

**FAITHFUL ACTIONS  
OF AUTOMORPHISM GROUPS  
OF FREE GROUPS  
ON ALGEBRAIC VARIETIES**

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*To the memory of J. E. Humphreys*

ABSTRACT. Considering a certain construction of algebraic varieties  $X$  endowed with an algebraic action of the group  $\text{Aut}(F_n)$ ,  $n < \infty$ , we obtain a criterion for the faithfulness of this action. It gives an infinite family  $\mathcal{F}$  of  $X$ 's such that  $\text{Aut}(F_n)$  embeds into  $\text{Aut}(X)$ . For  $n \geq 3$ , this implies nonlinearity, and for  $n \geq 2$ , the existence of  $F_2$  in  $\text{Aut}(X)$  (hence nonamenability of the latter) for  $X \in \mathcal{F}$ . We find in  $\mathcal{F}$  two infinite subfamilies  $\mathcal{N}$  and  $\mathcal{R}$  consisting of irreducible affine varieties such that every  $X \in \mathcal{N}$  is nonrational (and even not stably rational), while every  $X \in \mathcal{R}$  is rational and  $3n$ -dimensional. As an application, we show that the minimal dimension of affine algebraic varieties  $Z$ , for which  $\text{Aut}(Z)$  contains the braid group  $B_n$  on  $n \geq 3$  strands, does not exceed  $3n$ . This upper bound strengthens the one following from the paper by D. Krammer [Kr02], where the linearity of  $B_n$  was proved (this latter bound is quadratic in  $n$ ). The same upper bound also holds for  $\text{Aut}(F_n)$ . In particular, it shows that the minimal rank of the Cremona groups containing  $\text{Aut}(F_n)$ , does not exceed  $3n$ , and the same is true for  $B_n$  if  $n \geq 3$ .

**1. Introduction.** The exploration of abstract-algebraic, topological, algebro-geometric and dynamical properties of biregular automorphism groups and birational self-map groups of algebraic varieties has become the trend of the last decade. In terms of popularity, the Cremona groups are probably the leaders among the studied groups.

Below, algebraic varieties and algebraic groups are understood in the same sense as in [Se55], [Sh07], [Bo91], [Hu75] and are taken over an algebraically closed field  $k$ .

The subject of this paper are the following questions on the group embeddability related to automorphism groups of algebraic varieties.

(Q1) For a given group  $S$ , is there an algebraic variety  $Z$  such that  $S$  embeds in the group  $\text{Aut}(Z)$  of its biregular automorphisms?

(Q2) If yes, what are the properties of such  $Z$ ? Are there such  $Z$  in some distinguished classes of varieties (e.g., rational, nonrational, affine, complete, etc.)? What are the “extreme” values of the parameters of such  $Z$  (e.g., the minimum of their dimensions)? Etc.

(Q3) Conversely, in which groups can automorphism groups of algebraic varieties of some type be embedded (e.g., are these groups linear?)

Similar questions are also formulated in the context of birational self-map groups of algebraic varieties.

It is clear that question (Q1) (but not (Q2)) stands only for “large” groups  $S$ , in particular, nonlinear ones. Generally speaking, the answer to it is no.<sup>1</sup> Finding for a given  $S$  the varieties  $Z$  such that the answer is yes serves not only as a source of information about  $\text{Aut}(Z)$ , but also as the method of obtaining essential information about the structure of  $S$  (see [BL83], [Ma81], [CX18]).

In this paper, we explore the case  $S = \text{Aut}(F_n)$ , where  $F_n$  is a free group of rank  $n < \infty$ . To this end, we consider a general construction that assigns to any finitely generated group  $\Sigma$  a family of algebraic varieties  $Z$  endowed with an action of  $\text{Aut}(\Sigma)$  by biregular automorphisms. Our results concern each of questions (Q1), (Q2), (Q3). The main for us is question (Q1), i.e., that of faithfulness of the action of  $\text{Aut}(F_n)$  on  $Z$  which means that the homomorphism  $\text{Aut}(F_n) \rightarrow \text{Aut}(Z)$  defining the action is an embedding.

Here is the construction. Let  $\Sigma$  and  $G$  be the groups, and let

$$X := \text{Hom}(\Sigma, G). \quad (1)$$

For any  $\sigma \in \text{End}(\Sigma)$ ,  $\gamma \in \text{End}(G)$ , put

$$\sigma_X: X \rightarrow X, \quad x \mapsto x \circ \sigma, \quad \gamma_X: X \rightarrow X, \quad x \mapsto \gamma \circ x. \quad (2)$$

If  $\sigma \in \text{Aut}(\Sigma)$  and  $\gamma \in \text{Aut}(G)$ , then  $\sigma_X$  and  $\gamma_X$  are invertible (their inverses  $\sigma_X^{-1}$  and  $\gamma_X^{-1}$  are respectively  $(\sigma^{-1})_X$  and  $(\gamma^{-1})_X$ ), and the mapping

$$(\text{Aut}(\Sigma) \times \text{Aut}(G)) \times X \rightarrow X, \quad (\sigma\gamma, x) \mapsto (\sigma_X^{-1} \circ \gamma_X)(x) \quad (3)$$

is an action on  $X$  of the group  $\text{Aut}(\Sigma) \times \text{Aut}(G)$  (whose factors are naturally identified with its subgroups).

Below, when considering the action on  $X$  of a subgroup of this group, the restriction of action (3) on it is always meant. The actions of  $\text{Aut}(\Sigma)$  and  $\text{Aut}(G)$  commute with each other.

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<sup>1</sup>E.g., in view of [CX18, Thm. C], even in the context of birational self-map groups, the answer is negative if  $S$  is an infinite simple torsion group with Kazhdan’s property (T) (such a group exists, see [Ki94, Sect. 5]).

If  $\Sigma$  is a finitely generated group, and  $G$  is an algebraic group, then  $X$  is endowed with the structure of an algebraic variety so that all  $\sigma_X$  and  $\gamma_X$  lie in  $\text{Aut}(X)$ . Let  $R$  be an algebraic subgroup of  $\text{Aut}(G)$ , for whose action on  $X$  there is a categorical quotient

$$\pi_{X//R}: X \rightarrow X//R \quad (4)$$

in the sense of geometric invariant theory (see [MF82, Def. 05], [PV94, Def. 4.5]). The following two cases are the main examples when this quotient exists (see Proposition 4.1 below):

- (F)  $R$  is finite;
- (R)  $G$  is affine and  $R$  is reductive.

Since the actions of  $\text{Aut}(\Sigma)$  and  $R$  on  $X$  commute, it follows from the definition of categorical quotient that for every  $\sigma \in \text{Aut}(\Sigma)$ , the automorphism  $\sigma_X$  of the variety  $X$  descends to a uniquely defined automorphism  $\sigma_{X//R}$  of the variety  $X//R$  having the property

$$\pi_{X//R} \circ \sigma_X = \sigma_{X//R} \circ \pi_{X//R}. \quad (5)$$

The map

$$\text{Aut}(\Sigma) \rightarrow \text{Aut}(X//R), \quad \sigma \mapsto \sigma_{X//R}^{-1} \quad (6)$$

is a group homomorphism. It determines an action of  $\text{Aut}(\Sigma)$  on  $X//R$  by biregular automorphisms. In view of (5), the morphism  $\pi_{X//R}$  is  $\text{Aut}(\Sigma)$ -equivariant.

