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Matrix Capelli identities related to reflection equation algebra

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ABSTRACT

By using the notion of quantum double we introduce analogs of partial derivatives on a reflection equation algebra, associated with a Hecke symmetry of GL_N type. We construct the matrix $L = MD$, where M is the generating matrix of the reflection equation algebra and D is the matrix composed of the quantum partial derivatives and prove that the matrices M, D and L satisfy a matrix identity, called the matrix Capelli one. Upon applying quantum trace, it becomes a scalar relation, which is a far-reaching generalization of the classical Capelli identity. Also, we get a generalization of the some higher Capelli identities proved by A. Okounkov in [6].

1. Introduction

Let $M = \|m_i^j\|_{1\le i,j\le N}$ be a matrix with commutative entries and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be the matrix composed of the partial derivatives[1] $\partial_i^j = \partial/\partial m_j^i$. The famous Capelli identity reads

$$\mathrm{cdet}(MD + K) = \det M \det D, \tag{1.1}$$

where cdet is the so-called column-determinant and K is a diagonal matrix of the form: $K = \mathrm{diag}(N-1, N-2, \dots, 1, 0)$.

There are known many generalizations of this identity. We only mention the paper [5], where a quantum version of the Capelli identity was established, related to the Quantum Group (QG) $U_q(sl_N)$ and its dual algebra.

In the present note we exhibit another quantum version of the Capelli identity, which by contrast with [5] is related to Reflection Equation (RE) algebras. By definition, an RE algebra is a unital associative algebra $\mathcal{M}(R)$ generated by entries of the matrix $M = \|m_i^j\|_{1\le i,j\le N}$ subject to the following relation:

$$R\,M_1 R\,M_1 - M_1 R\,M_1 R = 0, \quad M_1 = M \otimes I, \tag{1.2}$$

where I is the unit matrix and R is a Hecke symmetry. The matrix M is called the *generating matrix* of the algebra $\mathcal{M}(R)$.

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[1] Note that $\partial_i^j m_k^s = \delta_i^s\,\delta_k^j$. Usually, in the Capelli identity one employs the matrix, transposed to our D.

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Let us precise that by a *Hecke symmetry* we mean a braiding, meeting the Hecke condition:

$$(q\, I\otimes I - R)(q^{-1} I\otimes I + R) = 0, \quad q\notin \{0, \pm 1\},$$

whereas by a *braiding* we mean a solution of the braid relation:

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R\otimes I, \quad R_{23} = I\otimes R.$$

Hereafter, R is treated to be an $N^2\times N^2$ numerical matrix.

The best known examples of the Hecke symmetries are those coming from the QG $U_q(sl_N)$. These Hecke symmetries are deformations of the usual flips P. Nevertheless, there exist other Hecke symmetries possessing this property (for instance, the Crammer-Gervais symmetries) as well as those which are not deformations of the usual (or super-)flips.

We impose two additional requirements on the Hecke symmetry R: it should be skew-invertible and even (see [2]). In such a case R will be called the GL_N type Hecke symmetry. Note that if R is a GL_N type Hecke symmetry, then for generating matrix M of the RE algebra $\mathcal{M}(R)$ one can define the quantum (or R-)trace $\mathrm{Tr}_R M$ and the quantum determinant $\det_R M$.

Besides, for any GL_N type Hecke symmetry R we define analogs of the partial derivatives ∂_i^j in such a way that the matrix $L = MD$, where $D = \|\partial_i^j\|_{1\le i,j\le N}$, meets the relation:

$$R\, L_1 R\, L_1 - L_1 R\, L_1 R = R\, L_1 - L_1 R. \tag{1.3}$$

An algebra $\hat{\mathcal{L}}(R)$, generated by entrees of the matrix $L = \|l_i^j\|_{1\le i,j\le N}$ is called a *modified RE algebra*. Note that as $R\to P$, the algebra $\mathcal{M}(R)$ tends to $\mathrm{Sym}(gl_N)$, whereas the algebra $\hat{\mathcal{L}}(R)$ tends to $U(gl_N)$. This is one of the reasons why we consider the algebras $\mathcal{M}(R)$ (resp., $\hat{\mathcal{L}}(R)$) for any GL_N type Hecke symmetry R as a quantum (or q-)analog of $\mathrm{Sym}(gl_N)$ (resp., $U(gl_N)$).

