EMBEDDINGS OF AUTOMORPHISM GROUPS OF FREE GROUPS INTO AUTOMORPHISM GROUPS OF AFFINE ALGEBRAIC VARIETIES

VLADIMIR L. POPOV

To the memory of A. N. Parshin

ABSTRACT. A new infinite series of rational affine algebraic varieties is constructed whose automorphism group contains the automorphism group $\operatorname{Aut}(F_n)$ of the free group F_n of rank n. The automorphism groups of such varieties are nonlinear and contain the braid group B_n on n strands for $n \geq 3$, and are nonamenable for $n \geq 2$. As an application, it is proved that for $n \geq 3$, every Cremona group of rank $\geq 3n-1$ contains the groups $\operatorname{Aut}(F_n)$ and B_n . This bound is 1 better than the one published earlier by the author; with respect to B_n the order of its growth rate is one less than that of the bound following from the paper by D. Krammer. The basis of the construction are triplets (G, R, n), where G is a connected semisimple algebraic group and R is a closed subgroup of its maximal torus.

1. Introduction

The trend of the last decade has been the study of abstract-algebraic, topological, algebro-geometric, and dynamical properties of automorphism groups of algebraic varieties. This paper is related to this topic and continues the research started in author's paper [11].

In [11], an infinite series of irreducible algebraic varieties is constructed in whose automorphism group embeds the automorphism group of a free group F_n of rank n. This has applications to the problems of linearity and amenability of automorphism groups of algebraic varieties and that of the embeddability of various groups into Cremona groups. To formulate the results obtained in this paper, we recall the construction introduced in [11].

Consider a connected algebraic group G. Denote

X is the group variety of the algebraic group

$$G^n := G \times \cdots \times G \quad (n \text{ times}).$$

We fix in F_n a free system of generators f_1, \ldots, f_n . For any $w \in F_n$ and

$$x = (g_1, \dots, g_n) \in X, \quad g_j \in G \text{ for all } j,$$
 (1)

denote by w(x) the element of G obtained from the word w in f_1, \ldots, f_n by replacing f_j with g_j for each j. For each $\sigma \in \operatorname{Aut}(F_n)$, the mapping

$$\sigma_X \colon X \to X, \quad x \mapsto (\sigma(f_1)(x), \dots, \sigma(f_n)(x)).$$
 (2)

is an automorphism of the algebraic variety X (but not, in general, of the group G^n). The mapping $\sigma \mapsto (\sigma^{-1})_X$ is a group homomorphism $\operatorname{Aut}(F_n) \to \operatorname{Aut}(X)$. It defines an action of the group $\operatorname{Aut}(F_n)$ by automorphisms of the variety X commuting with the diagonal action G on X by conjugation. Let us assume that for the restriction of this action to a closed subgroup R of the group G, there is a categorical quotient

$$\pi_{X/\!\!/R} \colon X \to X/\!\!/R \tag{3}$$

(for example, this property holds if R is finite, see [15, Prop. 19; p. 50, Expl. 2)], or if G is affine and R is reductive, see [12, 4.4]). Then it follows from the definition of categorical quotient (see [12, Def. 4.5]) that σ_X descends to a uniquely defined automorphism $\sigma_{X/\!\!/R}$ of $X/\!\!/R$, which has the property

$$\pi_{X/\!\!/R} \circ \sigma_X = \sigma_{X/\!\!/R} \circ \pi_{X/\!\!/R}. \tag{4}$$

In this case, there arises a group homomorphism

$$\operatorname{Aut}(F_n) \to \operatorname{Aut}(X/\!\!/R), \quad \sigma \mapsto (\sigma^{-1})_{X/\!\!/R},$$
 (5)

defining the action of the group $\operatorname{Aut}(F_n)$ by automorphisms of the variety $X/\!\!/R$. For some (but not all) G and R, homomorphism (5) is an embedding. Namely, in [11] it is proved that

- (a) in the following cases, homomorphism (5) is an embedding:
 - G is nonsolvable and R is finite;
 - G is reductive, R = G, n = 1 and G contains a connected simple normal subgroup of one of the following types:

$$A_{\ell} \text{ with } \ell \geqslant 2, \ D_{\ell} \text{ with odd } \ell, \ E_6;$$
 (6)

- (b) in the following cases, homomorphism (5) is not an embedding:
 - G is solvable, R is finite and $n \ge 3$,
 - G is reductive, R = G and either $n \ge 2$, or n = 1 and G does not contain a connected simple normal subgroup of either of types (6).