In the present paper, for  $\Sigma = F_n$ , we consider the problem of classifying pairs  $(G, R)$  such that the action of  $\text{Aut}(\Sigma)$  on  $X//R$  is *faithful*. Our main results concern case (F).<sup>2</sup> This problem is related to question (Q1). We apply our results to questions (Q2), (Q3) as well. These results consist of the following.

The first is the faithfulness criterion for the action of  $\text{Aut}(F_n)$  on  $X//R$  in case (F).

**Theorem 1.1.** *Let  $G$  be an algebraic group (not necessarily connected or affine),  $X = \text{Hom}(F_n, G)$ ,  $n \geq 2$ , and let  $R$  be a finite subgroup of  $\text{Aut}(G)$ . The following properties are equivalent:*

- (a) *the action of  $\text{Aut}(F_n)$  on  $X//R$  is faithful;*
- (b) *the connected component of the identity of the group  $G$  is non-solvable.*

Corollary 1.2 describes the applications of Theorem 1.1 to questions (Q1)–(Q3), namely, to the problem of linearity of automorphism groups

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<sup>2</sup>In [Po23], we consider the situation of case (R) where  $G$  is connected and semi-simple and  $R$  is the image in  $\text{Int}(G)$  of a closed subgroup of a maximal torus of  $G$ . We prove the faithfulness of the action of  $\text{Aut}(F_n)$  on  $X//R$  in this case.

of algebraic varieties (considered in [CD13, Prop. 5.1], [Co17], [Ca12]), and to the problem of describing subgroups of the Cremona groups.

**Corollary 1.2.** *In the notation of Theorem 1.1, let the connected component of the identity of the group  $G$  be nonsolvable. Then*

- (a)  $\text{Aut}(X//R)$  contains the following groups:
  - $\text{Aut}(F_n)$ ,
  - $F_2$ ,
  - the braid group  $B_n$  on  $n$  strands if  $n \geq 3$ ;
- (b)  $\text{Aut}(X//R)$  is nonamenable and, if  $n \geq 3$ , nonlinear.

Among the varieties  $X//R$  from Corollary 1.2 whose automorphism group contains  $\text{Aut}(F_n)$ ,  $F_2$  and  $B_n$ , there are both rational and nonrational (and even not stably rational), namely:

**Proposition 1.3.** *Let, in the notation of Theorem 1.1, the group  $G$  be connected and the group  $R$  be trivial. Then the variety  $X//R$  is rational if  $G$  is affine and nonunirational, if  $G$  is nonaffine.*

In general case, the rationality of the variety  $X//R$  for a connected reductive  $G$  and  $R = \text{Int}(G)$  is the old problem open even for  $n = 2$  and  $G = \text{GL}_d$  with  $d \geq 5$  (see [Po94, (1.5.2)], [DF04, pp. 190–191]).

In view of Proposition 1.3, if  $G$  is nonaffine, then the variety  $X//R$  with trivial  $R$  is not stably rational. For nontrivial finite  $R$ , the variety  $X//R$  with the faithful action of  $\text{Aut}(F_n)$  may be not stably rational even if  $G$  is affine. Our second main result is Theorem 1.4 giving the construction of such affine  $X//R$  with a connected reductive  $G$ . Its proof also uses Theorem 1.1.

**Theorem 1.4.** *For every prime number  $p \neq \text{char}(k)$ , there is a finite  $p$ -group  $K$ , having the following property. Let  $V$  be a finite-dimensional vector space over  $k$ , and let  $\iota: K \hookrightarrow \text{GL}(V)$  be a group embedding for which  $\iota(K)$  has no nontrivial center elements of the group  $\text{GL}(V)$  (such pairs  $(V, \iota)$  exist for any finite group  $K$ ). Let  $X = \text{Hom}(F_n, \text{GL}(V))$ ,  $n \geq 2$ , and let  $R$  be the image of the group  $\iota(K)$  under the canonical homomorphism  $\text{GL}(V) \rightarrow \text{Int}(\text{GL}(V))$ . Then  $X//R$  is nonrational (and even not stably rational) affine algebraic variety, on which the group  $\text{Aut}(F_n)$  acts faithfully.*

Examples of groups  $K$  from Theorem 1.4 can be explicitly specified using generators and relations (see Remark 10.2 below).

Our third main result concerns question (Q2) and, in particular, gives upper bounds for “extremal” parameter values for embeddings of  $\text{Aut}(F_n)$  and  $B_n$  into automorphism groups of algebraic varieties.

**Theorem 1.5.** *Keep the notation of Theorem 1.1. Let  $n \geq 2$ ,  $G = \mathrm{SL}_2$  or  $\mathrm{PSL}_2$ , and let  $R$  be finite. Then  $X//R$  is an irreducible rational affine  $3n$ -dimensional algebraic variety, whose automorphism group contains  $\mathrm{Aut}(F_n)$ .*

Note that the variety  $X//R$  from Theorem 1.5 in the case of trivial  $R$  and  $G = \mathrm{SL}_2$  (respectively,  $\mathrm{PSL}_2$ ) is the product of  $n$  copies of the smooth affine quadric  $Q$  in  $\mathbb{A}^4$  given by the equation  $x_1x_2 + x_3x_4 = 1$  (respectively,  $n$  copies of  $Q/I$ , where  $I$  is the group, generated by the automorphism  $(a, b, c, d) \mapsto (-a, -b, -c, -d)$ ).

**Definition 1.6.** For any group  $S$ , denote by  $\mathrm{Var}_k(S)$  (respectively,  $\mathrm{Crem}_k(S)$ ) the minimal dimension of (defined over  $k$ ) irreducible algebraic varieties  $Z$  (respectively, the ranks of the Cremona groups  $C$ ) such that  $S$  embeds into  $\mathrm{Aut}(Z)$  (respectively, into  $C$ ). If there are no such  $Z$  (respectively,  $C$ ), then set  $\mathrm{Var}_k(S) = \infty$  (respectively,  $\mathrm{Crem}_k(S) = \infty$ ).

Groups  $S$  with  $\mathrm{Var}_k(S) = \mathrm{Crem}_k(S) = \infty$  exist (see footnote<sup>1</sup>).

**Corollary 1.7.** *Let  $S = \mathrm{Aut}(F_n)$  with  $n \geq 1$  or  $B_n$  with  $n \geq 3$ . Then*

$$\mathrm{Var}_k(S) \leq 3n \quad \text{and} \quad \mathrm{Crem}_k(S) \leq 3n. \quad (7)$$

The upper bounds (7) for  $S = B_n$  strengthen the ones following from the paper by D. Krammer [Kr02], where embeddability of  $B_n$  into  $\mathrm{GL}_{n(n-1)/2}$  was proved (which yields the upper bound  $n(n-1)/2$ ).<sup>3</sup>

A special case of the described construction, where  $R = \mathrm{Int}(G)$  with reductive  $G$ , is explored in many publications, starting essentially with the paper by Vogt of 1889. The subjects of these studies are: (a) applications to the theory of deformations of hyperbolic structures on topological surfaces, see [Go09] (in this case,  $\Sigma$  is the fundamental group of the surface,  $G = \mathrm{SL}_2(\mathbb{C})$ , and  $X//R$  is the “variety of characters” of  $\Sigma$ ); (b) dynamic properties of the action of  $\mathrm{Aut}(\Sigma)$  on  $X//R$ , see [Go97], [Go06], [Ca13<sub>1</sub>]; (c) for  $\Sigma = F_n$ , finding the equations of the “variety of characters” and describing the kernel of the action of  $\mathrm{Aut}(F_n)$  on it, see [Ho72], [Ho75], [Ma80].

