A. Yu. Mikhaylenko, S. V. Sudoplatov, Conflict and conflict-free theories, *Sib. Èlektron. Mat. Izv.*, 2019, Volume 16, 1833–1842

DOI: https://doi.org/10.33048/semi.2019.16.130

S(e)MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 16, стр. 1833–1842 (2019) УДК 510.67

DOI 10.33048/semi.2019.16.130 MSC 03C30, 03C45, 03C15, 03C52

CONFLICT AND CONFLICT-FREE THEORIES

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ABSTRACT. We define and study λ-conflict theories and, in particular, conflict-free theories. A series of conflict-free theories is found. It is proved that there are λ-conflict theories for arbitrary λ. It is shown that λ-conflictness is not preserved under expansions of theories.

Keywords: conflict theory, conflict-free theory, generic structure, cardinality contradiction.

Considering syntactic approach to generic constructions and their limits [1, 2, 3, 4, 5, 6, 7, 8] we introduce and study λ-conflict theories. It is shown that there are no conflict theories in the classes of countable theories, theories of unary predicates and sequentially embedded equivalence relations. It is proved that there are λ-conflict theories for arbitrary λ. It is shown that λ-conflictness is not preserved under expansions of theories.

1. Preliminaries

We remind notions, notations and assertions of [1, 2, 3, 4, 5, 6, 7, 8] that will be used in the next section.

We consider collections of formulas in first order logic over a language Σ. Thus, as usual, $\vdash$ means proof from no hypotheses deducing $\vdash \varphi$ for a formula φ of language Σ, which may contain function symbols and constants. If deducing φ, hypotheses in a set Φ of formulas can be used, we write $\Phi \vdash \varphi$. Usually Σ will be fixed in context and not mentioned explicitly.

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This research was partially supported by the program of fundamental scientific researches of the SB RAS No. I.1.1, project No. 0314-2019-0002, Russian Foundation for Basic Researches (Project No. 17-01-00531-a), and Committee of Science in Education and Science Ministry of the Republic of Kazakhstan (Grant No. AP05132546).

Received November, 8, 2019, published December, 5, 2019.

Below we write $X, Y, Z, \dots$ for finite sets of variables, and denote by $A, B, C, \dots$ finite sets of elements, as well as finite sets in structures, or else the structures with finite universes themselves.

In diagrams, $A, B, C, \dots$ denote finite sets of constant symbols disjoint from the constant symbols in Σ and $\Sigma(A)$ is the vocabulary with the constants from A adjoined. $\Phi(A), \Psi(B), \mathrm{X}(C)$ stand for Σ-*diagrams* (of sets A, B, C), that is, *consistent sets of* $\Sigma(A)$-, $\Sigma(B)$-, $\Sigma(C)$-*sentences*, respectively.

Below we assume that for any considered diagram $\Phi(A)$, if a_1, a_2 are distinct elements in A then $\neg(a_1 \approx a_2) \in \Phi(A)$. This means that if c is a constant symbol in Σ, then there is at most one element $a \in A$ such that $(a \approx c) \in \Phi(A)$.

If $\Phi(A)$ is a diagram and B is a set, we denote by $\Phi(A)|_B$ the set $\{\varphi(\bar{a}) \in \Phi(A) \mid \bar{a} \in B\}$. Similarly, for a language Σ, we denote by $\Phi(A)|_\Sigma$ the restriction of $\Phi(A)$ to the set of formulas in the language Σ.

Definition [1, 2, 3, 4, 5, 6, 8]. We denote by $[\Phi(A)]^A_B$ the diagram $\Phi(B)$ obtained by replacing a subset $A' \subseteq A$ by a set $B' \subseteq B$ of constants disjoint from Σ and with $|A'| = |B'|$, where $A \setminus A' = B \setminus B'$. Similarly we call the consistent set of formulas denoted by $[\Phi(A)]^A_X$ the type $\Phi(X)$ if it is the result of a bijective substitution into $\Phi(A)$ of variables of X for the constants in A. In this case, we say that $\Phi(B)$ is a *copy* of $\Phi(A)$ and a *representative* of $\Phi(X)$. We also denote the diagram $\Phi(A)$ by $[\Phi(X)]^X_A$.

Remark 1.1. If the vocabulary contains functional symbols then diagrams $\Phi(A)$ containing equalities and inequalities of terms can generate both finite and infinite structures. The same effect is observed for purely predicate vocabularies if it is written in $\Phi(A)$ that the model for $\Phi(A)$ should be infinite. For instance, diagrams containing axioms for finitely axiomatizable theories have this property.

By the definition, for any diagram $\Phi(A)$, each constant symbol in Σ appears in some formula of $\Phi(A)$. Thus, $\Phi(A)$ can be considered as $\Phi(A \cup K)$, where K is the set of constant symbols in Σ.

We now give conditions on a partial ordering of a collection of diagrams which suffice for it to determine a structure. We modify some of the conditions for structures by d to signify they are conditions on diagrams not structures.