This leads to the following general question:

Question. Is it possible to classify the triples (G, R, n), where G is a connected reductive algebraic group, R is its closed subgroup, and n is a positive integer, for which homomorphism (5) is an embedding?

The main result of the present paper is Theorem 1.1, in which the next step after [11] is taken towards answering the question posed: we add one more class to the triples of the specified type found in [11]:

Theorem 1.1. Let G be a connected semisimple algebraic group and let R be a closed subgroup of its maximal torus. Then homomorphism (5) is an embedding.

As an application we obtain the following Theorem 1.2, in which B_n denotes the braid group on n strands.

Theorem 1.2. We keep the assumptions of Theorem 1.1. Then $X/\!\!/R$ is an affine algebraic variety such that

- (a) $\operatorname{Aut}(X/\!\!/R)$ contains $\operatorname{Aut}(F_n)$,
- (b) $\operatorname{Aut}(X/\!\!/R)$ is nonlinear for $n \ge 3$,
- (c) Aut $(X/\!\!/R)$ contains the braid group B_n for $n \ge 3$.
- (d) $\operatorname{Aut}(X/\!\!/R)$ is nonamenable for $n \ge 2$.

We also explore the rationality problem:

Theorem 1.3. The variety $X/\!\!/R$ from Theorems 1.1, 1.2 is rational.

As an application we strengthen by 1 the bounds obtained in [11, Cor. 9 and Rem. 10]:

Theorem 1.4. For any integer $n \ge 1$, the Cremona group of rank $\ge 3n-1$ contains the group $\operatorname{Aut}(F_n)$ and, for $n \ge 3$, the braid group B_n .

Remark. As is proved in [4] by D. Krammer, the braid group B_n embeds into $GL_{n(n-1)/2}$. Hence B_n embeds into the Cremona group of rank n(n-1)/2. The order of the growth rate of this bound for the minimal rank of the Cremona group containing B_n is one bigger than that of the bound from Theorem 1.4.

The proofs of Theorems 1.1–1.4 are given in Section 6.

2. Conventions and notation

In what follows, algebraic varieties are considered over an algebraically closed field k. We use the results of paper [6] and statement [12, Prop. 3.4] obtained under the condition char(k) = 0. Therefore, we also assume that this condition holds. With respect to algebraic geometry and algebraic groups we follow [1].

The identity element of a group considered in multiplicative notation is denoted by e (it will be clear from the context which group is meant).

The statement that a group A contains a group B means the existence of a group monomorphism $B \hookrightarrow A$, by which B is identified with its image.

 $\mathscr{C}(A)$ is the center of a group A.

 $A \cdot m$ and A_m are respectively, the orbit and the stabilizer of a point m with respect to a considered action of a group A (it will be clear from the context which action is meant).

The kernel of an action $\alpha \colon A \times M \to M$ of a group A on a set M is the following normal subgroup of A:

$$\ker(\alpha) := \{ a \in A \mid a \cdot m = m \text{ for all } m \in M \}.$$

By homomorphisms of algebraic groups we mean algebraic homomorphisms, and by their actions on algebraic varieties we mean algebraic actions. In particular, for an algebraic group A, we denote by $\operatorname{Aut}(A)$ the group of its algebraic automorphisms.

The multiplicative group k^{\times} of the field k is considered as the algebraic group \mathbb{G}_m , and its additive group is considered as \mathbb{G}_a .

3. Terminology and some general results

Recall (see [12]) the terminology and results used below, which concern an action of α of an algebraic group H on an irreducible algebraic variety Y.

- (a) An action α is said to be *stable* if there is a nonempty open set in Y, the H-orbits of whose points are closed in Y.
- (b) A subgroup H_* of H is called the *stabilizer in general posisiton* (s.g.p.) of the action α if there is a nonempty open set in Y such that for any its point y, the subgroups H_y and H_* are conjugate in H.
 - (c) If H is reductive and Y is smooth and affine, then
 - The s.g.p. exists.
- The varieties Y and $Y/\!\!/H$ are endowed with the Luna stratifications defined as follows. The fact that the points $a,b\in Y/\!\!/H$ belong to the same Luna stratum means that the normal vector bundles to the unique H-orbits closed in the fibers $\pi_{Y/\!\!/H}^{-1}(a)$ and $\pi_{Y/\!\!/H}^{-1}(b)$ orbits are H-equivariantly isomorphic. The Luna strata in Y are the sets of the form $\pi_{Y/\!\!/H}^{-1}(L)$, where L is a Luna stratum in $Y/\!\!/H$. The Luna stratifications have the following properties:
 - (i) the set of all Luna strata is finite;
 - (ii) all Luna strata in the varieties $Y/\!\!/H$ and Y are smooth locally closed subvarieties of these varieties;
 - (iii) for any Luna stratum L in $Y/\!\!/H$ there exists an affine variety F endowed with an action of H such that the restriction

