

Article

Traveling Waves in Shallow Seas of Variable Depths

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Abstract: The problem of the existence of traveling waves in inhomogeneous fluid is very important for enabling an explanation of long-distance wave propagations such as tsunamis and storm waves. The present paper discusses new solutions to the variable-coefficient wave equations describing traveling waves in fluid layers of variable depths (1D shallow-water theory). Such solutions are obtained by using the transformation methods when variable-coefficient equations can be reduced to the constant coefficient equation when the existence of traveling waves is evident. It is shown that there is a wide class of monotonic bottom profiles (discrete set) that allow the existence of traveling waves that are not reflected in a strongly inhomogeneous water medium. Their temporal shape changes with distance, mainly near the water–land boundary (shoreline). Traveling waves can transfer the wave energy over a long distance that is often observed at the transoceanic propagation of tsunami waves.

Keywords: wave equation; inhomogeneous media; shallow sea; traveling waves



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1. Introduction

The phenomenon of water waves propagating over long distances in the sea and ocean is well known, and it is enough to indicate tsunami waves in the ocean, undular bores in rivers and so on. This phenomenon is related to the weak effects of reflection, diffraction and scattering. The mathematical bright example of waves, which can propagate over long distances, is the solution to the wave equation in the form $u(x - t)$, which does not change with distance. Here x is the spatial coordinate and t is the time. Finding such traveling solutions in the framework of one-dimensional theory is carried out in the framework of ordinary differential equations (ODE), in the general case, nonlinear, by using dynamical system methods. Here it suffices to point out the traveling waves of the cnoidal wave type in the hierarchies of the Korteweg-de Vries equations, the nonlinear Schrödinger equation, the Whitham equation, and the equations of nonlinear field theory; the examples of these are well known, so we will not give references to classical works here. If the medium is inhomogeneous or non-stationary in the direction of wave propagation, when the equations contain variable coefficients, it is not usually possible to find rigorous wave equation solutions in the form of traveling waves due to the reflection, scattering and diffraction effects. At the same time, if the medium changes slowly enough over time or smoothly in space, traveling waves with variable amplitudes and phases can be found approximately using the WKB methods such as geometric optics or acoustics; see, for example, [1–5]. In the literature [1] it is mentioned that for the special coefficient change (with application to water waves this bottom profile is proportional to $x^{4/3}$), the WKB solution is rigorous. So, generalized traveling waves can exist in inhomogeneous media with special variability laws. They transmit the wave energy over a long distance with no reflection, but the wave amplitude and phase both change. A few examples are constructed for surface gravity waves in water layers of variable depths [6,7], waves in inhomogeneous flow [8,9], internal waves in a stratified fluid [10], and waves in the atmosphere of the Earth

and the Sun [11,12]. Some rigorous solutions demonstrating traveling wave features are known for acoustic waves [13,14], electromagnetic waves and plasma waves [15–17].

Mathematically, various methods are used to find them, including Lie algebra and transformation methods [7,8,18–28]. With application to water waves these methods confirm the existence of traveling waves in the basin with the bottom profile proportional to $x^{4/3}$ with a fixed waveshape, and some other waves which are very dispersive and, therefore, attenuated.

Consequently, the question arises how wide the range of changes in the medium parameters is, that allow the existence of traveling waves. In this paper, we consider the classical wave equation obtained in the linear theory for one-dimensional long waves in a variable depth fluid. The basic idea to obtain solutions in the form of traveling waves is related to the transformation of the wave equation with variable coefficients to the equation with constant coefficients, within which the existence of traveling waves becomes obvious. This model, developed in [6,27], is briefly described in Section 2. Another way of fulfilling transformations is also possible, when the equations with constant coefficients are obtained within the framework of a double transformation. In particular, the original one-dimensional wave equation with variable coefficients can be reduced to the equation for spherically symmetric waves, and then to the equation with constant coefficients (Section 3). The same approach can be applied to spherically symmetric waves in a high-dimensional space (Section 4) or even negative dimensions (Section 5). Mathematically, the transformed equations of this kind are known as the hierarchy of the Euler–Darboux–Poisson equations. As a result, a wide class of monotonous bottom profiles is found, allowing reflectionless wave propagation over long distances in shallow seas. The results obtained are summarized in the conclusion (Section 6). The main result here is the demonstration of a discrete set of bottom profiles providing the existence of traveling waves, which, although they change their form while propagating, do not disperse wider in space.