For the purposes of this paper, this special case is of little interest, since the group  $\mathrm{Int}(\Sigma)$  is always contained in the kernel of the action of  $\mathrm{Aut}(\Sigma)$  on  $X//\mathrm{Int}(G)$ , and therefore, for  $\Sigma = F_n$ , the faithfulness of this action is possible only for  $n = 1$  (when  $\mathrm{Aut}(F_n)$  is a group of order 2). For  $n = 1$  and connected  $G$ , the rare cases when this action is faithful are described the following theorem.

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<sup>3</sup>As the referee noted, a lower bound for  $\mathrm{Var}_k(\mathrm{Aut}(F_n))$  can be obtained using the methods of [CX18].

**Theorem 1.8.** *Let  $G$  be a connected reductive algebraic group,  $X = \text{Hom}(F_n, G)$  and  $R = \text{Int}(G)$ . The action of  $\text{Aut}(F_n)$  on  $X//R$  is faithful if and only if  $n = 1$  and  $G$  contains a connected simple normal subgroup any of the following types:*

$$A_\ell \text{ with } \ell \geq 2, \quad D_\ell \text{ with odd } \ell, \quad E_6. \quad (8)$$

The proof of Theorem 1.1 is given in Sections 7 and 8, of Corollary 1.2 in Section 9, of Theorem 1.4 in Section 10, of Theorem 1.5 and Corollary 1.7 in Section 11, and of Theorem 1.8 in Section 12.

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## 2. Conventions and notation.

If  $X$  is an algebraic variety (respectively, a differentiable manifold), then  $\text{Aut}(X)$  denotes the group its regular automorphisms (respectively, diffeomorphisms).

Groups are considered in multiplicative notation. The identity element of a group is denoted by  $e$  (it is clear from the context which group is meant).

The claim that the group  $G$  contains the group  $H$ , means the existence of a group embedding  $\iota: H \hookrightarrow G$ , by which  $H$  is identified with  $\iota(H)$ .

$\mathcal{C}(G)$  is the center of the group  $G$ .

$\mathcal{C}_G(g)$  is the centralizer in  $G$  of an element  $g \in G$ .

$\text{int}_g$  is the inner group automorphism determined by an element  $g$ .

$\langle g_1, \dots, g_m \rangle$  is the group generated by the elements  $g_1, \dots, g_m$ .

$G^0$  is the connected component of the identity of an algebraic group or a real Lie group  $G$ .

The Lie algebra of an algebraic group is denoted by the lowercase Gothic version of the letter denoting that group.

$\underline{G}$  is the underlying variety (or manifold) of an algebraic group (or real Lie group)  $G$ .

$\text{Aut}(G)$ ,  $\text{Int}(G)$ ,  $\text{Out}(G)$ , and  $\text{End}(G)$  are respectively the group of automorphisms, inner automorphisms, outer automorphisms, and the monoid of endomorphisms of a group  $G$ . If  $G$  is an algebraic group or a real Lie group, then by its automorphisms we mean automorphisms in the category of algebraic groups or real Lie groups, so that  $\text{Aut}(G)$  denotes the intersection of  $\text{Aut}(\underline{G})$  with the automorphism group of the

abstract group  $G$ . If an algebraic group  $H$  faithfully acts by automorphisms of an algebraic group  $G$  and the mapping  $H \times G \rightarrow G$  defining this action is a morphism of algebraic varieties, then  $H$  is called an algebraic subgroup of  $\text{Aut}(G)$ .

The reductivity of an affine algebraic group  $G$  does not assume its connectedness and is understood in the sense of [MF82], i.e., as the triviality of the unipotent radical of the group  $G^0$ .

**3. Fixing a system of generators of  $\Sigma$ .** Consider a group  $G$  and a finitely generated group  $\Sigma$ . Let  $s_1, \dots, s_n$  be a system of generators of  $\Sigma$  and let  $\varphi: F_n \rightarrow \Sigma$  be the epimorphism defined by the equalities  $\varphi(f_j) = s_j$  for every  $j$ .

For any group  $H$  and any  $w \in F_n$ ,  $h = (h_1, \dots, h_n) \in H^n$ , denote by  $w(h) = w(h_1, \dots, h_n)$  the image of  $w$  under the (unique) homomorphism  $F_n \rightarrow H$  mapping  $f_j$  to  $h_j$  for every  $j$ . In other words, if we write  $w$  as a word

$$f_{i_1}^{\varepsilon_1} \cdots f_{i_d}^{\varepsilon_d}, \quad \text{where } \varepsilon_j \in \mathbb{Z}, \quad (9)$$

(a noncommutative Laurent monomial in  $f_1, \dots, f_n$ ), then  $w(h)$  is obtained by replacing  $f_j$  with  $h_j$  in (9) for each  $j$ .

The map

$$X := \text{Hom}(\Sigma, G) \rightarrow G^n, \quad x \mapsto (x(s_1), \dots, x(s_n)) \in G^n \quad (10)$$

is an injection. Its image is the set

$$\{g \in G^n \mid w(g) = e \text{ for all } w \in \text{Ker}(\varphi)\}. \quad (11)$$

In the rest of this paper, if necessary and without reminders, we *identify*  $X$  with the set (11) using the injection (10). For  $\Sigma = F_n$  and  $s_j = f_j$  for all  $j$ , we have  $X = G^n$ , so in this case  $X$  is the group (with the componentwise multiplication).

Let  $g = (g_1, \dots, g_n) \in X \subseteq G^n$  and  $t \in \Sigma$ . It follows from (11) that the element  $w(g) \in G$  is the same for all  $w \in \varphi^{-1}(t)$ . Denote it by  $t(g)$ . In other words, writing  $t$  as a noncommutative Laurent monomial in  $s_1, \dots, s_n$  and replacing  $s_j$  in this monomial by  $g_j$  for each  $j$ , we obtain, regardless of the chosen monomial,  $t(g)$ . In this notation, for any  $\sigma \in \text{End}(\Sigma)$ ,  $\gamma \in \text{End}(G)$ , formulas (2) are rewritten as follows:

$$\begin{aligned} \sigma_X: X &\rightarrow X, & g = (g_1, \dots, g_n) &\mapsto (\sigma(s_1)(g), \dots, \sigma(s_n)(g)), \\ \gamma_X: X &\rightarrow X, & (g_1, \dots, g_n) &\mapsto (\gamma(g_1), \dots, \gamma(g_n)). \end{aligned} \quad (12)$$

If  $G$  is an algebraic group, then  $X$  is closed in  $G^n$  and therefore endowed with the structure of an algebraic variety. This structure does not depend on the choice of systems of generators, and the maps (12)

are morphisms. If  $\Sigma = F_n$  and  $G$  is a real Lie group, then (12) are the differentiable mappings  $G^n \rightarrow G^n$ .

Some properties, selectively used below and in [Po21], [Po23], [Po22], are brought together in Proposition 3.1 for ease of reference.

**Proposition 3.1.** *We maintain the notation introduced above in Section 3. Let  $\sigma$  and  $\tau \in \text{End}(F_n)$ . Then the following hold.*

- (a)  $(\sigma \circ \tau)_X = \tau_X \circ \sigma_X$ .
- (b)  $e_X = \text{id}$ .
- (c)  $\sigma_X(X \cap S^n) \subseteq X \cap S^n$  for any subgroup  $S$  of the group  $G$ .
- (d) Let  $\theta: G \rightarrow H$  be a group homomorphism and let  $Y := \text{Hom}(\Sigma, H) \subseteq H^n$ . Then the map

$$\theta_n: X \rightarrow Y, \quad (g_1, \dots, g_n) \mapsto (\theta(g_1), \dots, \theta(g_n))$$

is  $\text{End}(\Sigma)$ -equivariant, i.e.,  $\theta_n \circ \sigma_X = \sigma_Y \circ \theta_n$ .