of the morphism $\pi_{Y/\!\!/H}$ to the stratum $\pi_{Y/\!\!/H}^{-1}(L)$ (called the canonical morphism of the Luna stratum $\pi_{Y/\!\!/H}^{-1}(L)$) is an étale trivial bundle $\pi_{Y/\!\!/H}^{-1}(L) \to L$ with fiber F.

In view of (i) and (ii), there are (unique) open Luna strata in $Y/\!\!/H$ and Y. They are called the *principal strata* and denoted by $(Y/\!\!/H)_{pr}$ and Y_{pr} respectively.

Lemma 3.1. We keep the previous notation H, Y, α . Let $y \in Y$ be a point such that

- (y_1) the orbit $H \cdot y$ is closed in Y;
- (y_2) $H_y = \ker(\alpha)$.

Let β be an action of the group H on an algebraic variety Z such that

(k)
$$\ker(\alpha) = \ker(\beta)$$
,

and let $\varphi \colon Z \to Y$ be an H-equivariant morphism such that $\varphi^{-1}(Hy) \neq \emptyset$. Then for every point $z \in \varphi^{-1}(H \cdot y)$ the following properties hold:

- (z_1) the orbit $H \cdot z$ is closed in Z;
- (\mathbf{z}_2) $H_z = \ker(\beta)$.

Proof. In view of (y_1) , the nonempty H-invariant subset $\varphi^{-1}(H \cdot y)$ is closed in Z and hence contains the closure $\overline{H \cdot z}$ of the orbit $H \cdot z$. Assume that (z_1) fails, i.e., $\overline{H \cdot z} \setminus H \cdot z \neq \emptyset$. Let $v \in \overline{H \cdot z} \setminus H \cdot z$. Then

$$\dim(H \cdot v) < \dim(H \cdot z). \tag{7}$$

The restriction of the morphism φ to the orbit $H \cdot v$ is an H-equivariant and therefore a surjective morphism $H \cdot v \to H \cdot y$. So $\dim(H \cdot v) \geqslant \dim(H \cdot y)$, which together with (7) gives

$$\dim(H \cdot y) < \dim(H \cdot z) \tag{8}$$

On the other hand, since $H_z \supseteq \ker(\beta)$, from (y_2) and (k) it follows that $\dim(H \cdot y) \geqslant \dim(H \cdot z)$. This contradicts (8) and proves (z_1) .

The restriction of the morphism φ to the orbit $H \cdot z$ is an H-equivariant, and therefore a surjective morphism of $H \cdot z \to H \cdot y$. Hence, there exists $h \in H$ for which $\varphi(h \cdot z) = y$. Therefore,

$$\ker(\alpha) \stackrel{\text{(k)}}{=} \ker(\beta) \subseteq H_{h \cdot z} \subseteq H_y \stackrel{\text{(y_2)}}{=} \ker(\alpha);$$

whence, $H_{h\cdot z} = \ker(\beta)$. Since $H_{h\cdot z} = hH_zh^{-1}$ and $\ker(\beta)$ is normal in H, this proves (\mathbf{z}_2) .

Lemma 3.2. Let a commutative group H act transitively on a set M.

(e₁) For every H-equivariant mapping $\varphi \colon M \to M$ there is an element $h \in H$ such that

$$\varphi(m) = h \cdot m \text{ for every point } m \in M. \tag{9}$$

(e₂) For every element $h \in H$, the map $\varphi: M \to M$ defined by formula (9) is H-equivariant.

Proof. (e₁) Fix a point $m_0 \in M$. Since the action is transitive, for any point $m \in M$ there is an element $z \in H$ such that $m = z \cdot m_0$. In particular, there is $h \in H$ for which $\varphi(m_0) = h \cdot m_0$. Since φ is H-equivariant and H is commutative, we then have $\varphi(m) = \varphi(z \cdot m_0) = z \cdot \varphi(m_0) = z \cdot (h \cdot m_0) = zh \cdot m_0 = hz \cdot m_0 = h \cdot (z \cdot m_0) = h \cdot m$.

