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Dynamics of three-dimensional A-diffeomorphisms with two-dimensional attractors and repellers

Marina Barinova, Vyacheslav Grines and Olga Pochinka

HSE University, Nizhny Novgorod, Russian Federation

ABSTRACT

In this paper, we consider a class of A-diffeomorphisms given on a 3-manifold, assuming that all the basic sets of the diffeomorphisms are two dimensional. It is known that such basic sets are either attractors or repellers and they are two types only, surface or expanding (contracting). One of the results of the paper is the proof that different types of two-dimensional basic sets do not coexist in the non-wandering set of the same 3-diffeomorphism. Also, the existence of an energy function is constructively proved for systems of the class under consideration. It is illustrated by examples that the two-dimensionality of the basic sets is essential in this matter and a decrease in the dimension can lead to the absence of the energy function for a diffeomorphism.

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1. Introduction and formulation of results

Smale [19] in 1967 introduced so-called *A-diffeomorphisms* f on a compact manifold M^n, whose non-wandering set $NW(f)$ is hyperbolic and coincides with the closure of the periodic points of f. For such diffeomorphisms, he proves the Spectral Theorem: the non-wandering set $NW(f)$ of an A-diffeomorphism $f : M^n \to M^n$ can be represented uniquely as a disjoint union of a finite number of subsets

$$NW(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k,$$

where every Λ_i is a compact, f-invariant and topologically transitive (contains an orbit of f which is dense in Λ_i). Such subsets are called *basic sets* of the diffeomorphism f. A basic set Λ_i, which is a periodic orbit, is called a *trivial* basic set. Otherwise, Λ_i is called *non-trivial*, in this case it contains infinitely many periodic points and $\dim\ W^s_x \cdot \dim\ W^u_x \neq 0$ for $x \in \Lambda_i$.

Recall that for diffeomorphism $f : M^n \to M^n$ a compact f-invariant subset $\mathcal{A}$ of M^n is called an *attractor* if there exists a compact neighbourhood $U_{\mathcal{A}}$ (*a trapping neighbourhood*) of $\mathcal{A}$ such that $f(U_{\mathcal{A}}) \subset \operatorname{int} U_{\mathcal{A}}$ and $\mathcal{A} = \bigcap_{i=0}^{+\infty} f^i(U_{\mathcal{A}})$. A *repeller* is defined as an attractor for f^{-1}. An attractor $\mathcal{A}$ of a diffeomorphism f is called *expanding*, if $\dim \mathcal{A} = \dim\ W^u_x$, $x \in \mathcal{A}$. An expanding attractor for f^{-1} is called *a contracting repeller* for f.

CONTACT Marina Barinova mkbarinova@yandex.ru

Plykin [17, Theorem 1] proved that each hyperbolic attractor (repeller) consists of unstable (stable) manifolds of its points. Moreover, every basic set of an A-diffeomorphism $f : M^n \to M^n$ of topological dimension n or $n-1$ is either attractor or repeller. Herewith, if at least one basic set Λ_i has topological dimension n, then $M^n = \Lambda_i$ and f is an Anosov diffeomorphism.

In the present paper, we consider A-diffeomorphisms $f : M^3 \to M^3$, where M^3 is a connected closed manifold and $NW(f)$ consists of two-dimensional basic sets. It follows from [9, Theorems 1, 2] and [4, Corollary 1.2] that a basic set Λ of such a diffeomorphism has exactly one of the following properties:

- every connected component of Λ is homeomorphic to 2-torus (Λ is called *surfaced*);
- Λ is an expanding attractor (contracting repeller) (Λ is called *non-surfaced*), then it follows from [17, Theorem 2] Λ has a local structure of a direct product of a Cantor set and a 2-disc.

The first result of this paper shows that all basic sets of a diffeomorphism under considerations have the same type.

Theorem 1.1: *Let M^3 be a closed 3-manifold and $f : M^3 \to M^3$ be an A-diffeomorphism, whose non-wandering set consists of two-dimensional basic sets. Then either all basic sets are surfaced, or they are all non-surfaced.*

Ch. Conley [5] in 1978 proved an existence of a Lyapunov function[1] for Ω-stable diffeomorphisms[2] in sense of the following definition.

A Lyapunov function for an Ω-stable diffeomorphism $f : M^n \to M^n$ is a continuous function $\varphi : M^n \to \mathbb{R}$ with following properties:

- $\varphi(f(x)) < \varphi(x)$ if $x \notin NW(f)$;
- $\varphi(f(x)) = \varphi(x)$ if $x \in NW(f)$.

The central property of a Lyapunov function for a diffeomorphism $f : M^n \to M^n$ – the decreasing on the wandering set – naturally implies the expectation that a smooth Lyapunov function has no critical points (where the gradient vanishes) there. Such a function, being a Morse function out of non-trivial basic sets, was named *an energy function* for *f*. Indeed the condition of coinciding of the non-wandering set of the diffeomorphism with the set of the critical points of its Lyapunov function is quite strong and, starting from dimension $n = 2$, there are diffeomorphisms (both with and without non-trivial basic sets) that do not possess an energy function [1, 10, 14, 16]. In [3, 11, 12] were selected classes of Ω-stable 3-diffeomorphisms with non-trivial basic sets (one- and two-dimensional surfaced, unique expanding attractor or contracting repeller) which admit an energy function. The following theorem, which will be proven in Section 4, expands these classes.

