

# RATIONALITY OF ADJOINT ORBITS

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ABSTRACT. We prove that every orbit of the adjoint representation of any connected reductive algebraic group  $G$  is a rational algebraic variety. For complex simply connected semisimple  $G$ , this implies rationality of affine Hamiltonian  $G$ -varieties (which we classify).

## 1. Introduction

Let  $G$  be a connected affine algebraic group and let  $H$  be its closed subgroup. Whether the algebraic variety  $G/H$  is rational is a well-known old problem closely related to the rationality problem of invariant fields of linear representations of algebraic groups (see [Po94, 1.5], [Po13, Thm. 1, Cor. 2]).

As is shown in [Po11, p. 298], [Po13, Thm. 2], [Po94, Rem. 1.5.9], for some  $G$  and *finite*  $H$ , the variety  $G/H$  is nonrational (and even not stably rational). However, the existence of nonrational varieties  $G/H$  with a *connected* group  $H$  is still an intriguing open problem.

At the same time, for many pairs  $(G, H)$  with connected group  $H$  either rationality or stable rationality of the variety  $G/H$  is proved; for instance,  $G/H$  is rational whenever  $\dim(G/H) \leq 10$  (see [CZ15]).

The conference talk [Ba15] served for me as an impetus to explore rationality of orbits of the adjoint representations of connected reductive groups. Searching for some special rational coordinates on the adjoint orbits of  $\mathrm{GL}_n(\mathbb{C})$ ,  $\mathrm{SO}_n(\mathbb{C})$ ,  $\mathrm{Sp}_n(\mathbb{C})$ , the author of [Ba15] proved, as a byproduct, rationality of the majority of these orbits. He uses the “method of the canonical orbit parameterization” that dates back to the work of I. M. Gelfand and M. I. Naimark on unitary representations of classical groups (1950). The parameterization of (co)adjoint orbits was of interest to many researchers because of its connection with the problems of the theory of integrable systems (see introduction and related references in [Ba16]).

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In the present paper, is proved the following theorem announced in [Po16], which yields infinitely many new examples of rational varieties of the form  $G/H$ .

**Theorem 1.** *Let  $G$  be a connected reductive algebraic group. Every  $G$ -orbit of the adjoint representation of  $G$  is a rational algebraic variety.*

The  $G$ -orbits in Theorem 1 are exactly the varieties  $G/H$ , where  $H$  is the  $G$ -centralizer  $C_G(x)$  of an element  $x$  of the Lie algebra

$$\mathfrak{g} := \text{Lie}(G).$$

The groups  $C_G(x)$  have been thoroughly studied (see [CM93], [Hu95], [SS70]). Among them there are both connected and disconnected groups. Their dimensions are not less than  $r := \text{rk}(G)$ .

Note that Theorem 1 establishes a specific property of the adjoint representation: by Remark 2 below, there exist representations of some  $G$  not all of whose orbits are rational algebraic varieties.

Theorem 1 is applied in the proof of the following Theorem 2, which concerns the classification and properties of Hamiltonian  $G$ -varieties. In it, the following notation is used:

$\mathcal{H}$  is the set of isomorphism classes of affine Hamiltonian  $G$ -varieties,  
 $\mathcal{S}$  is the set of  $G$ -orbits of nonzero semisimple elements of  $\mathfrak{g}$ .

**Theorem 2.** *If  $G$  is a simply connected complex semisimple algebraic group and  $X$  is an affine Hamiltonian  $G$ -variety, then the following holds.*

- (a)  $X$  is isomorphic to a unique  $G$ -orbit  $\mathcal{O}_X \in \mathcal{S}$  endowed with the standard structure of a Hamiltonian  $G$ -variety (see [Ko70, 5.2]).
- (b) The map  $X \mapsto \mathcal{O}_X$  yields a bijection

$$\mathcal{H} \rightarrow \mathcal{C}. \tag{1}$$

- (c)  $X$  is a simply connected rational variety.
- (d) The  $G$ -stabilizer of any point of  $X$  is a Levi subgroup of a proper parabolic subgroup of  $G$ .
- (e) Every Levi subgroup of every proper parabolic subgroup of  $G$  is the  $G$ -stabilizer of a point of some affine Hamiltonian  $G$ -variety.

Recall that for the adjoint action of  $G$  on  $\mathfrak{g}$ , the categorical quotient  $\mathfrak{g} // G$  is isomorphic to the affine space  $\mathbb{A}^r$ , and every fiber of the quotient morphism  $G \rightarrow \mathfrak{g} // G$  contains a unique  $G$ -orbit from  $\mathcal{S}$  (see [Ko63], [PV94, 8.5]). Combined with the existence of bijection (1), this yields a parametrization of  $\mathcal{H}$  by  $\mathbb{A}^r \setminus \{0\}$ .