Definition [1, 2, 3, 4, 5, 6, 8]. Let Σ be a vocabulary. We say that $(\mathbf{D}_0; \leqslant)$ (or $\mathbf{D}_0$) is *generic*, or *generative*, if $\mathbf{D}_0$ is a class of Σ-diagrams of finite sets so that $\mathbf{D}_0$ is partially ordered by a binary relation $\leqslant$ such that $\leqslant$ is preserved by bijective substitutions, i. e., if $\Phi(A) \leqslant \Psi(B)$, and $A' \subseteq B'$ such that $[\Phi(A)]^A_{A'} = \Phi(A')$ and $[\Psi(B)]^B_{B'} = \Psi(B')$ are defined, then $[\Phi(A)]^A_{A'}$, $[\Psi(B)]^B_{B'}$ are in $\mathbf{D}_0$ and $[\Phi(A)]^A_{A'} \leqslant [\Psi(B)]^B_{B'}$.[1] Furthermore:

(i) if $\Phi(A) \in \mathbf{D}_0$ then for any quantifier free formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in A$ either $\varphi(\bar{a}) \in \Phi(A)$ or $\neg\varphi(\bar{a}) \in \Phi(A)$;

(ii) if $\Phi \leqslant \Psi$ then $\Phi \subseteq \Psi$;[2]

(iii) if $\Phi \leqslant \mathrm{X}$, $\Psi \in \mathbf{D}_0$, and $\Phi \subseteq \Psi \subseteq \mathrm{X}$, then $\Phi \leqslant \Psi$;

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[1] Note that $\mathbf{D}_0$ is closed under bijective substitutions since $\leqslant$ is preserved by bijective substitutions and $\leqslant$ is reflexive.

[2] Note that $\Phi(A) \leqslant \Psi(B)$ implies $A \subseteq B$, since if $a \in A$ then $(a \approx a) \in \Phi(A)$, so $\Phi(A) \leqslant \Psi(B)$ implies $\Phi(A) \subseteq \Psi(B)$ and we have $(a \approx a) \in \Psi(B)$, whence $a \in B$.

(iv) some diagram $\Phi_0(\emptyset)$ is the least element of the system $(\mathbf{D}_0;\leqslant)$, and $\mathbf{D}_0 \setminus \{\Phi_0(\emptyset)\}$ is nonempty;

(v) (the *d-amalgamation property*) for any diagrams $\Phi(A)$, $\Psi(B)$, $\mathrm{X}(C) \in \mathbf{D}_0$, if there exist injections $f_0\colon A \to B$ and $g_0\colon A \to C$ with $[\Phi(A)]^A_{f_0(A)} \leqslant \Psi(B)$ and $[\Phi(A)]^A_{g_0(A)} \leqslant \mathrm{X}(C)$, then there are a diagram $\Theta(D) \in \mathbf{D}_0$ and injections $f_1\colon B \to D$ and $g_1\colon C \to D$ for which $[\Psi(B)]^B_{f_1(B)} \leqslant \Theta(D)$, $[\mathrm{X}(C)]^C_{g_1(C)} \leqslant \Theta(D)$ and $f_0 \circ f_1 = g_0 \circ g_1$; the diagram $\Theta(D)$ is called the *amalgam* of $\Psi(B)$ and $\mathrm{X}(C)$ over the diagram $\Phi(A)$ and witnessed by the four maps (f_0, g_0, f_1, g_1);

(vi) (the *local realizability property*) if $\Phi(A) \in \mathbf{D}_0$ and $\Phi(A) \vdash \exists x\, \varphi(x)$, then there are a diagram $\Psi(B) \in \mathbf{D}_0$, $\Phi(A) \leqslant \Psi(B)$, and an element $b \in B$ for which $\Psi(B) \vdash \varphi(b)$;

(vii) (the *d-uniqueness property*) for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ if $A \subseteq B$ and the set $\Phi(A) \cup \Psi(B)$ is consistent then $\Phi(A) = \{\varphi(\bar{b}) \in \Psi(B) \mid \bar{b} \in A\}$.

A diagram Φ is called a *strong subdiagram* of a diagram Ψ if $\Phi \leqslant \Psi$.

A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a diagram $\Psi(B)$ if there is an injection $f\colon A \to B$ such that $[\Phi(A)]^A_{f(A)} \subseteq \Psi(B)$ $([\Phi(A)]^A_{f(A)} \leqslant \Psi(B))$. The injection f, in this instance, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in diagram $\Psi(B)$ and is denoted by $f\colon \Phi(A) \to \Psi(B)$. A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a structure $\mathcal{M}$ if $\Phi(A)$ is (strongly) embeddable in some diagram $\Psi(B)$, where $\mathcal{M} \models \Psi(B)$. The corresponding embedding $f\colon \Phi(A) \to \Psi(B)$, in this case, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in structure $\mathcal{M}$ and is denoted by $f\colon \Phi(A) \to \mathcal{M}$.