Theorem 1.2: *Let M^3 be a closed 3-manifold and $f : M^3 \to M^3$ be an A-diffeomorphism, whose non-wandering set consists of two-dimensional basic sets. Then f possesses an energy function.*

The first example of a 3-diffeomorphism with a non-trivial basic set and without an energy function is constructed in Section 5.

2. On two-dimensional basic sets of axiom A 3-diffeomorphism

As was mentioned in the introduction, every two-dimensional basic set Λ of an A-diffeomorphism $f : M^3 \to M^3$ has exactly one of the following types:

- every connected component of Λ is homeomorphic to 2-torus (Λ is called *surfaced*);
- Λ is an expanding attractor (contracting repeller) (Λ is called *non-surfaced*).

Below we describe the topological structure of the basin of such basic sets.

2.1. Surfaced basic sets

Lemma 2.1: *If $\mathcal{A}$ is a two-dimensional surfaced attractor of an A-diffeomorphism $f : M^3 \to M^3$, then every connected component of the set $W^s_{\mathcal{A}} \setminus \mathcal{A}$ is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$.*

Proof: It was proved in [9, Theorem 1] that any surfaced attractor of an A-diffeomorphism $f : M^3 \to M^3$ is cylindrically embedded in M^3. So there exists a compact neighbourhood U of the attractor $\mathcal{A}$ and a homeomorphism $h : U \to \mathbb{T}^2 \times [-1, 1]$ such that $h(\mathcal{A}) = \mathbb{T} \times \{0\}$. Let us prove that there exists a number k such that $f^k(U) \subset \operatorname{int} U$.

Without loss of generality, we will assume that each connected component of the set $W^s_{\mathcal{A}} \setminus \mathcal{A}$ is f-invariant, in the opposite case we can consider a suitable degree of f. Let $\tilde{U}$ be a trapping neighbourhood for the attractor $\mathcal{A}$. As $\bigcap_{n\in\mathbb{N}} f^n(\tilde{U}) = \mathcal{A}$ then we can assume that $\tilde{U} \subset \operatorname{int} U$, in the opposite case we can consider instead $\tilde{U}$ a suitable iteration of $\tilde{U}$ by f. Let us consider an open cover of $W^s_{\mathcal{A}}$ by sets $P = \{f^n(\tilde{U}) : n \in \mathbb{Z}\}$. Since U is a compact set covered by P then there exists a finite subcover $\tilde{P}$ of P such that $U \subset \tilde{P}$. By the construction of P there exists a number $k > 0$ such that $U \subset f^{-k}(\tilde{U})$ so $f^k(U) \subset \operatorname{int} U$.

Let $V_{\mathcal{A}}$ be a connected component of $W^s_{\mathcal{A}} \setminus \mathcal{A}$. Without loss of generality, we will assume that $h(V_{\mathcal{A}} \cap U) = \mathbb{T}^2 \times (0, 1]$. Let $U_{\mathcal{A}} = h^{-1}(\mathbb{T}^2 \times (0, 1])$, $T_{\mathcal{A}} = h^{-1}(\mathbb{T}^2 \times \{1\})$, and $T^*_{\mathcal{A}} = f^k(T_{\mathcal{A}})$. Notice that $T^*_{\mathcal{A}} \subset U_{\mathcal{A}}$. Moreover, $\mathcal{A}$ and $T_{\mathcal{A}}$ belong to different connected components of the set $cl(U_{\mathcal{A}}) \setminus T^*_{\mathcal{A}}$. Indeed, if we assume the contrary, then, according to [8, Lemma 3.1], $T^*_{\mathcal{A}} = \partial D$ for some compact set $D \subset U_{\mathcal{A}}$. As $V_{\mathcal{A}} = U_{\mathcal{A}} \sqcup U^*_{\mathcal{A}}$ for $U^*_{\mathcal{A}} = V_{\mathcal{A}} \setminus U_{\mathcal{A}}$, $\partial U_{\mathcal{A}} = \mathcal{A} \sqcup T_{\mathcal{A}}$ and $f^k(\mathcal{A}) = \mathcal{A}$ then (see Figure 1) $f^k(U^*_{\mathcal{A}}) = D$ and $f^k(U_{\mathcal{A}}) = V_{\mathcal{A}} \setminus D$, that contradicts the fact that $f^k(U_{\mathcal{A}}) \subset U_{\mathcal{A}}$.

