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PAPER

Stability of KdV solitons with respect to transverse perturbations: Absolute and convective instabilities

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Abstract

We study the stability of one-dimensional solitons propagating in an anisotropic medium. We derived the Kadomtsev-Petviashvili equation for nonlinear waves propagating in an anisotropic medium. By a proper variable substitution this equation reduces either to the KPI or to the KPII equation. In the former case solitons are unstable with respect to the normal modes of transverse perturbations, and in the latter they are stable. We only consider the case when the solitons are unstable. We formulated the linear stability problem. Using the Laplace–Fourier transform, we found the solution describing the evolution of an initial perturbation. Then, using Briggs' method we studied the absolute and convective instabilities. We found that a soliton is convectively unstable unless it propagates at an angle smaller then critical with respect to a critical direction defined by the condition that the group velocity is parallel to the phase velocity. The critical angle is proportional to the ratio of the dispersion length to the soliton width, which is a small parameter. The coefficient of proportionality is expressed in terms of the phase speed and its second derivative with respect to the angle between the propagation direction and the critical direction. As an example we consider the stability of solitons propagating in Hall plasmas.

1. Introduction

More than a century ago Korteweg and de Vries derived their KdV equation to describe nonlinear waves on the water surface [1]. They also obtained the most famous solution to this equation, the soliton. Then this equation was practically forgotten for more then sixty decades before it gloriously resurrected in early sixties of the last century when it turned out that it describes various types of wave modes in plasmas. The interest of researchers in this equation was further boosted when it became the first nonlinear equation that can be completely integrated using the inverse scattering transform [2–4].

Kadomtsev and Petviashvili formulated the problem of stability of the KdV solitons with respect to transverse perturbations. To solve this problem they derived the Kadomtsev-Petviashvili (KP) equation [5]. Then they used the regular perturbation method to study the soliton stability with respect to very long transverse normal modes. They obtained a very simple result: solitons are stable if they propagate in a medium with the negative dispersion, in which case the Kadomtsev-Petviashvili equation is called the KPII equation, and unstable if they propagate in a medium with positive dispersion, in which case the Kadomtsev-Petviashvili equation is called the KPI equation [5, 6]. This result was then confirmed in [7–9], where it was also shown that in case of the KPI euqation only normal modes with the wavelengths above a threshold value are unstable, while those with the wavelengths below the threshold value are neutrally stable. Recently it was proved in [10] that the KdV solitons are stable with respect to nonlinear transverse perturbations in the case when they are described by the KPII equation.

However, the conclusion that the soliton stability is described by the KPI equation when the dispersion is positive, and by the KPII equation when the dispersion is negative, is only valid when solitons propagate in an isotropic medium where the propagation speed is the same in all directions. The problem of stability of solitons propagating in an anisotropic medium was addressed by Ostrovskii and Shrira [11]. They used the nonlinear geometrical optics to study the soliton evolution with time. In particular, they showed that a soliton propagating in an anisotropic medium is stable unless it propagates at a small angle with respect to a critical direction, which is the direction where the propagation speed takes extremal value. Recently Ruderman [12] derived the KP equation for nonlinear waves propagating in a Hall plasma. In this plasma the wave dispersion is related to the account of the Hall current in the induction equation. The plasma is anisotropic because of the presence of the equilibrium magnetic field. The critical directions are those that are parallel and perpendicular to the magnetic field. There are two wave modes that can propagate in this plasma, fast and slow magnetosonic waves. The fast waves are characterised by the positive dispersion, and slow by the negative dispersion. It was shown that fast solitons are unstable with respect to the normal modes no matter what is the propagation direction. Slow solitons are unstable if the angle between the propagation direction and the equilibrium magnetic field is smaller than a critical value, and stable otherwise.

The results obtained by Ruderman [12] contradict to those obtained by Ostrovskii and Shrira [11]. To reconcile the results Ruderman made a conjecture that, in fact, the result obtained by Ostrovskii and Shrira concerns the stability with respect to bounded perturbations rather than the normal modes. In that case we must distinct between the absolute and convective instabilities. In accordance with the conjecture suggested by Ruderman the result obtained in [11] should be formulated as follows:

If a soliton propagating in an anisotropic medium is unstable with respect to normal modes, then the instability is always convective unless the soliton propagates at a small angle with respect to a critical direction.

In this article we aim to prove the conjecture suggested in [12]. The article is organised as follows: In the next section we derive the general KP equation for nonlinear waves propagating in an anisotropic medium. In Section 3 we formulate the linear stability problem and introduce the Laplace–Fourier transform to obtain its solution. In Section 4 we obtain the solution to the linear stability problem. In Section 5 we study the absolute and convective instability of a soliton with respect to transverse perturbations. In Section 6 we apply the general results to the soliton stability propagating in Hall plasmas. Section 7 contains the summary of the results and our conclusions.

2. Kadomtsev-Petviashvili equation in an anisotropic medium

We consider two-dimensional waves propagating in a homogeneous anisotropic medium. In the linear approximation the dispersion relation for these waves is $\omega = \omega(q, r)$, where $q = k_x(1 + k_y^2/k_x^2)^{1/2}$, $r = \mathbf{n}\cdot\mathbf{k}$, $\mathbf{k} = (k_x, k_y)$, $\mathbf{n} = (\cos\alpha,\ \sin\alpha)$, and α is a free parameter. Now we introduce the scaled quantities $k_x = \epsilon\tilde{k}_x$ and $k_y = \epsilon^2\tilde{k}_y$. Then we obtain

$$q = \epsilon\tilde{k}_x + \epsilon^3\frac{\tilde{k}_y^2}{2\tilde{k}_x} + \mathcal{O}(\epsilon^5),\quad r = \epsilon\tilde{k}_x\cos\alpha + \epsilon^2\tilde{k}_y\sin\alpha. \tag{1}$$

Next we assume that $\omega(0, 0) = 0$ and use the expansion

$$\begin{aligned}\omega(q,\ r) = {} & \frac{\partial\omega}{\partial q}q + \frac{\partial\omega}{\partial r}r + \frac{1}{2}\frac{\partial^2\omega}{\partial q^2}q^2 + \frac{\partial^2\omega}{\partial q\partial r}qr \\ & + \frac{1}{2}\frac{\partial^2\omega}{\partial r^2}r^2 + \frac{1}{6}\frac{\partial^3\omega}{\partial q^3}q^3 + \frac{1}{2}\frac{\partial^3\omega}{\partial q^2\partial r}q^2 r \\ & + \frac{1}{2}\frac{\partial^3\omega}{\partial q\partial r^2}qr^2 + \frac{1}{6}\frac{\partial^3\omega}{\partial r^3}r^3 + \dots,\end{aligned} \tag{2}$$

where all derivatives are calculated at $q = r = 0$. We assume that we use the reference frame where the unperturbed medium is at rest. In this case the wave properties must be invariant with respect to the substitution $-\mathbf{k} \to \mathbf{k}$. In accordance with this we impose the condition that ω changes the sign when $\mathbf{k}$ changes the sign. This condition implies that $\omega(-q, -r) = -\omega(q, r)$. It follows from this condition and equation (2) that

$$\frac{\partial^2\omega}{\partial q^2} = \frac{\partial^2\omega}{\partial q\partial r} = \frac{\partial^2\omega}{\partial r^2} = 0. \tag{3}$$

Then, substituting equation (1) in equation (2) we obtain

$$\omega = \epsilon \tilde{k}_x\left(\frac{\partial\omega}{\partial q} + \frac{\partial\omega}{\partial r}\cos\alpha\right) + \epsilon^2\tilde{k}_y\frac{\partial\omega}{\partial r}\sin\alpha + \epsilon^3\left[\frac{\partial\omega}{\partial q}\frac{\tilde{k}_y^2}{2\tilde{k}_x} + \frac{1}{2}\tilde{k}_x^3\left(\frac{1}{3}\frac{\partial^3\omega}{\partial q^3} + \cos\alpha\frac{\partial^3\omega}{\partial q^2\partial r} + \cos^2\alpha\frac{\partial^3\omega}{\partial q\partial r^2} + \frac{1}{3}\cos^3\alpha\frac{\partial^3\omega}{\partial r^3}\right)\right] + \mathcal{O}(\epsilon^4). \tag{4}$$