2. Mathematical Model

Let us consider a water layer, the depth of which varies along the x -axis (Figure 1). The long linear wave dynamics in the ideal fluid approximation is described by the equations of the shallow water theory [2,3]

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[h(x)u] = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = 0, \quad (2)$$

where $u(x,t)$ is the depth-averaged current velocity; $\eta(x,t)$ is the water surface displacement axis; $h(x)$ is the variable basin depth; and g is the gravitational acceleration.

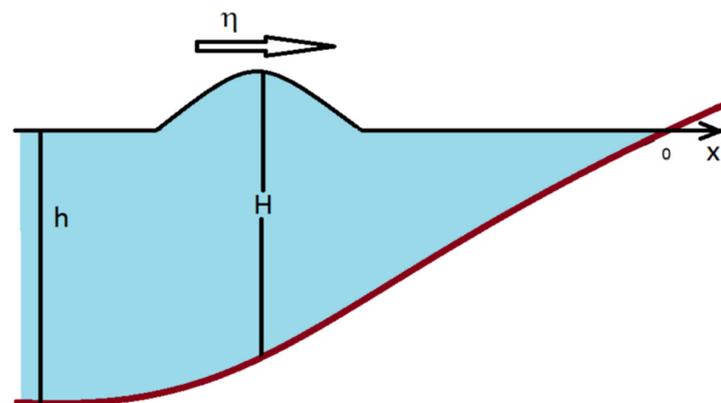


Figure 1. Geometry sketch.

Equations (1) and (2) are trivially reduced to the wave equation for surface displacement with variable coefficients

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x} \left(c^2(x) \frac{\partial \eta}{\partial x} \right) = 0, \quad (3)$$

where $c(x)$ is the long-wave speed determined by the water depth

$$c(x) = \sqrt{gh(x)} \quad (4)$$

To find the solutions to Equation (3) in the form of a traveling wave, we will use the transformation technique (mapping) of reducing the variable-coefficient wave equation to the constant-coefficient wave equation [19,23,27]. To do this, we can make the following substitution in Equation (3)

$$\eta(x, t) = A(x)G[t, \tau(x)], \quad (5)$$

where $A(x)$, $G(t, \tau)$ and $\tau(x)$ are three unknown functions to be determined. Then, the given Equation (3) is transformed into the Klein–Gordon equation with variable coefficients

$$A \left[\frac{\partial^2 \Phi}{\partial t^2} - c^2 \left(\frac{d\tau}{dx} \right)^2 \frac{\partial^2 G}{\partial \tau^2} \right] - \left[\frac{d}{dx} \left(c^2 A \frac{d\tau}{dx} \right) + c^2 \frac{dA}{dx} \frac{d\tau}{dx} \right] \frac{\partial G}{\partial \tau} - \frac{d}{dx} \left[c^2 \frac{dA}{dx} \right] G = 0. \quad (6)$$

Since this equation contains three unknown functions, we can impose three conditions for their unique definition. In the literature [6,27], the following choice of these conditions was suggested in the three equation form

$$c^2 \left(\frac{d\tau}{dx} \right)^2 = 1, \quad (7)$$

$$\frac{d}{dx} \left(c^2 A \frac{d\tau}{dx} \right) + c^2 \frac{dA}{dx} \frac{d\tau}{dx} = 0, \quad (8)$$

$$\frac{d}{dx} \left[c^2 \frac{dA}{dx} \right] = PA, \quad (9)$$

where P is an arbitrary constant. When these conditions are satisfied, Equation (6) is reduced to the constant-coefficient Klein–Gordon equation

$$\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial \tau^2} - PG = 0, \quad (10)$$

and the existence traveling waves (in particular, monochromatic traveling waves) becomes evident. If the wave is not monochromatic, we may use the Fourier or the Laplace transformation to analyze the wave shape. Nevertheless, the waves propagating in opposite directions do not interact between them.