Thus the torus $T^*_{\mathcal{A}}$ separates $T_{\mathcal{A}}$ and $\mathcal{A}$ from each other in the set $cl(U_{\mathcal{A}})$. In such a case, by [8, Theorem 3.2], the closure of each connected component of $U_{\mathcal{A}} \setminus T^*_{\mathcal{A}}$ is homeomorphic to $\mathbb{T}^2 \times [0, 1]$. Let $p_1 : \mathbb{T}^2 \times [0, 1] \to \mathbb{T}^2$, $p_2 : \mathbb{T}^2 \times [0, 1] \to [0, 1]$ be canonical projections. Let $K_{\mathcal{A}} = U_{\mathcal{A}} \setminus \operatorname{int} f^k(U_{\mathcal{A}})$, then $V_{\mathcal{A}} = \bigcup_{n\in\mathbb{Z}} f^{kn}(K_{\mathcal{A}})$. Let $\psi_0 : K_{\mathcal{A}} \to \mathbb{T}^2 \times [0, 1]$ be a homeomorphism and $\phi_{01} : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism given by the formula $\phi_0(w) = p_1(\psi_0(f^k(\psi_0^{-1}(w, 0))))$, $w \in \mathbb{T}^2$. Let us extend ψ_0 to a homeomorphism $\psi_1 : f^k(K_{\mathcal{A}}) \to \mathbb{T}^2 \times [1, 2]$ by the formula

$$\psi_1(x) = (\phi_0(p_1(\psi_0(f^{-k}(x)))), \quad p_2(\psi_0(f^{-k}(x))) + 1),\ x \in f^k(K_{\mathcal{A}}).$$

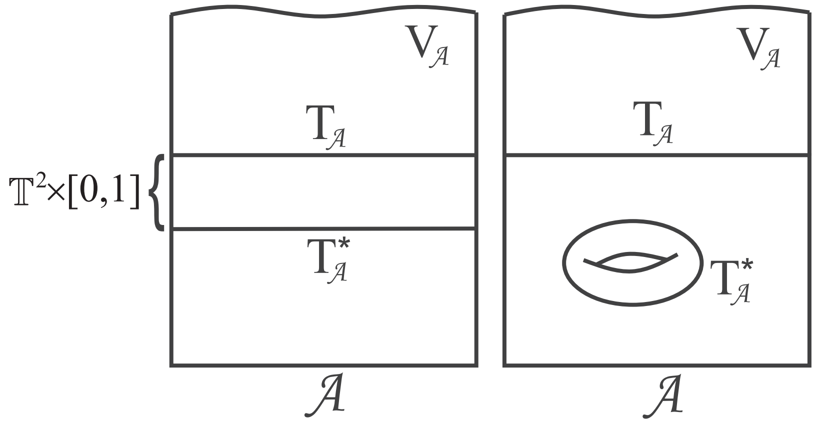

Figure 1. Mutual arrangement of tori $T_{\mathcal{A}}$, $T^*_{\mathcal{A}}$ and $\mathcal{A}$.

Similarly we can extend ψ_1 to $\psi_2 : f^{2k}(K_{\mathcal{A}}) \to \mathbb{T}^2 \times [2, 3]$, etc. in the positive side. Also we can extend ψ_0 in the negative side. As a result, we get a homeomorphism $\psi : V_{\mathcal{A}} \to \mathbb{T}^2 \times \mathbb{R}$. ■

2.2. Non-surfaced basic set

Now let $\mathcal{A}$ be a co-dimension 1 expanding attractor of an A-diffeomorphism $f : M^n \to M^n$. For any point $x \in \mathcal{A}$, $\dim\ W^s_x = 1$ allows one to introduce the notation $(y, z)^s$ $([y, z]^s)$ for an open (closed) arc of the stable manifold W^s_x bounded by points $y, z \in W^s_x$. The set $W^s_x \setminus \{x\}$ consists of two connected components, and by virtue of [6, Lemma 2.1], [7, Lemma 1.5] at least one of them has a non-empty intersection with the set $\mathcal{A}$. A point $p \in \mathcal{A}$ is called *boundary* if one of the connected components of the set $W^s_p \setminus \{p\}$ does not intersect $\mathcal{A}$. Let's denote this component $W^{s\varnothing}_p$. The set of boundary points of the basic set is finite. The union of the unstable manifolds of all boundary points $p_1, \dots, p_r$ of the attractor $\mathcal{A}$ whose components $W^{s\varnothing}_{p_1}, \dots, W^{s\varnothing}_{p_r}$ belong to the same connected component W_B of the set $W^s_{\mathcal{A}} \setminus \mathcal{A}$ is called a *bunch* B of the attractor $\mathcal{A}$, corresponded to W_B, the number r is called a *degree of the bunch*. It follows from [18, Theorem 2.1] that every expanding attractor $\mathcal{A}$ of an A-diffeomorphism $f : M^3 \to M^3$ has bunches of degrees 1 or 2 only.