(e₂) This follows directly from (9) in view of the commutativity of H.

4. Reduction

The proof of Theorem 1.1 is based on the following geometric description of the kernel of homomorphism (5):

Lemma 4.1. Let G be a connected affine algebraic group and let R be its closed reductive subgroup. The following properties of an element $\sigma \in \operatorname{Aut}(F_n)$ are equivalent:

- (a) σ lies in the kernel of homomorphism (5);
- (b) $\sigma_X(\mathcal{O}) = \mathcal{O}$ for every closed R-orbit \mathcal{O} in X.

Proof. In this case, the variety X is affine, which implies (see [9, §2 and Append. 1B]) that the morphism π is surjective, its fibers are R-invariant, and for each point $b \in X/\!\!/R$, the fiber $\pi^{-1}(b)$ contains a unique closed R-orbit \mathcal{O}_b . It follows from (4) that the restriction of the morphism σ_X to the fiber $\pi^{-1}(b)$ is its R-equivariant isomorphism with the fiber $\pi^{-1}(\sigma_{X/\!\!/R}(b))$. In view of the uniqueness of closed orbits in the fibers, this means that $\sigma_X(\mathcal{O}_b) = \mathcal{O}_{\sigma_{X/\!\!/R}(b)}$. Therefore, $\sigma_{X/\!\!/R}(b) = b$ if and only if $\sigma_X(\mathcal{O}_b) = \mathcal{O}_b$.

Under the conditions of Lemma 4.1, the algebra $k[X]^R$ of all R-invariant elements of the algebra k[X] of regular functions on X is finitely generated, $X/\!\!/R$ is the affine algebraic variety with the algebra of regular functions $k[X/\!\!/R] = k[X]^R$, and the comorphism corresponding to morphism (3) is the identity embedding $k[X]^R \hookrightarrow k[X]$. This implies that for any reductive closed subgroup S of G containing R, the identity embedding $k[X]^S \hookrightarrow k[X]^R$ determines a dominant morphism $X/\!\!/R \to X/\!\!\!/S$. This morphism is $Aut(F_n)$ -equivariant. Therefore, the

kernel of the action of the group $\operatorname{Aut}(F_n)$ on $X/\!\!/R$ lies in the kernel of its action on $X/\!\!/S$.

In the situation considered in Theorem 1.1, this gives the following. By the assumption, in it, R is a subgroup of some maximal torus T of the group G. Therefore, it follows from what has been said that it suffices to prove Theorem 1.1 for

$$R = T. (10)$$

In what follows, we assume that the group G satisfies the conditions of Theorem 1.1, i.e., is connected and semisimple. Note that the kernel of the action of the group T on X is $\mathscr{C}(G)$, since $\mathscr{C}(G) \subset T$ (see [1, 13.17, Cor. 2(d)]).

5. Principal Luna stratum for action of T on X

The variety X is smooth, and the group T is reductive. Therefore, the diagonal action of the torus T on X by conjugation determines the Luna stratifications of the varieties X and $X/\!\!/T$. In what follows, $X_{\rm pr}$ denotes the principal stratum of this stratification of the variety X.

Theorem 5.1. Let G be a connected semisimple algebraic group with a maximal torus T acting diagonally by conjugation on the group variety X of the group G^n .

- (a) The kernel of the specified action is $\mathscr{C}(G)$.
- (b) The following properties of a point $x \in X$ are equivalent:
 - $(b_1) x \in X_{pr};$
 - (b₂) the orbit $T \cdot x$ is closed in X, and $T_x = \mathscr{C}(G)$.
- (c) Each fiber of the canonical morphism of the Luna stratum X_{pr} is a T-orbit equivariantly isomorphic to $T/\mathscr{C}(G)$.
- (d) $\operatorname{codim}_X(X \setminus X_{\operatorname{pr}}) \geqslant n$.

Proof. Statement (a) is obvious.