The first Equation (7) determines the transition to the wave phase (for definiteness, the wave propagating to the right is taken) or the travel time

$$\tau = \int_{x_0}^x \frac{dy}{c(y)}. \quad (11)$$

The second Equation (8) determines the relationship between the wave amplitude and the propagation celerity (the basin depth)

$$A(x) = \frac{\text{const}}{\sqrt{c(x)}} \sim h^{-1/4}, \quad (12)$$

and this formula coincides with the famous Green's law, known from the energy flux conservation in media with slowly varying parameters [1,2]. We emphasize that here we do not impose the condition on the slowness of the change in the basin depth.

The third Equation (9) determines the water depth for which the traveling waves exist

$$\frac{d^2}{dx^2} h^{3/4} = -\frac{3P}{gh^{1/4}}. \quad (13)$$

It is a second-order ordinary differential equation and, therefore, it depends on two arbitrary constants. It can be once integrated

$$\left(\frac{dh^{3/4}}{dx}\right)^2 + \frac{9P}{2g}h^{-1/2} = E, \quad (14)$$

where E is the additional constant. Here, we give only monotonic solutions of this equation which are important for the analysis of the wave shoaling. The first of them can be obtained when $P = E = 0$

$$h(x) \sim x^{4/3}, \quad A(x) \sim h^{-1/4} \sim x^{-1/3}. \quad (15)$$

The second one exists when $E = 0$ and $P < 0$

$$h(x) \sim x^{8/5}, \quad A(x) \sim h^{-1/4} \sim x^{-2/5}. \quad (16)$$

In the first case, the wave is the solution to the constant-coefficient wave equation and presents two waves of permanent form propagating in opposite directions. The wave field dynamics over such bottom profiles with oceanic applications was studied in detail in [6]. In the second example, we have the solution of the constant-coefficient Klein-Gordon Equation (10), and the wave shape changes with time. Shape changing is related to geometrical dispersion and can lead to the wave focusing with a rogue wave formation or defocusing

The results obtained here are rigorous and do not require verification. However, it is very useful to mention that such monotonic bottom profiles with exponents in the range 1–2 are very often observed in the coastal zone, see for instance [29].

3. Mapping the Spherical Symmetric Wave Equation

The imposed conditions in Equations (7)–(9) are not the only possible conditions for the existence of traveling waves. It is known, for example, that the wave equation for spherical waves after the introduction of the amplitude factor r^{-1} is also reduced to the one-dimensional wave equation with constant coefficients [30]. Equation (6) under the conditions in Equation (7) is reduced to the spherically symmetric wave equation

$$\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial \tau^2} - \frac{2}{\tau} \frac{\partial G}{\partial \tau} = 0, \quad (17)$$

if these two conditions are satisfied

$$\frac{d}{dx}(cA) + c \frac{dA}{dx} = \frac{2}{\tau} A, \quad (18)$$

$$\frac{d}{dx} \left[c^2 \frac{dA}{dx} \right] = 0. \quad (19)$$

Equation (17) after the transformation $G(t, \tau) = H(t, \tau)/\tau$ is reduced to the classic wave equation

$$\frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 H}{\partial \tau^2} = 0, \quad (20)$$

and the existence of travelling waves is evident.

In fact, the system in Equations (18) and (19) representing the integral–differential equations for c and A are not analyzed here in detail because of our focusing on the wave Equation (17). We indicate here only an important particular solution to this system, which can be obtained by setting $A = 1$. Then, Equation (19) is automatically satisfied, and Equation (18) is reduced to

$$\frac{dc}{dx} = \frac{2}{\tau}, \quad (21)$$

and with the use of Equation (7) to

$$\tau \frac{d^2\tau}{dx^2} = -2 \left(\frac{d\tau}{dx} \right)^2. \quad (22)$$

The last equation is trivially integrated

$$h(x) \sim x^{4/3}, \tau(x) \sim x^{1/3}, \eta(x, t) = \Phi(t, \tau) \sim \tau^{-1} f(t - \tau) \sim x^{-1/3} f(t - \tau), \quad (23)$$

which completely coincides with Formula (15) obtained in a different way. Formally speaking, the point $x = 0$ is the shoreline (as in Figure 1) and the positive values of the coordinate equate to the wave propagating away from the coast. Due to the wave equation symmetry, we can choose any direction of wave propagation.