Lemma 2.2: *Let $\mathcal{A}$ be a two-dimensional non-surfaced attractor of an A-diffeomorphism $f : M^3 \to M^3$ and W_B be a connected component of the set $W^s_{\mathcal{A}} \setminus \mathcal{A}$ corresponded to a bunch B. Then there are two possible cases for the structure of W_B:*

- *if B has degree 2, then W_B is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$;*
- *if B has degree 1, then W_B is homeomorphic to $\mathbb{R}P^2 \times \mathbb{R}$, where $\mathbb{R}P^2$ is a real projective plane.*

Proof: Let p, q be boundary periodic points of the bunch B ($p = q$ in the case when B has the degree 1) and W^u_p, W^u_q be unstable manifolds of the points p and q respectively. Without loss of generality we can assume that p and q are fixed points of the diffeomorphism f. Then

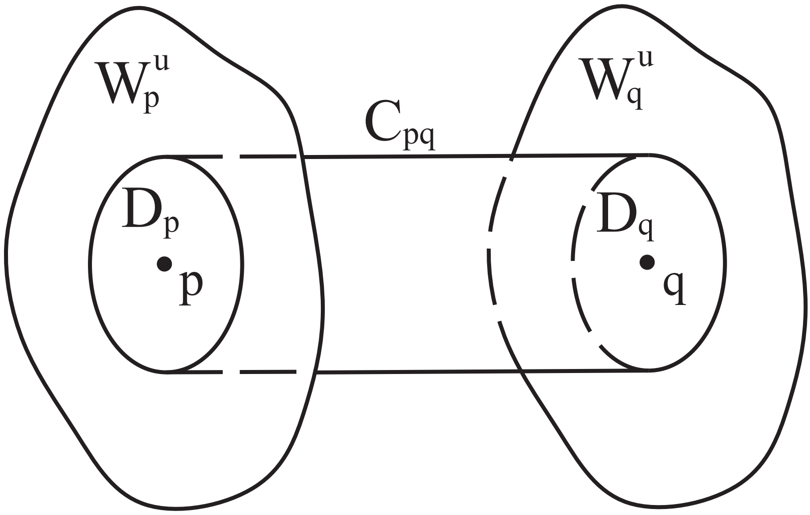

Figure 2. Connecting cylinder.

by [7, Section 2] and [20, Lemmas 2.1, 2.3] there exists a homeomorphism $\varphi : W_p^u \setminus \{p\} \to W_q^u \setminus \{q\}$ which assigns to a point $x \in W_p^u \setminus \{p\}$ the point $\varphi(x) \in W_q^u \cap W_x^s$ such that the arc $(x, \varphi(x))^s \subset W^s(x)$ does not intersect $\mathcal{A}$. It is directly verified that $\varphi \circ f|_{W_p^u \setminus \{p\}} = f \circ \varphi|_{W_p^u \setminus \{p\}}$ and, hence, φ is continuously extended up to $\varphi : W_p^u \to W_q^u$ by $\varphi(p) = q$.

Further let us consider cases (a) $p \neq q$ and (b) $p = q$ separately.

(a) Let B be the 2-bunch of the attractor $\mathcal{A}$. The restriction $f|_{W_p^u}$ has exactly one hyperbolic repelling fixed point p, hence there exists a smooth 2-disc $D_p \subset W_p^u$ such that $p \in f(D_p) \subset \operatorname{int} D_p$. Then the set $C_{pq} = \bigcup_{x \in \partial D_p} [x, \varphi(x)]^s$ is diffeomorphic to a closed cylinder $\mathbb{S}^1 \times [0,1]$. The set C_{pq} is called the *connecting cylinder*. The circle $\varphi(\partial D_p)$ bounds in W_q^u a 2-disc D_q such that $q \in D_q \subset \operatorname{int}(f(D_q))$. The set $S_{pq} = D_p \cup C_{pq} \cup D_q$ is homeomorphic to a 2-dimensional sphere and it is called the *characteristic sphere* corresponding to the bunch B (see Figure 2).

Let us construct a foliation of W_B on 2-spheres. For this aim let us consider a point $x_\alpha \in W_\alpha^{s\varnothing}, \alpha \in \{p, q\}$ and the fundamental segment $I_\alpha \subset W_\alpha^{s\varnothing}$ with the endpoints x_α and $f(x_\alpha)$. Choose a tubular neighbourhood $V(I_\alpha)$ diffeomorphic to $\mathbb{D}^2 \times [0,1]$ by means a diffeomorphism $h_\alpha : \mathbb{D}^2 \times [0,1] \to V(I_\alpha)$ and such that $f(h_\alpha(\mathbb{D}^2 \times \{0\})) \subset h_\alpha(\mathbb{D}^2 \times \{1\})$. Due to λ-lemma one can assume that every disc $h_\alpha(\mathbb{D}^2 \times \{0\})$ intersects $W^s(x)$ for every $x \in (D_p \setminus p)$ exactly once (in the opposite case one can choose an appropriate iteration of $V(I_\alpha)$).

The set $K_p = C_{pq} \setminus \operatorname{int} f^{-1}(C_{pq})$ is diffeomorphic to $\mathbb{S}^1 \times [0,1] \times [0,1]$ by means a diffeomorphism $h : \mathbb{S}^1 \times [0,1] \times [0,1] \to K_p$. Denote by $S_t, t \in [0,1]$ the two-dimensional sphere bounded by the discs $h_p(\mathbb{D}^2 \times \{t\})$, $h_q(\mathbb{D}^2 \times \{t\})$ and the cylinder $h(\mathbb{S}^1 \times [0,1] \times \{t\})$. Each S_t can be considered smooth (see, e.g. [13, Statement 10.58]). Iterations of these spheres by $f^k, k \in \mathbb{Z}$ give the desired foliation on W_B.