- (b) Denote by \mathcal{V} the trivial $\operatorname{codim}_X(T)$ -dimensional vector bundle over $T/\mathscr{C}(G)$. Note that $\dim(T/\mathscr{C}(G)) = \dim(T)$ since the group $\mathscr{C}(G)$ is finite (see [1, 14.2. Cor.(a)]).
 - It suffices for us to prove that
 - (i) the action of the torus T on X under consideration is stable;
 - (ii) $\mathscr{C}(G)$ is its stabilizer in general position,

or, in other words, that there is a nonempty open subset of X, for all points x of which property (b_2) holds. Indeed, suppose this subset exists. Due to its openness, its intersection with the open set X_{pr} is nonempty. Let x be a point of this intersection. Since $\mathscr{C}(G)$ is the kernel of the action of the group T on X, from condition (b_2) it follows

that the normal bundle of the orbit $T \cdot x$ is equivariantly isomorphic to \mathcal{V} . From this and from the definition of the Luna strata it follows that a closed T-orbit from X lies in $X_{\rm pr}$ if and only if its normal bundle is equivariantly isomorphic to \mathcal{V} . In particular, the dimension of this orbit is $\dim(T)$. It remains to note that the T-orbit of any point $y \in X_{\rm pr}$ is closed. Indeed, if this were not the case, then the unique closed T-orbit in the fiber $\pi_{X/\!\!/T}^{-1}(\pi_{X/\!\!/T}(y)) \subseteq X_{\rm pr}$ lying in its closure had dimension strictly less than $\dim(T \cdot y) \leqslant \dim(T)$, which contradicts the $\dim(T)$ -dimensionality of this closed orbit.

- Let us now prove that properties (i) and (ii) indeed hold. It suffices to prove them for n=1. Indeed, suppose that for n=1 they are proved, i.e., there is a nonempty open subset of G such that T-orbit of every its point x is closed in G and $T_x = \mathscr{C}(G)$. Then, as explained above, $G_{\rm pr}$ is the set of all such points x. Let $\pi_i \colon X = G^n \to G$ be the natural projection onto the ith factor. Applying Lemma 3.1 to it, we infer that property (b₂) holds for each point x of a nonempty set $\pi_i^{-1}(G_{\rm pr})$, which means that properties (i) and (ii) hold.
- It remains to prove that (i) and (ii) hold for n = 1. In [16, 6.11], it is proved that the action of G on itself by conjugation is stable and its s.g.p. is T. From [5, Thm. and Sect. 3] and the reductivity of T, it follows that the natural action of T on G/T is stable. These two facts imply, according to [10, Prop. 6], that (i) holds for n = 1.
- Let Φ be the root system of the group G with respect to the torus T in which subsystems of positive and negative roots with respect to some base in Φ are fixed. For any $\alpha \in \Phi$, there is an embedding of algebraic groups $\varepsilon_{\alpha} \colon \mathbb{G}_a \hookrightarrow G$, such that

$$t\varepsilon_{\alpha}(x)t^{-1} = \varepsilon_{\alpha}(\alpha(t)x)$$
 for all $t \in T, x \in \mathbb{G}_a$ (11)

(see [3, 26.3. Thm.], [16, 2.1]). Consider in G the "big cell" Θ (see [3, 28.5 Prop.]), i.e., the set of all elements of the form

$$\prod_{\alpha<0} \varepsilon_{\alpha}(x_{\alpha})t \prod_{\alpha>0} \varepsilon_{\alpha}(x_{\alpha}), \quad x_{\alpha} \in \mathbb{G}_{a}, t \in T,$$
(12)

where the factors in the products are taken with respect to some fixed orders on the sets of positive and negative roots. The set Θ is open in G and each of its elements can be uniquely written as (12) (see [1, 14.5. Prop.(2), 14.14. Cor.], [16, 2.2, 2.3]). In view of (11), it is T-invariant. The set Θ^0 of all elements of the form (12) with $x_{\alpha} \neq 0$ for each $\alpha \in \Phi$ has the same properties. Let $a \in \Theta^0$ and $c \in T$. It follows from (11) and the indicated uniqueness that the condition $c \in T_a$ is equivalent to the condition

$$c \in \ker(\alpha) \text{ for all } \alpha \in \Phi.$$
 (13)

In turn, it follows from (11), (12) and the openness of Θ^0 that (13) is equivalent to the property that c belongs to the kernel of the action of T on G, i.e., (13) is equivalent to the inclusion $c \in \mathscr{C}(G)$. This proves that $T_a = \mathscr{C}(G)$. Hence, (ii) holds for n = 1. This completes the proof of (b).

- (c) This follows from (b), since each fiber of the canonical morphism of any Luna stratum in X contains a unique orbit closed in X.
- (d) As is explained in the proof of statement (b), the set X_{pr} contains the set $\bigcup_{i=1}^{n} \pi_i^{-1}(G_{pr})$, from where we get

$$X \setminus X_{\mathrm{pr}} \subseteq (G \setminus G_{\mathrm{pr}}) \times \cdots \times (G \setminus G_{\mathrm{pr}}) \quad (n \text{ factors}).$$
 (14)

From (14) it follows that $\dim(X \setminus X_{\operatorname{pr}}) \leq n(\dim(G) - 1) = \dim(X) - n$. This proves (d).

