



Quantum doubles of Fock type and bosonization

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ABSTRACT

We introduce analogs of creation and annihilation operators, related to involutive and Hecke symmetries R , and perform bosonic and fermionic realization of the modified Reflection Equation algebras in terms of the so-called Quantum Doubles of Fock type. Also, we introduce Quantum Doubles of Fock type, associated with Birman-Murakami-Wenzl symmetries coming from orthogonal or symplectic Quantum Groups and exhibit the algebras obtained by means of the corresponding bosonization (fermionization). Besides, we apply this scheme to current braidings arising from Hecke symmetries R via the Baxterization procedure.

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1. Introduction

By an associative double we mean a couple (A, B) of associative algebras A and B endowed with the so-called permutation map

$$\sigma : A \otimes B \rightarrow B \otimes A, \quad (1.1)$$

which satisfies certain requirements (see Section 3).

By a quantum double (QD) we mean an associative double where the map σ is defined by means of a braiding – constant or current (i.e. depending on spectral parameters), different from a (super-)flip. A typical example is the so-called Heisenberg double constructed of an RTT algebra and a reflection equation (RE) one, associated with the same braiding R . A number of other examples of QD can be found in [7].

By a bosonic QD of Fock type we mean a double composed of the algebras $\text{Sym}_R(V)$ and $\text{Sym}_R(V^*)$, where $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ is a braiding, while $\text{Sym}_R(V)$ and $\text{Sym}_R(V^*)$ are R -analogs of the symmetric algebras of a vector space V and of its dual V^* . Let us precise that by a braiding $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ we mean a solution of the braid relation

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$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

where $R_{12} = (R \otimes I)$, $R_{23} = (I \otimes R)$ and I stands for the identity operator.

A fermionic version of a QD of Fock type can be defined in a similar way by using R -analogs $\Lambda_R(V)$ and $\Lambda_R(V^*)$ of the skew-symmetric algebras $\Lambda(V)$ and $\Lambda(V^*)$.

In the classical case, i.e. when R is the usual flip P , the algebra $\text{Sym}(V)$ often plays the role of a Fock space. Consequently, its generators give rise to creation operators, whereas generators of the algebra $\text{Sym}(V^*)$ give rise to annihilation ones. The permutation relations between the generators of these two algebras allow one to define an action of the annihilation operators on the algebra $\text{Sym}(V)$ as partial derivatives. These two algebras form a classical version of a Fock double. By combining the annihilation and creation operators in a proper way we can perform a bosonic realization (or simply, a bosonization) of the Lie algebra $gl(N)$ and consequently its enveloping algebra $U(gl(N))$.

In the present paper we construct R -analogs of these well-known objects, provided braidings R come from the quantum groups (QG) $U_q(\mathfrak{g})$, corresponding to the classical simple Lie algebras \mathfrak{g} . More precisely, the role of such a braiding R is played by the composition of the image of the universal R -matrix in $V^{\otimes 2}$ and the usual flip (here V is the first fundamental space, called basic).

If $\mathfrak{g} \in A_n$, then R is a Hecke symmetry. Recall that by a Hecke symmetry we mean a braiding R satisfying an additional relation

$$(R - qI)(R + q^{-1}I) = 0, \quad q \neq \pm 1.$$

If \mathfrak{g} belongs to other series of classical Lie algebras, then R is a Birman-Murakami-Wenzl symmetry (see Section 4). For all corresponding QG $U_q(\mathfrak{g})$ the R -symmetric algebra $\text{Sym}_R(V)$ and R -skew-symmetric algebras $\Lambda_R(V)$ of the space V are well defined. Similar algebras for the dual space V^* are also well defined. Hopefully, all these algebras are deformations of their classical counterparts (see Remark 6). The first aim of the present paper is to construct the QD from the above algebras.

The second aim consists in studying the following problem: which algebras can be constructed by means of the bosonization (or the fermionization) procedure in a way similar to the classical one. It should be emphasized that the final algebras are not the corresponding Quantum Groups themselves but they are covariant with respect to their actions.¹ Moreover, the final algebras differ drastically from each other for different classical series. If R comes from the QG $U_q(sl(N))$ (in what follows we call it the standard symmetry), we get the so-called modified reflection equation algebra $\mathcal{L}(R)$, which is generated by the unit² and elements l_i^j , subject to the following system of relations:

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} = R_{12}L_1 - L_1R_{12}, \quad L = \|l_i^j\|_{1 \leq i, j \leq N}, \quad L_1 = L \otimes I. \tag{1.2}$$

This algebra is a deformation of the universal enveloping algebra $U(gl(N))$. Observe that the algebra $\mathcal{L}(R)$ itself can be treated as an enveloping algebra of a generalized (or braided) Lie algebra defined in the space $\text{End}(V)$ of internal endomorphisms. Moreover, a similar algebra $\mathcal{L}(R)$ can be associated with any Hecke symmetry R (or with an involutive symmetry, i.e. such that $R^2 = I$), provided R is skew-invertible (see the next section for definition). A big family of such involutive and Hecke symmetries has been constructed in [1]. For each of these symmetries the corresponding algebra $\mathcal{L}(R)$ can be constructed by means of the bosonization or the fermionization procedures and can be given a meaning of the enveloping algebra of a generalized Lie algebra. Moreover, $\mathcal{L}(R)$ can be endowed with the trace Tr_R , which is coordinated with the mentioned generalized Lie algebra structure.

Especially we are interested in the following question: to what extent the above construction, associated with Hecke symmetries R , can be generalized to the case, when R is a BMW symmetry coming from the orthogonal or symplectic QG. In this case we construct some QD associated with such symmetries and introduce algebras arising from the bosonization or fermionization procedure. It turns out that the final algebras differ from $\mathcal{L}(R)$ and do not have any meaning of enveloping algebras. This result shows that the BMW symmetries are not well adapted to constructing Lie-algebraic objects. By contrast, they fit well for introducing group-like objects. We refer the reader to the end of Section 4, where we discuss this phenomenon.

Besides, we apply a similar scheme to the case when R is a current braiding constructed by means of the so-called Baxterization procedure from an involutive or Hecke symmetry. We construct the QD of Fock type in the spirit of the Zamolodchikov-Faddeev algebras and exhibit the algebra arising from the corresponding bosonic realization. It turns out that this algebra is inhomogeneous counterpart of the braided Yangians, introduced in [2].

The paper is organized as follows. In section 2 we recall some basic notions and constructions, related to braidings. In section 3 we introduce the corresponding QD of Fock type and perform the bosonic and fermionic realizations of the algebras $\mathcal{L}(R)$. In section 4 we consider the algebras, associated with BMW symmetries coming from the QG of the B_n , C_n and D_n series. In section 5 we exhibit QD of Zamolodchikov-Faddeev type as well as the corresponding algebras obtained by the bosonization procedure.