(b) Let B be a bunch of the degree 1. Then $p = q$ and $\varphi : W_p^u \to W_p^u$ is an orientation-preserving periodic homeomorphism of period 2 with a unique fixed point p. As $f|_{W_p^u \setminus \{p\}}$

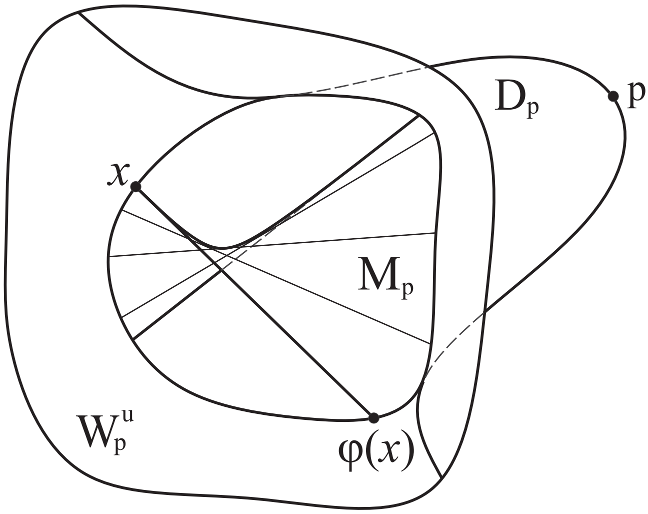

Figure 3. Connecting mobius band.

is smoothly conjugated with the linear expansion then the orbit space $(W^u_p \setminus \{p\})/f$ is diffeomorphic to the 2-torus $\mathbb{T}^2$. Denote by $\rho : W^u_p \setminus \{p\} \to \mathbb{T}^2$ the natural projection. Since the homeomorphism φ commutes with f then it can be projected to $\mathbb{T}^2$ as $\hat{\varphi} = \rho\varphi\rho^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$. Thus $\hat{\varphi}$ is a periodic diffeomorphism whose all points have the period 2. Then it is topologically conjugate to the shift on the torus $\Psi(e^{i2x\pi}, e^{i2y\pi}) = (e^{i2\pi(x+\frac{1}{2})}, e^{i2y\pi})$ (see, e.g. [2, Theorem 1.2]). Hence there exists a foliation $\hat{F}$ by circles $\hat{F} = \{\hat{l}\}$ on $\mathbb{T}^2$ such that $\hat{\varphi}|_{\hat{l}}$ is conjugated with the rotation on 180 degrees. As φ is a periodic map, then $F_p = \{\rho^{-1}(\hat{l})\}$ is a foliation by circles on $W^u_p \setminus \{p\}$ with the following properties:

- each leaf $l \in F_p$ is homeomorphic to $\mathbb{S}^1$ and bounded a disc D_p containing p such that $f^{-1}(D_p) \subset \text{int}\, D_p$;
- $\varphi|_l$ is conjugated with the rotation on 180 degrees.

Let $l \in F_p$, $x \in l$ and $c \subset l$ be a closed segment of l bounded by the points x and $\varphi(x)$. Let $M_p = \bigcup_{x \in c}[x, \varphi(x)]^s$ (see Figure 3). Then M_p is diffeomorphic to a mobius band, which we will called *connecting mobius band*. Thus $M_p \cup D_p$ is homeomorphic to the real projective plane $\mathbb{R}P^2$. Then a foliation of W_B on $\mathbb{R}P^2$ can be obtained analogically to the case a) so W_B is diffeomorphic to $\mathbb{R}P_2 \times \mathbb{R}$.

■

3. On *A*-diffeomorphisms with two-dimensional basic sets

In this section, we prove Theorem 1.1.

Proof: First notice that due to [15, Proposition 3.11], the non-wandering set $NW(f)$ of considered diffeomorphism f contains at least one attractor and at least one repeller. Let $NW(f) = \Lambda_1 \cup \dots \cup \Lambda_k \cup \Lambda_{k+1} \cup \dots \cup \Lambda_{k+m}$, where Λ_i, $i = 1, \dots, k$ is two-dimensional attractor and Λ_{k+j}, $j=1,\dots,m$ is two-dimensional repeller. Introduce the following denotations: $V_i^s = W^s_{\Lambda_i}\setminus\Lambda_i$, $i=1,\dots,k$, and $V^u_{k+j}=W^u_{\Lambda_{k+j}}\setminus\Lambda_{k+j}$, $j=1,\dots,m$. Then V_i^s is a basin of the attractor Λ_i without the attractor and V^u_{k+j} is a basin of the repeller Λ_{k+j} without the repeller. Denote unions of such sets by $V^s=\bigcup_{i=1}^k V_i^s$ and $V^u=\bigcup_{j=1}^m V^u_{k+j}$. Due to S. Smale [19], $M^3=\bigcup_{i=1}^m W^s_{\Lambda_i}=\bigcup_{j=1}^m W^u_{\Lambda_{k+j}}$ and, hence, $V^s=V^u=M^3\setminus NW(f)$. Then the number of connected components of V^s is equal to the number of connected components of V^u and they are pairwise coincide.