6. Proofs of Theorems 1.1–1.4

Proof of Theorem 1.1. As is explained in Section 4, we can (and shall) assume that equality (10) holds. Arguing by contradiction, suppose that the kernel of homomorphism (5) contains an element $\sigma \in \text{Aut}(F_n)$, $\sigma \neq e$. The cases n = 1 and $n \geq 2$ will be considered separately: in each of them the proof is based on the properties that do not hold in the other.

Case n=1.

The order of $\operatorname{Aut}(F_1)$ is 2 and $\sigma(f_1) = f_1^{-1}$, so

$$\sigma_X(g) = g^{-1}$$
 for each $g \in G = X$. (15)

For any element $t \in T$ we have $T \cdot t = t$. In view of Lemma 4.1, this implies that $\sigma_X(t) = t$. Together with (15) this shows that $t^2 = e$ for any $t \in T$. This conclusion contradicts the fact that the set of orders of elements of the torsion subgroup of any torus of positive dimension is not upper bounded (see [1, 8.9. Prop.]).

Case $n \geqslant 2$.

• Since the kernel of the considered action of the torus T on X is $\mathscr{C}(G)$ (see Theorem 5.1(a)), this action defines a faithful (that is, with trivial kernel) action on X of the torus

$$S := T/\mathscr{C}(G). \tag{16}$$

The orbits of this action of the torus S, and hence the categorical quotient and the Luna stratifications are the same as those of the action of the torus T. Below, instead of the original action of the torus T, we consider the indicated action of the torus S.

• Theorem 5.1(b) and Lemmas 4.1, 3.2(e₁) imply the existence of a set-theoretic mapping $\psi: (X/\!\!/T)_{pr} \to S$ such that

$$\sigma_X(x) = \psi(\pi_{X/T}(x)) \cdot x$$
 for each point $x \in X_{\text{pr}}$. (17)

Let us prove that the set-theoretic mapping

$$X_{\rm pr} \to S, \quad x \mapsto \psi(\pi_{X/\!\!/T}(x))$$
 (18)

is a morphism of algebraic varieties. According to Theorem 5.1(b) and what was said in part (iii) of Section 3, the canonical morphism $X_{\rm pr} \to (X/\!\!/T)_{\rm pr}$ is an étale trivial bundle with fiber S. Since algebraic tori are special groups in the sense of Serre (see [14, Prop. 14]), this bundle is locally trivial in the Zariski topology. Hence, $X_{\rm pr}$ is covered by S-invariant open sets for which there are S-equivariant isomorphisms of them with varieties of the form $U \times S$, where U is an open subset of $(X/\!\!/T)_{\rm pr}$, and the torus S acts through translations of the second factor. If we identify them by these isomorphisms, then the restriction of mapping (18) to any of these open sets has the form

$$\alpha: U \times S \to S, \quad (u, s) \mapsto \psi(u).$$

The issue therefore boils down to proving that α is a morphism of algebraic varieties. To this end, note that since σ_X is a morphism, then

$$U\times S\to U\times S,\quad (u,s)\mapsto (u,\psi(u)s)$$

is also a morphism in view of (17). Hence

$$\beta \colon U \times S \to S \times S, \quad (u, s) \mapsto (s, \psi(u)s)$$

is a morphism as well. Moreover,

$$\gamma \colon S \times S \to S, \quad (s_1, s_2) \mapsto s_1^{-1} s_2.$$
 (19)

is a morphism too. It remains to note that $\alpha = \gamma \circ \beta$.

• Thus, there exists a rational mapping

$$\theta \colon X \dashrightarrow S$$
,

which is defined everywhere on the open set X_{pr} and coincides on it with morphism (18). Since $n \ge 2$, it follows from Theorem 5.1(d) that

$$\operatorname{codim}_{X}(X \setminus X_{\operatorname{pr}}) \geqslant 2. \tag{20}$$

The torus S can be identified with the product of several copies of the group k^{\times} . Then θ is given by a set of rational functions $\theta_i \colon X \dashrightarrow k$, which are compositions of the mapping θ with projections of this product onto the factors. Each θ_i is regular and does not vanish on X_{pr} . Since X is smooth, it follows from this and (20) that the divisor of θ_i on X is zero, that is, θ_i is regular and does not vanish on the whole of X. Thus, we have a morphism $\theta_i \colon X \to k^{\times}$. Since X is the group

variety of the connected algebraic group G^n , it follows from this and from [13, Thm. 3] that θ_i is the product of a character of this group and a constant. But due to semisimplicity, G^n has no nontrivial characters. Hence θ_i is a constant. This means that there is an element $s \in S$ for which $\theta(X) = s$.

