u(1)=0$.

We now fix an arbitrary $\alpha \in(3 / 4,1)$. Then we have $\mathcal{H}^{2 \alpha} \hookrightarrow C^{1}(J)$ and $X^{\alpha} \hookrightarrow$ $C^{1}(J)$, where the symbol $\hookrightarrow$ denotes a linear continuous embedding of function spaces. We will use several required embedding theorems [3,13]. For an arbitrary $C^{\infty}$-function $z: J \times \mathbb{R}^{m} \rightarrow \mathbb{R}$, the mapping $\psi: u \rightarrow z(x, u)$ is a function of class $W^{2}$ (see (2.2)) from $C^{s}\left(J, \mathbb{R}^{m}\right)$ to $C^{s}(J)$ for all $s \in \mathbb{N}$. This implies that $\psi \in W^{2}\left(\mathcal{H}^{2 \alpha}\left(J, \mathbb{R}^{m}\right), C^{1}(J)\right)$. Using the embeddings $\mathcal{H}^{s+1} \hookrightarrow C^{s}(J) \hookrightarrow \mathcal{H}^{s}$, we can conclude that $\psi \in W^{2}\left(\mathcal{H}^{s}\left(J, \mathbb{R}^{m}\right), \mathcal{H}^{s}(J)\right)$. We thus obtain $F \in W^{2}\left(X^{1}, X^{1 / 2}\right)$ for the nonlinear part $F: u \rightarrow f(x, u) \partial_{x} u+g(u)$ of system (1.1). Moreover, $X^{\alpha} \hookrightarrow C^{1}\left(J, \mathbb{R}^{m}\right) \hookrightarrow C\left(J, \mathbb{R}^{m}\right) \hookrightarrow X$, and hence $F \in$ $W^{2}\left(X^{\alpha}, X\right)$. We also note that $X^{3 / 2} \hookrightarrow C^{2}\left(J, \mathbb{R}^{m}\right)$.

We choose $X^{\alpha}$ as the phase space of system (1.1). Then the phase dynamics of (1.1) in $X^{\alpha}$ is dissipative, and there exists a global attractor $\mathcal{A} \subset X^{\alpha}$. Since $F \in W^{2}\left(X^{1}, X^{1 / 2}\right)$, system (1.1) also generates a smooth dissipative phase semiflow in the space $X^{1}$ and the attractor $\mathcal{A}$ is compact in $X^{3 / 2}$. As above, we denote $\mathcal{N}=\mathcal{A} \times \mathcal{A}$.

Remark 2.3. The phase dynamics of system (1.1) has the following property: if $Y$ is a Banach space, then each vector field $\Pi: \mathcal{N} \rightarrow Y$ continuous in the $\left(X^{\alpha} \times X^{\alpha}\right)$-metric and extendable to a $C^{1}$-mapping $X^{1} \times X^{1} \rightarrow Y$ is regular in the sense of Definition 2.1.

Indeed, the smoothness of the semiflow in $X^{1}$ means that the mapping $(t, u) \rightarrow \Phi_{t} u$ : $(0,+\infty) \times X^{1} \rightarrow X^{1}$ is smooth. This ensures that the identity embedding $\mathcal{N} \rightarrow X^{1} \times X^{1}$ is regular, and hence the field $\Pi$ on $\mathcal{N}$ is also regular.

## 3. Decomposition of a vector field on an attractor

We want to apply Theorem 2.2 to SPE (1.1) with $D=\operatorname{diag}$ and the phase space $X^{\alpha}$, $\alpha \in(3 / 4,1)$. We let $\mathbb{M}^{m}$ denote the algebra of numerical $m \times m$ matrices with Euclidean norm, and let $Y\left(J, \mathbb{M}^{m}\right)$ denote linear spaces of such matrices with elements from some Banach space $Y$ of scalar functions on $J=[0,1]$. Similarly [11; pp.13412-13413], we assume that

$$
\begin{gather*}
B_{0}(x ; u, v)=\int_{0}^{1}\left(f_{u}(x, w(x)) w_{x}(x)+g_{u}(x, w(x)) d \tau\right.  \tag{3.1.1}\\
B(x ; u, v)=\int_{0}^{1} f(x, w(x)) d \tau \tag{3.1.2}
\end{gather*}
$$

for $u, v \in X^{\alpha}, w(x)=\tau u(x)+(1-\tau) v(x), x \in J$. The elements of the matrices $B_{0}, B$ are continuous functions, and for $u, v \in \mathcal{A}$, they are functions of class $C^{2}$ on $J$.