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¹ In this connection we want to remark that there are numerous papers (we only mention the pioneering paper [12]), where the QG are realized via the q -counterparts of harmonic oscillators.

² All algebras below are assumed to be unital.

2. Hecke symmetries and corresponding algebras

Let $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ be a braiding. By a successive application of R it is possible to transpose arbitrary tensor powers $V^{\otimes k}$ and $V^{\otimes l}$, that is to define a linear map

$$V^{\otimes k} \otimes V^{\otimes l} \rightarrow V^{\otimes l} \otimes V^{\otimes k}, \quad k, l \in \mathbb{N}.$$

However, we also need a transposition rule of the space V and its dual V^* . For this purpose we should extend the action of the operator R on spaces $V \otimes V^*$ and $V^* \otimes V$. A method of defining such an extension belongs to V. Lyubashenko [10,11].

We fix a basis $\{x_i\}_{1 \leq i \leq N}$ of the space V and represent the operator R in the corresponding basis of the space $V^{\otimes 2}$ by an $N^2 \times N^2$ matrix $\|R_{ij}^{kl}\|$:

$$R(x_i \otimes x_j) = R_{ij}^{kl} x_k \otimes x_l.$$

A summation over the repeated indices is always understood.

Let us fix a nondegenerate bilinear form $\langle, \rangle_r : V \otimes V^* \rightarrow \mathbb{C}$ and choose a *right dual* basis³ $\{x^i\}_{1 \leq i \leq N}$ of V^* with respect to this form: $\langle x_i, x^j \rangle_r = \delta_i^j$. The pairing of the spaces $V^{\otimes 2}$ and $(V^*)^{\otimes 2}$ is defined by

$$\langle x_i \otimes x_j, x^k \otimes x^l \rangle_r = \langle x_i, x^l \rangle_r \langle x_j, x^k \rangle_r.$$

The operator R on the space $(V^*)^{\otimes 2}$ is defined as an operator adjoint to R , and its action on the right dual basis vectors is given by the rule:

$$R(x^i \otimes x^j) = R_{lk}^{ji} x^k \otimes x^l.$$

We keep the notation R for this adjoint operator.

A braiding R is called *skew-invertible* if there exists an operator $\Psi : V^{\otimes 2} \rightarrow V^{\otimes 2}$ such that

$$\text{Tr}_2 R_{12} \Psi_{23} = P_{13} \Leftrightarrow R_{ij}^{kl} \Psi_{lm}^{jn} = \delta_m^k \delta_i^n.$$

As usual, the bottom indexes in the notation like Ψ_{23} indicate the position of components of a tensor product where the operator acts. The aforementioned extension of the braiding R on the spaces $V \otimes V^*$ and $V^* \otimes V$ reads (see [6] for detail):

$$R(x_i \otimes x^j) = x^k \otimes x_l (R^{-1})_{ki}^lj, \quad R(x^i \otimes x_j) = x_l \otimes x^k \Psi_{kj}^{li}.$$

If R is skew-invertible we can define the following operators (matrices)

$$B = \text{Tr}_1 \Psi_{12} \Leftrightarrow B_i^j = \Psi_{ki}^{kj}, \quad C = \text{Tr}_2 \Psi_{12} \Leftrightarrow C_i^j = \Psi_{ik}^{jk}. \tag{2.1}$$

If the matrices B and C are nonsingular, the symmetry R is called *strictly skew-invertible*.

Introduce now the second bilinear form $\langle, \rangle_l : V^* \otimes V \rightarrow \mathbb{C}$ as the composition of the linear maps:

$$\langle, \rangle_l = \langle, \rangle_r \circ R.$$

Then we can introduce a *left dual* basis $\{\tilde{x}^i\}_{1 \leq i \leq N}$ of V^* by the requirement

$$\langle \tilde{x}^i, x_j \rangle_l = \delta_j^i, \quad 1 \leq \forall i, j \leq N.$$

If R is strictly skew-invertible, then it is not difficult to establish the following relations

$$\langle x^j, x_i \rangle_l = B_i^j, \quad \langle x_i, \tilde{x}^j \rangle_r = (B^{-1})_i^j. \tag{2.2}$$

Note that in the standard case the pairings \langle, \rangle_r and \langle, \rangle_l are $U_q(\mathfrak{sl}(N))$ -covariant.

Now, consider the space of the internal endomorphisms

$$\text{End}(V) \cong V \otimes V^* \tag{2.3}$$

equipped with the basis $l_i^j = x_i x^j$. Hereafter, we omit the sign \otimes , if it does not lead to misunderstanding.

By using the identification (2.3), we get the following multiplication table in this basis

$$l_i^j \circ l_k^m = B_k^j l_i^m.$$

³ The term "right" reflects the position of the argument x^j in the bilinear form \langle, \rangle_r .

By contrast with the matrix B , determining the product \circ , the matrix C enters the definition of the so-called R -trace. Namely, for any $N \times N$ matrix M (even with noncommutative entries) its R -trace is defined by the following formula $\text{Tr}_R M = \text{Tr} CM$.

Now, consider the following map

$$\mathcal{R} : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)^{\otimes 2}, \quad R_{12}L_1R_{12}L_1 \xrightarrow{\mathcal{R}} L_1R_{12}L_1R_{12}, \tag{2.4}$$

where $L = \|l_i^j\|_{1 \leq i, j \leq N}$ and $L_1 = L \otimes I$. A more explicit expression of this map can be found in [6].

Using the map \mathcal{R} , we can define an analog of the Lie bracket in the space $\text{End}(V)$ by putting

$$[X, Y]_R = X \circ Y - \circ \mathcal{R}(X \otimes Y), \quad X, Y \in \text{End}(V). \tag{2.5}$$

The data $(\text{End}(V), \mathcal{R}, [,]_R)$ is called a *generalized (or braided) Lie algebra* and is denoted $gl(V_R)$.

Observe that for this $gl(V_R)$ -bracket the following analog of the Jacobi identity is valid on $\text{End}(V)^{\otimes 3}$ (here we omit the subscript R)

$$[,] [,]_{23}(I - \mathcal{R})_{23} = [,] [,]_{12}.$$

A proof of this relation is given in [6].

Note that if R is an involutive symmetry, the corresponding Jacobi identity can be cast in the form similar to the classical one:

$$[,] [,]_{23}(I + \mathcal{R}_{12} \mathcal{R}_{23} + \mathcal{R}_{23} \mathcal{R}_{12}) = 0.$$

Let us explain the reason why the algebra $\mathcal{L}(R)$ plays the role of the enveloping algebra of the braided Lie algebra $gl(V_R)$. Indeed, the defining system of the algebra $\mathcal{L}(R)$ can be cast under the following form

$$l_i^j l_k^l - \mathcal{R}(l_i^j l_k^l) = [l_i^j, l_k^l]_R, \tag{2.6}$$

where \mathcal{R} is the operator introduced in (2.4). Thus, by applying the product \circ to the left hand side of the relation (2.6), where we treat the generators l_i^j as elements of $\text{End}(V)$, we get the element $[l_i^j, l_k^l]_R$.