Note that for the generating matrix L of the algebra $\hat{\mathcal{L}}(R)$ the quantum trace $\mathrm{Tr}_R L$ and the quantum determinant $\det_R L$ are defined in the same way as for the matrix M. Namely, the quantum determinant of $L = MD$ with a proper shift enters our quantum Capelli identity. It should be emphasized that this identity is valid for the whole class of RE algebras $\mathcal{M}(R)$, associated with GL_N type Hecke symmetries R. Note that if $R\to P$ in the limit $q\to 1$ our quantum Capelli identity turns into the classical one expressed as in [6].

The note is organized as follows. In section 2 we exhibit the quantum double (QD) construction enabling us to introduce q-analogs of the partial derivatives in the entries of the matrix M. In Theorem 3 we present the matrix factorization identities which are called the *matrix Capelli identities*. Upon applying the R-trace, they turn into a quantum version of the Capelli identity and some its generalizations which are the quantum counterparts of the *higher Capelli identities* (see Theorem in [6]), corresponding to one-column and one-row Young diagrams. In section 3 we give a proof of these identities. In section 4 we reduce the Capelli identity to a more conventional form, based on the use of quantum determinants. Also, we compare our version of the Capelli identity with that from the article [5].

2. Quantum partial derivatives and matrix Capelli identities

In this section we deal with a skew-invertible Hecke symmetry R without assuming it to be even.

Consider two unital associative algebras A and B equipped with an invertible linear map $\sigma : A\otimes B\to B\otimes A$ which satisfies the following relations:

$$\sigma\circ(\mu_A\otimes \mathrm{id}_B) = (\mathrm{id}_B\otimes \mu_A)\circ\sigma_{12}\circ\sigma_{23} \quad \text{on} \quad A\otimes A\otimes B,$$
$$\sigma\circ(\mathrm{id}_A\otimes \mu_B) = (\mu_B\otimes \mathrm{id}_A)\circ\sigma_{23}\circ\sigma_{12} \quad \text{on} \quad A\otimes B\otimes B,$$
$$\sigma(1_A\otimes b) = b\otimes 1_A, \quad \sigma(a\otimes 1_B) = 1_B\otimes a \qquad \forall a\in A,\ \forall b\in B,$$

where $\mu_A : A\otimes A\to A$ is the product in the algebra A, 1_A is its unit, and similarly for B. We call the data (A, B, σ) a quantum double, if the map σ is defined in terms of a braiding R (see [3] for more detail).

Also, the map σ defines permutation relations $a\otimes b = \sigma(a\otimes b)$, $a\in A$, $b\in B$ and due to this fact σ is referred to as the *permutation map*. If the algebra A is equipped with a counit $\varepsilon : A\to C$, then it becomes possible to define an action of the algebra A onto B.

Below we deal with the QD (A, B, σ), where $B = \mathcal{M}(R)$ with the generating matrix M obeying (1.2), the algebra $A = \mathcal{D}(R^{-1})$ is the RE algebra with the generating matrix $D = \|\partial_i^j\|$ satisfying the relation[2]

$$R^{-1} D_1 R^{-1} D_1 - D_1 R^{-1} D_1 R^{-1} = 0 \tag{2.1}$$

and the permutation map is

[2] The matrix R^{-1} is also a Hecke symmetry but with q replaced by q^{-1}.

$$\sigma: \quad D_1 R\, M_1 R \to R\, M_1 R^{-1} D_1 + R\, 1_B 1_A.$$

Below we omit the factors 1_A and 1_B. The corresponding permutation relations can be written in the form:

$$D_1 R\, M_1 = R\, M_1 R^{-1} D_1 R^{-1} + I. \tag{2.2}$$

Remark 1. The quantum double $(\mathcal{D}(R^{-1}), \mathcal{M}(R), \sigma)$ with the permuttation relations (2.2) was obtained in [1] from the representation theory of the RE algebra.