Let $\mathbf{D}_0$ be a class of diagrams, $\mathbf{P}_0$ be a class of structures of some language, and $\mathcal{M}$ be a structure in $\mathbf{P}_0$. The class $\mathbf{D}_0$ is *cofinal* in the structure $\mathcal{M}$ if for each finite set $A \subseteq M$, there are a finite set B, $A \subseteq B \subseteq M$, and a diagram $\Phi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Phi(B)$. The class $\mathbf{D}_0$ is *cofinal* in $\mathbf{P}_0$ if $\mathbf{D}_0$ is cofinal in every structure of $\mathbf{P}_0$. We denote by $\mathbf{K}(\mathbf{D}_0)$ the class of all structures $\mathcal{M}$ with the condition that $\mathbf{D}_0$ is cofinal in $\mathcal{M}$, and by $\mathbf{P}$ a subclass of $\mathbf{K}(\mathbf{D}_0)$ such that each diagram $\Phi \in \mathbf{D}_0$ is true in some structure in $\mathbf{P}$.

Now we extend the relation $\leqslant$ from the generative class $(\mathbf{D}_0;\leqslant)$ to a class of subsets of structures in the class $\mathbf{K}(\mathbf{D}_0)$.

Let $\mathcal{M}$ be a structure in $\mathbf{K}(\mathbf{D}_0)$, A and B be finite sets in $\mathcal{M}$ with $A \subseteq B$. We call A a *strong subset* of the set B (in the structure $\mathcal{M}$), and write $A \leqslant B$, if there exist diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, for which $\Phi(A) \leqslant \Psi(B)$ and $\mathcal{M} \models \Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0 \subseteq M$ (in the structure $\mathcal{M}$), where $A \subseteq M_0$, if $A \leqslant B$ for any finite set B such that $A \subseteq B \subseteq M_0$ and $\Phi(A) \subseteq \Psi(B)$ for some diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ with $\mathcal{M} \models \Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A \leqslant M_0$. If $A \leqslant M$ in $\mathcal{M}$ then we refer to A as a *self-sufficient set* (in $\mathcal{M}$).

Notice that, by the d-uniqueness property, the diagrams $\Phi(A)$ and $\Psi(B)$ specified in the definition of strong subsets are defined uniquely. A diagram $\Phi(A) \in \mathbf{D}_0$, corresponding to a self-sufficient set A in $\mathcal{M}$, is said to be a *self-sufficient diagram* (in $\mathcal{M}$).

Definition [1, 2, 3, 4, 5, 6, 8]. A class $(\mathbf{D}_0;\leqslant)$ possesses the *joint embedding property* (JEP) if for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, there is a diagram $\mathrm{X}(C) \in \mathbf{D}_0$ such that $\Phi(A)$ and $\Psi(B)$ are strongly embeddable in $\mathrm{X}(C)$.

Clearly, every generative class has JEP since JEP means the d-amalgamation property over the empty set.

Definition [1, 2, 3, 4, 5, 6, 8]. A structure $\mathcal{M} \in \mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0; \leqslant)$, or is *finitely generated over* Σ, if any finite set $A \subseteq M$ is contained in some finite self-sufficient set in $\mathcal{M}$, i. e., there is a finite set B with $A \subseteq B \subseteq M$ and $\Psi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Psi(B)$ and $\Psi(B) \leqslant \mathrm{X}(C)$ for any $\mathrm{X}(C) \in \mathbf{D}_0$ with $\mathcal{M} \models \mathrm{X}(C)$ and $\Psi(B) \subseteq \mathrm{X}(C)$. A class $\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0; \leqslant)$, or is *finitely generated over* Σ, if each structure in $\mathbf{P}$ has finite closures (with respect to $(\mathbf{D}_0; \leqslant)$).

Clearly, an at most countable structure $\mathcal{M}$ has finite closures with respect to $(\mathbf{D}_0; \leqslant)$ if and only if $M = \bigcup\limits_{i\in\omega} A_i$ for some self-sufficient sets A_i with $A_i \leqslant A_{i+1}$, $i \in \omega$.

Note that the finite closure property is defined modulo Σ and does not correlate with the cardinalities of algebraic closures. For instance, if Σ contains infinitely many constant symbols then $\mathrm{acl}(A)$ is always infinite whereas a finite set A can or can not be extended to a self-sufficient set.

Besides, for the finite closures of sets A we consider finite self-sufficient extensions B in a given structure $\mathcal{M}$ with respect to $(\mathbf{D}_0; \leqslant)$ only and B can be both a universe of a substructure of $\mathcal{M}$ or not. Moreover, it is permitted that corresponding diagrams $\Psi(B)$ can have only finite, finite and infinite, or only infinite models.

Thus, for instance, a finitely axiomatizable theory without finite models and with a generative class $(\mathbf{D}_0; \subseteq)$, containing diagrams for all finite sets and with axioms in diagrams, has identical finite closures whereas each diagram in $\mathbf{D}_0$ has only infinite models.