Using the $C^{1}$-smoothness of the mappings $(u, v) \rightarrow f_{u}(x, w) w_{x}+g_{u}(x, w),(u, v) \rightarrow$ $f(x, w), \quad X^{\alpha} \times X^{\alpha} \rightarrow C\left(J, \mathbb{M}^{m}\right)$ for a fixed $\tau \in[0,1]$ and differentiating the integrands in the expressions for $B_{0}$ and $B$ with respect to the parameters $(u, v)$, we see that the mappings

$$
\begin{equation*}
(u, v) \rightarrow B_{0}(\cdot ; u, v), \quad(u, v) \rightarrow B(\cdot ; u, v) \tag{3.2}
\end{equation*}
$$

are of class $C^{1}\left(X^{\alpha} \times X^{\alpha}, C\left(J, \mathbb{M}^{m}\right)\right)$. We use the integral mean value theorem for nonlinear operators to write the decomposition of the vector field of (1.1) on the attractor $\mathcal{A} \subset X^{\alpha}$ as

$$
\begin{aligned}
& G(u)-G(v)=-A h+\left(\int_{0}^{1} F^{\prime}(\tau u+(1-\tau) v) d \tau\right) h \\
& =D h_{x x}+B_{0}(x ; u, v) h+B(x ; u, v) h_{x}, \quad u, v \in \mathcal{A}
\end{aligned}
$$

where $h=u-v, \tau u+(1-\tau) v \in \operatorname{co} \mathcal{A}$, and $(\cdot)^{\prime}$ is the Frechet differentiation. To eliminate the dependence on $h_{x}$, we apply (following [4]) the transformation $h=U \eta$, where the $m \times m$ matrix function $U(x)=U(x ; u, v), x \in[0,1]$, is the solution of the linear Cauchy problem

$$
\begin{equation*}
U_{x}=-\frac{1}{2} D^{-1} B(x) U, \quad U(0)=E \tag{3.3}
\end{equation*}
$$

As a result, we obtain relation (2.3) with linear operators

$$
\begin{align*}
T_{0}(u, v) h= & \left(B_{0}(x)-\frac{1}{2} B_{x}(x)-\frac{1}{4} B(x) D^{-1} B(x)\right) h,  \tag{3.4.1}\\
& T(u, v) h=-D U \partial_{x x} U^{-1} h . \tag{3.4.2}
\end{align*}
$$

We note that the change of variable $h=U \eta$ does not change the Dirichlet boundary conditions for the linear part of (1.1). In the expressions for the matrices $B_{0}, B, U, U^{-1}$, we often omit the dependence on $u, v$, and sometimes on $x$.

Lemma 3.1. The field of operators $T_{0}$ on $\mathcal{N}$ is regular with values in $\mathcal{L}\left(X^{\alpha}, X\right)$ and bounded with values in $\mathcal{L}\left(X^{\alpha}\right)$.

Proof. We assume that $T_{0} h=Q(x ; u, v) h$ in (3.4.1) with $h \in \mathcal{A}-\mathcal{A} \subset X^{\alpha}$. The convex hull of the attractor $\mathcal{A}$ is bounded in the $X^{3 / 2}$-norm equivalent to the $\mathcal{H}^{3}\left(J, \mathbb{R}^{m}\right)$ norm, and hence the matrix functions $B, B D^{-1} B$ and $B_{0}$ are bounded uniformly in $(u, v) \in \mathcal{N}$ in $\mathcal{H}^{3}\left(J, \mathbb{M}^{m}\right)$ and $\mathcal{H}^{2}\left(J, \mathbb{M}^{m}\right)$, respectively. Thus, the matrix functions $B_{x}$ and $Q$ are bounded on $\mathcal{N}$ in the norm of $\mathcal{H}^{2}\left(J, \mathbb{M}^{m}\right)$ and $T_{0}$ is the operator of multiplication
of vector functions from $X^{\alpha} \subset \mathcal{H}^{2 \alpha}\left(J, \mathbb{R}^{m}\right)$ by the matrix $Q \in \mathcal{H}^{2 \alpha}\left(J, \mathbb{M}^{m}\right)$ with $2 \alpha \in$ $(3 / 2,2)$. Since $\mathcal{H}^{2 \alpha}(J)$ is a Banach algebra, we obtain $T_{0}(u, v) \in \mathcal{L}\left(X^{\alpha}\right)$ and $\left\|T_{0}(u, v)\right\|_{\alpha} \leq$ const on $\mathcal{N}$.

With regard to Remark 2.3 and the above-noted smoothness of mappings (3.2), the regularity of the field of the operators $T_{0}: \mathcal{N} \rightarrow \mathcal{L}\left(X^{\alpha}, X\right)$ can be proved as in the case of periodic boundary conditions in [11; Lemma 3.3].

The matrix function $U(x)$ in the Cauchy problem (3.3) can be treated as a bounded linear operator in $X$.

Lemma 3.2. The fields of the operators $U, U^{-1}: \mathcal{N} \rightarrow \mathcal{L}(X)$ are regular.
For the field of $U$, this can be proved as a similar assertion in the periodic case [11; Lemma 3.4]. At the same time, the regularity of $U$ implies the regularity of the field of the inverse operators $U^{-1}$.