The permutation relations (2.2) are compatible with the associative structures of the both algebras $\mathcal{M}(R)$ and $\mathcal{D}(R^{-1})$. To prove this we introduce the matrix notation:

$$M_{\overline{1}} = M_{\underline{1}} = M_1, \qquad M_{\overline{i+1}} = R_i M_{\overline{i}} R_i^{-1}, \qquad M_{\underline{i+1}} = R_i^{-1} M_{\underline{i}} R_i, \quad i \ge 1,$$

where $R_i := R_{i\,i+1} := I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(p-i-1)}$ is an embedding of R into the space of $N^p \times N^p$ matrices for any $p \ge i+1$. Then the braid relation on R allows one to prove the equivalence of two forms of defining relations of RE algebra $\mathcal{M}(R)$:

$$R\, M_1 R\, M_1 - M_1 R\, M_1 R = 0 \quad \Leftrightarrow \quad R\, M_1 M_{\overline{2}} - M_1 M_{\overline{2}}\, R = 0.$$

Note that the defining relations of RE algebra can be also written in terms of any higher copies of matrix M:

$$R_p\, M_{\overline{p}}\, M_{\overline{p+1}} - M_{\overline{p}}\, M_{\overline{p+1}}\, R_p = 0, \qquad \forall\, p \ge 1. \tag{2.3}$$

By a straightforward calculation with the use of (2.2) we find:

$$D_1 \left(R_2 M_{\overline{2}} M_{\overline{3}} - M_{\overline{2}} M_{\overline{3}} R_2\right) = \left(R_2 M_{\overline{2}} M_{\overline{3}} - M_{\overline{2}} M_{\overline{3}} R_2\right) D_1 R_1^{-1} R_2^{-2} R_1^{-1}. \tag{2.4}$$

The relation (2.4) entails that the defining ideal of the algebra $\mathcal{M}(R)$ is preserved by the permutation relations. In a similar way it is possible to check that the defining ideal of the algebra $\mathcal{D}(R^{-1})$ is also preserved by the permutation relations.

In order to get an action of the algebra $\mathcal{D}(R^{-1})$ onto $\mathcal{M}(R)$ we introduce a counit in the algebra $A = \mathcal{D}(R^{-1})$ in the classical way:

$$\varepsilon(1_A) = 1_C, \quad \varepsilon(\partial_i^j) = 0 \quad \forall\, i, j, \quad \varepsilon(a_1 a_2) = \varepsilon(a_1)\, \varepsilon(a_2) \quad \forall\, a_1, a_2 \in A.$$

With this counit the action of ∂_i^j on the generators m_s^k reads:

$$D_1 \triangleright M_{\overline{2}} = R_{12}^{-1}.$$

The permutation relations (2.2) together with the counit map allow one to extend this action on the whole algebra $\mathcal{M}(R)$. The elements ∂_i^j with the above action are treated as the quantum analogs of the usual partial derivatives in the commutative variables m_i^j. As was mentioned above, this action is compatible with the algebraic structure of $\mathcal{M}(R)$ (see (2.4)). However, below we do not use the operator treatment of the quantum partial derivatives ∂_i^j.

Remark 2. If R is a Hecke symmetry coming from $U_q(sl_N)$ (the so-called Drinfeld-Jimbo R-matrix) then at the classical limit $q \to 1$ the permutation relations (2.2) turns into the usual Leinbniz rule for the commutative partial derivatives $\partial_i^j = \partial/\partial m_j^i$.

With any Hecke symmetry R we associate the idempotents $A^{(k)}$ and $S^{(k)}$ called the R-skew-symmetrizers and R-symmetrizers respectively. They are defined by the following recursion:

$$A^{(1)} = I, \quad A^{(k)}_{1\dots k} = \frac{1}{k_q} A^{(k-1)}_{1\dots k-1} \left(q^{(k-1)}\, I^{\otimes k} - (k-1)_q\, R_{k-1}\right) A^{(k-1)}_{1\dots k-1}, \quad k \ge 2. \tag{2.5}$$

$$S^{(1)} = I, \quad S^{(k)}_{1\dots k} = \frac{1}{k_q} S^{(k-1)}_{1\dots k-1} \left(q^{-(k-1)}\, I^{\otimes k} + (k-1)_q\, R_{k-1}\right) S^{(k-1)}_{1\dots k-1}, \quad k \ge 2.$$

If R is a GL_N type Hecke symmetry, then $\dim \mathrm{Im}\, A^{(N)} = 1$ and $A^{(N+1)} \equiv 0$.

Now we are ready to formulate the main result of the paper. We establish a series of matrix factorization identities which leads to the quantum versions of the Capelli identity and some its generalizations called by A. Okounkov the "higher Capelli identities" in [6].