Definition [1, 2, 3, 4, 5, 6, 8]. A structure $\mathcal{M} \in \mathbf{K}(\mathbf{D}_0)$ is $(\mathbf{D}_0; \leqslant)$-*generic*, or a *generic limit for the class* $(\mathbf{D}_0; \leqslant)$ and denoted by $\mathrm{glim}(\mathbf{D}_0; \leqslant)$, if it satisfies the following conditions:

(a) $\mathcal{M}$ has finite closures with respect to $\mathbf{D}_0$;

(b) if $A \subseteq M$ is a finite set, $\Phi(A), \Psi(B) \in \mathbf{D}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leqslant \Psi(B)$, then there exists a set $B' \leqslant M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

Theorem 1.2 [1, 2, 3, 6, 8]. *For any generative class $(\mathbf{D}_0; \leqslant)$ with at most countably many diagrams whose copies form $\mathbf{D}_0$, there exists a $(\mathbf{D}_0; \leqslant)$-generic structure.*

Theorem 1.3 [4, 6, 8]. *Every ω-homogeneous structure $\mathcal{M}$ is $(\mathbf{D}_0; \leqslant)$-generic for some generative class $(\mathbf{D}_0; \leqslant)$.*

Thus any first-order theory has a generic model and therefore can be represented by it.

Definition [1, 2, 3, 4, 5, 6, 8]. A generative class $(\mathbf{D}_0; \leqslant)$ is *self-sufficient* if the following *axiom of self-sufficiency* holds:

(viii) if $\Phi, \Psi, \mathrm{X} \in \mathbf{D}_0$, $\Phi \leqslant \Psi$, and $\mathrm{X} \subseteq \Psi$, then $\Phi \cap \mathrm{X} \leqslant \mathrm{X}$.

Theorem 1.4 [1, 2, 3, 6, 8]. *Let $(\mathbf{D}_0; \leqslant)$ be a self-sufficient class, $\mathcal{M}$ be at most countable $(\mathbf{D}_0; \leqslant)$-generic structure, and $\mathbf{K}$ be the class of all models of $T = \mathrm{Th}(\mathcal{M})$ which has finite closures. Then the generic structure $\mathcal{M}$ is homogeneous.*

Thus, since any ω-homogeneous structure can be considered as generic with respect to a generic class with complete diagrams, a countable structure $\mathcal{M}$ is homogeneous if and only if it is generic for an appropriate self-sufficient generative class $(\mathbf{D}_0; \leqslant)$.

Recall the following notations and properties for links between definable sets.

Definition [8]. If X and Y are definable sets in a structure $\mathcal{M}$, $X = \phi(\mathcal{M}, \bar{a})$, $Y = \psi(\mathcal{M}, \bar{b})$, $\bar{a}, \bar{b} \in M$, $|X| = \lambda$, $|Y| = \mu$, then we write $X \Rightarrow_{\mu,\mathcal{M}} Y$, $X\,_{\lambda,\mathcal{M}}\!\Leftarrow Y$, and $X\,_{\lambda,\mathcal{M}}\!\Leftrightarrow_{\mu,\mathcal{M}} Y$. If $X \Rightarrow_{\mu,\mathcal{N}} Y'$ (respectively, $X'\,_{\lambda,\mathcal{N}}\!\Leftarrow Y$; $X'\,_{\lambda,\mathcal{N}}\!\Leftrightarrow_{\mu,\mathcal{N}} Y'$) for any $\mathcal{N}$ such that $\bar{a}, \bar{b} \in N$, $\mathcal{M} \prec \mathcal{N}$ or $\mathcal{M} \succ \mathcal{N}$, $X = \phi(\mathcal{N}, \bar{a})$, $Y' = \psi(\mathcal{N}, \bar{b})$ ($X' = \phi(\mathcal{N}, \bar{a})$, $Y = \psi(\mathcal{N}, \bar{b})$; $X' = \phi(\mathcal{N}, \bar{a})$, $Y' = \psi(\mathcal{N}, \bar{b})$, and $|X'| = \lambda$ or $|Y'| = \mu$), then we say that X *forces* the cardinality μ for Y (Y *forces* the cardinality λ for X; X and Y mutually force cardinalities λ and μ), written $X \Rightarrow_{\mu} Y$ ($X\,_{\lambda}\!\Leftarrow Y$; $X\,_{\lambda}\!\Leftrightarrow_{\mu} Y$). Here X' (respectively, Y') is called a *copy* of X (Y).

Replacing λ by $\leq \lambda$ or $\geq \lambda$ or $< \lambda$ or $> \lambda$, and/or μ by $\leq \mu$ or $\geq \mu$ or $< \mu$ or $> \mu$, we get a series of related notions and notations, for instance, $X\,_{\leq\lambda,\mathcal{N}}\!\Leftrightarrow_{>\mu,\mathcal{N}} Y$.

Having $X \Rightarrow_{\mu,\mathcal{M}} Y$, $X \Rightarrow_{\leq\mu,\mathcal{M}} Y$, or $X \Rightarrow_{\geq\mu,\mathcal{M}} Y$ for any X we write $\Rightarrow_{\mu,\mathcal{M}} Y$, $\Rightarrow_{\leq\mu,\mathcal{M}} Y$, or $\Rightarrow_{\geq\mu,\mathcal{M}} Y$ respectively.