Now we assume that $d_{-}=\min _{1 \leq j \leq m} d_{j}$ and $d_{+}=\max _{1 \leq j \leq m} d_{j}$ for $D=\operatorname{diag}\left\{d_{j}\right\}$. Assume also that $\left\{\lambda_{n}: \lambda_{1}<\lambda_{2}<\ldots\right\}$ are eigenvalues of the linear operator $A=-D \partial_{x x}$. Since

$$
\begin{equation*}
\operatorname{spec} A=\left\{d_{j} \pi^{2} \nu^{2}, \quad \nu \in \mathbb{N}, \quad j \in \overline{1, m}\right\} \tag{3.5}
\end{equation*}
$$

we have $\lambda_{n} \leq \pi^{2} d_{+} n^{2}$. Using the counting function for spec $A$, we obtain

$$
n \leq \sum_{j=1}^{m} \frac{\sqrt{\lambda_{n}}}{\pi \sqrt{d_{j}}} \leq \frac{m}{\pi \sqrt{d_{-}}} \sqrt{\lambda_{n}}
$$

and hence

$$
\begin{equation*}
\frac{\pi^{2} d_{-}}{m^{2}} n^{2} \leq \lambda_{n} \leq \pi^{2} d_{+} n^{2}, \quad n \in \mathbb{N} \tag{3.6}
\end{equation*}
$$

Lemma 3.3. The estimate $\limsup _{n \rightarrow \infty} n^{-1}\left(\lambda_{n+1}-\lambda_{n}\right)>0$ holds.
Proof. If, on the contrary, $\lambda_{n+1}-\lambda_{n}=\beta_{n} n$ with $\beta_{n} \xrightarrow{n \rightarrow \infty} 0$, then

$$
\begin{aligned}
n^{-2} \lambda_{n} & =n^{-2}\left(\lambda_{1}+\sum_{k=1}^{n-1}\left(\lambda_{k+1}-\lambda_{k}\right)=n^{-2}\left(\lambda_{1}+\sum_{k=1}^{n-1} \beta_{k} k\right)\right. \\
& \leq n^{-2}\left(\lambda_{1}+\sum_{k=1}^{n-1} \beta_{k} n\right) \leq n^{-2} \lambda_{1}+n^{-1} \sum_{k=1}^{n} \beta_{k} .
\end{aligned}
$$

However, this implies the relation $\lambda_{n}=o\left(n^{2}\right)$ which contradicts the left inequality in (3.6).

## 4. Main results

By the assumptions of Theorem 2.1, it is necessary to prove the "uniform" similarity of the operators $T(u, v)$ in (3.4.2) to positive definite operators of the form (2.4), as well as the required sparsity $(2.5)$ of their total spectrum $\Sigma_{T}$. We assume that the regularity conditions $\mathbf{( H )}$ are satisfied for the functions $f$ and $g$ in (1.1).

Theorem 4.1. If the matrix $D=\operatorname{diag}\left\{d_{j}\right\}$ with $d_{j}>0$ and condition (1.2) is satisfied, then the phase dynamics on the attractor is finite-dimensional.

Proof. The operator $A=-D \partial_{x x}$ with Dirichlet condition is self-adjoint and positive definite in $X$. Assumption (1.2) (for any $x \in J$ and $u, v \in \mathcal{A}$ ) implies the relation $D B(x)=B(x) D$ for the matrices $B(x)=B(x ; u, v)$ in (3.1.2). Thus, the matrices $B(x)$ and $D^{-1} B(x)$ inherit the block (with respect to equal $d_{j}$ ) structure of the diffusion matrix $D=\operatorname{diag}\left\{d_{1}, \ldots, d_{m}\right\}$. Therefore, the same also holds for the solutions $U(x)$ of the Cauchy problem (3.3), and hence, $D U(x)=U(x) D, x \in J$, and

$$
T(u, v)=U(u, v)\left(-D \partial_{x x}\right) U^{-1}(u, v)
$$

in (3.4.2). Thus, representation (2.4) with $S(u, v)=U^{-1}(u, v)$ and $H(u, v) \equiv A$ holds for $T(u, v)$. The total spectrum $\Sigma_{T}$ coincides with $\operatorname{spec}(A)$ in (3.5). By Lemma 3.3, there exists $\varepsilon>0$ and an increasing sequence of indices $n(k)$ such that $\lambda_{n(k)+1}-\lambda_{n(k)}>\varepsilon n(k)$ for $k \geq k_{0}$. We put $a_{k}=\left(\lambda_{n(k)+1}+\lambda_{n(k)}\right) / 2, \xi_{k}=\left(\lambda_{n(k)+1}-\lambda_{n(k)}\right) / 3$ and $M=\pi^{2} d_{+}$. From the right inequality in (3.6) we obtain