It follows from Lemmas 2.1 and 2.2 that every connected component of the set V_i^s (V^u_{k+j}) is homeomorphic to $\mathbb{T}^2\times\mathbb{R}$ if and only if Λ_i (Λ_{k+j}) is a surface attractor (repeller). Then an attractor Λ_i and a repeller Λ_{k+j}, such that $W^s_{\Lambda_i}\cap W^u_{\Lambda_{k+j}}\neq 0$, have the same type.

Let $\Lambda_1\cup\dots\cup\Lambda_p\cup\Lambda_{k+1}\cup\dots\cup\Lambda_{k+q}$, $p\leqslant k$, $q\leqslant m$ be a union of all surface basic set of diffeomorphism f. It follows from explanation above that $\bigcup_{i=1}^p(W^s_{\Lambda_i}\setminus\Lambda_i)=\bigcup_{j=1}^q(W^u_{\Lambda_{k+j}}\setminus\Lambda_{k+j})$ and, hence,

$$\bigcup_{i=1}^p W^s_{\Lambda_i}\cup\bigcup_{j=1}^q\Lambda_{k+j}=\bigcup_{j=1}^q W^u_{\Lambda_{k+j}}\cup\bigcup_{i=1}^p\Lambda_i=\bigcup_{i=1}^p W^s_{\Lambda_i}\cup\bigcup_{j=1}^q W^u_{\Lambda_{k+j}}.$$

Let $\tilde M^3=\bigcup_{i=1}^p W^s_{\Lambda_i}\cup\bigcup_{j=1}^q\Lambda_{k+j}$. Due to [19] $cl(W^s_{\Lambda_i})\setminus W^s_{\Lambda_i}=\bigcup_{\Lambda_{k+j}:W^u_{\Lambda_{k+j}}\cap W^s_{\Lambda_i}\neq\emptyset} W^s_{\Lambda_{k+j}}$ that implies that $\tilde M^3$ is closed. On the other side $W^s_{\Lambda_i}$ and $W^u_{\Lambda_{k+j}}$ are open as they are basins of an attractor and repeller, accordingly. Thus $\tilde M^3$ is closed and open subset of M^3 simultaneously and, hence, coincides with M^3 as it is connected.

Thus the presence of surface basic sets in $NW(f)$ leads to the fact that all basic sets of f are surface, which completes the proof of the theorem. ■

3.1. Example of a 3-diffeomorphism with two-dimensional basic sets of the both types

The homogeneous property proved in the previous section can be broken if the non-wandering set contains a basic set of dimensions 0 or 1. Notice that all such examples are not structurally stable due to results of Grines and Zhuzhoma [7, Theorem 7.1]: if a non-wandering set of a structurally stable diffeomorphism of closed 3-manifold contains an expanding attractor or a contracting repeller then all other basic sets are trivial.

Let us construct an example of an Ω-stable diffeomorphism $f:\mathbb{T}^3\#\mathbb{T}^3$ with both types of two-dimensional basic sets. We will start with an Anosov diffeomorphism $f_C:\mathbb{T}^2\to\mathbb{T}^2$ induced by a hyperbolic matrix $C\in GL(2,\mathbb{Z})$ and a sink-source system $f_{NS}:\mathbb{S}^1\to\mathbb{S}^1$ on a circle $\mathbb{S}^1$. Then the direct product $F=f_C\times f_{NS}:\mathbb{T}^3\to\mathbb{T}^3$ of the diffeomorphisms f_C and f_{NS} has the non-wandering set consisting of one attracting torus $\mathbb{T}_A$ and one repelling torus $\mathbb{T}_R$ (see Figure 4).

Due to [3, Lemma 1] there exists a diffeotopic to the identity diffeomorphism $b:\mathbb{T}^2\to\mathbb{T}^2$ such that $b\circ f_C$ is a DA-diffeomorphism with one-dimensional repeller and a sink. Let $h_t:\mathbb{T}^2\to\mathbb{T}^2$, $t\in[0,1]$ be a diffeotopy from b to the identity map, $\mathbb{S}^1=\{e^{i\pi\varphi},\ \varphi\in[-1,1]\}$ and $F_b:\mathbb{T}^2\times\mathbb{S}^1\to\mathbb{T}^2\times\mathbb{S}^1$ be a diffeomorphism given by the formula

$$F_b(w,\varphi)=(h_{|\varphi|}(w),\varphi).$$

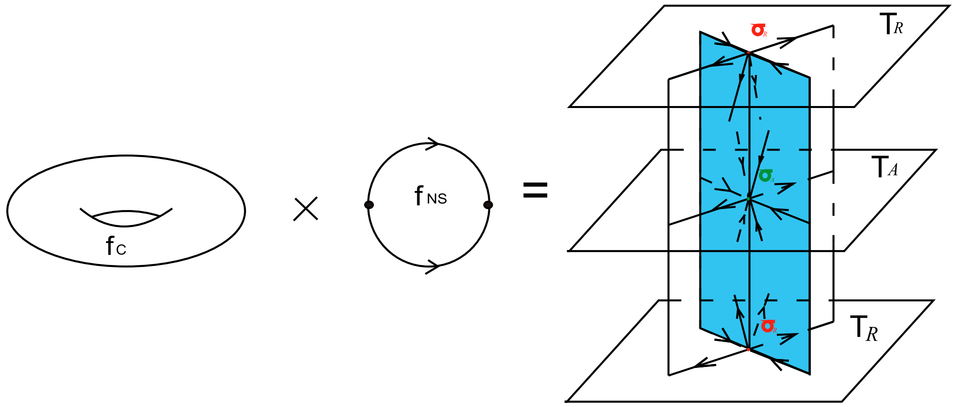

Figure 4. Diffeomorphism $F = f_C \times f_{NS}$.