Now, returning to non-scaled quantities we obtain the approximate dispersion equation

$$k_x\omega = ck_x^2 + \beta k_x^4 + \Upsilon k_x k_y + \Lambda k_y^2, \tag{5}$$

where

$$c = \frac{\partial\omega}{\partial q} + \frac{\partial\omega}{\partial r}\cos\alpha, \quad \Upsilon = \frac{\partial\omega}{\partial r}\sin\alpha, \quad \Lambda = \frac{1}{2}\frac{\partial\omega}{\partial q},$$
$$\beta = \frac{1}{6}\frac{\partial^3\omega}{\partial q^3} + \frac{1}{2}\cos\alpha\frac{\partial^3\omega}{\partial q^2\partial r} + \frac{1}{2}\cos^2\alpha\frac{\partial^3\omega}{\partial q\partial r^2} + \frac{1}{6}\cos^3\alpha\frac{\partial^3\omega}{\partial r^3}. \tag{6}$$

We recall that all derivatives are calculated at $q = r = 0$. It is not difficult to see that Υ is the y-component of the wave group velocity. Substituting $\partial/\partial t$ for $-i\omega$, $\partial/\partial x$ for ik_x, $\partial/\partial y$ for ik_y, and assuming quadratic nonlinearity we obtain that the Kadomtsev-Petviashvili equation has the form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \Upsilon\frac{\partial u}{\partial y} + Nu\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3}\right) = -\Lambda\frac{\partial^2 u}{\partial y^2}. \tag{7}$$

Now we use the variable substitution

$$T = \frac{t}{t_0}, \quad U = \frac{\chi t_0 N u}{6}, \quad X = \chi(x - ct),$$
$$Y = (y - \Upsilon t)\sqrt{\frac{3\chi\sigma^2}{t_0\Lambda}}, \tag{8}$$

where $\chi = -1/\sqrt[3]{t_0\beta}$ and t_0 is an arbitrary positive constant with the dimension of time. As a result we reduce equation (7) to

$$\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial T} + 6U\frac{\partial U}{\partial X} + \frac{\partial^3 U}{\partial X^3}\right) + 3\sigma^2\frac{\partial^2 U}{\partial Y^2} = 0, \tag{9}$$

where $\sigma^2 = \mathrm{sgn}(\chi\Lambda) = \mathrm{sgn}(\beta\Lambda)$. This is the KPI equation when $\sigma^2 = -1$ corresponding to solitons unstable with respect to transverse perturbations, while this is the KPII equation when $\sigma^2 = 1$ corresponding to solitons stable with respect to transverse perturbations.

3. Formulation of the linear stability problem and introducing the Laplace–Fourier transform

The concept of absolute and convective instabilities was first introduced in plasma physics [13, 14], and then in hydrodynamics (see e.g. Drazin and Reid [15]). An instability is absolute if perturbations grow at any fixed spatial position, while it is convective if perturbations grow, however they are so quickly swapped out of any finite spatial domain that eventually they decay at any fixed spatial position. Obviously, the distinction between the absolute and convective instabilities depends on a reference frame. An instability can be absolute in one reference frame and convective in another (see figure 1). Equation (5) is derived under the assumption that the unperturbed state is static, that is the unperturbed velocity is zero. In accordance with this equation (7) can be used to study the soliton stability in the reference frame where the medium is at rest far from the soliton. In contract, equation (9) describes the soliton in the reference frame moving with the velocity c in the x-direction and Υ in the y-direction with respect to the reference frame where the medium is at rest far from the soliton. Below we study the convective and absolute soliton stability in the latter reference frame and then explain how to translate this study to any other reference frame.

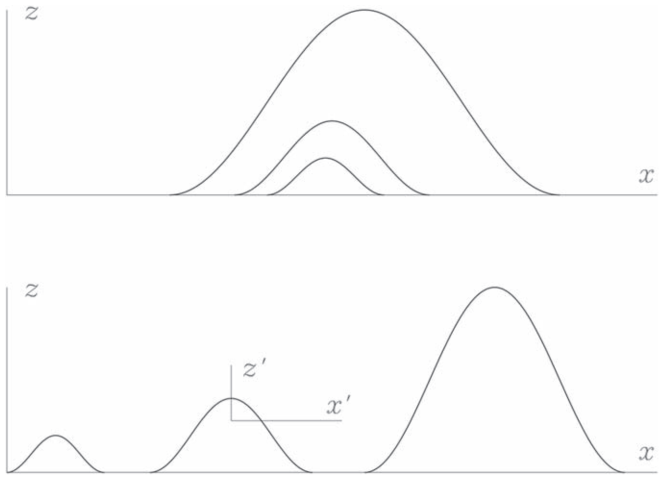

Figure 1. Temporary evolution of localised disturbance. In the upper panel the disturbance amplitude tends to infinity at any fixed spatial position as $t \to \infty$. In the lower panel the disturbance amplitude also tends to infinity as $t \to \infty$. However, simultaneously it moves out of any finite spatial domain so fast that the perturbation amplitude tends to zero as $t \to \infty$. The upper panel illustrates the absolute instability, and the lower the convective instability. The distinction between the absolute and convective instability is frame dependent. In the lower panel the instability is absolute in the reference frame $x'z'$ moving together with the wave packet.

Studying absolute and convective instabilities is based on the solution to the initial value problem. We consider a solution to equation (9) describing a soliton. It reads

$$U = U_0(\xi) \equiv \frac{2\nu^2}{\cosh^2(\nu\xi)}, \qquad \xi = X - 4\nu^2 T. \tag{10}$$

where $\nu > 0$ is a free parameter. Below we only consider the case where the soliton is unstable with respect to transverse perturbations. In this case equation (9) is the KPI equation and $\sigma^2 = -1$. Now we substitute $U = U_0 + v$ in equation (9) and linearise the obtained equation with respect to v. In addition we make the variable substitution $\xi = X - 4\nu^2 T$. As a result we obtain

$$\frac{\partial}{\partial\xi}\left(\frac{\partial v}{\partial T} - 4\nu^2\frac{\partial v}{\partial\xi} + 6\frac{\partial(U_0 v)}{\partial\xi} + \frac{\partial^3 v}{\partial\xi^3}\right) - 3\frac{\partial^2 v}{\partial Y^2} = 0. \tag{11}$$

Then we consider the initial value problem for this equation defined by

$$\begin{aligned} &T \in \mathbb{R}^+, \quad (\xi, Y) \in \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |v|^2 d\xi\, dY < \infty, \\ &v = v_0(\xi, Y) \ \text{ at } \ T = 0, \end{aligned} \tag{12}$$

and assume that the derivatives of $v(\xi, T, Y)$ that are present in equation (12) exist. To solve the initial value problem for equation (12) we use the Fourier transform with respect to Y,

$$\begin{aligned} \hat{f}(K) &= \int_{-\infty}^{\infty} f(Y) e^{-iKY} dY, \\ f(Y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(K) e^{iKY} dK, \end{aligned} \tag{13}$$

and the Laplace transform with respect to T,

$$\begin{aligned} \tilde{f}(\Omega) &= \int_0^{\infty} f(T) e^{i\Omega T}\, dT, \\ f(T) &= \frac{1}{2\pi}\int_{i\varsigma-\infty}^{i\varsigma+\infty} \tilde{f}(\Omega) e^{-i\Omega T}\, d\Omega, \end{aligned} \tag{14}$$

where ς is a real number such that the Bromwich integration contour,

$$\mathcal{B} = \{\Omega \mid \Omega \in \mathbb{C}, \ \Im(\Omega) = \varsigma\}, \tag{15}$$

lies above all singularities of function $\tilde{f}(\Omega)$. In equation (15) $\Im$ indicate the imaginary part of a quantity. We also denote the Fourier-Laplace transform of function $f(T, Y)$ as $\check{f}(\Omega, K)$. Applying the Fourier-Laplace transform to equation (11) yields

$$\frac{d}{d\xi}\left(-i\Omega\breve{v} - 4\nu^2\frac{d\breve{v}}{d\xi} + 6\frac{d(U_0\breve{v})}{d\xi} + \frac{d^3\breve{v}}{d\xi^3}\right) + 3K^2\breve{v} = \frac{d\hat{v}_0}{d\xi}. \tag{16}$$