$$
a_{k} \leq M\left(n^{2}(k)+n(k)+\frac{1}{2}\right) \leq 3 M n^{2}(k) \leq \frac{3 M}{\varepsilon^{2}}\left(\lambda_{n(k)+1}-\lambda_{n(k)}\right)^{2} \leq \frac{27 M}{\varepsilon^{2}} \xi_{k}^{2}
$$

for $k \geq k_{0}$, i.e., $a_{k}=O\left(\xi_{k}^{2}\right)$ as $k \rightarrow \infty$. Since $a_{k}^{\alpha / 2}=o\left(\xi_{k}\right)$ for $\alpha \in(3 / 4,1)$ and $k \rightarrow \infty$, the sought assertion follows from Lemmas 3.1 and 3.2 and Theorem 2.2.

Remark 4.2. Parabolic systems (1.1) with $D=$ diag demonstrate a finite-dimensional dynamics on the attractor for any admissible nonlinearities $f$ and $g$ in the case of scalar diffusion and under the condition $f=$ diag in the case of $m$ distinct diffusion coefficients $d_{j}$. In the case of $s$ distinct diffusion coefficients with $1<s<m$, the dynamics on the attractor is finite-dimensional under the condition that the matrix function $f$ inherits the block (with respect to the same $d_{j}$ ) structure of the matrix $D=\operatorname{diag}\left\{d_{j}\right\}$.

Now we formulate the main result. We assume that the matrix $D$ in system (1.1) has the form $D=C \bar{D} C^{-1}$, where the matrix $C$ is nondegenerate and $\bar{D}=\operatorname{diag}\left\{d_{1}, \ldots, d_{m}\right\}$ with $d_{j}>0$. The linear operator $-D \partial_{x x}=-C\left(\bar{D} \partial_{x x}\right) C^{-1}$ is sectorial in $X=L^{2}\left(J, \mathbb{R}^{m}\right)$. The change of variable $u=C v$ reduces (1.1) to the system of equations

$$
\begin{gather*}
\partial_{t} v=\bar{D} \partial_{x x} v+\bar{f}(x, v) \partial_{x} v+\bar{g}(x, v), \quad v(0)=v(1)=0, \\
\bar{f}(x, v)=C^{-1} f(x, C v) C, \quad \bar{g}(x, v)=C^{-1} g(x, C v) . \tag{4.2}
\end{gather*}
$$

The matrix function $\bar{f}$ and the vector function $\bar{g}$ inherit the regularity properties ( $\mathbf{H}$ ) of the original functions $f$ and $g$. The system of equations (4.2) is dissipative in $X^{\alpha}$, and hence, the same is also true for system (1.1). The attractors $\mathcal{A}$ of system (1.1) and $\overline{\mathcal{A}}$ of system (4.2) are related by the formula $\mathcal{A}=C \overline{\mathcal{A}}$. By the definition of the finite-dimensionality of the final phase dynamics (Section 1), systems (4.2) and (1.1) simultaneously demonstrate this property.

Theorem 4.3 (main theorem). If the matrix $D$ is similar to $\operatorname{diag}\left\{d_{j}\right\}$ for $d_{j}>0$ and consistency condition (1.2) is satisfied, then the final dynamics of system (1.1) is finite-dimensional.

Proof. Since $D f(x, u)=f(x, u) D$ on $J \times \operatorname{co} \mathcal{A}$, we have

$$
\begin{gathered}
\bar{D} \bar{f}(x, v)=C^{-1} D C \cdot C^{-1} f(x, C v) C=C^{-1} D f(x, u) C \\
\quad=C^{-1} f(x, u) D C=C^{-1} f(x, C v) C \bar{D}=\bar{f}(x, v) \bar{D}
\end{gathered}
$$

on $J \times \operatorname{co} \overline{\mathcal{A}}$. Here $u \in \operatorname{co} \mathcal{A}$ and $v \in \overline{\mathcal{A}}$. So we see that condition (1.2) is satisfied for the matrix function $\bar{f}$ and, by Theorem 4.1, the dynamics of system (4.2) on the attractor $\overline{\mathcal{A}} \subset X^{\alpha}$ is finite-dimensional. This also implies that the dynamics of system (1.1) is finite-dimensional on the attractor $\mathcal{A} \subset X^{\alpha}$.

Remark 4.4. Under consistency condition (1.2), the final dynamics of system (1.1) is finite-dimensional if all eigenvalues of the matrix $D$ are distinct and positive. In particular, condition (1.2) is satisfied for $f=D_{1} \varphi$, where the numerical matrix $D_{1}$ commutes with $D$ and $\varphi=\varphi(x, u)$ is a smooth scalar function finite in $u$.

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