• Fix an element $t \in T$ that maps to s under the natural surjection $T \to S$ (see (16)). We have proven that $\sigma_X(x) = t \cdot x$ for every point $x \in X_{\rm pr}$. Since $X_{\rm pr}$ is open in X, this means that

$$\sigma_X(x) = t \cdot x$$
 for every point $x \in X$. (21)

Since $\sigma \neq e$, it follows from [11, Thm. 2(b₁)] that $\sigma_X \neq id_X$. In view of (21) and Theorem 5.1(a), this gives

$$t \notin \mathscr{C}(G). \tag{22}$$

It follows from (21), (1), and (2) that for each $i \in \{1, ..., n\}$ the following group identity holds

$$\sigma(f_i)(g_1, \dots, g_n) = tg_i t^{-1} \quad \text{for any } g_1, \dots, g_n \in G.$$
 (23)

In particular, for each $g \in G$ the equality obtained by substituting $g_1 = \ldots = g_n = g$ into (23) holds. Since $\sigma(f_i)$ is a noncommutative Laurent monomial in $f_1, \ldots f_n$, this means that there exists an integer d such that the following group identity holds:

$$g^d = tgt^{-1}$$
 for each $g \in G$. (24)

Notice that

$$d \neq 1$$
 and $d \neq -1$. (25)

Indeed, in view of (24), if d = 1, then $t \in \mathcal{C}(G)$ contrary to (22). If d = -1, then for any $g, h \in G$ the following equality holds:

$$h^{-1}g^{-1} = (gh)^{-1} \stackrel{(24)}{=} t(gh)t^{-1} = tgt^{-1}tht^{-1} \stackrel{(24)}{=} g^{-1}h^{-1}$$

which means that the group G is commutative and contradicts its semisimplicity.

Further, if r is a positive integer, then the following group identity holds:

$$t^r g t^{-r} = g^{d^r} \quad \text{for each } g \in G.$$
 (26)

Indeed, (26) becomes (24) for r=1. Arguing by induction, from $t^{r-1}gt^{-r+1}=g^{d^{r-1}}$ we get

$$t^r g t^{-r} = t (t^{r-1} g t^{-r+1}) t^{-1} = t g^{d^{r-1}} t^{-1} \stackrel{(24)}{=} (g^{d^{r-1}})^d = g^{d^r},$$

as stated.

Substituting g = t into (24) and taking into account (25), we conclude that t is an element of finite order. Let r in (26) be equal to this order. Then (26) turns into the group identity

$$e = g^{d^r - 1}$$
 for every $g \in G$. (27)

In view of (25), we have $d^r - 1 \neq 0$. Hence the group identity (27) implies that G, and therefore also T, is a torsion group the orders of whose elements are upper bounded. We have arrived at the same contradiction as when considering the case n = 1. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Theorem 1.1 implies (a). For $n \geq 3$, the group $\operatorname{Aut}(F_n)$ is nonlinear (see [2]) and contains the group B_n (see [8, Sect. 3.7]). In view of (a), this implies (b) and (c). Since the group $\mathscr{C}(F_n)$ is trivial for $n \geq 2$ (see [7, Chap. I, Prop. 2.19]), the group $\operatorname{Int}(F_n)$ is isomorphic to F_n and hence is not amenable. From this it follows (d).

Proof of Theorem 1.3. Let us prove that the group variety of the group G is birationally T-equivariantly isomorphic to some T-module. To this end consider the open set Θ in G introduced in the proof of Theorem 5.1 and fix the following objects:

• one-dimensional T-module L_{α} for every $\alpha \in \Phi$ on which T acts by the formula

$$t \cdot \ell = \alpha(t)\ell, \quad t \in T, \ell \in L_{\alpha},$$
 (28)

- nonzero element $\ell_{\alpha} \in L_{\alpha}$,
- a trivial T-module F of dimension $\dim(T)$,
- an open embedding of algebraic varieties

$$\iota \colon T \hookrightarrow F$$

(it exists because T is a torus).