4. Solution to the initial-value problem

It follows from the result obtained in Appendix A that the complementary function of equation (16) is

$$\breve{v}_{\rm com} = \frac{d}{d\xi}\sum_{j=1}^{4} C_j h_j(\xi), \tag{17}$$

where C_j are arbitrary constants,

$$h_j(\xi) = e^{-\kappa_j\xi}\left[\frac{K^2}{4\nu^2\kappa_j^2} - \left(\tanh(\nu\xi) + \frac{\kappa_j}{2\nu}\right)^2\right], \tag{18}$$

and $\kappa_1,\ldots,\kappa_4$ are the four roots of the algebraic equation

$$\kappa^4 - 4\nu^2\kappa^2 + i\Omega\kappa + 3K^2 = 0. \tag{19}$$

Using equations (18) and (19) we can study the soliton stability with respect to normal modes. We temporarily drop the subscript j. An eigenvalue Ω corresponding to a normal mode is determined by equation (16) with $\hat{v}_0 = 0$ and the condition $\breve{v} \to 0$ as $|\xi| \to \infty$. When $\Re(\kappa) > 0$, where $\Re$ indicates the real part of a quantity, we have $h(\xi) \to 0$ as $\xi \to \infty$. The condition that $h(\xi) \to 0$ as $\xi \to -\infty$ is satisfied when

$$\left(1 - \frac{\kappa}{2\nu}\right)^2 = \frac{K^2}{4\nu^2\kappa^2}, \quad \Re(\kappa) < 2\nu. \tag{20}$$

Eliminating K from equations (19) and (20) yields

$$i\Omega = 4\kappa(2\nu - \kappa)(\kappa - \nu). \tag{21}$$

It follows from equation (20) that

$$\kappa^2 - 2\nu\kappa + |K| = 0, \quad \text{or} \quad \kappa^2 - 2\nu\kappa - |K| = 0. \tag{22}$$

The condition $\Re(\kappa) > 0$ is satisfied by both roots of the first equation when $|K| < \nu^2$, and both of them satisfy the condition $\Re(\kappa) < 2\nu$. Only the positive root of the second equation satisfies this condition. However, it is straightforward to see that this positive root is real and larger than 2ν. Hence, the second equation must be rejected. Now, using the first equation in equation (22) we obtain

$$i\Omega_{1,2} = \pm 4|K|\sqrt{\nu^2 - |K|}. \tag{23}$$

The value of κ corresponding to Ω_1 defined by this equation with the plus sign is given by the larger root of the left quadratic equation in equation (22), and the value of κ corresponding to Ω_2 defined by equation (23) with the minus sign is given by the smaller root of the same quadratic equation.

A normal mode is unstable when $i\Omega < 0$, which corresponds to Ω_2. Hence, we conclude that there is the unstable normal mode with the increment

$$\Gamma = -i\Omega_2 = 4|K|\sqrt{\nu^2 - |K|} \tag{24}$$

when $|K| < \nu^2$. This result was previously obtained in [7] and [8]. The increment takes its maximum equal to $\frac{8}{9}\nu^3\sqrt{3}$ at $|K| = \frac{2}{3}\nu^2$. Since the instability growth rate is bounded the stability problem is well-posed. The two values of Ω determined by equation (23) with $|K| < \nu^2$ constitute the point spectrum of the eigenvalues problem defined by equation (16) with the zero right-hand side and the condition that the solution is square integrable on the real axis.

Next we consider the case where $\Re(\kappa) < 0$. In this case $h(\xi) \to 0$ when $\xi \to -\infty$. The condition that $h(\xi) \to 0$ as $\xi \to \infty$ is satisfied when

$$\left(1 + \frac{\kappa}{2\nu}\right)^2 = \frac{K^2}{4\nu^2\kappa^2}, \quad \Re(\kappa) > -2\nu. \tag{25}$$

The analysis similar to that carried out in the case where $\Re(\kappa) > 0$ shows that $\Omega = \Omega_2$ is defined by the same equation (23), while the corresponding values of κ are defined by the equation

$$\kappa^2 + 2\nu\kappa + |K| = 0. \tag{26}$$

Again the value of κ corresponding to Ω defined by equation (23) with the plus sign is given by the larger root of this equation, and the value of κ corresponding to Ω defined by equation (23) with the minus sign is given by the smaller root of this equation.

Now we use the method of variation of arbitrary constants to obtain the solution to equation (16) tending to zero as $|\xi| \to \infty$. In accordance with this method we look for the solution in the from given by equation (17), however with C_j being functions of ξ. Then, substituting this expression in equation (16) and imposing standard conditions on the derivatives of $C_j(\xi)$ we obtain the system of four linear algebraic equations determining these derivatives,

$$\sum_{j=1}^{4} \frac{dC_j}{d\xi} h_j(\xi) = 0, \quad \sum_{j=1}^{4} \frac{dC_j}{d\xi}\frac{dh_j}{d\xi} = 0, \tag{27a}$$

$$\sum_{j=1}^{4} \frac{dC_j}{d\xi}\frac{d^2h_j}{d\xi^2} = 0, \quad \sum_{j=1}^{4} \frac{dC_j}{d\xi}\frac{d^3h_j}{d\xi^3} = \hat{v}_0, \tag{27b}$$

where the prime denotes the derivative with respect to ξ. Using Cramer's rule we obtain that the solution to the system of equations (27) is given by

$$\frac{dC_j}{d\xi} = (-1)^j \frac{\hat{v}_0 W_j}{W_0}, \quad j = 1,\ldots,4, \tag{28}$$

where

$$W_0 = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 \\ h'_1 & h'_2 & h'_3 & h'_4 \\ h''_1 & h''_2 & h''_3 & h''_4 \\ h'''_1 & h'''_2 & h'''_3 & h'''_4 \end{vmatrix}, \tag{29a}$$

$$W_1 = \begin{vmatrix} h_2 & h_3 & h_4 \\ h'_2 & h'_3 & h'_4 \\ h''_2 & h''_3 & h''_4 \end{vmatrix}, \quad W_2 = \begin{vmatrix} h_1 & h_3 & h_4 \\ h'_1 & h'_3 & h'_4 \\ h''_1 & h''_3 & h''_4 \end{vmatrix}, \tag{29b}$$

$$W_3 = \begin{vmatrix} h_1 & h_2 & h_4 \\ h'_1 & h'_2 & h'_4 \\ h''_1 & h''_2 & h''_4 \end{vmatrix}, \quad W_4 = \begin{vmatrix} h_1 & h_2 & h_3 \\ h'_1 & h'_2 & h'_3 \\ h''_1 & h''_2 & h''_3 \end{vmatrix}. \tag{29c}$$

W_0 is the Wronskian of equation (16). Hence, $W'_0(\xi)$ is equal to $W_0(\xi)$ times the coefficient at the third derivative in equation (16) [16]. However, this coefficient is zero, so that $W'_0(\xi) = 0$ and $W_0(\xi) = \text{const} = W_0(0)$. The determinant W_0 is calculated in Appendix B. It is given by

$$W_0 = \frac{\Delta}{9\nu^8 K^2} \prod_{1 \leqslant j < l \leqslant 4} (\kappa_l - \kappa_j), \tag{30}$$

where

$$\Delta = K^6 - \left(\frac{\Omega^2}{16} + \nu^2 K^2\right)^2. \tag{31}$$

We aim to obtain the solution to equation (16) that tends to zero as $|\xi| \to \infty$. In accordance with this we take

$$C_j(\xi) = \frac{\widetilde{C}_j(\xi)}{W_0}, \tag{32a}$$

$$\widetilde{C}_j(\xi) = \begin{cases} (-1)^j \int_{-\infty}^{\xi} \hat{v}_0(\xi_1) W_j(\xi_1) d\xi_1, & \Re(\kappa_j) > 0, \\ (-1)^{j+1} \int_{\xi}^{\infty} \hat{v}_0(\xi_1) W_j(\xi_1) d\xi_1, & \Re(\kappa_j) < 0. \end{cases} \tag{32b}$$