Theorem 3. *Let $L = MD$, where M and D are the generating matrices of the algebras $\mathcal{M}(R)$ and $\mathcal{D}(R^{-1})$ from the quantum double defined by* (1.2), (2.1) *and* (2.2)*. Then the following matrix factoriazation identities take place for* $\forall\, k \ge 1$*:*

$$A^{(k)} L_{\overline{1}} (L_{\overline{2}} + qI) \dots (L_{\overline{k}} + q^{k-1}(k-1)_q I\,)\, A^{(k)} = q^{k(k-1)} A^{(k)} M_{\overline{1}} \dots M_{\overline{k}}\, D_{\overline{k}} \dots D_{\overline{1}} \tag{2.6}$$

$$S^{(k)} L_{\overline{1}} \left(L_{\overline{2}} - \frac{1}{q} I \right) \dots \left(L_{\overline{k}} - \frac{(k-1)_q}{q^{k-1}} I \right) S^{(k)} = q^{-k(k-1)} S^{(k)} M_{\overline{1}} \dots M_{\overline{k}}\, D_{\overline{k}} \dots D_{\overline{1}}. \tag{2.7}$$

Observe that Theorem 3 is valid for *any* skew-invertible Hecke symmetry R. In the case when R is a GL_N type symmetry, the right hand side of (2.6) for $k = N$ can be presented as the product of quantum determinants of the matrices M and D (see the last section).

Definition 4. Let M be the generating matrix of an RE algebra $\mathcal{M}(R)$. The quantities[3]

$$e_k(M) = \langle A^{(k)}\, M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{k}} \rangle_{1\dots k}$$

are called the elementary (q-)symmetric polynomials in the matrix M.

By definition, the quantum determinants of the matrices M and D are proportional to the highest elementary symmetric polynomials e_N (similarly to the classical matrix analysis):

$$\det{}_R M := q^{N^2} \langle A^{(N)}\, M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{N}} \rangle_{1\dots N}, \quad \det{}_{R^{-1}} D := q^{N^2} \langle A^{(N)}\, D_{\overline{N}} \dots D_{\overline{1}} \rangle_{1\dots N}. \tag{2.8}$$

The normalizing factior q^{N^2} is introduced to simplify the formulae below. Note that in the definition of $\det_{R^{-1}} D$ the inverse order of the matrix copies $D_{\overline{k}}$ is used. This is motivated by the relations (2.1) imposed on D.

So, as a corollary of Theorem 3, we have the following version of the generalized quantum Capelli identities.

Corollary 5. *Under the assumption of Theorem 3 the following identities hold for* $\forall\, k \ge 1$*:*

$$\langle A^{(k)} L_{\overline{1}} (L_{\overline{2}} + qI) \dots (L_{\overline{k}} + q^{k-1}(k-1)_q I\,) \rangle_{1\dots k} = q^{k(k-1)} \langle A^{(k)} M_{\overline{1}} \dots M_{\overline{k}}\, D_{\overline{k}} \dots D_{\overline{1}} \rangle_{1\dots k} \tag{2.9}$$

$$\langle S^{(k)} L_{\overline{1}} \left(L_{\overline{2}} - \frac{1}{q} I \right) \dots \left(L_{\overline{k}} - \frac{(k-1)_q}{q^{k-1}} I \right) \rangle_{1\dots k} = q^{-k(k-1)} \langle S^{(k)} M_{\overline{1}} \dots M_{\overline{k}}\, D_{\overline{k}} \dots D_{\overline{1}} \rangle_{1\dots k}. \tag{2.10}$$

Formulae (2.9) and (2.10) are generalizations of the higher Capelli identities from [6], corresponding to one-column and one-row Young diagrams respectively.

Corollary 6. *Under the assumption of Theorem 3 the following quantum Capelli identity holds true:*

$$\langle A^{(N)} L_{\overline{1}} (L_{\overline{2}} + qI) \dots (L_{\overline{N}} + q^{N-1}(N-1)_q I) \rangle_{1\dots N} = q^{-N} \det{}_R M \, \det{}_{R^{-1}} D. \tag{2.11}$$

In the last section we consider the quantum determinants in more detail and complete the proof of this Capelli identity.

3. Proof of Theorem 3

We only prove the identity (2.6). The identity (2.7) can be proven in the same way.

Let us apply the induction in k. The base of induction for $k = 1$ is tautological. We assume the identity (2.6) to be true up to $k - 1$ for some integer $k \ge 2$. Consider the matrix:

$$F(\alpha) = A^{(k)} L_1 (L_{\overline{2}} + q\, I)(L_{\overline{3}} + q^2\, 2_q\, I) \dots (L_{\overline{k-1}} + q^{k-2}\, (k-2)_q\, I)(L_{\overline{k}} + \alpha\, I) A^{(k)},$$

where α is a numerical parameter.