Now, we want to show that the R -trace Tr_R is coordinated with the bracket $[,]_R$ in the following sense

$$\text{Tr}_R [X, Y]_R = 0, \quad \forall X, Y \in \text{End}(V),$$

which is similar to the classical relation for the trace of the commutator of two operators. To this end we first compute $\text{Tr}_R l_i^j$. Since $l_i^j \triangleright x_k = B_k^j x_i$, then applying the R -trace to the image of the element l_i^j we get $\text{Tr}_R l_i^j = \text{Tr} B_k^j C_k^j = \alpha \delta_i^j$, where α is a nontrivial numerical factor, whose value is determined by the bi-rank of R (see [6] for detail). Thus, taking $X = l_i^j$ and $Y = l_k^m$, we have to show $\text{Tr}_R [l_i^j, l_k^m]_R = 0$. This equality is a consequence of the following chain of relations:

$$\text{Tr}_R \circ (R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12}) = \text{Tr}_R (R_{12}L_1 - L_1R_{12}) = \alpha(R_{12}I - I R_{12}) = 0.$$

With any Hecke symmetry R we associate the R -symmetric $\text{Sym}_R(V)$ and R -skew-symmetric $\Lambda_R(V)$ algebras of the spaces V and V^* by setting

$$\text{Sym}_R(V) = T(V) / \langle \text{Im}(qI - R) \rangle = T(V) / \langle q x_i x_j - R_{ij}^{kl} x_k x_l \rangle, \tag{2.7}$$

$$\Lambda_R(V) = T(V) / \langle \text{Im}(q^{-1}I + R) \rangle = T(V) / \langle q^{-1} x_i x_j + R_{ij}^{kl} x_k x_l \rangle,$$

$$\text{Sym}_R(V^*) = T(V^*) / \langle \text{Im}(qI - R) \rangle = T(V^*) / \langle q x^j x^i - R_{kl}^{ij} x^j x^k \rangle, \tag{2.8}$$

$$\Lambda_R(V^*) = T(V^*) / \langle \text{Im}(q^{-1}I + R) \rangle = T(V^*) / \langle q^{-1} x^j x^i + R_{kl}^{ij} x^j x^k \rangle,$$

where

$$T(V) = \bigoplus_k V^{\otimes k}, \quad T(V^*) = \bigoplus_k V^{*\otimes k}$$

are the free tensor algebras and $\langle J \rangle$ stands for a two-sided ideal generated by a subset J in the given algebra. If R is an involutive symmetry, we put $q = 1$ in all these formulae.

It should be emphasized that all these algebras are deformations of their classical counterparts, provided R is a deformation of the usual flip P , i.e. dimensions of their homogenous components are classical (for a generic q if R is Hecke). A similar statement is valid, if a Hecke symmetry is a deformation of another involutive symmetry, for instance, a super-symmetry.

To complete this section, we also mention two quantum matrix algebras, associated with any braiding R . One of them is the so-called RTT algebra, defined by the system

$$R_{12}T_1T_2 = T_1T_2R_{12}, \quad T = \|t_i^j\|_{1 \leq i, j \leq N}, \quad T_1 = T \otimes I, \quad T_2 = I \otimes T. \tag{2.9}$$

The other one is the RE algebra defined via (1.2) but with the vanishing right hand side. We do not use the former algebra. The latter one appears only under its modified form. However, it is worth noting that if R is a Hecke symmetry, the RE algebra and its modified version are isomorphic to each other.

3. Quantum doubles and bosonic (fermionic) realization of $\mathcal{L}(R)$

Our next aim is to construct some quantum doubles (QD) from the above R -symmetric and R -skew-symmetric algebras and use them in order to perform bosonic and fermionic realizations of the algebra $\mathcal{L}(R)$.

First, we recall the definition of doubles of associative algebras from [3]. Consider two associative unital algebras A and B equipped with a linear map

$$\sigma : A \otimes B \rightarrow B \otimes A,$$

which satisfies the following conditions:

$$\begin{aligned} \sigma \circ (\mu_A \otimes \text{id}) &= (\text{id} \otimes \mu_A) \circ \sigma_{12} \sigma_{23} \quad \text{on } A \otimes A \otimes B, \\ \sigma \circ (\text{id} \otimes \mu_B) &= (\mu_B \otimes \text{id}) \circ \sigma_{23} \sigma_{12} \quad \text{on } A \otimes B \otimes B, \\ \sigma(1_A \otimes b) &= b \otimes 1_A, \quad \sigma(a \otimes 1_B) = 1_B \otimes a, \quad \forall a \in A, \forall b \in B, \end{aligned}$$

where $\mu_A : A \otimes A \rightarrow A$ is the multiplication operation in the algebra A , 1_A is the unit element of A and the symbols μ_B and 1_B have the same meaning in the algebra B . The symbol “id” stands for the identity operator.

It is not difficult to see that the bilinear map $* : (B \otimes A)^{\otimes 2} \rightarrow B \otimes A$ defined by the rule

$$(b \otimes a) * (b' \otimes a') := (\mu_B \otimes \mu_A) \circ (\text{id}_B \otimes \sigma_{23} \otimes \text{id}_A)(b \otimes a \otimes b' \otimes a')$$

endows the space $B \otimes A$ with the structure of a unital associative algebra and its unit element is $1_B \otimes 1_A$.

Definition 1. We call the data (A, B, σ) and the corresponding algebra $(B \otimes A, *)$ a *double of associative algebras* and denote it (A, B) .

An associative double (A, B) is called *quantum* if the map σ is constructed with the use of a braiding, different from a (super-)flip.

Observe that if the both algebras A and B are introduced via some systems of relations on the generators, the above conditions on the data (A, B, σ) mean that the permutation map preserves these relations. In this sense we say that the permutation map and the defining relations of the algebras A and B are *compatible*.

Now, assume the algebra A to be equipped with a counit (an algebra homomorphism)⁴ $\varepsilon_A : A \rightarrow \mathbb{C}$. Then it is possible to define a linear action of the algebra A on B by setting

$$a \triangleright b = (\text{id}_B \otimes \varepsilon_A) \circ \sigma(a \otimes b), \quad \forall a \in A, \forall b \in B.$$

Identifying $b \otimes 1_{\mathbb{C}}$ and b , we get that $a \triangleright b \in B$. Thus, each element $a \in A$ defines a linear operator

$$Op(a) : B \rightarrow B.$$

It is easy to check that the map $a \mapsto Op(a)$ defines a representation of the algebra A in the algebra B :

$$Op(ab) = Op(a)Op(b), \quad Op(1_A) = \text{id}_B.$$

Besides, the aforementioned Heisenberg double, we can also mention QD composed from two copies of the RTT algebras. The permutation relations between them are also similar to the system (2.9). A counit ε defined on one component of such a double is usually defined by $\varepsilon(t_i^j) = \delta_i^j$. Note that the corresponding representation of the component A , defined according to the above scheme, is a deformation of the trivial one $\pi(T) = I$ and consequently is not interesting indeed. The QD of Fock type, which we are dealing with, are more meaningful. We introduce these doubles, assuming R to be a Hecke symmetry.