When studying the absolute and convective instabilities we can always assume that the perturbation has a finite support, that is $v_0(\xi) = 0$ for $|\xi| > \xi_0$. Then it is obvious that $C_j(\xi) \to 0$ as $|\xi| \to \infty$. The solution to equation (16) decaying at infinity is given by

$$\check{v} = \frac{1}{W_0}\frac{d\widetilde{\Psi}}{d\xi}, \quad \widetilde{\Psi}(\xi) = \sum_{j=1}^{4} h_j(\xi) \widetilde{C}_j(\xi). \tag{33}$$

The case where $\Re(\kappa_j) = 0$ for one value of j is special. Let us assume that there is a purely imaginary root of equation (19), $i\tau$, where τ is real. Then it follows from equation (19) that

$$\Omega = \tau^3 + 4\nu^2\tau + 3K^2\tau^{-1}. \tag{34}$$

It is easily shown that the range of function $\Omega(\tau)$ is

$$\Omega \in (-\infty, -\Omega_0] \bigcup [\Omega_0, \infty), \tag{35}$$

where

$$\Omega_0 = \frac{4}{3}(2\nu^2\theta_- + 3|K|\theta_+), \tag{36a}$$

$$\theta_\pm = \sqrt{\frac{\sqrt{4\nu^4 + 9K^2} \pm 2\nu^2}{3}}. \tag{36b}$$

When $\kappa_j = i\tau$ we have $h_j(\xi)$ behaving as $e^{\pm i\tau\xi}$ for large $|\xi|$ implying that Ω is in the continuum spectrum of the operator defined by equation (16) with the zero right-hand side and the condition that the solution is square integrable on the real axis. Hence, the continuous spectrum is defined by equation (35). When considering $\check{v}$ as a function of Ω we make the cuts along the two intervals constituting the continuous spectrum.

We have $W_0 = 0$ when $\kappa_l = \kappa_j$ for some j and l, that is when equation (19) has a double root when $\Omega = \Omega_d$. There are four values of Ω_d. These values of Ω_d with the corresponding double roots κ_d are given by

$$\Omega_d = \pm\frac{4i}{3}(3|K|\theta_- - 2\nu^2\theta_+), \quad \kappa_d = \pm\theta_+, \tag{37}$$

or by

$$\Omega_d = \pm\Omega_0, \quad \kappa_d = \pm i\theta_-. \tag{38}$$

When there is one double root of equation (19) it is easy to find two other roots and show that they are different and none of them coincides with the double root.

Now we take $\Omega = \Omega_d + \epsilon^2$, where Ω_d is given by equation (37) and ϵ^2 can be either positive or negative. Without loss of generality we can denote the roots that tend to κ_d as κ_1 and κ_2. It is easy to show using equation (19) that $\kappa_1 = \kappa_d + \mathcal{O}(\epsilon)$ and $\kappa_2 = \kappa_d + \mathcal{O}(\epsilon)$. It follows from equation (18) that $h_1(\xi)$ and $h_2(\xi)$ are analytic functions of κ_1 and κ_2, respectively, with the only pole at zero. Then it follows that $h_1(\xi) = h_d(\xi) + \mathcal{O}(\epsilon)$ and $h_2(\xi) = h_d(\xi) + \mathcal{O}(\epsilon)$, where $h_d(\xi)$ is equal to $h(\xi)$ with $\kappa = \kappa_d$. Using this result we obtain that

$$W_1 = W_d + \mathcal{O}(\epsilon), \quad W_2 = W_d + \mathcal{O}(\epsilon),$$
$$W_3 = \mathcal{O}(\epsilon), \quad W_4 = \mathcal{O}(\epsilon), \tag{39}$$

where W_d is obtained by substituting $h_d(\xi)$ for $h_2(\xi)$ in the expression for W_1. It follows from these results that the numerator in equation (32) is of the order of ϵ. Since $W_0 \sim \kappa_2 - \kappa_1 = \mathcal{O}(\epsilon)$ it follows that there is a finite limit of $\check{v}$ as $\epsilon \to 0$. Hence the singularity of $\check{v}$ considered as a function of Ω at $\Omega = \Omega_d$ is removable.

We do not consider the double roots defied by equation (38) because they correspond to the values of Ω that are on the cuts. Hence, all the singularities of $\check{v}$ on the Ω-plane with cuts are related to zeros of Δ.

We now change the notation and write $\widetilde{\Psi}(\xi;\, K,\, \Omega)$ to show explicitly that the solution to equation (16) depends on K and Ω. We now prove that $\widetilde{\Psi}(\xi;\, K,\, \Omega) \equiv 0$ when Ω is given by

$$i\Omega_\pm = \pm 4|K|\sqrt{\nu^2 + |K|}. \tag{40}$$

Substituting $\check{v}$ given by equation (33) in equation (16) we obtain

$$-i\Omega_\pm\frac{d\widetilde{\Psi}}{d\xi} - 4\nu^2\frac{d^2\widetilde{\Psi}}{d\xi^2} + 6\frac{d}{d\xi}\left(U_0\frac{d\widetilde{\Psi}}{d\xi}\right) + \frac{d^4\widetilde{\Psi}}{d\xi^4} + 3K^2\widetilde{\Psi} = W_0(\Omega)\hat{v}_0. \tag{41}$$

Since $D(K, \Omega_\pm) = 0$ and $\widetilde{\Psi}(\xi;\, K,\, \Omega) \to 0$ as $|\xi| \to \infty$ it follows from equation (41) that $\widetilde{\Psi}(\xi;\, K,\, \Omega_\pm)$ is an eigenfunction unless $\widetilde{\Psi}(\xi;\, K,\, \Omega_\pm) \equiv 0$. However, we have proved that $\Omega_\pm$ are not eigenvalues, which implies that $\widetilde{\Psi}(\xi;\, K,\, \Omega_\pm) \equiv 0$.

Noticing that

$$\Delta = -\left(\frac{(i\Omega)^2}{16} - \nu^2K^2 - |K|K^2\right) \times \left(\frac{(i\Omega)^2}{16} - \nu^2K^2 + |K|K^2\right), \tag{42}$$

and summarising the results obtained in this section we obtain that

$$\check{v} = \frac{\Psi(\xi;\, K,\, \Omega)}{D(K,\, \Omega)}, \tag{43}$$

where

$$D(K,\,\Omega) = \frac{\Omega^2}{16} + K^2(\nu^2 - |K|), \tag{44}$$

$$\Psi(\xi;\,K,\,\Omega) = 9\nu^8K^2\left[\frac{(i\Omega)^2}{16} - K^2(\nu^2 + |K|)\right]^{-1} \times \left(\prod_{1\leqslant j<l\leqslant 4}(\kappa_l - \kappa_j)\right)^{-1}\frac{d\widetilde{\Psi}}{d\xi}. \tag{45}$$

In accordance with the results obtained in this section, for any ξ function $\Psi(\xi; K, \Omega)$ is a regular function of Ω on the complex Ω-plane with the cuts along the intervals $(-\infty, -\Omega_0]$ and $[\Omega_0, \infty)$. Hence, $\breve{v}$ is a meromorphic function in the complex Ω-plane with the cuts. It has exactly two singularities that coincide with the zeros of function $D(K, \Omega)$. Their positions are given by equation (23). Both of these singularities are simple poles. When $|K| > \nu^2$ these poles are on the real axis. It is straightforward to see that $|\Omega_{1,2}| < \Omega_0$ meaning that the poles are always on the complex Ω-plane with the cuts.