Consider the T-module

$$V := \bigoplus_{\alpha > 0} L_{\alpha} \oplus F \oplus \bigoplus_{\alpha < 0} L_{\alpha},$$

where the summands in the direct sums are taken with respect to some fixed orders on the sets of positive and negative roots. In view of (11), (28), the mapping $\tau : \Theta \to V$, sending each element (12) to the vector

$$\bigoplus_{\alpha>0} x_{\alpha}\ell_{\alpha} \oplus \iota(t) \oplus \bigoplus_{\alpha<0} x_{\alpha}\ell_{\alpha},$$

is the searched for birational morphism (see [1, 14.4. Rem.]). Consequently,

$$\tau^n := \tau \times \dots \times \tau : \Theta^n := \Theta \times \dots \times \Theta \to V^n := V \oplus \dots \oplus V \quad (n \text{ components})$$

is also a T-equivariant, and therefore, an R-equivariant birational morphism.

Since R and V^n are respectively a diagonalizable group and an R-module, the field $k(V)^R$ is rational over k (see [12, Sect 2.9]). Since Θ^n is open in X, this implies that the field $k(X)^R$ is also rational over k. But the action of T on X is stable, and $\mathscr{C}(G)$ is the s.g.p. for it by Theorem 5.1(b). Since R is reductive, from [5, Thm. and Sect. 3] it follows that the natural action of R on $T/\mathscr{C}(G)$ is stable. Hence, according to [10, Prop. 6], the action of R on X is stable. In view of [12, Prop. 3.4], this implies that $k(X)^R$ is the field of fractions of the algebra $k[X]^R = k[X/\!\!/R]$. This is what the rationality of the variety $X/\!\!/R$ means.

Proof of Theorem 1.4. Let $G = \operatorname{SL}_2$, so that $\dim(G) = 3$ and $\dim(T) = 1$. It follows from here and from Theorem 5.1(c) that $\dim(X/\!\!/T) = 3n-1$. Hence, in view of the rationality of the variety $X/\!\!/T$ (Theorem 1.3), the group $\operatorname{Aut}(X/\!\!/T)$ embeds into the Cremona group of rank 3n-1. The claim of the theorem now follows from Theorem 1.2 and the fact that every Cremona group embeds into any Cremona group of a higher rank.

References

- [1] A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Vol. 126, Springer-Verlag, New York, 1991.
- [2] E. Formanek, C. Procesi, The automorphism group of a free group is not linear,
 J. Algebra 149 (1992), 494–499.
- [3] J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1975.
- 4 D. Krammer, Braid groups are linear, Annals of Math. 155 (2002), 131–156.
- [5] D. Luna, Sur les orbites fermées des groupes algébriques rédictifs, Invent. math. 16 (1972), 1–5.
- [6] D. Luna, Slices étalés, Bull. Soc. math. France 33 (1973), 81–105.
- [7] R. C. Lyndon, P. E. Schupp, *Combinatorial Group Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 89, Springer-Verlag, Berlin, 1977.
- [8] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, Interscience, New York, 1966.
- [9] D. Mumford, J. Fogarty, Geometric Invariant Theory, Second Edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 34, Springer-Verlag, Berlin, 1982
- [10] V. L. Popov, Closed orbits of Borel subgroups, Math. USSR-Sb. **63** (1989), no. 2, 375–392.
- [11] V. L. Popov, Embeddings of groups $Aut(F_n)$ into automorphism groups of algebraic varieties, arXiv:2106.02072 (2021).
- [12] V. L. Popov, E. B. Vinberg, *Invariant Theory*, in: *Algebraic Geometry* IV, Encycl. Math. Sci., Vol. 55, Springer Verlag, Berlin, 1994, pp. 123–284.

- [13] M. Rosenlicht, Toroidal algebraic groups, Proc. Amer. Math. Soc. 12 (1961), 984–988.
- [14] J.-P. Serre, Espaces fibrés algébriques, in: Anneaux de Chow et Applications, Séminaire C. Chevalley, Exp. 1, 1958.
- [15] J.-P. Serre, Algebraic Groups and Class Fields, Graduate Texts in Mathematics, Vol. 117, Springer, New York, 1997.
- [16] R. Steinberg, Regular elements in semisimple algebraic groups, Publ. Math. l'IHES 25 (1965), 49–80.

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina 8, Moscow 119991, Russia

Email address: popovvl@mi-ras.ru