Let $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ be a skew-invertible Hecke symmetry, $A = \text{Sym}_R(V^*)$ and $B = \text{Sym}_R(V)$. Construct a quantum double (A, B) by introducing the following permutation relations⁵:

$$x^a R_{ai}^{bj} x_b = q^{-1} x_i x^j + \delta_i^j \Leftrightarrow x^l x_k = q^{-1} x_i x^j \Psi_{jk}^{il} + B_k^l. \tag{3.1}$$

⁴ In general we do not consider any coalgebraic structure in the algebra A . So, the counit is only coordinated with the algebraic structure.

⁵ By permutation relations we mean the equality of the form $a \otimes b = \sigma(a \otimes b)$.

Remark 2. Recall, that we use the right dual basis in the space V^* . For the left dual basis $\{\tilde{x}^j\}_{1 \leq j \leq N}$, the permutation relations read

$$x_b R_{ia}^{jb} \tilde{x}^a = q^{-1} \tilde{x}^j x_i + \delta_i^j \Leftrightarrow x_k \tilde{x}^l = q^{-1} \tilde{x}^j x_i \Psi_k^{li} + C_k^l.$$

Proposition 3. The defining systems of the algebras $\text{Sym}_R(V)$ and $\text{Sym}_R(V^*)$ are compatible with the permutation relations (3.1).

Proof. Below, we use the Dirac’s “bra” and “ket” notation:

$$x_i \rightarrow x_{|1\rangle}, \quad x_i x_j \rightarrow x_{|1\rangle} x_{|2\rangle}, \quad x^i \rightarrow x^{(1)}, \quad x^i x^j \rightarrow x^{(1)} x^{(2)}, \dots$$

We call this form of notation the *matrix* one. Some typical examples are

$$x^{(1)} x_{|1\rangle} = \sum_{i=1}^N x^i x_i, \quad x_{|1\rangle} x^{(1)} = x_i x^i, \quad x^{(2)} R_{|12\rangle}^{(12)} x_{|2\rangle} = \sum_{a,b=1}^N x^a R_{ia}^{jb} x_b.$$

In what follows the notation $R_{|12\rangle}^{(12)}$ will be simplified to R_{12} .

Thus, the defining relations of the algebras (2.7), (2.8) and the permutation relations (3.1) can be respectively rewritten as:

$$\begin{aligned} R_{ij}^{kl} x_k x_l &= q x_i x_j & \rightarrow & \quad R_{12} x_{|1\rangle} x_{|2\rangle} = q x_{|1\rangle} x_{|2\rangle} \\ x^l x^k R_{kl}^{ij} &= q x^j x^i & \rightarrow & \quad x^{(2)} x^{(1)} R_{12} = q x^{(2)} x^{(1)} \\ x^a R_{ai}^{bj} x_b &= q^{-1} x_i x^j + \delta_i^j & \rightarrow & \quad x^{(1)} R_{12} x_{|1\rangle} = q^{-1} x_{|2\rangle} x^{(2)} + I_{|2\rangle}^{(2)} \\ & & \text{or} & \quad x_{|2\rangle} x^{(2)} = q x^{(1)} R_{12} x_{|1\rangle} - q I_{|2\rangle}^{(2)}. \end{aligned}$$

Let us verify, for example, that the defining relations of the algebra $\text{Sym}_R(V)$ (2.7) are not destroyed by the permutation with vectors from V^* . For this purpose we permute an arbitrary basis vector x^i with elements of the generating set of the ideal in (2.7):

$$\begin{aligned} (R_{23} - qI_{23}) x_{|2\rangle} x_{|3\rangle} x^{(3)} &= (R_{23} - qI_{23}) x_{|2\rangle} \left(q x^{(2)} R_{23} x_{|2\rangle} - q I_{|3\rangle}^{(3)} \right) \\ &= -q (R_{23} - qI_{23}) x_{|2\rangle} + q (R_{23} - qI_{23}) \left(q x^{(1)} R_{12} x_{|1\rangle} - q I_{|2\rangle}^{(2)} \right) R_{23} x_{|2\rangle} \\ &= q^2 x^{(1)} (R_{23} - qI_{23}) R_{12} R_{23} x_{|1\rangle} x_{|2\rangle} = q^2 x^{(1)} R_{12} R_{23} (R_{12} - qI_{12}) x_{|1\rangle} x_{|2\rangle}. \end{aligned}$$

Here, by passing to the third line we have taken into account the consequence of the Hecke condition $(R_{23} - qI_{23}) R_{23} = -q^{-1} (R_{23} - qI_{23})$, while the last equality is due to the braid relation for R .

So, we finally get the equality:

$$(R_{23} - qI_{23}) x_{|2\rangle} x_{|3\rangle} x^{(3)} = q^2 x^{(1)} R_{12} R_{23} (R_{12} - qI_{12}) x_{|1\rangle} x_{|2\rangle},$$

which means the compatibility of the defining relations of the algebra $\text{Sym}_R(V)$ with the permutation relations (3.1).

The compatibility condition for the defining relations of $\text{Sym}_R(V^*)$ and the permutation relations can be verified in the same way. \square

Now, we introduce the counit similarly to the classical case:

$$\varepsilon_A : A = \text{Sym}_R(V^*) \rightarrow \mathbb{C}, \quad \varepsilon_A(1_A) = 1_{\mathbb{C}}, \quad \varepsilon_A(x^j) = 0, \quad 1 \leq \forall j \leq N.$$

According to the above scheme all elements of the algebra $A = \text{Sym}_R(V^*)$ can be represented by operators acting on the algebra $B = \text{Sym}_R(V)$. Compute the action of elements x^j onto some monomials in x_i :

$$x^j \triangleright 1_B = 0, \quad x^j \triangleright x_i = B_i^j, \quad x^j \triangleright (x_i x_k) = B_i^j x_k + q^{-1} B_k^l \Psi_{li}^{mj} x_m, \quad \text{etc.} \tag{3.2}$$

Note that using the properties of Ψ and R we can rewrite the above actions in the *equivalent* matrix form:

$$x^{(1)} \triangleright \left(R_{12} x_{|1\rangle} \right) = I_{|2\rangle}^{(2)}, \quad x^{(1)} \triangleright \left(R_{12} R_{23} x_{|1\rangle} x_{|2\rangle} \right) = (R_{23} + q^{-1} I_{23}) x_{|2\rangle}.$$

As we noticed above, the generators of the algebra $A = \text{Sym}_R(V^*)$ are analogs of the usual annihilation operators, whereas the generators of $B = \text{Sym}_R(V)$ are analogs of the creation operators: the result of the action of x_i on $a \in \text{Sym}_R(V)$ is the product $x_i a$. The above double (A, B) is said to be *quantum bosonic double of Fock type*.