Now we can calculate $v(T, \xi, Y)$ using the inverse Laplace and Fourier transforms:

$$v(\xi,\,T,\,Y) = \frac{1}{4\pi^2}\int_{i\varsigma-\infty}^{i\varsigma+\infty}\left[\int_{-\infty}^{\infty}\frac{\Psi(\xi;\,K,\,\Omega)}{D(K,\,\Omega)} \times e^{iKY}dK\right]e^{-i\Omega T}d\Omega. \tag{46}$$

When studying the absolute and convective instabilities we consider $\breve{v}$ as a function of two complex variables, Ω and K. To do this we make the extension of this function from the real K-axis to the complex K-plane. However, we cannot do this using the expression for $v(T, \xi, Y)$ given by equation (43) because of the presence of $|K|$ in this expression. Ruderman *et al* [17] and Ruderman [18] encountered a similar problem when studying the absolute and convective instabilities of a tangential discontinuity in an incompressible fluid viscous at one side of the discontinuity and ideal at the other side. Hence, to overcome the problem related to the presence of $|K|$ in the expression for the dispersion function D we closely follow the analysis by Ruderman *et al.* We transform equation (46) in such a way that the integration with respect to K is only over the positive real axis. Using equation (44) we obtain

$$\overline{D(K,\,\Omega)} = D(-K,\,-\overline{\Omega}), \tag{47}$$

where the bar indicates the complex conjugate. The roots of equation (19) are functions of Ω and K. It is easy to verify that the roots of this equation satisfy

$$\overline{\kappa_j(K,\,\Omega)} = \kappa_j(-K,\,-\overline{\Omega}), \tag{48}$$

where $j = 1,\ldots,4$. Using this result we obtain from equation (18)

$$\overline{h_j(\xi;\,K,\,\Omega)} = h_j(\xi;\,-K,\,-\overline{\Omega}). \tag{49}$$

Now it follows that

$$\overline{W_j(\xi;\,K,\,\Omega)} = W_j(\xi;\,-K,\,-\overline{\Omega}), \quad j = 0,\ldots,4, \tag{50}$$

and, consequently,

$$\overline{C_j(\xi;\,K,\,\Omega)} = C_j(\xi;\,-K,\,-\overline{\Omega}), \quad j = 1,\ldots,4. \tag{51}$$

Continuing this analysis we eventually arrive at

$$\overline{\Psi(\xi;\,K,\,\Omega)} = \Psi(\xi;\,-K,\,-\overline{\Omega}). \tag{52}$$

Now we split the inner integral in two: one from $-\infty$ to 0, and the other from 0 to ∞, so that now $v(\xi, T, Y)$ is equal to the sum of two integrals. Next we make the variable substitution $K \to -K$, $\Omega \to -\overline{\Omega}$ in the first integral and use equations (47) and (51). This yields

$$\begin{aligned}
&\int_{i\varsigma-\infty}^{i\varsigma+\infty}\left[\int_{-\infty}^{0}\frac{\Psi(\xi;\,K,\,\Omega)}{D(K,\,\Omega)}\,e^{iKY}dK\right]e^{-i\Omega T}d\Omega \\
&= \int_{i\varsigma+\infty}^{i\varsigma-\infty}\left[\int_{\infty}^{0}\frac{\Psi(\xi;\,-K,\,-\overline{\Omega})}{D(-K,\,-\overline{\Omega})}\,e^{-iKY}dK\right]e^{i\overline{\Omega}T}d\overline{\Omega} \\
&= \overline{\int_{i\varsigma-\infty}^{i\varsigma+\infty}\left[\int_{0}^{\infty}\frac{\Psi(\xi;\,K,\,\Omega)}{D(K,\,\Omega)}\,e^{iKY}dK\right]e^{-i\Omega T}d\Omega}.
\end{aligned} \tag{53}$$

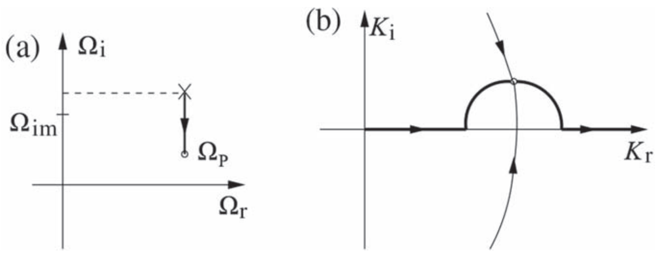

Figure 2. (a) Movement of Ω down to Ω_p. (b) The trajectories of colliding K-roots of the dispersion equation $\widetilde{D}(K, \Omega) = 0$ when Ω moves down to Ω_p (thin curves), the deformed integration contour in the complex K-plane (thick curve), and pinching of this contour by two K-roots are shown.

Using this result we transform equation (46) to

$$v(\xi, T, Y) = \frac{1}{2\pi^2}\Re\int_{i\varsigma-\infty}^{i\varsigma+\infty} \Phi(\xi, Y; \Omega)e^{-i\Omega T}d\Omega, \tag{54}$$

where

$$\Phi(\xi, Y; \Omega) = \int_0^\infty \frac{\Psi(\xi; K, \Omega)}{D(K, \Omega)} e^{iKY}dK. \tag{55}$$

Since now the integration with respect to K is along the positive real axis we have $|K| = K$ in equation (44) and thus $D(K, \Omega)$ is a holomorphic function of K.

5. Studying absolute and convective instabilities

To study the absolute and convective instability we use the method developed by Briggs [13] (see also Bers [14] and Brevdo [19]). However, similar to the analysis in [17] and [18], the function $\Psi(\xi, \Omega)$ is given by the integral from 0 to ∞ and not from $-\infty$ to ∞ as in [13]. Hence, we use the modification of Briggs' method introduced in [17].

Below we study the absolute and convective instabilities in various reference frames. We consider a reference frame moving with the velocity $\widetilde{V}$ in the y-direction with respect to the reference frame where equation (9) is valid. This reference frame, in turn, moves in the y-direction with the velocity Υ with respect to the reference frame where the medium is at rest far from the soliton. In the moving reference frame the frequency Ω is Doppler-shifted by VK and, as a result, we have to substitute $\Omega + VK$ for Ω in all expressions containing Ω, where

$$V = \widetilde{V}\sqrt{-\frac{3\chi t_0}{\Lambda}}. \tag{56}$$

In particular we have to substitute

$$\widetilde{D}(K, \Omega) = D(K, \Omega + VK) \tag{57}$$

for $D(K, \Omega)$ in equation (55). For any K function $\widetilde{D}(K, \Omega)$ is a meromorphic function on the Ω-complex plane with the cuts along the intervals $(-\infty, -\Omega_0 + VK]$, and $[\Omega_0 + VK, \infty)$, and with two poles at $\Omega_{1,2} + VK$.

Since the maximum growth rate of the instability does not exceed $\frac{8}{9}\nu^3\sqrt{3} = \Gamma_m$ it follows that $\widetilde{D}(K, \Omega)$ is a holomorphic function in the half-plane $\Im(\Omega) > \Gamma_m$. Hence, it is enough to take $\varsigma > \Gamma_m$ in equation (54). It is also obvious that $\widetilde{D}(K, \Omega)$ considered as a cubic polynomial with respect to K cannot have a real root when $\Im(\Omega) > \Gamma_m$ and, consequently, the integrand in equation (55) is non-singular. Now in accordance with Briggs' method [13] we starts to move the integration contour in equation (54). We do it point by point (see figure 2a) reducing the imaginary part of Ω while keeping constant its real part. As soon as $\Im(\Omega) = \Gamma_m$ while $\Im(\Omega) = -KV$ we have $\widetilde{D}(K, \Omega) = 0$ at $K = \frac{2}{3}\nu^2$, and the integrand in equation (55) is singular. However this problem can be easily solved. We deform the integration contour in equation(55) in such a way that $K = \frac{2}{3}\nu^2$ is not on the new contour. The only conditions that we need to observe are that the left end of the contour remains attached to the origin of the K-complex plane and the contour approaches the real K-axis at infinity. We keep deforming the contour to make the integrand in equation (55) non-singular every time when a root of cubic equation $\widetilde{D}(K, \Omega) = 0$ hits the contour. We emphasise that the shape of the deformed contour depends on the real part of Ω. In principle, it is even possible that, for some values of Ω we do not need to deform the K-contour at all. Using the above procedure we obtain the analytical continuation of $\Phi(\xi, Y; \Omega)$ below the line $\Im(\Omega) = \Gamma_m$. It is stated in the standard Brigg' method [13] method that we need to make the analytical continuation of $\Phi(\xi, Y; \Omega)$ slightly below the real axis to have perturbations decaying with time. However, in fact, it is enough to have the analytical continuation to the upper half of the complex Ω-plane and continuous up to the real axis. Really, if we can take $\varsigma = 0$ in equation (54) then simple integration by parts shows that $v(\xi, T, Y) \to 0$ as $T \to \infty$ at least as fast as $1/T$. This implies that the instability is convective.