Thus, the reduction of the original wave Equation (3) to the spherically symmetric wave equation did not allow us to obtain new solutions, but made it possible to establish an important connection between solutions for waves in inhomogeneous media and for spherically symmetric waves in the case of reflectionless wave propagation.

4. The Symmetric Wave Equation Reduction on a Sphere in the High-Dimensional Space

Let us now discuss a more general case of spherically symmetric waves in a space of odd dimensions. Equation (6), when the condition (7) is imposed, is reduced to the equation

$$\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial \tau^2} - \frac{2m}{\tau} \frac{\partial G}{\partial \tau} = 0 \quad (24)$$

(m is an integer) when the following conditions are imposed:

$$\frac{d}{dx}(cA) + c \frac{dA}{dx} = \frac{2m}{\tau} A, \quad (25)$$

$$\frac{d}{dx} \left[c^2 \frac{dA}{dx} \right] = 0. \quad (26)$$

When we set $A = 1$, Equation (26) is automatically satisfied, and Equation (25) is reduced to

$$\frac{dc}{dx} = \frac{2m}{\tau}, \quad (27)$$

and with the help of Equation (7) it is reduced to

$$\tau \frac{d^2\tau}{dx^2} = -2m \left(\frac{d\tau}{dx} \right)^2. \quad (28)$$

Equation (27) is integrated in the general form (for any value of the parameter m)

$$h(x) = h_0(1 + x/L)^{\frac{4m}{1+2m}}, \tau(x) = \tau_0(1 + x/L)^{\frac{1}{1+2m}}. \quad (29)$$

where we introduced constants for convenience. Naturally, for $m = 1$, the solution (29) goes over to (23). Additionally, $m = 2$ leads to the exponent $8/5$ in (29) as in the solution (16). We should note that as m increases, the depth function tends to a parabolic profile (Figure 2).

Thus, all the reflectionless bottom profiles lie between the $x^{4/3}$ and x^2 curves (lower and higher curves in Figure 2), and there are their countable sets.

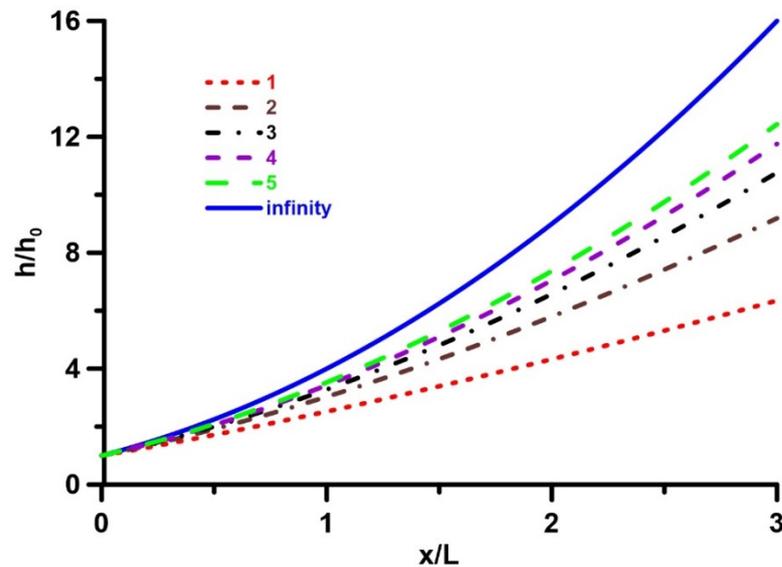


Figure 2. Reflectionless bottom profiles for various m .