In a similar manner it is possible to construct a *quantum fermionic double of Fock type* by setting $A = \Lambda_R(V^*)$, $B = \Lambda_R(V)$ and

$$x^a R_{ai}^{bj} x_b = -q x_i x^j + \delta_i^j \Leftrightarrow x^j x_k = -q x_i x^j \Psi_{jk}^{il} + B_k^j. \tag{3.3}$$

In this case all defining relations are also compatible.

Now, compose the matrix $L = \|\|l_i^j\|$, $l_i^j = x_i x^j$, where $x_i \in B$, $x^j \in A$ and (A, B) is a bosonic or fermionic double of Fock type. Then the following proposition is valid.

Proposition 4. *The matrix L meets the relations (1.2).*

Proof. In the bosonic case this proposition was proved in [2]. In the fermionic case the proof is similar. \square

Thus, representing the factors of the product $x_i x^j$ as the creation and annihilation operators acting on the algebra B , we get the so-called *bosonic (or fermionic) realization* of the algebra $\mathcal{L}(R)$ similar to that of the algebra⁶ $U(\mathfrak{gl}(N))$.

Moreover, similarly to the classical case the generators l_i^j preserve any homogenous component $\text{Sym}_R^k(V)$, $k \geq 1$. Thus, we get a series of finite-dimensional representations of the algebra $\mathcal{L}(R)$. For instance, the action of the elements $l_i^j = x_i x^j$ onto the first component $\text{Sym}_R^1(V) = V$ is as follows

$$l_i^j \triangleright x_k = x_i \triangleright (x^j \triangleright x_k) = B_k^j x_i. \tag{3.4}$$

Remark 5. Note that on the role of the algebra B we can assign the free tensor algebra $T(V)$ instead of $\text{Sym}_R(V)$ and keep the same permutation relations. By considering all invariant subspaces with respect to the action \triangleright , it is possible to construct a more rich representation category of the algebra $\mathcal{L}(R)$. Such a category was constructed [6] by means of a “braided bi-algebra structure” of $\mathcal{L}(R)$. Our present method based on bosonization is a sense close to the method of constructing differential calculus on the quantum hyperplane from [16,8].

4. Quantum doubles associated with BMW symmetries

Let us recall that by a BMW symmetry we mean a braiding R which comes from a QG belonging to one of the orthogonal (B_n , D_n) or symplectic (C_n) series. An explicit form of these symmetries is exhibited in [15]. Each of them is subject to some relations. We need only one of them:

$$(R - qI)(R + q^{-1}I)(R - \mu I) = 0. \tag{4.1}$$

Note that if R comes from the QG corresponding to an orthogonal group then $\mu = q^{1-N}$ and if R comes from the QG corresponding to a symplectic group, then $\mu = -q^{-1-N}$. Here N is the dimension of the basic vector space. Below, for the sake of simplicity we speak about orthogonal and symplectic QG.

Also, there are known examples of the BMW symmetries coming from super-QG with other values of μ (see [9]). Below, we only deal with the BMW symmetries, coming from orthogonal or symplectic QG. Consequently, the parameter μ is assumed to take one of the above values. Thus, as $q \rightarrow 1$, this parameter tends to 1 for the orthogonal QG and to -1 for the symplectic ones.

In [13] there were considered the so-called BMW algebras, which are defined by similar relations but with the parameter μ independent on q .

In virtue of (4.1) there exist three complementary idempotents \mathcal{P}^q , $\mathcal{P}^{-q^{-1}}$ and \mathcal{P}^μ such that

$$R = q \mathcal{P}^q - q^{-1} \mathcal{P}^{-q^{-1}} + \mu \mathcal{P}^\mu.$$

Each of them is a projector onto the corresponding eigenspace and can be explicitly written in terms of R . For instance, the projector

$$\mathcal{P}^{-q^{-1}} = \frac{(R - qI)(R - \mu I)}{(q + q^{-1})(q^{-1} + \mu)} \tag{4.2}$$

maps the space $V^{\otimes 2}$ onto the eigenspace of the operator R corresponding to the eigenvalue $-q^{-1}$. Note that for a generic q the denominator of this formula does not vanish.

⁶ Note that if $R \rightarrow P$ as $q \rightarrow 1$ (for instance, if R is a standard symmetry) the system (1.2) turns into the matrix equality

$$PL_1 PL_1 - L_1 PL_1 P = PL_1 - L_1 P \Leftrightarrow l_i^j l_k^m - l_k^m l_i^j = l_i^m \delta_k^j - l_k^j \delta_i^m,$$

valid for the usual generators of the algebra $U(\mathfrak{gl}(N))$.

Now, let us define the R -symmetric and R -skew-symmetric algebras of the space V by setting

$$\text{Sym}_R(V) = T(V)/\langle \text{Im } \mathcal{P}^{-q^{-1}} \rangle, \quad \Lambda_R(V) = T(V)/\langle \text{Ker } \mathcal{P}^{-q^{-1}} \rangle, \tag{4.3}$$

provided R comes from an orthogonal QG and

$$\text{Sym}_R(V) = T(V)/\langle \text{Ker } \mathcal{P}^q \rangle, \quad \Lambda_R(V) = T(V)/\langle \text{Im } \mathcal{P}^q \rangle,$$

provided R comes from a symplectic QG.

Remark 6. Note that according to [15] R -symmetric algebras have the classical Poincaré series, i.e. these series are equal to the series corresponding to the algebras $\text{Sym}(V)$. However, any proof of this claim has never been published. We do not use this property. We want only to observe that the space $\text{Ker } \mathcal{P}^\mu$ belongs to the second homogenous R -symmetric component of the space $V^{\otimes 2}$ if the QG is orthogonal and to R -skew-symmetric component if QG is symplectic.

In a similar manner we define the R -symmetric and R -skew-symmetric algebras of the space V^* . For instance, we put (in the orthogonal case)

$$\text{Sym}_R(V^*) = T(V^*)/\langle \text{Im } \mathcal{P}^{-q^{-1}} \rangle, \tag{4.4}$$

where the idempotents \mathcal{P}^q , $\mathcal{P}^{-q^{-1}}$ and \mathcal{P}^μ acting in the space $(V^*)^{\otimes 2}$ are expressed in terms of R by the same formulae as above. We only have to take into account the way of extending the braiding R onto the space $V^{*\otimes 2}$, namely, $R(x^k \otimes x^l) = R_{ji}^{lk}(x^i \otimes x^j)$.

Let us define the permutation relations by formula (3.1) for an orthogonal QG and by (3.3) for a symplectic QG. Then the following claim is valid.