The proofs of Theorems 1 and 2 are given in Section 3.

*Conventions and notation.* Our basic reference for algebraic groups and algebraic geometry is [Bo91] and we follow the conventions therein. Unless otherwise stated, all algebraic groups and algebraic varieties are taken over an algebraically closed field  $k$  whose characteristic is not a bad prime for reductive  $G$  (see [SS70, Chap. I, Def. 4.1]).

We use the following notation:

$C_G(M)$  is the  $G$ -centralizer of a subset  $M$  of  $\mathfrak{g}$  or  $G$ .

$\text{Rad}_u(Q)$  is the unipotent radical of an affine algebraic group  $Q$ .

$\mathbb{A}^n$  is the  $n$ -dimensional affine space.

## 2. Birational complements

**Definition 1.** Let  $G$  and  $H$  be as in Section 1. A sequence  $S_1, \dots, S_m$  of locally closed subsets of  $G$  is called a *birational complement to  $H$  in  $G$*  if the morphism

$$\lambda: S_1 \times \cdots \times S_m \times H \rightarrow G, \quad (s_1, \dots, s_m, h) \mapsto s_1 \cdots s_m h \quad (2)$$

is an open embedding.

*Remark 1.* One can show that this notion is order-sensitive, i.e., reshuffling the terms of a sequence that is a birational complement to  $H$  in  $G$ , one obtains a sequence that, generally speaking, is not a birational complement to  $H$  in  $G$ .

If  $S_1, \dots, S_m$  is a birational complement to  $H$  in  $G$ , then Definition 1 implies, in view of the connectedness of  $G$ , that  $H$  is connected as well. It also implies that open embedding (2) is  $H$ -equivariant with respect to the action of  $H$  on  $G$  by right translations and on  $S_1 \times \cdots \times S_m \times H$  by right translations of the last factor. Thus, the image of open embedding (2) is an  $H$ -invariant open subset of  $G$  that is  $H$ -equivariantly isomorphic to  $S_1 \times \cdots \times S_m \times H$ .

**Example 1.** Let  $G$  be a semidirect product  $A \rtimes B$  of closed subgroups  $A$  and  $B$ . Then the one-term sequence  $A$  (resp.  $B$ ) is a birational complement to  $B$  (resp.  $A$ ) in  $G$ . For instance, one can take  $A$  and  $B$  to be respectively a Levi subgroup of  $G$  and the unipotent radical  $\text{Rad}_u(G)$ .

**Example 2.** Let  $G$  be a reductive algebraic group and let  $P$  be a parabolic subgroup of  $G$ . Let  $P^-$  be the parabolic subgroup of  $G$  opposite to  $P$ . Then the one-term sequence  $\text{Rad}_u(P^-)$  is a birational complement in  $G$  to  $P$  (see [Bo91, Prop. 14.21(iii)]).

**Lemma 1.** *Let  $G$  and  $H$  as in Section 1 and let  $Q$  be a closed subgroup of  $H$ . Let  $S_1, \dots, S_m$  be a birational complement to  $H$  in  $G$ .*

(a) *If  $Z_1, \dots, Z_n$  is a birational complement to  $Q$  in  $H$ , then*

$$S_1, \dots, S_m, Z_1, \dots, Z_n$$

*is a birational complement to  $Q$  in  $G$ .*

(b) *The variety  $G/Q$  contains an open subset isomorphic to*

$$S_1 \times \cdots \times S_m \times (H/Q). \quad (3)$$

*Proof.* (a) Let  $X := S_1 \times \cdots \times S_m$  and  $Y := Z_1 \times \cdots \times Z_n$ . By Definition 1,

$$\mu: Y \times Q = Z_1 \times \cdots \times Z_n \times Q \rightarrow H, \quad (z_1, \dots, z_n, q) \mapsto z_1 \cdots z_n q$$

is open embedding, therefore,  $\nu := \text{id}_X \times \mu: X \times (Y \times Q) \rightarrow X \times H$  is an open embedding. Since  $\lambda$  (see (2)) is also an open embedding, this implies that the morphism

$$\begin{aligned} \lambda \circ \nu: X \times (Y \times Q) &= S_1 \times \cdots \times S_m \times Z_1 \times \cdots \times Z_n \rightarrow G, \\ (s_1, \dots, s_m, z_1, \dots, z_n, q) &\mapsto s_1 \cdots s_m z_1 \cdots z_n q \end{aligned}$$

is an open embedding as well. This proves (a).