Example 1.5 [8]. Taking a structure $\mathcal{M}$ with infinite disjoint unary predicates P_0 and P_1 of cardinalities λ and μ, respectively, and without any links we have $X\,_{\lambda,\mathcal{M}}\!\Leftrightarrow_{\mu,\mathcal{M}} Y$ for $X = P_0(\mathcal{M})$ and $Y = P_1(\mathcal{M})$, whereas $X\,_{\lambda}\!\not\Leftrightarrow_{\mu} Y$, even $X \not\Rightarrow_{\mu} Y$ and $X\,_{\lambda}\!\not\Leftarrow Y$. If $\lambda \geq \mu$ we can extend the language for $\mathcal{M}$ by a function $f\colon P_0 \to P_1$ which guarantee $X \Rightarrow_{\mu} Y$.

The example confirms that the relation $X \Rightarrow_{\mu} Y$ is not preserved under language restrictions.

The following properties for definable sets are obvious.

1. If Y is finite then $X \Rightarrow_n Y$ for some unique $n \in \omega$ and for any/some X. Conversely, if Y is infinite then $X \not\Rightarrow_n Y$ for any $n \in \omega$ and for any/some X. Thus we have $\Rightarrow_{<\omega} Y$ for finite Y and $\not\Rightarrow_{<\omega} Y$ for infinite one.

2. If Y is infinite then $X \Rightarrow_{\geq\omega} Y$ for any/some X. Conversely, if Y is finite then $X \not\Rightarrow_{\geq\omega} Y$ for any/some X. Thus we have $\Rightarrow_{\geq\omega} Y$ for infinite Y and $\not\Rightarrow_{\geq\omega} Y$ for finite one.

3. (Monotony) If $X \Rightarrow_{\leq\lambda} Y$, $\lambda \leq \mu$ and $Y \supseteq Z$, then $X \Rightarrow_{\leq\mu} Z$.

4. (Transitivity) If X, Y, Z are definable sets in a structure $\mathcal{M}$, $X \Rightarrow_{\lambda} Y'$ and $Y' \Rightarrow_{\mu} Z'$ for any copies Y' and Z' of Y and Z, respectively, then $X \Rightarrow_{\mu} Z$. The same is true replacing λ by $\leq \lambda$ or $\geq \lambda$, and μ by $\leq \mu$ or $\geq \mu$.

5. If X_i are disjoint subsets of Y and $\Rightarrow_{\geq\lambda_i} X_i$, $i \in \kappa$, then $\Rightarrow_{\geq \sum\limits_{i\in\kappa} \lambda_i} Y$. In particular, if Y is implied by λ disjoint nonempty definable sets then $\Rightarrow_{\geq\lambda} Y$.

6. If X and Y have a definable function $f\colon X \to Y$ and $|Y| = \lambda$ then $X\,_{\geq\lambda}\!\Leftarrow Y$. In particular, if X and Y have a definable bijection $f\colon X \leftrightarrow Y$ then for any λ, $X\,_{\lambda}\!\Leftrightarrow_{\lambda} Y$.

Property 6 can be generalized taking, for instance, an infinite Y and a definable relation $R \subset X \times Y$ such that each $a \in X$ has *uniformly* finitely many R-images, i.e., the sets $R(a, \mathcal{M})$ have bounded finite cardinalities. In such a case we have $X\,_{\geq|Y|}\!\Leftarrow Y$. Similarly, having a definable *almost bijection* $R \subset X \times Y$ with uniformly finitely many R-images and R-preimages, then, for infinite X and Y, we get $|X| = |Y|$ and, moreover, $X\,_{|X|}\!\Leftrightarrow_{|X|} Y$.

Note that we have the similar effect, with $X\,{}_\lambda\!\Leftrightarrow_\lambda Y$, replacing f by a relation $R \subset (X \times Y) \cup (Y \times X)$ with infinite $R^{-1}(a)$ and uniformly bounded finite $R(a)$, $a \in X \cup Y$.

For definable sets $X = \phi(\mathcal{M}, \bar{a})$ and $Y = \psi(\mathcal{M}, \bar{b})$ we denote by $X \vee Y$ the set of solutions, in $\mathcal{M}$, of a formula $\phi(\bar{x}, \bar{a}) \vee \psi(\bar{y}, \bar{b})$, by $X \wedge Y$ — of $\phi(\bar{x}, \bar{a}) \wedge \psi(\bar{y}, \bar{b})$, by $\neg X$ — of $\neg\phi(\bar{x}, \bar{a})$, by $\forall x X$ — of $\forall x \phi(\bar{x}, \bar{a})$, by $\exists x X$ — of $\exists x \phi(\bar{x}, \bar{a})$.