Proposition 7. *The permutation relations are compatible with the defining systems of the algebras $\text{Sym}_R(V)$ and $\text{Sym}_R(V^*)$ for the orthogonal series and with these of the algebras $\Lambda_R(V)$ and $\Lambda_R(V^*)$ for the symplectic series.*

Proof. The proposition is proved by a straightforward calculation similar to the case of the Hecke symmetries (see Proposition 3 above). Nevertheless, we give a short sketch of the proof in order to stress the difference with the Hecke case.

Consider the algebra $\text{Sym}_R(V)$ for the orthogonal QG defined by the quotient (4.3). Taking into account the permutation relations (3.1) we get the following transformation:

$$x_{|2}x_{|3}x^{(3)} = q^2x^{(1)}R_{12}R_{23}x_{|1}x_{|2} - q^2(R_{23} + q^{-1}I_{23})x_{|2}.$$

Then for the generating set of the ideal $\langle \text{Im } \mathcal{P}^{-q^{-1}} \rangle$ we have:

$$\mathcal{P}^{-q^{-1}}(R_{23}x_{|2}x_{|3})x^{(3)} = q^2x^{(1)}R_{12}R_{23}\mathcal{P}^{-q^{-1}}(R_{12})x_{|1}x_{|2} - q^2\mathcal{P}^{-q^{-1}}(R_{23})(R_{23} + q^{-1}I_{23})x_{|2}.$$

The first term in the right hand side is a consequence of the braid relation on R :

$$\mathcal{P}^{-q^{-1}}(R_{23})R_{12}R_{23} = R_{12}R_{23}\mathcal{P}^{-q^{-1}}(R_{12}),$$

while the second term is equal to zero due to the definition (4.2) of the projector $\mathcal{P}^{-q^{-1}}$ and the cubic minimal polynomial (4.1) of the symmetry R . So, we finally get

$$\underline{\mathcal{P}^{-q^{-1}}(R_{23}x_{|2}x_{|3})x^{(3)}} = q^2x^{(1)}R_{12}R_{23}\underline{\mathcal{P}^{-q^{-1}}(R_{12})x_{|1}x_{|2}},$$

which means the compatibility of the permutation relations (3.1) with the defining relations of the algebra $\text{Sym}_R(V)$ for the orthogonal case. All other compatibilities are verified by the similar calculations. \square

Now, consider the QD (A, B) of Fock type, corresponding to a BMW symmetry R . For R corresponding to the orthogonal QG we set $A = \text{Sym}_R(V^*)$, $B = \text{Sym}_R(V)$, and for R corresponding to a symplectic QG we set $A = \Lambda_R(V^*)$, $B = \Lambda_R(V)$. By introducing the same counit as in the algebra $\mathcal{L}(R)$, we get an action of the algebra A on the algebra B . The creation and annihilation operators are similar to these from the bosonic (resp., fermionic) realization of the algebra $\mathcal{L}(R)$.

Now, we introduce a matrix $L = \|l_i^j\|$ with entries $l_i^j = x_i x^j$.

Proposition 8. *For all series the matrix L meets the following system*

$$\mathcal{P}_{12}L_1R_{12}L_1 - L_1R_{12}L_1\mathcal{P}_{12} = \mathcal{P}_{12}L_1 - L_1\mathcal{P}_{12}, \tag{4.5}$$

where $\mathcal{P} = \mathcal{P}^q + \mathcal{P}^\mu$ for the orthogonal series and $\mathcal{P} = \mathcal{P}^{-q^{-1}} + \mathcal{P}^\mu$ for the symplectic one.

Proof. Consider the case of R corresponding to an orthogonal QG in detail. For a symplectic QG all considerations are analogous. Below, we use the matrix notation introduced above.

Since the idempotents \mathcal{P}^q , $\mathcal{P}^{-q^{-1}}$ and \mathcal{P}^μ are complementary, we have

$$\mathcal{P}^{-q^{-1}} + \mathcal{P}^q + \mathcal{P}^\mu = I.$$

Also, according to the definitions of the algebras $\text{Sym}_R(V)$ and $\text{Sym}_R(V^*)$ and the permutation relations the complete system on the generators x_i and x^i are

$$\mathcal{P}_{12}x_{|1}x_{|2} = x_{|1}x_{|2}, \quad x^{(2)}x^{(1)}\mathcal{P}_{12} = x^{(2)}x^{(1)}, \quad x^{(1)}R_{12}x_{|1} = q^{-1}x_{|2}x^{(2)} + I_2,$$

where $\mathcal{P} = \mathcal{P}^q + \mathcal{P}^\mu$.

Now, substitute the matrix $L_1 = x_{|1}x^{(1)}$ into the left hand side of (4.5). Using the relations on the generators we transform the first summand to the following expression:

$$\mathcal{P}_{12}x_{|1}x^{(1)}R_{12}x_{|1}x^{(1)} = q^{-1}\mathcal{P}_{12}x_{|1}x_{|2}x^{(2)}x^{(1)} + \mathcal{P}_{12}x_{|1}x^{(1)} = q^{-1}x_{|1}x_{|2}x^{(2)}x^{(1)} + \mathcal{P}_{12}L_1.$$

Here, underlined are the terms which undergo changes. By the same steps of transformations the second summand can be rewritten in the form:

$$x_{|1}x^{(1)}R_{12}x_{|1}x^{(1)}\mathcal{P}_{12} = q^{-1}x_{|1}x_{|2}x^{(2)}x^{(1)}\mathcal{P}_{12} + x_{|1}x^{(1)}\mathcal{P}_{12} = q^{-1}x_{|1}x_{|2}x^{(2)}x^{(1)} + L_1\mathcal{P}_{12}.$$

By taking the difference of the above expressions, we get (4.5). \square

Observe that the algebra defined by (4.5) cannot be treated as the enveloping algebra $U(\mathfrak{g})$ of a generalized Lie algebra, since the relation (4.5) does not enable us to define any operator $\text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)^{\otimes 2}$ similar to \mathcal{R} .⁷

Moreover, though the algebras $\text{Sym}_R(V)$ and $\Lambda_R(V)$ are well defined in this case and are (hopefully!) deformations of their classical counterparts, the algebras obtained via the bosonic (fermionic) realization are not deformations of $U(\mathfrak{g})$.

Nevertheless, proceeding in a similar way as described above we can construct finite dimensional representations of the algebra defined by (4.5) in homogenous components of the algebras $\text{Sym}_R(V)$ or $\Lambda_R(V)$ depending on the series.

Concluding this section we note that if R is an involutive or Hecke symmetry, deforming the usual flip P , the both algebras $\text{Fun}(GL(N))$ and $U(gl(N))$ admit quantum deformations. Moreover, there are two different deformations of the algebra $\text{Fun}(GL(N))$: the corresponding RTT algebra or the RE one. However, a deformation of the algebra $U(gl(N))$ can be performed only via the modified RE algebra.