(b) As noted above,  $G$  contains an  $H$ -invariant open subset  $U$  that is  $H$ -equivariantly isomorphic to  $S_1 \times \cdots \times S_m \times H$ . Therefore, by [Bo91, II, Thm. 6.8 and Cor. 6.6], a geometric quotient  $U/Q$  exists and is isomorphic to variety (3). On the other hand,  $U/Q$  is isomorphic to an open subset of  $G/Q$  because the canonical morphism  $G \rightarrow G/Q$  is open (see [Bo91, II, 6.1]). This proves (b).  $\square$

**Example 3.** Taking  $H = Q$  in Lemma 1(b) yields that  $G/H$  contains an  $H$ -invariant open subset isomorphic to  $S_1 \times \cdots \times S_m$ . For instance,  $G/P$  in Example 2 contains an open subset isomorphic to the underlying variety of  $\text{Rad}_u(P^-)$ , i.e., to an affine space (see [Bo91, IV, 14.4, Rem.]); whence,  $G/P$  is rational.

**Example 4.** Let  $G$  be a reductive group, let  $P$  be its a parabolic subgroup and let  $L$  be a Levi subgroup of  $P$ . Then by Lemma 1 and Examples 1, 2, the two-term sequence  $\text{Rad}_u(P^-)$ ,  $\text{Rad}_u(P)$  is a birational complement to  $L$  in  $G$  and  $G/L$  contains an open subset isomorphic to the underlying variety of  $\text{Rad}_u(P^-) \times \text{Rad}_u(P)$ , i.e., to an affine space; whence  $G/L$  is a rational variety.

**Example 5.** Let  $B$  be a Borel subgroup of  $G$  and let  $Q$  be a closed subgroup of  $B$ . In view of Lemma 1 and Example 2, the variety  $G/Q$  contains an open subset isomorphic to  $\mathbb{A}^d \times (B/Q)$ . By [Ro62, Thm. 5], the variety  $B/Q$  is isomorphic to  $\mathbb{A}^s \times (\mathbb{A}^1 \setminus \{0\})^t$  for some  $s, t$ . Whence  $G/Q$  is a rational variety (this statement is Theorem 2.9 of [CZ15]).

Since every connected solvable subgroup of  $G$  lies in a Borel subgroup of  $G$ , this implies that the variety  $G/H$  is rational if  $H$  is connected solvable.

### 3. Proofs of Theorems 1 and 2

In the proof of Theorems 1 and 2, we shall use the following

#### Theorem 3.

• Let  $G$  be a connected reductive algebraic group and let  $x$  be an element of  $\mathfrak{g}$ . The following properties are equivalent:

- (a)  $x$  is semisimple;
- (b) the  $G$ -orbit  $\mathcal{O}$  of  $x$  is an affine variety;
- (c)  $C_G(x)$  is reductive;
- (d)  $\mathcal{O}$  is a closed subset of  $\mathfrak{g}$ .

• Let  $G$  be a simply connected semisimple algebraic group. Then properties (a)–(d) are equivalent to the property

- (e)  $C_G(x)$  is a Levi subgroup of a parabolic subgroup of  $G$ .

For any parabolic subgroup  $P$  of  $G$  and any Levi subgroup  $L$  of  $P$ , there is a semisimple element  $s \in \mathfrak{g}$  such that  $C_G(s) = L$ .

*Proof.*

(a) $\Leftrightarrow$ (d) is well-known (see [Ko63, Rem. 11], [PV94, 8.5]).

(d) $\Rightarrow$ (b) is clear.

(b) $\Leftrightarrow$ (c) follows from Matsushima's criterion (see [PV94, Thm. 4.17], [Ri75]).

(b) $\Rightarrow$ (d) Let  $\mathcal{O}$  be affine and let  $\overline{\mathcal{O}}$  be the closure of  $\mathcal{O}$  in  $\mathfrak{g}$ . The set  $X := \overline{\mathcal{O}} \setminus \mathcal{O}$  is closed in  $\mathfrak{g}$  (see [PV94, 1.3]). Arguing on the contrary, assume that  $X \neq \emptyset$ . Since  $\mathcal{O}$  is affine, this yields  $\text{codim}_{\overline{\mathcal{O}}}(X) = 1$  (see [Go69]). On the other hand,  $\text{codim}_{\overline{\mathcal{O}}}(X) \geq 2$  by [Ko63, Cor. 1 of Thm. 3]. This contradiction shows that  $\overline{\mathcal{O}} = \mathcal{O}$ .