7. If $\Rightarrow_{<\omega} X$ and $\Rightarrow_{<\omega} Y$ then $\Rightarrow_{<\omega} X \wedge Y$, $\Rightarrow_{<\omega} X \wedge \neg Y$, $\Rightarrow_{<\omega} X \vee Y$.
8. If $X \Rightarrow_{\leq\lambda} Y$ and $X \Rightarrow_{\leq\lambda} Z$ for some infinite λ, then $X \Rightarrow_{\leq\lambda} Y \vee Z$. If $X \Rightarrow_\lambda Y$ and $X \Rightarrow_{<\lambda} Z$ for some infinite λ, then $X \Rightarrow_\lambda Y \vee Z$.
9. If $X \Rightarrow_\lambda Y$ for some infinite λ and $\Rightarrow_{<\omega} Z$, then $X \Rightarrow_\lambda Y \vee Z$ and $X \Rightarrow_\lambda Y \wedge \neg Z$.
10. For every variable x, if $X \Rightarrow_{\leq\lambda} Y$ then $X \Rightarrow_{\leq\lambda} \forall x Y$ and $X \Rightarrow_{\leq\lambda} \exists x Y$, and, by Monotony, if $X \Rightarrow_{\leq\lambda} \exists x Y$ then $X \Rightarrow_{\leq\lambda} \forall x Y$.

Definition [8]. We say that a generative class $(\mathbf{D}_0; \leqslant)$ *forces* the cardinality λ (respectively, $\leq \lambda$, $\geq \lambda$, $< \lambda$, $> \lambda$) for a (type-)definable set X, written $(\mathbf{D}_0; \leqslant) \Rightarrow_\lambda X$ ($(\mathbf{D}_0; \leqslant) \Rightarrow_{\leq\lambda} X$, $(\mathbf{D}_0; \leqslant) \Rightarrow_{\geq\lambda} X$, $(\mathbf{D}_0; \leqslant) \Rightarrow_{<\lambda} X$, $(\mathbf{D}_0; \leqslant) \Rightarrow_{>\lambda} X$) if the union of $\Phi(A)$-fragments for X, where $\Phi(A) \in \mathbf{D}_0$, has the cardinality λ (a cardinality $\leq \lambda$, $\geq \lambda$, $< \lambda$, $> \lambda$).

For a generative class $(\mathbf{D}_0; \leqslant)$, we say that a (type-)definable set X *meets a contradiction for its cardinality* if $(\mathbf{D}_0; \leqslant) \Rightarrow_{\leq\lambda} X$ and $(\mathbf{D}_0; \leqslant) \Rightarrow_{>\lambda} X$ for some cardinality λ. For the considered λ we say that X meets a contradiction *with respect to* λ.

Example 1.6 (cf. [9, Proof of Theorem 2.1]) Let $\mathcal{N}$ be a structure in the language with $\mu \geq \omega$ equivalence relations E_i such that $E_0 = N^2$, each E_i-class is divided into k_i E_{i+1}-classes, $k_i \in \omega \setminus \{0, 1\}$, $i \in \mu$, and every intersection of a $\subseteq$-chain of E_i-classes X_i, $i \in \mu$, has κ elements for some fixed $\kappa > 0$.

Clearly, $|\mathcal{N}| = 2^\mu \cdot \kappa$. In particular, if $\kappa \leq \omega$ then $|\mathcal{N}| = 2^\mu$.

Example 1.7 (cf. [9, Proof of Theorem 2.1]) Consider a structure $\mathcal{N}$ in Example 1.6 with $\mu \geq \omega$ sequential equivalence relations E_i, whose chains of E_i-classes, $i \in \mu$, have unique elements in intersections, and forming a unary predicate P_0. Now we extend P_0 and the language $\{P_0\} \cup \{E_i \mid i \in \mu\}$ by:

1) a disjoint unary predicate P_1 which is divided by $\lambda > 2^\mu$ disjoint infinite unary predicates Q_i;

2) a function $f\colon P_0 \to P_1$ such that $f^{-1}(a)$ is infinite for every $a \in P_1$.

The resulted hypothetic structure $\mathcal{N}'$ has the universe $P_0 \cup P_1$. Denote by $(\mathbf{D}_0; \subseteq)$ the generative class consisting of all diagrams being copies of quantifier free diagrams for finite subsets of $\mathcal{N}'$. As shown in Example 1.6, $|P_0| = 2^\mu$. Therefore $(\mathbf{D}_0; \subseteq) \Rightarrow_{2^\mu} P_0$.

At the same time by Property 5 for λ definable sets $X_i = Q_i$ and $Y = P_1$ we have $\Rightarrow_{>2^\mu} P_1$, and by Property 6 the definable function $f\colon P_0 \to P_1$ confirms that $P_0 \,{}_{>2^\mu}\!\Leftarrow P_1$.

Having $(\mathbf{D}_0; \subseteq) \Rightarrow_{2^\mu} P_0$ and $(\mathbf{D}_0; \subseteq) \Rightarrow_{>2^\mu} P_0$ we observe that X meets a contradiction for its cardinality. Hence the $(\mathbf{D}_0; \subseteq)$-generic structure $\mathcal{N}'$ does not exist.

The following example modifies Example 1.7 producing a meeting of cardinality contradiction for type-definable sets.

Example 1.8 [8]. We take Example 1.7 and replace the structure of P_1 by $\mu' > 2^\mu$ sequential equivalence relations E'_j, whose chains of E'_j-classes, $j \in \mu$, have infinitely many elements in intersections. Now we observe that the formulas $P_0(x)$ and $P_1(x)$ isolate complete 1-types. Introducing a language function f as in Example 1.7 we again meet the cardinality contradiction for $X = P_0$ which is forced by $Y = P_1$.