The Equation (24) is known as the Euler–Poisson–Darboux equation [30–32]; its solution for the integer coefficient m was obtained by Leonardo Euler. This general solution is expressed in terms of the constant-coefficient wave equation solution (derived from (24) at $m = 0$, which we denote as V) as a finite sum

$$G(t, \tau) = \frac{1}{\tau^{2m-1}} \sum_{i=0}^{m-1} a_i \tau^i \frac{\partial^i V}{\partial t^i} \quad (30)$$

with the numerical coefficients a_i , which is found elementarily after substituting (30) into (24). The function V , as a solution to the constant-coefficient wave equation, can be represented as two waves propagating in opposite directions

$$V(t, \tau) = \theta(t - \tau) + \psi(t + \tau), \quad (31)$$

so that the solution (30) can be also represented as a sum of two traveling waves but with a variable amplitude and phase. Consequently, the existence of traveling waves under these conditions becomes obvious. For simplicity, we restrict ourselves here to only one wave running to the right.

In particular, for $m = 2$ (5D sphere) the wave field is

$$G(t, \tau) = \frac{1}{\tau^3} \theta[t - \tau] + \frac{1}{\tau^2} \frac{\partial \theta[t - \tau]}{\partial t}. \quad (32)$$

Taking into account that $\tau(x) \sim x^{1/5}$ and $h(x) \sim x^{8/5}$, we can see that the first term is proportional to $h^{-3/8}$, and the second one is proportional to $h^{-1/4}$. At a large distance from the coast, the second term prevails, and the wave amplitude varies according to Green's law. In the coastal vicinity the first term prevails, and the wave amplitude changes more quickly than Green's law. The wave length decreases when the wave approaches the coast (Figure 3 qualitatively). As a result, the wave shape changes when the wave approaches the coast (its shape is integrated), but there is no reflection, and such transformation is related to the shoaling process only. When the wave moves from the coast, its shape is differentiated. If the wave represents a solitary-like wave, we must require the vanishing tails of the function $\theta(t)$, and the class of the traveling waves is bounded on the shape.

Of course, the point $x = 0$ (the shoreline) is singular in the linear theory, and we have to exclude the near-shore range from our consideration.

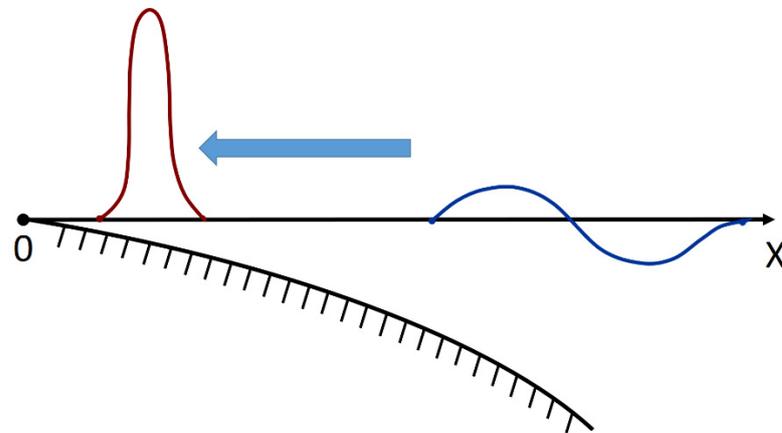


Figure 3. Wave shoaling on the bottom profile $x^{8/5}$.

We also give the solution in the case $m = 3$ (7D sphere)

$$G(t, \tau) = \frac{1}{\tau^5} \theta(t - \tau) + \frac{1}{\tau^4} \frac{\partial \theta}{\partial t} + \frac{1}{3\tau^3} \frac{\partial^2 \theta}{\partial t^2}, \quad h(x) \sim x^{12/7}, \quad \tau \sim x^{1/7}. \tag{33}$$

As we can see, the first term is the most significant near the edge (its amplitude is proportional to $h^{-5/12}$), and the last term is far from it, and its amplitude goes to Green’s law. In this case, the traveling wave shape is even more strongly transformed than is shown in Figure 3. This means that the vanishing tail requirement for the solitary-like wave is stricter.