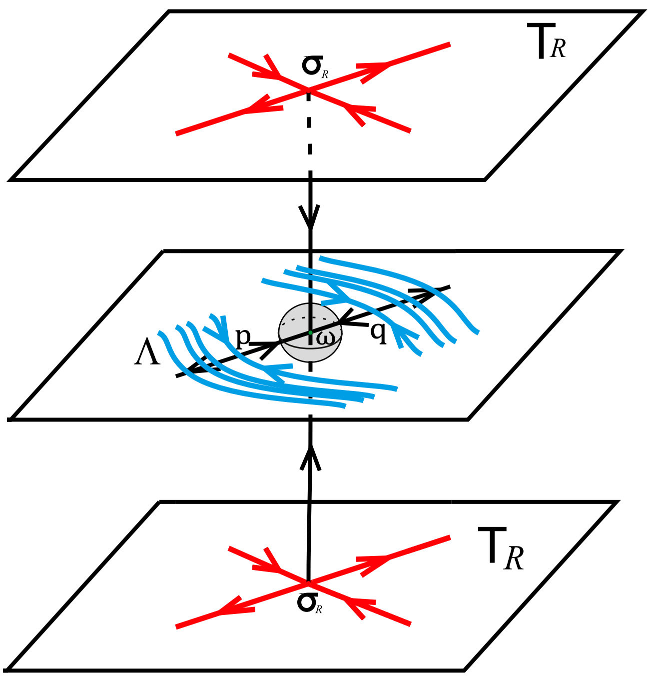

Figure 5. Diffeomorphism F_1.

Then the diffeomorphism $F_1 = F_b \circ F : \mathbb{T}^3 \to \mathbb{T}^3$ has three basic sets (see Figure 5): a two-dimensional repelling torus $\mathbb{T}_R$, a fixed sink ω and one-dimensional saddle basic set Λ.

Let $f_D : \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov diffeomorphism induced by a hyperbolic matrix $D \in SL(3, \mathbb{Z})$. Similar to [3, Lemma 2], there exists a diffeotopic to the identity diffeomorphism $B : \mathbb{T}^3 \to \mathbb{T}^3$ such that the diffeomorphism $F_2 = B \circ f_D$ is a DA-diffeomorphism with two-dimensional attractor and a source α (see Figure 6).

As ω and α are hyperbolic points then there are 3-balls B_ω, B_α around their such that $F_1(B_\omega) \subset B_\omega$, $F_2^{-1}(B_\alpha) \subset B_\alpha$. By the operation of the connected sum of two copies of $\mathbb{T}^3$

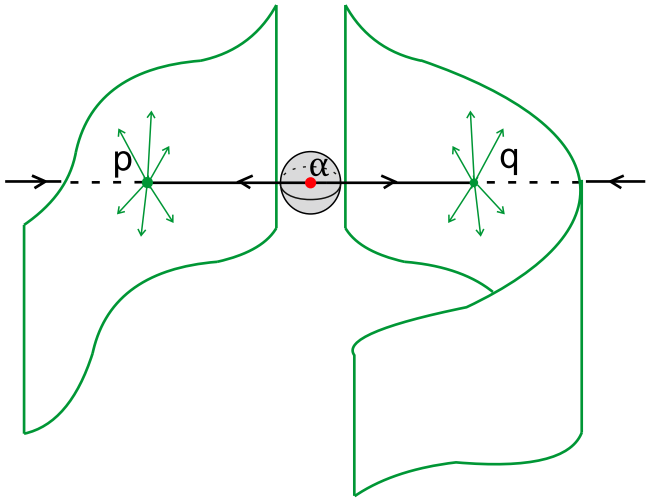

Figure 6. Diffeomorphism F_2.

along B_ω, B_α we get $f : \mathbb{T}^3\#\mathbb{T}^3 \to \mathbb{T}^3\#\mathbb{T}^3$ with the dynamic F_1 on $\mathbb{T}^3 \setminus B_\omega$ and with the dynamic F_2 on $\mathbb{T}^3 \setminus B_\alpha$.

4. Energy function for 3-diffeomorphisms with expanding attractors and contracting repellers

In this section we will prove Theorem 1.2. Namely we consider a closed 3-manifold M^3, Ω-stable diffeomorphism $f : M^3 \to M^3$, all of whose basic sets are two-dimensional and construct an energy function for f.

Proof: Let $NW(f) = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l \cup \mathcal{R}_1 \ldots \cup \mathcal{R}_m$, where $\mathcal{A}_i$, $i \in \{1, \ldots, l\}$, is a two-dimensional attractor and $\mathcal{R}_j, j \in \{1, \ldots, m\}$ is a two-dimensional repeller. It follows from Theorem 1.1 that $\mathcal{A}_i$ and $\mathcal{R}_j$ simultaneously are either surfaces or non-surfaces basic sets. The first case was considered in [11] and an energy function was constructed there for such diffeomorphisms. Let us prove the existence of an energy function for the second type of diffeomorphisms.