- (e) If  $\sigma = \text{int}_t$  for  $t \in \Sigma$ , then the following properties of an element

$$x = (g_1, \dots, g_n) \in X \subseteq G^n \tag{13}$$

are equivalent:

- (e<sub>1</sub>)  $\sigma_X(x) = x$ ;
- (e<sub>1</sub>)  $t(x) \in \bigcap_{i=1}^n \mathcal{C}_G(g_i)$ .

In statements (f) and (g), it is assumed that  $\Sigma = F_n$ .

- (f) The following properties of element (13) are equivalent:
  - (f<sub>1</sub>)  $\sigma_X(x) = x$  for each  $\sigma \in \text{Aut}(F_n)$ ;
  - (f<sub>2</sub>) if  $n > 1$ , then  $g_1 = \dots = g_n = e$ , and if  $n = 1$ , then  $g_1^2 = e$ .
- (g) The multiplication in  $X = G^n$  has the property:

$$\sigma_X(xz) = \sigma_X(x)\sigma_X(z) \text{ for all } x \in X = G^n, z \in \mathcal{C}(X).$$

In particular, the restriction of  $\sigma_X$  to the group  $\mathcal{C}(X)$  is its endomorphism.

*Proof.* Statement (f) follows from the fact that for  $n = 1$ , the only nonidentity element of  $\text{Aut}(F_n)$  maps  $f_1$  to  $f_1^{-1}$ , and for  $n \geq 2$ , for any  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ , the element  $\sigma_{ij} \in \text{End}(F_n)$  defined by the formula

$$\sigma_{ij}(f_l) = \begin{cases} f_l & \text{for } l \neq i, \\ f_i f_j & \text{for } l = i, \end{cases}$$

lies in  $\text{Aut}(F_n)$ .

The rest of the statements follow directly from the definitions and the fact that each element of  $\Sigma$  is written as a noncommutative Laurent monomial in  $s_1, \dots, s_n$ .  $\square$



**4. The existence of categorical quotient.**

**Proposition 4.1.** *Let  $\Sigma$  be a finitely generated group, let  $G$  be an algebraic group (not necessarily connected or affine), let  $R$  be an algebraic subgroup of  $\text{Aut}(G)$ , and let  $X = \text{Hom}(\Sigma, G)$ . The categorical quotient (4) exists in each of the following two cases:*

- (F)  $R$  is finite;
- (R)  $G$  is affine and  $R$  is reductive.

*If (F) holds, then the categorical quotient (4) is the geometric quotient. If (R) holds, then the variety  $X//R$  is affine. In each of cases (F) and (R), the morphism  $\pi_{X//R}$  is surjective.*

*Proof.* According to [Ba54], the variety  $\underline{G}$  is quasi-projective. Hence,  $X$ , being closed in the product of several copies of  $\underline{G}$ , is quasi-projective as well. This implies the existence of the geometric factor in case (F) (see [Se97, Chap. III, Sect. 12, Prop. 19, Ex. 2]). This factor is automatically categorical with the surjective morphism  $\pi_{X//R}$  (see [PV94, 4.3], [Bo91, Sect. II, §6]).

In case (R), the variety  $\underline{G}$  is affine. In view of the remark on closedness made in the previous paragraph,  $X$  is affine as well. According to [MF82, Chap. 1, §2], from this and the reductivity of  $R$  it follows the existence of the categorical quotient (4), the affineness of  $X//R$ , and the surjectivity of  $\pi_{X//R}$ .  $\square$

**5. The kernel of the action of  $\text{Aut}(\Sigma)$  on  $\text{Hom}(\Sigma, G)//R$  in cases (F) and (R): geometric description.** Let  $\Sigma$  be a finitely generated group and let  $G$  be an algebraic group (not necessarily connected or affine). Having fixed a system of  $n$  generators in  $\Sigma$ , we identify  $X = \text{Hom}(\Sigma, G)$  with a closed subset of  $G^n$  as described in Section 3. For any  $w \in \Sigma$ ,  $\gamma \in \text{Aut}(G)$  and  $i \in \{1, \dots, n\}$ , the closed set

$$X_{w,\gamma,i} := \{x = (g_1, \dots, g_n) \in X \mid w(x) = \gamma(g_i)\} \quad (14)$$

is the fiber over  $e$  of the morphism

$$X \rightarrow G, \quad x = (g_1, \dots, g_n) \mapsto w(x)\gamma(g_i)^{-1}.$$

As it contains  $(e, \dots, e)$ , it is nonempty.

From (12) and (14) it follows that for any  $\sigma \in \text{Aut}(\Sigma)$  we have

$$\bigcap_{i=1}^n X_{\sigma(f_i),\gamma,i} = \{x \in X \mid \sigma_X(x) = \gamma(x)\}. \quad (15)$$

The following Lemmas 5.1 and 5.2 describe the kernel of the action of  $\text{Aut}(\Sigma)$  on  $\text{Hom}(\Sigma, G)//R$  respectively in cases (F) and (R).

**Lemma 5.1.** *We retain the notation and conventions introduced in this section. Let  $R$  be a finite subgroup of  $\text{Aut}(G)$ . The following properties of an element  $\sigma \in \text{Aut}(\Sigma)$  are equivalent:*

- (a)  $\sigma$  lies in the kernel of the action of  $\text{Aut}(\Sigma)$  on  $X//R$ ;
- (b)  $\sigma_X(\mathcal{O}) = \mathcal{O}$  for every  $R$ -orbit  $\mathcal{O}$  in  $X$ ;
- (c) for every irreducible component  $Y$  of the variety  $X$  there is an element  $\gamma \in R$  such that

$$Y \subseteq \bigcap_{i=1}^n X_{\sigma(f_i), \gamma, i}. \quad (16)$$

*Proof.* In view of Proposition 4.1, each fiber of the morphism  $\pi_{X//R}$  is an  $R$ -orbit in  $X$  and vice versa. Since the actions of  $\text{Aut}(\Sigma)$  and  $R$  on  $X$  commute, it follows from (5) that for each point  $b \in X//R$ , the restriction of the morphism  $\sigma_X$  to the orbit  $\pi_{X//R}^{-1}(b)$  is its  $R$ -equivariant isomorphism with the orbit  $\pi_{X//R}^{-1}(\sigma_{X//R}(b))$ . This proves the equivalence of the conditions (a) and (b) and, given (15), their equivalence to the equality

$$X = \bigcup_{\gamma \in R} \left( \bigcap_{i=1}^n X_{\sigma(f_i), \gamma, i} \right). \quad (17)$$

(a) $\Rightarrow$ (c) If the equality (17) holds, then each irreducible component  $Y$  of  $X$  is the union of closed subsets of the form

$$Y \cap \left( \bigcap_{i=1}^n X_{\sigma(f_i), \gamma, i} \right), \quad \text{where } \gamma \in R. \quad (18)$$

Since the group  $R$  is finite, there are finitely many of these subsets. The irreducibility of  $Y$  therefore implies that  $Y$  coincides with one of them. Hence, (16) holds for some  $\gamma \in R$ .