The singularity in the integrand in equation (55) cannot be removed when the contour is pinched by two K-roots that come from two different sides of the contour (see figure 2b). We call a collision of two roots on the deformed contour that causes its pinching a pinching collision. If a pinching collision occurs when $\Omega = \Omega_p$ then $\Phi(\xi, Y; \Omega)$ has a branch-point singularity of the form $(\Omega - \Omega_p)^{-1/2}$ [13, 14, 19]. We denote the double root that appears as a result of collision of the two K-roots as K_p. The dominant term of the contribution to the asymptotics of the expression for $v(\xi, T, Y)$ at fixed ξ and Y, and $T \to \infty$ coming from this singularity has the form

$$A(\xi; K_p, \Omega_p) T^{-1/2} \exp(iK_p Y - i\Omega_p T), \tag{58}$$

where the factor $A(\xi; K_p, \Omega_p)$ is independent of T and Y. Hence, the instability is absolute when $\Im(\Omega_p) > 0$.

A necessary condition for a pinching collision of two K-roots to occur is that, at the collision point (K_p, Ω_p), the dispersion function $\widetilde{D}(K, \Omega)$ has a double root in K, that is

$$\widetilde{D}(K_p, \Omega_p) = 0, \quad \frac{\partial \widetilde{D}}{\partial K}(K_p, \Omega_p) = 0. \tag{59}$$

Using equation (44) and (57) we obtain from equation (59) the following system of equations:

$$(\Omega + VK)^2 + 16K^2(\nu^2 - K) = 0, \tag{60a}$$

$$V(\Omega + VK) + 8K(2\nu^2 - 3K) = 0. \tag{60b}$$

The solution to this system of equations is straightforward. It reads

$$K_{p\pm} = \frac{48\nu^2 + V^2 \pm V\sqrt{V^2 - 48\nu^2}}{72}, \tag{61a}$$

$$\Omega_{p\pm} = \frac{1}{216}[-V(144\nu^2 + V^2) \pm (48\nu^2 - V^2)\sqrt{V^2 - 48\nu^2}]. \tag{61b}$$

We see that $\Omega_{p\pm}$ are real when $|V| \geqslant 4\nu\sqrt{3}$. In this case pinching does not occur for any value of Ω in the upper Ω-plane, and the instability is convective. Hence, $|V| > 4\nu\sqrt{3}$ is a sufficient condition for the instability to be convective.

We also can conclude that $|V| < 4\nu\sqrt{3}$ is a necessary condition for the instability to be absolute. We have $\Im(\Omega_{p\pm}) > 0$ and the possible pinching occurs in the upper Ω-plane. However, this condition is not sufficient. It is also necessary to verify that the two colliding K-roots come from different sides of the integration contour in the K-complex plane. Ruderman et al. [17] showed that this condition reduces to the following: We take $\Omega = \Re(\Omega_{p+}) + i\varsigma$, where ς is slightly larger that Γ_m, and then start to reduce the imaginary part of Ω until $\Omega = \Omega_{p+}$. When the imaginary part of Ω decreases the two K-roots that collide when $\Omega = \Omega_{p+}$ move in the complex K-plane. Then the colliding roots pinch the K-integration contour if and only if the union of their trajectories crosses the positive part of the real K-axis odd number of times. It is shown in Appendix C that the colliding roots are always pinching if $|V| < 4\nu\sqrt{3}$. Therefore, the instability is absolute when $|V| < 4\nu\sqrt{3}$.

The most interesting are the stability properties in the reference frame where the medium is at rest far from the soliton. In this reference frame nonlinear waves are described by equation (7). To obtain this reference frame we have to take $\widetilde{V} = -\Upsilon$. Hence, using equation (56), we obtain that in this reference frame the instability is absolute if $|\Upsilon| < 4\nu\sqrt{\Lambda/\chi t_0}$ and convective otherwise. Now we connect the condition for the absolute instability with the direction of soliton propagation. We define a critical direction of wave propagation as a direction at which the group velocity is parallel to the phase velocity. We choose one of the critical directions and introduce the angle φ between this direction and the wave vector $\mathbf{k}$. Defining the unit vector in the direction of the wave vector, $\hat{\mathbf{k}} = \mathbf{k}/k$, we obtain that the phase velocity is given by $\mathbf{V}_{ph} = c(\varphi)\hat{\mathbf{k}}$. The wave frequency is $\omega = c(\varphi)k$. Introducing the unit vector in the critical direction, $\hat{\mathbf{e}}$, we obtain $k\cos\varphi = \hat{\mathbf{e}} \cdot \mathbf{k}$. Differentiating this relation yields

$$\hat{\mathbf{k}}\cos\varphi - k\frac{\partial\varphi}{\partial\mathbf{k}}\sin\varphi = \hat{\mathbf{e}}. \tag{62}$$

Using this identity we obtain for the group velocity

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = c(\varphi)\hat{\mathbf{k}} + c'(\varphi)\hat{\mathbf{k}}_\perp, \tag{63}$$

where the prime indicates the derivative and

$$\hat{\mathbf{k}}_\perp = \frac{\hat{\mathbf{k}}\cos\varphi - \hat{\mathbf{e}}}{\sin\varphi}. \tag{64}$$

It is easy to see that $\hat{\mathbf{k}}_\perp$ is a unit vector and $\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}_\perp = 0$. The condition that the direction the angle φ is measured from is critical is written as $\mathbf{V}_g \parallel \mathbf{k}$. It follows from equation (63) that this condition reduces to $c'(0) = 0$.

Now we introduce Cartesian coordinates x, y with the angle between the x-axis and the critical direction equal to φ. Then we consider a wave propagating at a small angle ϕ with respect to the x-axis. We obtain $k_x = k\cos\phi \approx k\left(1 - \frac{1}{2}\phi^2\right)$ and $k_y = k\sin\phi \approx k\phi$. The angle between the critical direction and the wave vector is $\varphi + \phi$. Using these relations and $k \approx k_x\left(1 + \frac{1}{2}\phi^2\right)$ we write the dispersion equation as

$$\begin{aligned}\omega = c(\varphi + \phi)k &\approx c(\varphi)k + c'(\varphi)\phi k + \frac{1}{2}c''(\varphi)\phi^2 k\\ &\approx c(\varphi)k_x + c'(\varphi)k_y + [c(\varphi) + c''(\varphi)]\frac{k_y^2}{2k_x}.\end{aligned} \tag{65}$$

The assumption that $|k_y| \ll |k_x|$ is equivalent to $|\phi| \ll 1$. Comparing equation (65) with equation (5) with the second term on the right-hand side describing the wave dispersion dropped we obtain $\Upsilon = c'(\varphi)$ and $\Lambda = \frac{1}{2}[c(\varphi) + c''(\varphi)]$. Hence, the condition for the absolute instability can be rewritten as $|c'(\varphi)| < 4\nu\sqrt{\Lambda/\chi t_0}$.

The KP equation was derived under the assumption that the perturbation amplitude is small and the characteristic spatial scale of perturbations is much larger than the dispersion length. The dispersion length is defined by the condition that the second term on the right-hand side of equation (5) describing the wave dispersion is of the order of the first term. Hence, we can define this length as $\ell = \sqrt{|\beta|/c}$.