Since $A^{(k-1)}$ is a polynomial in R_i for $i \le k-2$, then $L_{\overline{k}} A^{(k-1)} = A^{(k-1)} L_{\overline{k}}$ as a consequence of the braid relation on R. Using this fact as well as the identity $A^{(k)} = A^{(k)} A^{(k-1)} = A^{(k-1)} A^{(k)}$, we can rewrite $F(\alpha)$ in the form:

$$F(\alpha) = A^{(k)} \underline{A^{(k-1)} L_1 (L_{\overline{2}} + q\, I) \dots (L_{\overline{k-1}} + q^{k-2}(k-2)_q\, I) A^{(k-1)}} (L_{\overline{k}} + \alpha\, I)\, A^{(k)}.$$

---

[3] Hereafter, we use the notation $\langle X \rangle_{1\dots k} := \mathrm{Tr}_{R(1\dots k)} X := \mathrm{Tr}_{R(1)} \dots \mathrm{Tr}_{R(k)} X$, where X is an $N^k \times N^k$ matrix.

We transform the underlined expression in accordance with the induction hypothesis and get:

$$F(\alpha)=q^{(k-1)(k-2)}A^{(k)}M_1\dots M_{\overline{k-1}}D_{\overline{k-1}}\dots D_1(L_{\overline{k}}+\alpha I)A^{(k)}. \tag{3.1}$$

It remains to check that for $\alpha=q^{k-1}(k-1)_q$ the expression $F(\alpha)$ turns into the right hand side of (2.6). Expanding the brackets in (3.1) we obtain:

$$q^{-(k-1)(k-2)}F(\alpha)=A^{(k)}M_1\dots M_{\overline{k-1}}D_{\overline{k-1}}\dots D_1L_{\overline{k}}A^{(k)}+\alpha A^{(k)}M_1\dots M_{\overline{k-1}}D_{\overline{k-1}}\dots D_1A^{(k)}.$$

Now, in the first summand we permute step by step all factors $D_{\overline{i}}$ with the element $L_{\overline{k}}$. Taking into account that $L_{\overline{k}}=R_{k-1\to 1}L_1R^{-1}_{1\to k-1}$, where $R^{\pm}_{1\to m}:=R^{\pm}_1\dots R^{\pm}_m$ (and similarly for $R^{\pm}_{m\to 1}$), we find at the first step:

$$D_1L_{\overline{k}}=D_1R_{k-1\to 1}M_1D_1R^{-1}_{1\to k-1}=R_{k-1\to 2}D_1R_1M_1D_1R^{-1}_{1\to k-1}.$$

In the last expression we replace the product $D_1R_1M_1$ with the use of (2.2):

$$R_{k-1\to 2}\underline{D_1R_1M_1}D_1R^{-1}_{1\to k-1}=R_{k-1\to 1}M_1\underline{R_1^{-1}D_1R_1^{-1}D_1}R^{-1}_{1\to k-1}+D_1\,R_{k-1\to 2}\,R^{-1}_{1\to k-1},$$

then we change $R_1^{-1}D_1R_1^{-1}D_1$ for $D_1R_1^{-1}D_1R_1^{-1}$ according to (2.1) and finally get:

$$D_1L_{\overline{k}}=L_{\overline{k}}D_1R_{k-1\to 2}R_1^{-2}R^{-1}_{2\to k-1}+D_1R_{k-1\to 2}R_1^{-1}R^{-1}_{2\to k-1}.$$

So, the first summand in the above expression for $q^{-(k-1)(k-2)}F(\alpha)$ takes the form:

$$\begin{aligned}&A^{(k)}M_1\ \dots\ M_{\overline{k-1}}D_{\overline{k-1}}\dots D_1L_{\overline{k}}A^{(k)}=\\&(-q)^2\ A^{(k)}\ M_1\dots M_{\overline{k-1}}D_{\overline{k-1}}\dots D_2L_{\overline{k}}D_1A^{(k)}+(-q)\ A^{(k)}M_1\dots M_{\overline{k-1}}D_{\overline{k-1}}\dots D_1A^{(k)}.\end{aligned}$$