(c) $\Rightarrow$ (a) If (c) holds, then the union of all irreducible components of  $X$  lies on the right-hand side of the equality (17), i.e., this right-hand side contains  $X$ . The reverse inclusion is obvious. Hence, the equality (17) holds.  $\square$

**Lemma 5.2.** *Retaining the notation and conventions introduced in this section, we assume that the group  $G$  is affine. Let  $R$  be a reductive algebraic subgroup of  $\text{Aut}(G)$ . The following properties of an element  $\sigma \in \text{Aut}(\Sigma)$  are equivalent:*

- (a)  $\sigma$  lies in the kernel of the action of  $\text{Aut}(\Sigma)$  on  $X//R$ ;
- (b)  $\sigma_X(\mathcal{O}) = \mathcal{O}$  for every closed  $R$ -orbit  $\mathcal{O}$  in  $X$ ;
- (c) each closed  $R$ -orbit in  $X$  belongs to the set

$$\bigcup_{\gamma \in R} \left( \bigcap_{i=1}^n X_{\sigma(f_i), \gamma, i} \right). \quad (19)$$

*Proof.* For every  $b \in X//R$ , the fiber  $\pi_{X//R}^{-1}(b)$  of the surjective (see Proposition 4.1) morphism  $\pi_{X//R}$  is an  $R$ -invariant closed subset of  $X$ , which contains a unique closed  $R$ -orbit  $\mathcal{O}_b$  (see [MF82, §2 and Append. 1B]). The restriction of  $\sigma_X$  to  $\pi_{X//R}^{-1}(b)$  is an  $R$ -equivariant isomorphism with the fiber  $\pi_{X//R}^{-1}(\sigma_{X//R}(b))$ . In view of the uniqueness of closed orbits in the fibers, this means that  $\sigma_X(\mathcal{O}_b) = \mathcal{O}_{\sigma_{X//R}(b)}$ . Therefore, the

equalities  $\sigma_{X//S}(b) = b$  and  $\sigma_X(\mathcal{O}_b) = \mathcal{O}_b$  are equivalent. This proves (a) $\Leftrightarrow$ (b). In turn, this and (15) imply (a) $\Leftrightarrow$ (c).  $\square$

**Corollary 5.3.** *If the conditions of Lemma 5.2 hold and  $R = \text{Int}(G)$ , then  $\text{Int}(\Sigma)$  lies in the kernel of the action of  $\text{Aut}(\Sigma)$  on  $X//R$ .*

*Proof.* If  $\sigma \in \text{Int}(\Sigma)$ , then (12) implies that  $x$  and  $\sigma_X(x)$  lie in the same  $R$ -orbit for each  $x \in X$ . The assertion therefore follows from the equivalence of conditions (a) and (b) in Lemma 5.2.  $\square$

**6. The faithfulness of the action of  $\text{Aut}(F_n)$  on  $\text{Hom}(F_n, G)$ : algebraic criterion.**

**Theorem 6.1.** *Let  $G$  be a group and let  $X = \text{Hom}(F_n, G)$ . The following properties are equivalent:*

- (a) *the action of  $\text{Aut}(F_n)$  on  $X$  is faithful;*
- (b) *if  $n \geq 2$ , then in an  $n$ -letter alphabet there is no nonempty irreducible word that is the identity in  $G$ , and if  $n = 1$ , then  $G$  contains an element of order  $\geq 3$ .*

*Proof.* We use the notation of Section 3 with  $\Sigma = F_n$  and  $s_j = f_j$  for all  $j$ .

For  $n = 1$ , the equivalence of (a) and (b) follows from Proposition 3.1(f). Consider the case  $n \geq 2$ .

(a) $\Rightarrow$ (b) Suppose, arguing by contradiction, that (a) holds, but in an  $n$ -letter alphabet there exists a nonempty irreducible word that is the identity in  $G$ . So there is a nonidentity element  $w \in F_n$  such that

$$w(x) = e \text{ for each } x \in X. \tag{20}$$

The element  $\sigma := \text{int}_w \in \text{Aut}(F_n)$  is different from the identity because the group  $\mathcal{C}(F_n)$  is trivial for  $n \geq 2$  (cf. [LS77, Chap. I, Prop. 2.19]), and  $w \neq e$ . However, from (20) it follows that  $\sigma_X(x) = x$  for each  $x \in X$ , i.e., that  $\sigma$  lies in the kernel of the action of  $\text{Aut}(F_n)$  on  $X$ . This contradicts (a).

(b) $\Rightarrow$ (a) Suppose, arguing by contradiction, that (b) holds, but the kernel of the action of  $\text{Aut}(F_n)$  on  $X$  contains a nonidentity element  $\sigma \in \text{Aut}(F_n)$ , so that we have (see (12)),

$$\sigma(f_i)(x) = f_i(x) \text{ for all } x \in X \text{ and } i. \tag{21}$$

In view of  $\sigma \neq e$ , there exists  $f_j$  for which  $\sigma(f_j) \neq f_j$ , i.e.,  $w := \sigma(f_j)f_j^{-1}$  is a nonidentity element of the group  $F_n$ . At the same time, (21) implies that this  $w$  satisfies condition (20). Therefore, in the alphabet  $f_1, \dots, f_n$  there is a nonempty irreducible word that is the identity in  $G$ . This contradicts (b).  $\square$

**Corollary 6.2.** *For each virtually solvable group  $G$ , the action of  $\text{Aut}(F_n)$  on  $X := \text{Hom}(F_n, G)$ ,  $n \geq 2$ , is nonfaithful.*

*Proof.* By the definition of virtual solvable group,  $G$  has a solvable subgroup  $S$  of a finite index  $d$ . We can (and shall) assume that  $S$  is normal, replacing it with the intersection of all subgroups conjugate to it. Since  $S$  is solvable, in the alphabet of two letters  $x, y$  there exists a nonempty irreducible word  $r(x, y)$ , which is the identity in  $S$  (see [Ne67, 14.65]). It follows from the normality of  $S$  that  $g^d \in S$  for each  $g \in G$ . Hence, the nonempty irreducible word  $r(x^d, y^d)$  is the identity in  $G$ . The claim now follows from Theorem 6.1.  $\square$

**7. Proof of Theorem 1.1: the case of trivial subgroup  $R$ .** To prove Theorem 1.1, we first need to consider a special case of trivial subgroup  $R$ . We will prove a more general statement concerning not only algebraic groups, but also real Lie groups.

**Theorem 7.1.** *Let  $X = \text{Hom}(F_n, G)$ ,  $n \geq 2$ , and let  $G$  be either an algebraic group (not necessarily connected or affine) or a real Lie group with a finite number of connected components. Then the following properties are equivalent:*

- (a) *the action of  $\text{Aut}(F_n)$  on  $X$  is faithful;*
- (b) *the group  $G^0$  is nonsolvable.*

*If  $G$  is a real Lie group, then the implication (b) $\Rightarrow$ (a) is true even without the condition that the number of its connected components is finite.*

*Proof.* If  $G$  is a real Lie group, then

$$[G : G^0] < \infty \tag{22}$$

by the condition. If  $G$  is an algebraic group, then (22) is satisfied automatically. It follows from (22) that if  $G^0$  is solvable, then  $G$  is virtually solvable. Together with Corollary 6.2, this proves implication (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) Let the group  $G^0$  be nonsolvable. In view of Theorem 6.1, it is required to prove that in an alphabet of  $n$  letters there is no nonempty irreducible word that is the identity relation in  $G$ . Arguing by contradiction, suppose that such a word exists. Hence, there is a nontrivial element  $w \in F_n$  with the property (20).