Now, using equation (8) we rewrite equation (10) as

$$\chi^2 U = \frac{2(\nu\chi)^2}{\cosh^2(\nu\chi[x - (c + 4\nu^2/\chi t_0)t])}. \tag{66}$$

Then $2(\nu\chi)^2$ can be considered as the soliton amplitude and its width can be defined as

$$L = \frac{1}{\nu|\chi|} = \frac{1}{\nu}\sqrt[3]{t_0|\beta|}. \tag{67}$$

Equation (7) was derived under the assumption that the characteristic spatial scale of perturbations in the x-direction is much larger than the dispersion length. This condition can be written as $\ell \ll L$. Using equation (67) we reduce the condition for the absolute instability to

$$|c'(\varphi)| < \frac{2\ell}{L}\sqrt{2c|c + c''|}. \tag{68}$$

In this equation not only c and its derivatives can depend on φ but also ℓ. We will see this in the example considered in the next section. Below we assume that the ratio ℓ/L is independent of φ. It follows from equation (68) that $|c'(\varphi)/c(\varphi)| \ll 1$. Since $c'(0) = 0$ this condition is equivalent to $|\varphi| \ll 1$. This result shows that the conjecture made in [12] and formulated in the Introduction is correct. We make a viable assumption that $c(\varphi) \approx c(0)$ and $c''(\varphi) \approx c''(0)$ and recoll that ℓ/L is assumed to be independent of φ. Then the criterion of absolute instability given by equation (68) can be rewritten in the approximate form as

$$|\varphi| < \varphi_c = \frac{2\ell}{L|c''|}\sqrt{2c|c + c''|}, \tag{69}$$

where c and c' are now calculated at $\varphi = 0$. We can see that φ_c is inversely proportional to the soliton width, or, what is the same, it is proportional to the square root of the soliton amplitude. When the inequality equation (69) is not satisfied the soliton is only convectively unstable.

6. Example: Soliton stability in Hall plasmas

In [12] the KP equation was derived for nonlinear waves propagating in a Hall plasma. In such a plasma two kinds of nonlinear waves can propagate: fast and slow magnetosonic. One critical direction for both kinds of magnetosonic waves is along the equilibrium magnetic field. For fast magnetosonic waves there is the second critical direction. It is perpendicular to the equilibrium magnetic field. We measure the angle φ from the direction of the equilibrium magnetic field. Below we consider a two-dimensional problem where all the variables depend on the coordinates x and y with the x-axis in the direction of the wave propagation and the y axis in the plane defined by the x-axis and the equilibrium magnetic field. The KP equation was derived in [12] for the perturbation of the y-component of the magnetic field. The coefficients of this equation are given by

$$c_\pm^2 = \frac{1}{2}\left(c_s^2 + V_A^2 \pm \sqrt{(c_s^2 + V_A^2)^2 - 4c_s^2 V_A^2 \cos^2\varphi}\right), \tag{70a}$$

$$\Upsilon_\pm = -\frac{c_s^2 V_A^2 \cos^2\varphi}{2c_\pm[2c_\pm^2 - (c_s^2 + V_A^2)]}, \tag{70b}$$

$$N_\pm = \frac{c_\pm V_A^2 \sin\varphi}{2B_0[2c_\pm^2 - (c_s^2 + V_A^2)]} \times \left(3 + (\gamma + 1)\frac{c_\pm^2 - V_A^2}{c_\pm^2 - c_s^2}\right), \tag{70c}$$

$$\beta_\pm = \frac{\ell^2 V_A^3 (c_s^2 - c_\pm^2)\cos^2\varphi}{2(c_\pm^2 - V_A^2\cos^2\varphi)[2c_\pm^2 - (c_s^2 + V_A^2)]}, \tag{70d}$$

$$\Lambda_\pm = \frac{c_\pm^2(c_s^2 + V_A^2) - c_s^2 V_A^2 + \Upsilon_\pm^2(c_s^2 + V_A^2 - 6c_\pm^2)}{2c_\pm[2c_\pm^2 - (c_s^2 + V_A^2)]}. \tag{70e}$$

The sound speed c_s, Alfvén speed V_A, and dispersion length ℓ are defined by

$$c_s^2 = \frac{\gamma p_0}{\rho_0}, \quad V_A^2 = \frac{B_0}{\mu_0\rho_0}, \quad \ell = \frac{m_i B_0}{e\mu_0\rho_0 V_A}, \tag{71}$$

where γ is the adiabatic index, p_0, ρ_0, and B_0 are the equilibrium plasma pressure and density, and the magnetic field, m_i is the ion mass, e the elementary charge, and μ_0 the magnetic permeability of free space. We note that in equation (7) N is dimensionless, while in equation (70c) its dimension is c/B. This difference is related to the fact that in equation (7) the variable u has the dimension of velocity, while in [12] the KP equation was derived for the magnetic field perturbation.

It is shown in [12] that the KP equation for fast magnetosonic waves reduces to the KPI equations by the corresponding variable substitution. This implies that fast solitons are unstable with respect to normal modes of transverse perturbations. Using equation (70a) we obtain from equation (69) that solitons propagating at angles with respect to the equilibrium magnetic field satisfying the inequality $|\varphi| > \varphi_{c+}$ are only convectively unstable, where

$$\varphi_{c+} = \frac{2\ell c_M}{L c_m^2}\sqrt{2|V_A^2 - c_s^2|}, \tag{72}$$

and $c_M = \max(c_s, V_A)$ and $c_m = \min(c_s, V_A)$. We note that we cannot claim that a soliton propagating at an angle φ with respect to the equilibrium magnetic field satisfying the inequality $|\varphi| < \varphi_{c+}$ is absolutely unstable. The reason is that solitons propagating at very small angles with respect to the equilibrium magnetic field are not described by the KdV equation. It is immediately follows from the fact that $\beta_+ \to 0$ as $\varphi \to 0$ when $c_s > V_A$, and $\beta_+ \to \infty$ when $c_s < V_A$. In fact, they are described by the Derivative Nonlinear Schr ö dinger equation (DNLS) equation (e.g. [20–22]). The multidimensional generalisation of this equation was derived in [23, 24]. This multidimensional DNLS equation was used in [24] to study the soliton stability with respect to normal modes of transverse perturbations.

In fact, the condition $|\varphi| > \varphi_{c+}$ does not guaranty that the soliton is only convectively unstable. Since the direction orthogonal to the equilibrium magnetic field is critical the soliton can be absolutely unstable if it propagates at a small angle with respect to this direction. To obtain the condition for convective instability for a soliton propagating quasi-perpendicular with respect to the equilibrium magnetic field we introduce $\tilde{\varphi} = \pi/2 - \varphi$. Then we obtain that the instability is convective when $|\tilde{\varphi}| > \tilde{\varphi}_c$, where

$$\tilde{\varphi}_c = \frac{2\ell(c_s^2 + V_A^2)}{L c_s^2 V_A^2}\sqrt{2(c_s^4 + c_s^2 V_A^2 + V_A^4)}. \tag{73}$$

Hence, the soliton is only convectively unstable if $|\varphi| > \varphi_{c+}$ and $\left|\frac{\pi}{2} - \varphi\right| > \tilde{\varphi}_c$. Again, we cannot claim that a soliton propagating at an angle $\tilde{\varphi}$ with respect to the equilibrium magnetic field satisfying the inequality $|\tilde{\varphi}| < \tilde{\varphi}_c$ is absolutely unstable. It can be immediately seen from the fact that $\beta_+ \to 0$ as $\varphi \to \frac{\pi}{2}$. Solitons propagating quasiperpendicular with respect to the equilibrium magnetic field are still described by the KdV equation. However, for proper description of these solitons one must add the term taking the electron inertial into account in the induction equation. The account of electron inertia is important when $\left|\frac{\pi}{2} - \varphi\right| \lesssim \sqrt{m_e/m_i}$, where m_i is the electron mass.

Now we proceed to slow magnetosonic waves. It is shown in [12] that the variable substitution reduces the KP equation for slow magnetosonic waves to the KPI equation when $|\varphi| < \varphi_0$ and to KPII equation when $|\varphi| < \varphi_0$. The angle φ_0 decreases from $\pi/6$ to 0 as c_s/V_A increases from 0 to 1, and then increases from 0 to $\pi/6$ as c_s/V_A further increases from 1 to infinity. Hence, slow solitons are unstable with respect to normal modes of transverse perturbations when they propagate at small angles with respect to the equilibrium magnetic field unless c_s/V_A is very close to 1. The soliton is convectively unstable when $|\varphi| < \varphi_{c-}$ and φ is not very close to $\frac{\pi}{2}$, where

$$\varphi_{c-} = \frac{2\ell c_m}{L c_M^2}\sqrt{2|V_A^2 - c_s^2|}. \tag{74}$$

Again we cannot claim that a soliton is absolutely unstable when $|\varphi| > \varphi_{c-}$; the reason is the same as in the case of fast solitons.