This example can be easily transformed replacing definable sets P_0 and P_1 by correspondent type-definable sets with non-isolated $p_0(x), p_1(x) \in S(\emptyset)$. For this aim it suffices to introduce two sequences of predicates $P_{0,n}, P_{1,n}$, $n \in \omega$, satisfying the following conditions:

i) $P_{k,0} = P_k$, $k \in \{0, 1\}$;

ii) $P_{k,n} \supset P_{k,n+1}$, $k \in \{0, 1\}$, where $P_{0,n} \setminus P_{0,n+1}$ consists of infinitely many E_0-classes and $P_{1,n} \setminus P_{1,n+1}$ consists of infinitely many E'_0-classes;

iii) if $a \in P_{1,n} \setminus P_{1,n+1}$ then $f^{-1}(a) \in P_{0,n} \setminus P_{0,n+1}$;

iv) $\bigcap\limits_{n\in\omega} P_{kn}$ has infinitely many E_0-classes.

We denote by $p_k(x)$ the (unique) complete nonisolated 1-type which is isolated by the set $\{P_{k,n}(x) \mid n \in \omega\}$, $k \in \{0, 1\}$.

The formula $f(x) \approx y$ defines links between a type-definable set X of realizations of $p_0(x)$ and a type-definable set Y of realizations of $p_1(y)$. As in Example 1.7 we have $|X| = 2^\mu$, $|Y| = 2^{\mu'}$, $|X| < |Y|$ by choice of μ', but the links with respect to $f(x) \approx y$ imply $|X| \geq |Y|$. Thus, X meets the cardinality contradiction.

Note that since formula-definable sets consist of type-definable sets, lower cardinality bounds for type-definable sets imply similar bounds for formula-definable ones.

Theorem 1.9 [8]. *For any generative class* $(\mathbf{D}_0; \leqslant)$ *the following conditions are equivalent:*

(1) *there exists a* $(\mathbf{D}_0; \leqslant)$*-generic structure;*

(2) *there are no type-definable sets* X *constructed with respect to* $(\mathbf{D}_0; \leqslant)$ *such that these* X *meet contradictions for their cardinality;*

(3) *there are no definable sets* X *constructed with respect to* $(\mathbf{D}_0; \leqslant)$ *such that these* X *meet contradictions for their cardinality.*

2. λ-CONFLICT AND CONFLICT-FREE THEORIES

Definition. Let λ be a cardinality. A theory T is λ*-conflict* if it has a generic model $\mathcal{M}$ with λ independent definable sets X which meet contradictions for their cardinality, and this cardinality λ is maximal with respect to generic models of T. Here, independence means that each X has an extension in $\mathcal{N} \succ \mathcal{M}$ without meeting of contradictions for its cardinality whereas correspondent extensions of other considered definable sets again meet these contradictions.

A theory T is called *conflict* if T is λ-conflict for some $\lambda > 0$. If T is 0-conflict, it is called *conflict-free.*

Remark 2.1. Clearly, if a (type-)definable set X meets a cardinality contradiction with respect to μ then for any (type-)definable set Y of same arity as X and with $|Y| < \mu$, $X \cup Y$ and $X \setminus Y$ meet cardinality contradictions with respect to μ. It means that a cardinality contradiction can generate a family $\mathcal{F}$ of (type-)definable sets with cardinality contradictions. At the same time all these sets are dependent:

if X is saturated by at least μ elements then all sets in $\mathcal{F}$ lose the cardinality contradiction with respect to μ.

The argument above show that, for instance, 1-conflict theories can have many dependent sets with cardinality contradictions.

By Theorems 1.2 and 1.9 we have the following:

Theorem 2.2. *Any countable theory is conflict-free.*

Theorem 2.3. *Any theory T of unary predicates is conflict-free.*

Proof. Since types of T are defined by estimations for cardinalities of sets of solutions for the formulas $P_{i_1}^{\delta_1}(x) \wedge \ldots \wedge P_{i_k}^{\delta_k}(x)$, where P_{i_j} are language symbols and $\delta_{i_j} \in \{0,1\}$, it suffices, for the conflict-free of T, to note that all links between definable sets in T are exhausted by the relation $\subseteq$ and unions of (type-)definable subsets, i.e., if X is a disjoint union of some (type-)definable sets X_i then $|X| = \sum\limits_i |X_i|$ producing the cardinality of X depending just on number of indexes i and on cardinalities of X_i. Therefore constructing definable sets in a generic limit these sets do not meet cardinality contradictions. Thus, T is conflict-free. $\square$

Theorem 2.4. *Any theory T of sequentially embedded equivalence relations is conflict-free.*

Proof. Since types of T are defined by estimations for cardinalities of E_i-classes and of their quotients E_i/E_j, where E_i, E_j are language symbols, it suffices, for the conflict-free of T, to note that all links between definable sets in T are exhausted, again as in the proof of Theorem 2, by the relation $\subseteq$ and unions of (type-)definable subsets, i.e., if X is a disjoint union of some (type-)definable sets X_i then $|X| = \sum\limits_i |X_i|$ producing the cardinality of X depending just on number of indexes i and on cardinalities of X_i. Thus, constructing definable sets in a generic limit these sets do not meet cardinality contradictions, and T is conflict-free. $\square$

The following example shows that equivalence relations which are not sequentially embedded can generate conflict theories.