Let  $G$  be a connected real Lie group. Then, due to nonsolvability,  $G^0$  contains a free subgroup of rank  $n$  (see [Ep71, Thm.]). Let  $g_1, \dots, g_n$  be its free system of generators. Then  $w(g_1, \dots, g_n) = e$  due to (20), which contradicts the absence of nontrivial relations between  $g_1, \dots, g_n$ .

Let now  $G$  be an algebraic group. By Chevalley's theorem, the algebraic group  $G^0$  contains the largest connected affine normal subgroup

$G_{\text{aff}}^0$ , and  $G^0/G_{\text{aff}}^0$  is an Abelian variety. Since the group  $G^0$  is nonsolvable and the group  $G^0/G_{\text{aff}}^0$  is commutative (and therefore solvable), the group  $G_{\text{aff}}^0$  is nonsolvable. Hence,  $G_{\text{aff}}^0$  does not coincide with its radical  $\text{Rad}(G_{\text{aff}}^0)$ , and therefore,  $G_{\text{aff}}^0/\text{Rad}(G_{\text{aff}}^0)$  is a nontrivial connected semi-simple algebraic group. This reduces the proof to the case where  $G$  is a nontrivial connected semisimple algebraic group. We will therefore further assume that this condition is met. In view of [Bo83, Thm. B], from it and the inequality  $n \geq 2$  it follows that the morphism

$$X \rightarrow G, \quad x \mapsto w(x)$$

is dominant. In view of (20), this means that the group  $G$  is trivial, which is a contradiction.  $\square$

**Remark 7.2.** If  $G$  is an nonsolvable algebraic group and the field  $k$  is uncountable, then  $G$  contains a free subgroup of any finite rank (see [BGGT12, Thm. 1.1], [BGGT15, App. D]), which means that the same proof of the implication (b) $\Rightarrow$ (a) in Theorem 7.1 as in the case of a real Lie group goes through. This proof is given in the first version [Po21] of the present paper. However, in the general case,  $G$  may not contain a free subgroup (for example, this is the case for  $G = \text{SL}_d$  if  $k$  is the algebraic closure of a finite field, since then the order of every element of  $\text{SL}_d$  is finite).

**Remark 7.3.** Without the condition that the number of connected components is finite, the implication (a) $\Rightarrow$ (b) in Theorem 7.1 is false. Indeed, take as  $G$  the group  $F_n$  considered as a real Lie group with  $G^0 = \{e\}$ . Then  $\text{id}_{F_n} \in \text{Hom}(F_n, F_n) = X$  and, for any  $\sigma \in \text{Aut}(F_n)$ , we have  $\sigma_X(\text{id}_{F_n}) = \sigma$  (see (2)). Therefore, (a) holds, but (b) does not.

**8. Proof of Theorem 1.1: general case.** In view of the surjectivity and  $\text{Aut}(F_n)$ -equivariance of the morphism  $\pi_{X//R}$  (see (4)), the implication (a) $\Rightarrow$ (b) follows from Theorem 7.1.

(b) $\Rightarrow$ (a) Let the group  $G^0$  be nonsolvable. Arguing by contradiction, suppose that a nonidentity element  $\sigma \in \text{Aut}(F_n)$  lies in the kernel of the action of  $\text{Aut}(F_n)$  on  $X//R$ . The variety  $X$  is isomorphic to  $\underline{G}^n$ . It is clear that  $(\underline{G}^0)^n$  is one of the irreducible components of the variety  $\underline{G}^n$ . By virtue of what was said in Section 3, this implies that  $X^0 := \text{Hom}(F_n, G^0)$  is an  $\text{Aut}(F_n)$ -invariant irreducible component of the variety  $X$ . In turn, in view of Lemma 5.1 and formulas (15), (12), this implies the existence of an element  $\gamma \in R$  such that for every  $i \in \{1, \dots, n\}$ , the following group identity holds in  $G^0$ :

$$\sigma(f_i)(g_1, \dots, g_n) = \gamma(g_i) \quad \text{for any } g_1, \dots, g_n \in G^0. \quad (23)$$

In particular, for every  $g \in G^0$ , the equality obtained by substituting  $g_1 = \cdots = g_n = g$  in (23) holds. Since  $\sigma(f_i)$  has the form (9), this means the existence of an integer  $d$  such that the following group identity holds:

$$g^d = \gamma(g) \quad \text{for each } g \in G^0. \quad (24)$$

Notice that

$$d \neq 1 \quad \text{and} \quad d \neq -1. \quad (25)$$

Indeed, if  $d = 1$  then from (24) and (23) it follows that  $\sigma_{X^0} = \text{id}_{X^0}$ , i.e.,  $\sigma$  lies in the kernel of the action of  $\text{Aut}(F_n)$  on  $X^0$ . Since  $\sigma$  is a nonidentity element, and the group  $G^0$  is nonsolvable, this contradicts Theorem 7.1.

If  $d = -1$ , then for any  $g, h \in G^0$ , the equality

$$h^{-1}g^{-1} = (gh)^{-1} \stackrel{(24)}{=} \gamma(gh) = \gamma(g)\gamma(h) \stackrel{(24)}{=} g^{-1}h^{-1}$$

holds, meaning that the group  $G^0$  is commutative contrary to its nonsolvability.

Further, for any positive integer  $m$ , we obtain from (24) by induction the following group identity:

$$g^{d^m} = \gamma^m(g) \quad \text{for each } g \in G^0. \quad (26)$$

Since the group  $R$  is finite, the order of  $\gamma$  is finite. Let  $m$  in (26) be equal to this order. Then (26) becomes the group identity

$$g^{d^m - 1} = e \quad \text{for every } g \in G^0. \quad (27)$$

Since  $d^m - 1 \neq 0$  due to (25), from (27) we infer that  $G^0$  is a torsion group whose element orders are bounded from above. Let us show that this contradicts the properties of the group  $G^0$ .

Indeed, as in the proof of Theorem 7.1 (whose notation we retain), the affine algebraic group  $G_{\text{aff}}^0$  is nonsolvable. Hence, it contains a nontrivial semisimple element, and therefore, a torus of positive dimension (see [Bo91, Thms. 4.4, 11.10]). But the set of orders of elements of the torsion subgroup of any torus of positive dimension is not bounded (see [Bo91, Prop. 8.9]). This gives the required contradiction.  $\square$

## 9. Proofs of Corollary 1.2 and Proposition 1.3.

*Proof of Corollary 1.2.* Statements (a) and (b) follow from Theorem 1.1 and the next Proposition 9.1.  $\square$

**Proposition 9.1.** *Assume that a group  $H$  contains  $\text{Aut}(F_n)$ . Then*

- (i)  $H$  contains  $F_2$  if  $n \geq 2$ ;
- (ii)  $H$  contains  $B_n$  if  $n \geq 3$ ;
- (iii)  $H$  is not amenable if  $n \geq 2$ ;

(iv)  $H$  is nonlinear if  $n \geq 3$ .

*Proof.*

If  $n \geq 2$ , then  $\mathcal{C}(F_n)$  is trivial and therefore,  $\text{Int}(F_n)$  is isomorphic to  $F_n$ . This gives (i).

If  $n \geq 3$ , then  $\text{Aut}(F_n)$  contains  $B_n$  (see [MKS66, Chap. 3, 3.7]) and is nonlinear (see [FP92]). This gives (ii) and (iv).