Let $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_l$, $\mathcal{R} = \mathcal{R}_1 \ldots \cup \mathcal{R}_m$ and $V = M^3 \setminus (\mathcal{A} \cup \mathcal{R})$. We will construct an energy function $\varphi : M^3 \to [0, 1]$ for f such that $\varphi(\mathcal{A}) = 0$ and $\varphi(\mathcal{R}) = 1$. Let us explain how to construct it on the set V. Let v be a connected component of V, denoted by k_v its period.

Lemma 2.2 implies that v is diffeomorphic to $S \times \mathbb{R}$, where S is either 2-sphere or real projective plane. Moreover, there exists a diffeomorphism $\chi : S \times (0, 1) \to V$ such that a foliation on 2-spheres or real projective planes $\{S_t = \chi(\mathbb{S}^2 \times \{t\}), t \in (0, 1)\}$ is f^{k_v}-invariant. Let us define a function φ on v by the formula $\varphi(S_t) = t$. After that let us extend it to the set $\bigcup_{i=0}^{k_v-1} f^i(v)$ by the formula

$$\varphi(f^i(S_t)) = t + \frac{i(\varphi(f^{k_v}(S_t)) - t)}{k_v}.$$

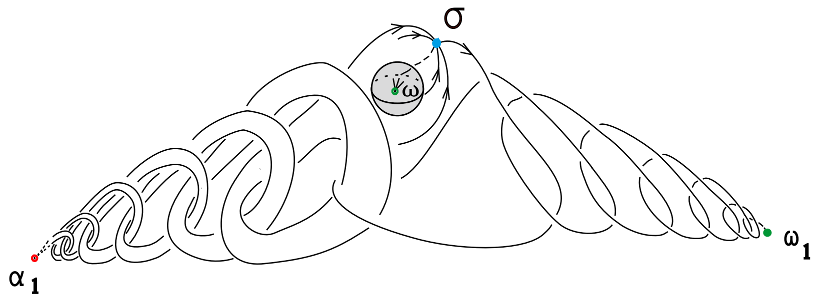

Figure 7. Diffeomorphism F_3.

Carrying out similar reasoning, we can extend the function φ to the whole manifold M^3. The obtained function $\varphi : M^3 \to [0,1]$ is a Lyapunov function for f, smooth on $M^3 \setminus NW(f)$), continuous on M^3 and has no wandering critical points.

Due to the lemma [3, Lemma 5], there exist functions $g_A : [0,1] \to [0,1]$ and $g_R : [0,1] \to [0,1]$ such that the function $\psi = (1 - g_R) \circ g_A \circ \varphi$ is an smooth energy function for the diffeomorphism f. ■

5. Example of an Ω-stable 3-diffeomorphism with an expanding two-dimensional attractor without an energy function

As it was mentioned in the introduction the first example of a diffeomorphism without an energy function is a Pixton example (see Figure 7). We will show how to make an example of a 3-diffeomorphism with non-trivial basic set which does not possess an energy function from the Pixton example and a DA-diffeomorphism.

For this aim it is enough to consider the Pixton example – a Morse–Smale diffeomorphism $F_3 : \mathbb{S}^3 \to \mathbb{S}^3$ with 4 fixed points: two sinks ω and ω_1, a source α_1, and a saddle σ such that the saddle separatrices are wild, except one 1-dimensional. The system does not possess an energy function. The second diffeomorphism is F_2 from Section 3.1. Analogically to 3.1 the hyperbolic points ω and α have 3-balls B_ω, B_α around them such that $F_3(B_\omega) \subset B_\omega$, $F_2^{-1}(B_\alpha) \subset B_\alpha$. By the operation of the connected sum of $\mathbb{S}^3$ and $\mathbb{T}^3$ along B_ω, B_α we get $f : \mathbb{T}^3 \to \mathbb{T}^3$ with the dynamic F_3 on $\mathbb{S}^3 \setminus B_\omega$ and with the dynamic F_2 on $\mathbb{T}^3 \setminus B_\alpha$.

Then the diffeomorphism f does not possess an energy function which is a Morse function outside the expanding attractor.

Notes

1. A fact of the existence of a Lyapunov function for a wider class of dynamical systems is called the Fundamental Theorem of Dynamical Systems.
2. Let $Diff(M^n)$ is a space of diffeomorphisms given on a manifold M^n. A diffeomorphism $f \in Diff(M^n)$ is called *Ω-stable* if there exists $\varepsilon > 0$ such that every diffeomorphism $g \in Diff(M^n)$ such that $||g - f||_{C^1} < \varepsilon$ possesses the property $g|_{NW(g)}$ and $f|_{NW(g)}$ are topologically conjugated.

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ORCID

Marina Barinova http://orcid.org/0000-0002-4406-583X
Vyacheslav Grines http://orcid.org/0000-0003-4709-6858
Olga Pochinka http://orcid.org/0000-0002-6587-5305

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