To get this expression, we "evaluate" the chains of R-matrices on the rightmost R-skew-symmetrizer $A^{(k)}$ in accordance with the rules:

$$A^{(k)}R_i^{\pm 1}=R_i^{\pm 1}A^{(k)}=-q^{\mp 1}A^{(k)},\qquad 1\le\forall i\le k-1. \tag{3.2}$$

At the second step we permute $D_{\overline{2}}$ and $L_{\overline{k}}$. In the same way as above we find:

$$D_{\overline{2}}L_{\overline{k}}=L_{\overline{k}}D_{\overline{2}}R_{k-1\to 3}R_2^{-2}R^{-1}_{3\to k-1}+D_{\overline{2}}R_{k-1\to 3}R_2^{-1}R^{-1}_{3\to k-1}.$$

Note that all R-matrices in this formula commute with D_1 and therefore they can be moved to the right $A^{(k)}$ and converted to powers of q according to (3.2).

By induction in p one can prove the general formula:

$$D_{\overline{p}}L_{\overline{k}}=L_{\overline{k}}D_{\overline{p}}R_{k-1\to p+1}R_p^{-2}R^{-1}_{p+1\to k-1}+D_{\overline{p}}R_{k-1\to p+1}R_p^{-1}R^{-1}_{p+1\to k-1}.$$

Here also all terms $R_i^{\pm 1}$ $i\ge p$ commute with $D_{\overline{p-1}}\dots D_1$ and can be evaluated at the R-skew-symmetrizer $A^{(k)}$.

Finally, we get the following formula:

$$\begin{aligned}q^{-(k-1)(k-2)}F(\alpha)&=q^{2(k-1)}A^{(k)}M_1\dots M_{\overline{k-1}}M_{\overline{k}}D_{\overline{k}}D_{\overline{k-1}}\dots D_1A^{(k)}\\&+(\alpha-q-q^3-\dots-q^{2k-3})A^{(k)}M_1\dots M_{\overline{k-1}}D_{\overline{k-1}}\dots D_{\overline{1}}A^{(k)},\end{aligned}$$

where we substituted $L_{\overline{k}}=M_{\overline{k}}D_{\overline{k}}$.

At last, by setting $\alpha=q+q^3+\dots+q^{2k-3}=q^{k-1}(k-1)_q$ we kill the second term and get:

$$F(q^{k-1}(k-1)_q)=q^{k(k-1)}A^{(k)}M_1\dots M_{\overline{k-1}}M_{\overline{k}}D_{\overline{k}}D_{\overline{k-1}}\dots D_1A^{(k)}. \tag{3.3}$$

To complete the proof it remains to note that due to algebraic relations (2.3) the R-skew-symmetrizer $A^{(k)}$ commute with the chain of M-matrices

$$A^{(k)}M_1M_{\overline{2}}\dots M_{\overline{k}}=M_1M_{\overline{2}}\dots M_{\overline{k}}A^{(k)},$$

and the same is true for the corresponding chain of D matrices. Since $A^{(k)}A^{(k)}=A^{(k)}$, then in the right hand side of (3.3) one can leave only one element $A^{(k)}$:

$$A^{(k)}M_1\dots M_{\overline{k}}D_{\overline{k}}\dots D_1A^{(k)}\equiv A^{(k)}M_1\dots M_{\overline{k}}D_{\overline{k}}\dots D_1\equiv M_1\dots M_{\overline{k}}D_{\overline{k}}\dots D_1\,A^{(k)}.$$

This completes the inductive proof of (2.6).

4. Some aspects of quantum determinants

It should be emphasized that the order m of the highest non-trivial skew-symmetrizer $A^{(m)}$ can be different from N, where $N^2 \times N^2$ is the matrix size of R.

Definition 7. We say that a skew-invertible Hecke symmetry R is of rank m if the R-skew-symmetrizers (2.5) satisfy the condition:

$$\dim \mathrm{Im}\, A^{(m)}(R) = 1, \qquad A^{(m+1)}(R) \equiv 0.$$

Note that Corollary 6 remains valid, if in (2.11) we replace N by m assuming the initial GL_N type Hecke symmetry R to be of rank m.