If \mathfrak{g} is an orthogonal or symplectic Lie algebra, the space $\text{Fun}(G)$ of functions on the corresponding group G can be deformed as was done in [13]. Note that the authors of the cited paper use Quantum Matrix Algebras of a general form including the RTT and RE algebras. By contrary, the algebra $U(\mathfrak{g})$ does not have any deformation, covariant with respect to the adjoint action of the corresponding QG $U_q(\mathfrak{g})$.

As we have noticed in Introduction, the BMW symmetries are not well adapted to constructing R -analogs of Lie algebras.

Remark 9. Now, we would like to discuss the Poisson counterparts of the algebras obtained by the bosonization. If R is a Hecke symmetry deforming the usual flip P (for instance, that coming from the QG $U_q(sl(N))$ or the Crammer-Gervais one), then the Poisson counterpart of the algebra $\mathcal{L}(R)$ (more precisely, of the algebra, obtained by introducing a second deformation parameter \hbar in the front of the right hand side of (1.2)) is a pencil defined on the commutative algebra $\text{Sym}(gl(N))$. This pencil is generated by the linear Poisson-Lie bracket, associated with the Lie algebra $gl(N)$, and by a quadratic bracket, corresponding to the case $\hbar = 0$. It is interesting to observe that almost all brackets of this pencil are not unimodular. Nevertheless, the quantum counterpart of this pencil (namely, the algebra $\mathcal{L}(R)$) is endowed with a trace, but this trace is not usual.

If R is a BMW symmetry coming from an orthogonal or symplectic QG, a similar Poisson pencil does not exist. However, a Poisson bracket, corresponding to the deformation of the algebra $\text{Fun}(G)$ can be defined.

5. Quantum double of Zamolodchikov-Faddeev type

In this section we deal with algebras associated with current braidings. By a current braiding we mean an operator $R(u, v)$ depending on spectral parameters subject to the relation

$$R_{12}(u, v)R_{23}(u, w)R_{12}(v, w) = R_{23}(v, w)R_{12}(u, w)R_{23}(u, v).$$

The current braidings, we are dealing with, arise from the Baxterization procedure applied to an involutive (resp., Hecke) symmetry. These current braidings are of the form

⁷ Observe that if R is a Hecke symmetry, the corresponding modified RE algebra can be cast in the form (4.5), but with $\mathcal{P} = \mathcal{P}^q$.

$$R(u, v) = R - \frac{I}{u - v}, \quad \text{resp.}, \quad R(u, v) = R - \frac{(q - q^{-1})uI}{u - v}, \tag{5.1}$$

where R is any skew-invertible involutive (resp., Hecke) symmetry. The former braiding $R(u, v)$ is called *rational*, the latter one is called *trigonometric*. The reader is referred to [2] for details.

Let us pass to the normalized braidings $\mathbf{R}(u, v) = g(u, v)^{-1}R(u, v)$, where

$$g(u, v) = 1 - \frac{1}{u - v}, \quad \text{resp.} \quad g(u, v) = q - \frac{(q - q^{-1})u}{u - v}$$

provided $R(u, v)$ is a rational (resp., trigonometrical) braiding (5.1).

It should be emphasized that these operators are still subject to the braid relation (5.1) and are involutive in the following sense

$$\mathbf{R}(u, v)\mathbf{R}(v, u) = I.$$

In virtue of this relation the operators $\mathbf{R}(u, v)$ have two eigenvalues ± 1 . So, we introduce an analog of the algebra $\text{Sym}_R(V)$ by the following relations:

$$\mathbf{R}_{ij}^{kl}(u, v)x_k(u)x_l(v) = x_l(v)x_k(u) \Leftrightarrow R_{ij}^{kl}(u, v)x_k(u)x_l(v) = g(u, v)x_l(v)x_k(u). \tag{5.2}$$

Write this system in a more detailed form for a trigonometric braiding

$$q x_i(v)x_j(u) - R_{ij}^{kl}x_k(u)x_l(v) = \frac{(q - q^{-1})u}{u - v}(x_i(v)x_j(u) - x_i(u)x_j(v)).$$

(The defining relations of the algebra $\Lambda_R(V)$ can be written in a similar way.)

However, if we try to represent the currents $x_i(u)$ as a formal series in u with Fourier coefficients $x_i[m]$ as follows

$$x_i(u) = \sum_{m \in \mathbb{Z}} x_i[m]u^{-m-1},$$

we find that it is not possible to rewrite the defining system via polynomial relations on these coefficients. By following the classical pattern, we present the algebra in terms of the so-called *half-currents*

$$x_i^+(u) = \sum_{m \in \mathbb{Z}, m < 0} x_i[m]u^{-m-1}, \quad x_i^-(u) = \sum_{m \in \mathbb{Z}, m \geq 0} x_i[m]u^{-m-1}, \quad x_i(u) = x_i^+(u) + x_i^-(u).$$

Then by imposing the relations

$$\begin{aligned} g(u, v)x_i^\pm(v)x_j^\pm(u) &= R_{ij}^{kl}(u, v)x_k^\pm(u)x_l^\pm(v), \\ g(u, v)x_i^-(v)x_j^+(u) &= R_{ij}^{kl}(u, v)x_k^+(u)x_l^-(v), \end{aligned} \tag{5.3}$$

and by using the expansion $\frac{1}{u-v} = \sum_{p \geq 0} \frac{v^p}{u^{p+1}}$, we can rewrite (5.2) in terms of the Fourier coefficients. However, the system (5.2) for the currents $x_i(u)$ is preserved.

According to the scheme exhibited in Section 3, we introduce the current R -symmetric algebra of the dual space by the following system

$$g(u, v)x^i(u)x^j(v) = R_{lk}^{ji}(u, v)x^k(v)x^l(u). \tag{5.4}$$

This system can be also expressed via the Fourier coefficients if we pass to the corresponding half-currents $x_\pm^i(u)$. Besides, we introduce the permutation relations

$$x^a(u)R_{ai}^{bj}x_b(v) = q^{-1}x_i(v)x^j(u) + \delta_i^j\delta(u - v), \tag{5.5}$$

which are compatible with the relations on the currents

$$x_i(u) = x_i^+(u) + x_i^-(u) \text{ and } x^j(u) = x_+^j(u) + x_-^j(u).$$

Thus, we have again constructed a quantum double (A, B) of Fock type. The algebras A and B are generated by the Fourier coefficients of the currents $x^k(u)$ and $x_k(u)$, which respectively play the role of the annihilation and creation operators. Using the definition $\delta(u - v) = \sum_{p \in \mathbb{Z}} \frac{v^p}{u^{p+1}}$, we can express the system (5.5) in terms of the Fourier coefficients:

$$x^a[k]R_{ai}^{bj}x_b[l] = q^{-1}x_i[l]x^j[k] + \delta_i^j\delta_{k+l}^1 \Leftrightarrow x^a[k]x_b[l] = q^{-1}x_i[l]x^j[k]\Psi_{jb}^{ia} + B_b^a\delta_{k+l}^1.$$

Now, define the counit ε_A as above by setting

$$\varepsilon_A(1_A) = 1_{\mathbb{C}}, \quad \varepsilon_A(x^a[k]) = 0, \quad \forall a, k.$$

This allows us to get the action of the algebra A on B :

$$x^a[k] \triangleright x_b[l] = B_b^a \delta_{k+l}^1 \text{ and so on.}$$

We call the algebra, defined by the relations (5.2), (5.4), and (5.5), the QD of Zamolodchikov-Faddeev type.