(a) $\Rightarrow$ (e) Let  $x$  be a semisimple element. By [St75, Cor. 3.8], there is a torus  $S$  in  $G$  such that  $x \in \mathfrak{s} := \text{Lie}(G)$  and

$$C_G(x) = C_G(\mathfrak{s}). \quad (4)$$

By [DM20, Prop. 3.4.7], there is a parabolic subgroup  $P$  of  $G$  such that  $C_G(S)$  is a Levi subgroups of  $P$ . Since

$$C_G(S) = C_G(\mathfrak{s}) \quad (5)$$

(see [TY05, Thm. 24.4.8(ii)]), from (4) and (5) we infer that  $C_G(x)$  is a Levi subgroups of  $P$ .

(e) $\Rightarrow$ (c) is clear.

To prove the last statement of Theorem 3, note that since  $L$  is a Levi subgroup of a parabolic subgroup of  $G$ , there is a torus  $S$  in  $G$  such that

$$L = C_G(S) \tag{6}$$

(see [DM20, Prop. 3.4.6]). We identify  $S$  with  $(k^*)^d$  by means an isomorphism between them. Let  $\mathfrak{s} = k^d$  be the Lie algebra of  $S$ . If  $z = (z_i) \in \mathfrak{s}$ , let  $R_z := \{(m_i) \in \mathbb{Z}^d \mid \sum_i m_i z_i = 0\}$ . Then the minimal algebraic subalgebra  $\mathfrak{a}(z)$  of  $\mathfrak{s}$  containing  $z$  is  $\{(s_i) \in \mathfrak{s} \mid \sum_i m_i s_i = 0 \text{ for all } (m_i) \in R_z\}$  (see [Bo91, II, 7.3(2)]). Since the degree of  $k$  over its prime subfield is infinite, this implies the existence of  $x \in \mathfrak{s}$  such that  $\mathfrak{s} = \mathfrak{a}(x)$ . By [St75, Cor. 3.8], we then have  $C_G(x) = C(\mathfrak{s})$ . In view of (5) and (6), this yields  $L = C_G(x)$ .  $\square$

*Proof of Theorems 1.* Since the center of  $G$  lies in the kernel of the adjoint representation of  $G$ , without changing the  $G$ -orbits in  $\mathfrak{g}$ , we may (and shall) assume that  $G$  is a simply connected semisimple group.

Let  $x$  be an element of  $\mathfrak{g}$ . Our goal is to prove that  $G/C_G(x)$  is a rational variety. We may (and shall) assume that  $x \neq 0$ . We shall consider separately three cases:

- (i)  $x$  is nilpotent,
- (ii)  $x$  is semisimple,
- (iii)  $x$  is neither nilpotent, nor semisimple.

*Case (i).* Let  $x$  be nilpotent. Then by the Jacobson–Morozov theorem, there are elements  $h, y \in \mathfrak{g}$  such that  $\{x, h, y\}$  is an  $\mathfrak{sl}_2$ -triple, i.e.,  $[h, x] = 2x$ ,  $[h, y] = -2y$ ,  $[x, y] = h$ . For every  $i \in \mathbb{Z}$ , put

$$\mathfrak{g}(i) := \{z \in \mathfrak{g} \mid [h, z] = iz\}.$$

Then we have the decomposition  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ , which is a structure of a  $\mathbb{Z}$ -graded Lie algebra on  $\mathfrak{g}$ .

The subspace  $\mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}(i)$  is a parabolic subalgebra of  $\mathfrak{g}$ . Let  $P$  be the parabolic subgroup of  $G$  such that  $\text{Lie}(P) = \mathfrak{p}$ . Then the  $P$ -stable subspace  $\mathfrak{u} := \bigoplus_{i > 0} \mathfrak{g}(i)$  is  $\text{Lie}(\text{Rad}_u(P))$ .

We have  $x \in \mathfrak{g}(2) \subseteq \mathfrak{u}$ . By [SS70, Chap. III, Sect. 4.20(i)], the  $P$ -orbit of  $x$  is open in  $\mathfrak{u}$ , therefore

$$P/C_P(x) \text{ is isomorphic to an open subset of an affine space.} \tag{7}$$

By [SS70, Chap. III, Sect. 4.16], we have  $C_G(x) \subset P$ ; whence

$$C_P(x) = C_G(x). \tag{8}$$

In view of (8), we have the following tower of algebraic groups:

$$G \supset P \supset C_G(x), \tag{9}$$

By Example 2 and Lemma 1(b) applied to (9), we infer that  $G/C_G(x)$  contains an open subset isomorphic to  $(\text{Rad}_u(P^-)) \times (P/C_G(x))$ . Since the underlying variety of  $\text{Rad}_u(P^-)$  is isomorphic to an affine space, we infer from (7) that  $G/C_G(x)$  contains an open subset isomorphic to an open set of an affine space. Therefore,  $G/C_G(x)$  is a rational variety.