Since $A^{(m)}$ is an idempotent and $\dim \mathrm{Im}\, A^{(m)} = 1$, there exist two tensors $|u\rangle = \|u_{i_1 i_2 \dots i_m}\|$ and $\langle v| = \|v^{i_1 i_2 \dots i_m}\|$ such that

$$A^{(m)}{}_{i_1\dots i_m}^{j_1\dots j_m} = u_{i_1\dots i_m} v^{j_1\dots j_m} \qquad \text{and} \qquad \sum_i v^{i_1\dots i_m} u_{i_1\dots i_m} = 1.$$

Using the above "bra" and "ket" notations, we can present these formulae as follows:

$$A^{(m)} = |u\rangle\langle v| \qquad \text{and} \qquad \langle v|u\rangle = 1. \tag{4.1}$$

The quantum determinant is defined as in (2.8) but with N replaced by m:

$$\det{}_R M = q^{m^2} \langle A^{(m)} M_1 M_{\overline{2}} \dots M_{\overline{m}} \rangle_{1\dots m}.$$

With the use of (4.1) we can prove the following *matrix* identity:

$$\begin{aligned} A^{(m)} M_1 \dots M_{\overline{m}} &= A^{(m)} M_1 \dots M_{\overline{m}} A^{(m)} \\ &= |u\rangle\langle v| M_1 \dots M_{\overline{m}} |u\rangle\langle v| = A^{(m)} \langle v| M_1 \dots M_{\overline{m}} |u\rangle. \end{aligned} \tag{4.2}$$

Upon calculating the R-trace over all spaces and taking into account that $\langle A^{(m)} \rangle_{1\dots m} = q^{-m^2}$ (see [2]), we find that the quantum determinant is actually given by the *usual* trace of the form:

$$\det{}_R M = \langle v| M_1 M_{\overline{2}} \dots M_{\overline{m}} |u\rangle := \mathrm{Tr}_{(1\dots m)} (A^{(m)} M_1 M_{\overline{2}} \dots M_{\overline{m}}).$$

As for the quantum determinant of the matrix D we have:

$$\det{}_{R^{-1}} D := q^{m^2} \langle A^{(m)} D_{\overline{m}} \dots D_{\overline{2}} D_1 \rangle_{1\dots m} = \langle v| D_{\overline{m}} \dots D_{\overline{2}} D_1) |u\rangle.$$

Note that this quantum determinant is defined with the same tensors $|u\rangle$ and $\langle v|$ though the matrix D is subject to the RE with R replaced by R^{-1}. It can be explained by the fact that all skew-symmetrizers are invariant with respect to the replacement $R \to R^{-1}$ and $q \to q^{-1}$.

Consider now the identity (2.6) for the Hecke symmetry of rank m. With the use of (4.2) the matrix structure of the right hand side of (2.6) for $k = m$ can be transformed as follows:

$$q^{m(m-1)} A^{(m)} M_1 \dots M_{\overline{m}} D_{\overline{m}} \dots D_1 = q^{m(m-1)} A^{(m)} \det{}_R M \det{}_{R^{-1}} D.$$

Finally, by calculating the R-trace over all spaces of the both sides of (2.6), we come to the desired form (2.11) of the right hand side of the quantum Capelli identity:

$$\langle A^{(m)} L_{\overline{1}} (L_{\overline{2}} + qI) \dots (L_{\overline{m}} + q^{m-1} (m-1)_q I) \rangle_{1\dots m} = q^{-m} \det{}_R M \det{}_{R^{-1}} D.$$

This completes the proof of Corollary 6.

As follows from the results of [4], if a given Hecke symmetry R is a deformation of the usual flip P, each of the determinants entering the right hand side of the (2.11) can be written as column-determinant or row-determinant. We do not know whether it is possible to do the same with the left hand side of (2.11). Also observe, that if an *involutive symmetry* R (i.e. such that $R^2 = I$) is a limit of a Hecke symmetry $R(q)$ as $q \to 1$, the corresponding Capelli identity can be obtained from (2.11) by setting $q = 1$. Thus, it looks like the Capelli identity from [6], but the skew-symmetrizers and quantum determinants should be adapted to $R = R(1)$.

At conclusion, we want to shortly compare our result for quantum Capelli identity with that of the paper [5]. The authors of that paper deal with another quantum version of the Capelli identity, related to the QG $U_q(sl_N)$ and the corresponding RTT algebra. As for our results, we are working with quite different quantum algebra — the RE algebra and different quantum derivatives. Besides, we do not restrict ourselves with the $U_q(sl_N)$ R-matrix, our results are valid for the wide class of RE algebras defined via *arbitrary* skew-invertible Hecke symmetries.

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