(i) implies (iii). □

*Proof of Proposition 1.3.* If  $G$  is affine, then the rationality of  $X = G^n$  follows from the rationality of  $\underline{G}$  (see [Bo91, Cor. 14.14]).

Let  $G$  be nonaffine. Arguing by contradiction, suppose that  $X = G^n$  is unirational. Let us use the notation of the proof of Theorem 7.1. The variety  $G/G_{\text{aff}}$  is unirational in view of the surjectivity of the composition of the following morphisms

$$X = G^n \xrightarrow{\alpha} G \xrightarrow{\beta} G/G_{\text{aff}},$$

where  $\alpha$  is a projection onto some factor, and  $\beta$  is the canonical projection. By the condition,  $G_{\text{aff}} \neq G$ , so that  $G/G_{\text{aff}}$  is a nontrivial Abelian variety. Since such varieties are nonunirational (see [Sh07, Chap. 3, Sect. 6.2, 6.4]), we get a contradiction. □

**10. Proof of Theorem 1.4.** We use in the proof of Theorem 1.4 the following known statement (see, e.g., [Po13, Thm. 1]).

**Lemma 10.1.** *If the field of invariant rational functions of some faithful linear action of a finite group on a finite-dimensional vector space over  $k$  is stably rational over  $k$ , then the same property holds for any other such action of this group.*

*Proof of Theorem 1.4.* Consider one of the pairs  $(K, \iota)$  found in [Sa84], where  $K$  is a finite group, and

$$\iota: K \hookrightarrow \text{GL}(V),$$

is a group embedding, where  $V$  is a finite-dimensional vector space over  $k$ , for which the field of  $\iota(K)$ -invariant rational functions on  $V$  is not stably rational over  $k$ .

In view of Lemma 10.1, replacing  $V$  and  $\iota$  if necessary, we can (and shall) assume that

$$\iota(K) \cap \mathcal{C}(\text{GL}(V)) = \{\text{id}_V\}. \tag{28}$$

Indeed, let  $L$  be a one-dimensional vector space over  $k$ . Since we have  $\mathcal{C}(\text{GL}(V \oplus L)) = \{c \cdot \text{id}_{V \oplus L} \mid c \in k, c \neq 0\}$ , the group embedding

$$\iota': K \hookrightarrow \text{GL}(V \oplus L), \quad f \mapsto \iota(f) \oplus \text{id}_L,$$

has the property  $\iota'(K) \cap \mathcal{C}(\mathrm{GL}(V \oplus L)) = \{\mathrm{id}_{V \oplus L}\}$ .

It follows from (28) that the diagonal linear action of  $\iota(K)$  on the vector space  $\mathrm{End}(V)^{\oplus n}$  by conjugation is faithful. Therefore, in view of Lemma 10.1, the field of  $\iota(K)$ -invariant rational functions on  $\mathrm{End}(V)^{\oplus n}$  is not stably rational over  $k$ . But  $\mathrm{GL}(V)^n$  is a  $\iota(K)$ -invariant open subset of  $\mathrm{End}(V)^{\oplus n}$ . It is  $\iota(K)$ -equivariantly isomorphic to the algebraic variety  $\mathrm{Hom}(F_n, \mathrm{GL}(V))$ . Hence the field of  $R$ -invariant rational functions on  $X$  is not stably rational over  $k$ . But this field is isomorphic to the field of rational functions on  $X//R$ , because, by Proposition 4.1, the categorical factor (4) is geometric. Hence the variety  $X//R$  is not stably rational. It is affine due to the affineness of  $\mathrm{GL}(V)$  (see Proposition 4.1). Finally, by Theorem 1.1, from the nonsolvability of  $\mathrm{GL}(V)$  with  $n \geq 2$  it follows that the action of  $\mathrm{Aut}(F_n)$  on  $X//R$  is faithful.  $\square$

**Remark 10.2.** Found in [Sa84], the first examples of groups, which can be taken as  $K$  in Theorem 1.4, have order  $p^9$ . At present, all groups of order  $p^5$  with the specified property have been found (See details and references in [Po13, p. 414, Rem.]). For example, for  $p \geq 5$ , one of them is the group  $K = \langle g_1, g_2, g_3, g_4, g_5 \rangle$  of order  $p^5$  given by the following conditions (in which  $[a, b] := a^{-1}b^{-1}ab$ ):

$$\begin{aligned} \mathcal{C}(F) &= \langle g_5 \rangle, \quad g_i^p = e \text{ for each } i, \\ [g_1, g_1] &= g_3, \quad [g_3, g_1] = g_4, \quad [g_4, g_1] = [g_3, g_2] = g_5, \quad [g_4, g_2] = [g_4, g_3] = e. \end{aligned}$$

## 11. Proofs of Theorem 1.5 and Corollary 1.7.

*Proof of Theorem 1.5.* In view of the finiteness of  $R$ , it follows from Proposition 4.1 that (4) is the geometric quotient, and the variety  $X//R$  is affine (and irreducible due to the connectedness of  $G$ ). In particular, the fibers of the surjective morphism (4) are  $R$ -orbits and therefore zero-dimensional. This implies the claim about the dimension, since  $\dim(G) = 3$  and  $X = G^n$ . It remains to prove the rationality.

In this case,  $\mathrm{Aut}(G) = \mathrm{Int}(G)$ . Consider the adjoint action of  $\mathrm{Int}(G)$  on  $\mathfrak{g}$  and the diagonal actions of the group  $\mathrm{Int}(G)$  on  $X = G^n$  and  $\mathfrak{g}^n := \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$  ( $n$  summands). According to [LPR06, Thm. 1.28],  $\mathrm{SL}_2$  and  $\mathrm{PSL}_2$  are the Cayley groups. Therefore, there exists an  $\mathrm{Int}(G)$ -equivariant (hence,  $R$ -equivariant) birational mapping

$$X \dashrightarrow \mathfrak{g}^n. \quad (29)$$

Consequently, the fields of  $R$ -invariant rational functions on  $X$  and  $\mathfrak{g}^n$  are isomorphic. Hence the geometric quotients  $X//R$  and  $\mathfrak{g}^n//R$  are birationally isomorphic. But the linearity of the action of  $R$  on  $\mathfrak{g}^n$  and the decomposition  $\mathfrak{g}^n = \mathfrak{g} \oplus \mathfrak{g}^{n-1}$  with the  $R$ -invariant summands imply, in view of the No-name lemma (see [Po13, Lem. 1], [PV94, Thm. 2.13]),



that  $\mathfrak{g}^n // R$  is birationally isomorphic to  $(\mathfrak{g} // R) \times \mathbb{A}^{(n-1)\dim(\mathfrak{g})}$ . Since  $\dim(\mathfrak{g}) = 3$  implies the rationality of  $\mathfrak{g} // R$  (see [Mi71, Thm. 2]), this shows that  $\mathfrak{g}^n // R$ , and therefore, also  $X // R$ , is rational.  $\square$

*Proof of Corollary 1.7.* This claim follows from Definition 1.6, Theorem 1.5, and Corollary 1.2.  $\square$

**12. Proof of Theorem 1.8.** For  $n \geq 2$ , the group  $\text{Int}(F_n)$  is nontrivial, and hence, by Corollary 5.3, the action of  $\text{Aut}(F_n)$  on  $X // R$  is nonfaithful.