*Case (ii).* Let  $x$  be semisimple. Then  $C_G(x)$  is a Levi subgroup of a parabolic subgroup of  $G$  in view of Theorem 3. Hence the variety  $G/C_G(x)$  is rational by Example 4.

*Case (iii).* Let  $x$  be neither nilpotent, nor semisimple. Let  $x = x_s + x_n$  be the Jordan decomposition of  $x$ . By Theorem 3, the group  $C_G(x_s)$  is a Levi subgroup of a parabolic subgroup of  $G$ . We have  $x_n \in \text{Lie}(C_G(x_s))$  and it follows from the uniqueness of the Jordan decomposition that

$$C_G(x) = C_{C_G(x_s)}(x_n). \quad (10)$$

By Example 4, there is a two-term birational complement  $S_1, S_2$  to  $C_G(x_s)$  in  $G$  such that

$$S_i \text{ is isomorphic to an affine space for every } i. \quad (11)$$

Applying Lemma 1 to  $H = C_G(x_s)$ ,  $Q = C_{C_G(x_s)}(x_n)$ , we obtain from (10) that  $G/C_G(x)$  contains an open set isomorphic to

$$S_1 \times S_2 \times (C_G(x_s)/C_{C_G(x_s)}(x_n)).$$

By (i), the variety  $C_G(x_s)/C_{C_G(x_s)}(x_n)$  is rational. This and (11) imply that  $G/C_G(x)$  is rational.  $\square$

*Remark 2.* Theorem 1 establishes a specific property of the adjoint representation: for some connected reductive groups  $G$ , there are finite-dimensional algebraic representations not all of whose  $G$ -orbits are rational algebraic varieties.

Indeed, in [Po11, p. 298], [Po13, Thm. 2], [Po94, Rem. 1.5.9] are constructed connected reductive algebraic groups  $G$  with a finite subgroup  $H$  such that the algebraic variety  $G/H$  is nonrational.

Being finite,  $H$  is reductive; hence, by Matsushima's criterion, the variety  $G/H$  is affine. Therefore, by the embedding theorem (see [PV94, Thm. 1.5]), there exists a  $G$ -equivariant (with respect to the natural action of  $G$  on  $G/H$ ) closed embedding of  $G/H$  into some finite-dimensional algebraic  $G$ -module.

*Proof of Theorems 2.*

(a) By [Ko70, Thm. 5.4.1, Prop. 5.1.1], there are a unique  $G$ -orbit  $\mathcal{O}_X \subset \mathfrak{g}$  and a unique morphism  $\tau_X: X \rightarrow \mathcal{O}_X$  of Hamiltonian  $G$ -varieties. By [Ko70, Prop. 5.1.1],  $\tau_X$  is a covering, and for every  $x \in X$ , the identity component of the  $G$ -stabilizer  $G_x$  of  $x$  coincides with that of

the  $G$ -stabilizer  $G_{\tau_X(x)}$  of  $\tau_X(x)$ . Since  $G$  acts on  $X$  transitively and  $X$  is affine,  $G_x$  is reductive by Matsushima's criterion. Therefore,  $G_{\tau_X(x)}$  is reductive as well. Then from the equivalence of properties (a), (c), and (e) in Theorem 3 we infer that  $\mathcal{O}_X \in \mathcal{S}$  and  $G_{\tau_X(x)}$  is connected. Since  $G$  is simply connected, the latter implies that  $\mathcal{O}_X$  is simply connected as well. This, in turn, implies that  $\tau_X$  is an isomorphism since  $\tau_X$  is a covering.

(b) In view of (a), this follows from the fact that each  $G$ -orbit in  $\mathfrak{g}$  is endowed with the standard structure of a Hamiltonian  $G$ -variety.

(c) This follows from (a), since, as was proved above,  $\mathcal{O}_X$  is a simply connected and, by Theorem 1, rational variety.

(d) This follows from (a) and the equivalence of (a) and (e) in Theorem 3.

(e) This follows from the last statement of Theorem 3. □

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