As m increases, higher derivatives appear in the solution, but the nature of the solution remains qualitatively the same with a strong transformation of the waveform near the water edge. The higher derivative presence in the solution imposes certain conditions on the smoothness of the traveling wave shape (and its integrals), which are essentially absent in the framework of the constant-coefficient wave equation.

5. The Symmetric Wave Equation Reduction in the Space of the Negative Dimension

The wave Equation (24) also admits solutions in the form of traveling waves in the case of negative values of m [31,32], which can be interpreted as waves in the space of negative dimensions. Replacing m with $n = -m$, we rewrite the Formula (29) for the bottom relief characteristics

$$h(x) = h_0(1 + x/L)^{\frac{4n}{2n-1}}, \quad \tau(x) = \tau_0(1 + x/L)^{-\frac{1}{2n-1}}. \tag{34}$$

All bottom profiles that allow reflectionless wave propagation lie between the dependences x^2 and x^4 , extending the bottom profile class considered in the previous section. The distinctive property of such profiles is related to large values of the wave speed $c(x)$, and the wave moves to infinity for finite time. Of course, this is due to the rapid change in the basin depth at infinity (it becomes too large), which contradicts the conditions of the shallow water approximation. This contradiction can be removed by merging the variable depth region with a flat bottom or going beyond the shallow water approximation.

The general Equation (24) solution for negative integer values of m (the positive values of n) is also expressed as a finite sum

$$G(t, \tau) = \sum_{i=0}^n a_i \tau^i \frac{\partial^i V}{\partial t^i}. \tag{35}$$

Let us give here the simplest solution (35) for $n = 1$

$$G(t, \tau) = \theta[t - \tau] + \tau \frac{\partial \theta[t - \tau]}{\partial t}, \quad h(x) \sim x^4, \quad \tau(x) \sim 1/x. \quad (36)$$

Here the wave form varies greatly with distance as in the cases described above (similar to Figure 3). Far from the shore, the wave amplitude satisfies Green's law, but the wave cannot reach the shoreline (travelling time tends to infinity). In the shoreline vicinity, the linear shallow-water theory breaks.

As n increases, higher derivatives appear in the solution, so for $n = 2$ we have

$$G(t, \tau) = \theta(t - \tau) + \tau \frac{\partial \theta}{\partial t} + \frac{\tau^2}{3} \frac{\partial^2 \theta}{\partial t^2}, \quad h(x) \sim x^{8/3}, \quad \tau \sim 1/x^{1/3}. \quad (37)$$

Thus, here too there is a countable set of bottom profiles that ensures the reflectionless wave propagation. As in the previous section, the solution to describe the wave shape near the parabolic bottom profile is more complicated.

6. Conclusions

This paper develops an approach to finding traveling waves in water layers of variable depths. It was shown that there are several ways to obtain traveling waves. In the first method, the original wave equation with variable coefficients is reduced to the constant-coefficient Klein–Gordon equation, and the existence of solutions in traveling wave forms becomes obvious. In another approach, the original wave equation with variable coefficients is reduced to the equation for symmetric waves in a space of odd, and even negative, dimensions; it has the Euler–Darboux–Poisson equation form, which also has solutions in traveling wave forms. However, in this case, the waveform changes with distance, but nevertheless there is no reflection from the inhomogeneities and dispersion with the transformation in the attenuated wave packet. It was found in this paper that the number of reflectionless bottom profiles is countable and that they lie between two monotone profiles $x^{4/3}$ and x^4 . Such monotonic profiles are very often observed in the coastal zone (see, for instance [29]). The presence of a large number of reflectionless profiles indicates a relatively small wave reflection on monotonic profiles and ensures energy transfer over long distances as is observed for tsunami waves and obtained in numerical simulations of basic shallow-water equations. Such strong amplification of water waves is important for explanation of the rogue wave phenomenon [33]. The above-mentioned approach is not exhaustive, so we are sure that it is possible to find a variety of situations where the wave will propagate over long distances without reflection. We are planning to study this in the future.

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