Example 2.5. We modify Example 1.7 as follows. Let the universe M be a disjoint union of infinite P_0 and P_1, where P_i is an E_i^0-equivalence class such that elements of P_{1-i} are E_i^0-singletons, $i = 0, 1$. Then we introduce sequentially embedded equivalence relations E_i on P_0, $i < \mu$, as in Example 1.7, such that E_i form singleton classes on P_1. Similarly, we introduce sequentially embedded equivalence relations E'_j on P_1, $j < 2^\mu$, as in Example 1.7, such that E'_j form singleton classes on P_0. Finally we replace the function $f\colon P_0 \to P_1$ by new equivalence relation E'' being the symmetric and transitive closure of $\{(a, f(a)) \mid a \in P_0\}$. The arguments for Example 1.7 show that P_1 meets the cardinality contradiction producing the 1-conflict theory of the resulted structure $\mathcal{M}$ in the language of equivalence relations.

Definition [10]. The *disjoint union* $\bigsqcup\limits_{i\in I} \mathcal{M}_i$ of pairwise disjoint structures $\mathcal{M}_i$ for pairwise disjoint predicate languages Σ_i, $i \in I$, is the structure of language $\bigcup\limits_{i\in I} \Sigma_i \cup \{P_i^{(1)} \mid i \in I\}$ with the universe $\bigsqcup\limits_{i\in I} M_i$, $P_i = M_i$, and interpretations of predicate symbols in Σ_i coinciding with their interpretations in $\mathcal{M}_i$, $i \in I$. The *disjoint union of theories* T_i for pairwise disjoint languages Σ_i accordingly, $i \in I$, is the theory

$$\bigsqcup_{i\in I} T_i \rightleftharpoons \mathrm{Th}\left(\bigsqcup_{i\in I} \mathcal{M}_i\right),$$

where $\mathcal{M}_i \models T_i$, $i \in I$.

Clearly, the theory $\bigsqcup\limits_{i\in I} T_i$ does not depend on choice of disjoint union $\bigsqcup\limits_{i\in I} \mathcal{M}_i$ of models $\mathcal{M}_i \models T_i$, $i \in I$.

Note that if I is finite then all models of $\bigsqcup\limits_{i\in I} T_i$ are represented by $\bigsqcup\limits_{i\in I} \mathcal{M}_i$. Otherwise, if I is infinite, then $\bigsqcup\limits_{i\in I} \mathcal{M}_i$ can be considered as disjoint P-combinations with consistent $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ [11] and with P-closures [12], for the families $\mathcal{T} = \{\mathrm{Th}(\mathcal{M}_i) \mid i \in I\}$, which are equal to $\mathcal{T} \cup \{T_\kappa \mid \kappa \in \omega + 1\}$, where T_κ has κ-element models and whose all predicates unless "=" are empty.

Proposition 2.6. *If T_i is a λ_i-conflict theory, $i \in I$, then $\bigsqcup\limits_{i\in I} T_i$ is a $\sum\limits_{i\in I} \lambda_i$-conflict theory.*

Proof. It suffices to note that if the theories T_i have λ_i independent (type-)definable sets meeting cardinality contradictions then $\bigsqcup\limits_{i\in I} T_i$ has exactly $\sum\limits_{i\in I} \lambda_i$ independent (type-)definable sets meeting cardinality contradictions. The family of these sets is the union of the families for T_i. □

Since there are conflict-free theories and 1-conflict theories, Proposition 2.6 implies:

Corollary 2.7. *For any cardinality λ there is a λ-conflict theory.*

Proposition 2.8. *For any λ-conflict theory T and $\nu < \lambda$ there is a ν-conflict expansion T' of T.*

Proof. Consider definable sets X_i, $i < \lambda$, witnessing that T is λ-conflict such that X_i meets the cardinality contradiction with some maximal μ_i. By compactness the sets X_i, for $i \geq \nu$, can be extended till X'_i with $|X'_i| = \mu_i$. Expanding X'_i by μ_i language singletons P_j for all elements of X'_i we remove the cardinality contradictions for $i \geq \nu$, whereas it remains for X_i, $i < \nu$, since the sets X_i are independent. Thus, the expansion T' of T by the predicates P_j is ν-conflict. □

By Proposition 2.8 and Examples 1.7, 2.5 the property of λ-conflictness is not preserved under expansions. Similarly this property is not preserved under restrictions. Thus, we have the following:

Theorem 2.9. *For any cardinality λ there is no a property $\mathcal{P}_\lambda$ for formulas such that any theory T is λ-conflict if and only if any formula $\varphi \in T$ satisfies $\mathcal{P}_\lambda$.*

Proof. If a property $\mathcal{P}_\lambda$ exists then it should be satisfied for any restriction of a λ-conflict theory. But as noticed above, the property of λ-conflictness is not preserved under restrictions. □

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