Now, introduce the matrix $L(u) = \|l_i^j(u)\|_{1 \leq i, j \leq N}$ where $l_i^j(u) = x_i(u) x^j(u)$.

Proposition 10. *The matrix $L(u)$ meets the following relation*

$$R_{12}(u, v) L_1(u) R_{12} L_1(v) - L_1(v) R_{12} L_1(u) R_{12}(u, v) = (R_{12} L_1(u) - L_1(u) R_{12}) \delta(u - v). \tag{5.6}$$

Proof. Again, we use the matrix notation. Consider the first summand in the left hand side of (5.6) and substitute the matrices $L(u)$ expressed in terms of currents $x_i(u)$ and $x^j(u)$:

$$\begin{aligned} & R_{12}(u, v) x_{|1}(u) x_{|1}(v) x^{(1)}(u) R_{12} x_{|1}(v) x^{(1)}(v) \stackrel{(5.5)}{=} q^{-1} R_{12}(u, v) x_{|1}(u) x_{|2}(v) x^{(2)}(u) x^{(1)}(v) \\ & + R_{12}(u, v) x_{|1}(u) x^{(1)}(v) \delta(u - v) \stackrel{(5.2)}{=} q^{-1} g(u, v) x_{|1}(u) x_{|2}(v) x^{(2)}(u) x^{(1)}(v) \\ & + R_{12}(u, v) x_{|1}(u) x^{(1)}(v) \delta(u - v). \end{aligned}$$

In a similar way (the only difference is using (5.4) instead of (5.2)) we transform the second summand in the left hand side of (5.6):

$$L_1(v) R_{12} L_1(u) R_{12}(u, v) = q^{-1} g(u, v) x_{|1}(u) x_{|2}(v) x^{(2)}(u) x^{(1)}(v) + x_{|1}(v) x^{(1)}(u) R_{12}(u, v) \delta(u - v).$$

Taking the difference of the above expressions we get the desired result:

$$\begin{aligned} & R_{12}(u, v) L_1(u) R_{12} L_1(v) - L_1(v) R_{12} L_1(u) R_{12}(u, v) = \\ & (R_{12}(u, v) x_{|1}(u) x^{(1)}(v) - x_{|1}(v) x^{(1)}(u) R_{12}(u, v)) \delta(u - v) = (R_{12} L_1(u) - L_1(u) R_{12}) \delta(u - v). \end{aligned}$$

Note that the product $R(u, v) \delta(u - v)$ contains the ill-defined terms $\frac{(q-q^{-1})u}{u-v} \delta(u - v)$, but they cancel each other in the final expression. As a consequence, the right hand side of (5.6) contains only constant matrix R . \square

Concluding the paper we want to observe that the algebras analogous to those defined by (5.6), but with vanishing right hand side, were introduced in [2,5] under the name of the braided Yangians or generalized Yangians of RE type. Similar algebras but with current braidings in the middle positions and depending of a charge were considered in [4]. Note that the algebras from [4] are generalizations of those from [14], corresponding to the A_n series.

Nevertheless, the defining relations of the algebras from [4] cannot be expressed via the Fourier components. By contrast, the system (5.6) can be written in terms of these components. This is a consequence of the fact that we define the permutation relations (5.5) in the QD above by means of a constant braiding (namely, a Hecke symmetry), in contrast with the initial definition of the ZF algebras.

References

- [1] D. Gurevich, Algebraic aspects of the Yang-Baxter equation, Leningr. Math. J. 2 (1991) 801–828.
- [2] D. Gurevich, Saponov P. Braided Yangians, J. Geom. Phys. 138 (2019) 124–143.
- [3] D. Gurevich, P. Saponov, Doubles of associative algebras and their applications, Phys. Part. Nucl. Lett. 17 (5) (2020) 774–778.
- [4] D. Gurevich, P. Saponov, Centers in generalized Reflection Equation algebras, Theor. Math. Phys. 204 (2020) 363–373.
- [5] D. Gurevich, P. Saponov, From reflection equation algebra to braided Yangians, in: V.M. Buchstaber, et al. (Eds.), Proceedings of the 1st International Conference on Mathematical Physics, Grozny, Russia, 2017, Recent Developments in Integrable Systems and Related Topics of Mathematical Physics, in: Springer Proceedings in Mathematics and Statistics, vol. 273, Springer Nature Switzerland AG, 2019.
- [6] D. Gurevich, P. Pyatov, P. Saponov, Representation theory of (modified) reflection equation algebra of $GL(m|n)$ type, St. Petersburg Math. J. 20 (2008) 213–253.
- [7] D. Gurevich, P. Pyatov, Saponov P. Braided, Differential operators on quantum algebras, J. Geom. Phys. 61 (2011) 1485–1501.
- [8] L. Hlavatý, Yang-Baxter matrices and differential calculi on quantum hyperplane, J. Phys. A 25 (1992) 485–494.
- [9] P. Isaev, Quantum Groups and Yang-Baxter Equations, MPIM preprint 2004-63.
- [10] V. Lyubashenko, Hopf algebras and vector-symmetries (Russian), Usp. Mat. Nauk 41 (1986) 185–186.
- [11] V. Lyubashenko, Super-analysis and solutions to the equation of triangles, Thesis, 1986, Kiev (Russian).
- [12] A. Macfarlane, On q -analogues of the quantum harmonic oscillator and quantum group $SU(2)_q$, J. Phys. A, Math. Gen. 22 (1989) 4581–4588.
- [13] O. Ogievetsky, Pyatov P. Quantum, Matrix algebras of BMW type: structure of the characteristic subalgebra, arXiv:1910.08551v1.
- [14] N. Reshetikhin, M. Semenov-Tian-Shansky, Central extensions of quantum current groups, Lett. Math. Phys. 19 (2) (1990) 133–142.
- [15] N. Reshetikhin, L. Takhtadzhyan, L. Faddeev, Quantization of Lie groups and Lie algebras, Leningr. Math. J. 1 (1990) 193–225.
- [16] J. Wess, B. Zumino, Covariant differential calculus on the quantum hyperplane, Nucl. Phys. B, Proc. Suppl. 18B (1990) 302–312.