Now, let  $n = 1$ , so that  $X = G$ , and  $\text{Aut}(F_n)$  is the group of order two. Let  $\sigma \in \text{Aut}(F_n)$ ,  $\sigma \neq e$ . Then  $\sigma(f_1) = f_1^{-1}$ , so  $\sigma_X(g) = g^{-1}$  for any  $g \in X$ . Each fiber of the morphism (4) contains a single orbit consisting of semisimple elements, and it is the only closed orbit in this fiber (see [St65]). Since  $R = \text{Int}(G)$ , from here and Lemma 5.2 the equivalence of the following properties follows:

- (i)  $\sigma$  lies in the kernel of the action of  $\text{Aut}(F_n)$  on  $X // R$ ;
- (ii)  $g$  and  $g^{-1}$  are conjugate for each semisimple element  $g \in G$ .

Since the intersection of any semisimple conjugacy class with a fixed maximal torus  $T$  of  $G$  is nonempty (see [Bo91, Thm. 11.10]) and is the orbit of the normalizer of this torus (see [St65, 6.1]), property (ii) is equivalent to the fact that the Weyl group  $W$  of the group  $G$  considered as a subgroup of the group  $\text{GL}(\mathfrak{t})$ , contains  $-1$ . This, in turn, is equivalent to the fact that  $-1$  is contained in the Weil group of every nontrivial connected simple normal subgroup of the group  $G$ . Let  $C$  be a Weyl chamber in  $\mathfrak{t}$ . Since  $-C$  is also a Weyl chamber, the simple transitivity of the action of  $W$  on the set of all Weyl chambers implies the existence of a unique element  $w_0 \in W$  such that  $w_0(C) = -C$ . In view of  $(-1)(C) = -C$ , this means that the inclusion of  $-1 \in W$  is equivalent to the equality  $w_0 = -1$ . In [Bo68, Table. I-IX], the explicit description of the element  $w_0$  is given for every connected simple algebraic group. It follows from it that the equality  $w_0 = -1$  for such a group is equivalent to the fact that the type of this group is not contained in list (8). This completes the proof.  $\square$

**13. Final remarks.**

(a) Corollary 1.7 concerns, in particular, the subgroups of the Cremona groups. Taking this opportunity, we will supplement it here with a remark on S. Cantat’s question about these subgroups.

In [Co17], are given examples of finitely generated (and even finitely presented) groups nonembeddable into any Cremona group, which answers S. Cantat’s question about the existence of such groups (see also

[Ca13<sub>1</sub>]). These examples are based on the fact that the word problem ([Co13, Thm. 1.2]) is solvable in every finitely generated subgroup of any Cremona group. Let us indicate another way to answer this question (and even in a stronger form, with the addition of the group simplicity condition).

Namely, we recall [Po14, Def. 1] that a group  $H$  is called *Jordan* if there exists a finite set  $\mathcal{F}$  of finite groups such that every finite subgroup of  $H$  is an extension of an Abelian group by a group taken from  $\mathcal{F}$ . According to [Po14, p. 188, Exmp. 6], the R. Thompson group  $V$  is an example of a non-Jordan finitely presented group. Since any Cremona group is Jordan (see [Bi16, Cor. 1.5], [PS16]), the group  $V$  is nonembeddable into it. Furthermore, in addition to this property,  $V$  is simple, and therefore, every homomorphism of  $V$  into any Cremona group is trivial (unlike [Co13], this proves [Co13, Cor. 1.4] without using the obtained in [Mi81] amplification of the Boone–Novikov construction).

We note that, after [Co13], many unrelated to the word problem examples of finitely generated (and even finitely presented) groups nonembeddable into any Cremona group were obtained in [CX18]. However, basing on the currently available (July 2022) information, it is impossible to deduce from [CX18, Thms. C and 7.15] that  $V$  is nonembeddable into any Cremona group. Indeed, the group  $V$  does not have Kazhdan’s property (T) (see [BJ19]), and whether it has property  $(\tau^\infty)$  (see [CX18, Sect. 7.1.3]) is unknown [Co22].

(b) Among the irreducible affine varieties  $X//R$  whose automorphism group contains  $\text{Aut}(F_n)$ , there are open subsets of affine spaces. Indeed, by Theorem 1.1 for  $n \geq 2$ , such an example is  $X//R$  with  $G = \text{GL}_d$ ,  $d \geq 2$ , and trivial  $R$ . The following construction generalizes this example.

Consider a finite dimensional associative  $k$ -algebra  $A$  with identity. The group  $A^*$  of its invertible elements is a connected affine algebraic group whose underlying variety is open in  $A$ . If  $A^*$  is nonsolvable, then in view of Theorem 1.1, it can be taken instead of  $\text{GL}_d$  in the example from the previous paragraph.

(c) In [CX18, pp. 272], is obtained the lower bound

$$n - 2 \leq \text{Var}_{\mathbb{C}}(\text{Out}(F_n)).$$

The following theorem yields, among other things, an upper bound.

**Theorem 13.1.** *We retain the notation of Theorem 1.1. Let  $\text{char}(k) = 0$ ,  $n \geq 3$ ,  $G = \text{SL}_2$  or  $\text{PSL}_2$ , and  $R = \text{Int}(G)$ . Then  $X//R$  is an irreducible rational affine  $(3n - 3)$ -dimensional manifold, whose automorphism group contains  $\text{Out}(F_n)$ .*

*Proof.* The affineness of  $X//R$  follows from the reductivity of  $R$ . According to [Ho75], for  $n \geq 3$ , the kernel of the action of  $\text{Aut}(F_n)$  on  $X//R$  is  $\text{Int}(F_n)$ , so  $\text{Out}(F_n)$  is embedded in  $\text{Aut}(X//R)$ . From [Ri88, Lem. 3.3, Thm. 4.1] and  $\dim(G) = 3$  we infer the nonemptiness of the open subsets of  $X = G^n$  and  $\mathfrak{g}^n$  comprised by points whose  $R$ -orbits are three-dimensional and closed. This and the existence of the geometric quotients for the suitable open subsets of  $X = G^n$  and  $\mathfrak{g}^n$  (see [PV94, Thm. 4.4]) imply that the dimensions of the varieties  $G^n//R$  and  $\mathfrak{g}^n//R$  are equal to  $3n - 3$ , and their fields of rational functions coincide under the natural embedding with the fields of  $R$ -invariant rational functions on  $X$  and  $\mathfrak{g}^n$  respectively. As in the proof of Theorem 1.5, there is an  $R$ -equivariant birational mapping (29), and therefore, the specified fields of  $R$ -invariant rational functions are isomorphic. Hence the algebraic varieties  $G^n//R$  and  $\mathfrak{g}^n//R$  are birationally isomorphic. But, according to P. Katsylo, the field of invariant rational functions on any finite dimensional algebraic  $\text{SL}(2)$ -module is purely transcendental over  $k$  (see [PV94, Thm. 2.12]). Hence  $G^n//R = X//R$  is a rational algebraic variety.  $\square$

**Corollary 13.2.** *If  $\text{char}(k) = 0$  and  $n \geq 3$ , then*

$$\text{Var}_k(\text{Out}(F_n)) \leq 3n - 3 \quad \text{and} \quad \text{Crem}_k(\text{Out}(F_n)) \leq 3n - 3. \quad (30)$$

As noted in [CX18, pp. 272] (with reference to [MS75]), over  $\mathbb{C}$ , the minimal dimension in which  $\text{Out}(F_n)$  is the group of birational self-maps, does not exceed  $6n$ . The right-hand side inequality in (30) is the twice stronger upper bound.

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