7. Summary

In this article we studied the stability of one-dimensional solitons propagating in an anisotropic medium with respect to transverse perturbations. We derived the general form of the Kadomtsev-Petviashvili (KP) equation for nonlinear waves propagating in anisotropic media. By a proper variable substitution this equation reduces either to the KPI or to the KPII equation. It is well known that a one-dimensional soliton solution to the KPI equation is unstable with respect to normal modes of transverse perturbations, while a one-dimensional soliton solution to the KPII equation is stable with respect to these perturbations.

We only considered the case when the KP equation reduces to the KPI equation. We studied the absolute and convective instabilities of solitons. For this we solved the initial value problem for the linear stability using the Laplace–Fourier transform. Then we used Briggs' method. We showed that so-called critical directions of propagation play an important role in the soliton stability. A critical direction is defined by the condition that the group velocity equals to the phase velocity when the wave propagates in this direction. We proved that a one-dimensional soliton is only convectively unstable unless it propagates at an angle smaller than the critical with respect to a critical direction. The critical angle is proportional to a small parameter equal to the ratio of the dispersion length to the soliton width. The coefficient of proportionality is expressed in terms of the phase speed and its second derivative with respect to the propagation angle.

As an example we considered the stability of solitons propagating in a Hall plasma. In such a plasma two kinds of solitons can propagate: fast and slow. There are two critical directions for fast solitons, one along the equilibrium magnetic field and the other perpendicular to this field. For slow solitons there is only one critical direction along the equilibrium magnetic field. We expressed the coefficient of proportionality in the expression for critical angles in terms of the sound and Alfvén speed.

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Data availability statement

No new data were created or analysed in this study.

Conflict of interest

No potential conflict of interest in the research was reported by the authors

Appendix A. Complementary function of equation (16)

In this section we obtain the complementary function of equation (16), that is the general solution to the homogeneous counterpart of this equation. We put

$$v = \frac{d}{d\xi}[e^{-\kappa\xi}g(\xi)]. \tag{A1}$$

Substituting this expression in equation (16) with the right-hand side equal to zero yields

$$\begin{aligned}&\frac{d^4g}{d\xi^4} - 4\kappa\frac{d^3g}{d\xi^3} + (6U_0 - 4\nu^2 + 6\kappa^2)\frac{d^2g}{d\xi^2}\\&+\left(6\frac{dU_0}{d\xi} - 12\kappa U_0 - 4\kappa^3 - i\Omega + 8\nu^2\kappa\right)\frac{dg}{d\xi}\\&+\left(i\Omega\kappa - 4\nu^2\kappa^2 - 6\kappa\frac{dU_0}{d\xi} + 6\kappa^2U_0\right.\\&\left.+\kappa^4 + 3K^2\right)g = 0.\end{aligned} \tag{A2}$$

Next we make the variable substitution $s = \tanh(\nu\xi)$. As a result we transform equation (A2) to

$$\begin{aligned}&\nu^4(1 - s^2)^4\frac{d^4g}{ds^4} - 4\nu^3(1 - s^2)^3(3\nu s + \kappa)\frac{d^3g}{ds^3}\\&+6\nu^2(1 - s^2)^2(4\nu^2s^2 + 4\nu\kappa s + \kappa^2)\frac{d^2g}{ds^2}\\&+\nu(1 - s^2)[24\nu^3s(s^2 - 1) - 12\nu\kappa^2s - 8\nu^2\kappa\\&-4\kappa^3 - i\Omega]\frac{dg}{ds} + [12\nu^2\kappa(1 - s^2)(\kappa + 2\nu s)\\&+i\Omega\kappa - 4\nu^2\kappa^2 + \kappa^4 + 3K^2]g = 0.\end{aligned} \tag{A3}$$

Now we look for the solution to this equation in the form

$$g(s) = a + bs - s^2, \tag{A4}$$

where a and b are the constants to be determined. Substituting this expression in equation (A3) we should obtain the sixth order polynomial. However the coefficient at s^6 is identically zero, so that we obtain the fifth order polynomial. In order that $g(s)$ given by equation (A4) be a solution to equation (A3) all the coefficients of this polynomial must be zero. The conditions that the coefficients at s^5 and s^4 are zeros coincide. Hence, we obtain that a, b, and κ must satisfy five equations,

$$\nu b = -\kappa, \tag{A5a}$$

$$i\Omega = 28\nu^2\kappa - 4\kappa^3 - 12a\nu^2\kappa + 24b\nu^3, \tag{A5b}$$

$$\begin{aligned}&i\Omega(\nu b - \kappa) + 40\nu^2\kappa^2 - 12a\nu^2\kappa^2 + 32\nu^3\kappa b\\&+4\nu\kappa^3b - \kappa^4 - 3K^2 = 0,\end{aligned} \tag{A5c}$$

$$\begin{aligned}&i\Omega(2\nu + \kappa b) + 8\nu^3\kappa(3a - 4) - 24\nu^4b + 8\nu\kappa^3\\&+b(\kappa^4 - 4\nu^2\kappa^2 + 3K^2) = 0,\end{aligned} \tag{A5d}$$

$$\begin{aligned}&i\Omega(\kappa a - \nu b) - 12\nu^2\kappa^2 - 4\nu b\kappa(2\nu^2 + \kappa^2)\\&+a(\kappa^4 + 8\nu^2\kappa^2 + 3K^2) = 0.\end{aligned} \tag{A5e}$$

Using equations (A5a) and (A5b) to eliminate b and Ω from equations (A5c)–(A5e) we obtain that all these three equations reduce to

$$4a\nu^2\kappa^2 + \kappa^4 - K^2 = 0. \tag{A6}$$

Now it follows from equations (A5a), (A5b), and (A6) that κ satisfies the equation

$$\kappa^4 - 4\nu^2\kappa^2 + i\Omega\kappa + 3K^2 = 0, \tag{A7}$$

while a and b are given by

$$a = \frac{K^2 - \kappa^4}{4\nu^2\kappa^2}, \quad b = -\frac{\kappa}{\nu}. \tag{A8}$$

Now it follows from equations (A1) and (A4) that the function

$$v = \frac{d}{d\xi}\{e^{-\kappa\xi}[a + b\tanh(\nu\xi) - \tanh^2(\nu\xi)]\}. \tag{A9}$$

is a solution to equation (16) with the right-hand side equal to zero when κ satisfies equation (A7), and a and b are given by equation (A8).

Appendix B. Calculation of W_0

In this section we calculate the determinant W_0 given by equation (29a). We write $h(\xi) = e^{-\kappa\xi}g(s)$, where $s = \tanh(\nu\xi)$, $g(s) = a + bs - s^2$, and a and b are given by equation (A8). We temporarily drop the subscript j. Then we obtain

$$\frac{dh}{d\xi} = e^{-\kappa\xi}\left(\nu(1 - s^2)\frac{dg}{ds} - \kappa g\right), \tag{B1a}$$

$$\frac{d^2h}{d\xi^2} = e^{-\kappa\xi}\left(\nu^2(1 - s^2)^2\frac{d^2g}{ds^2} - 2\nu(1 - s^2)(\nu s + \kappa)\frac{dg}{ds} + \kappa^2 g\right), \tag{B1b}$$

$$\frac{d^3h}{d\xi^3} = e^{-\kappa\xi}\left(\nu^3(1 - s^2)^3\frac{d^3g}{ds^3} - 3\nu^2(1 - s^2)^2(2\nu s + \kappa)\frac{d^2g}{ds^2} + \nu(1 - s^2)(6\nu^2 s^2 + 6\nu\kappa s + 3\kappa^2 - 2\nu^2)\frac{dg}{ds} - \kappa^3 g\right). \tag{B1c}$$

Substituting $g(s) = a + bs - s^2$ in these equations and using the relation $\nu b = -\kappa$ we reduce these equations to

$$\frac{dh}{d\xi} = e^{-\kappa\xi}[2\nu s^3 + 2\kappa s^2 - (2\nu - \nu^{-1}\kappa^2)s - \kappa(a + 1)], \tag{B2a}$$

$$\frac{d^2h}{d\xi^2} = e^{-\kappa\xi}[-6\nu^2 s^4 - 6\nu\kappa s^3 + (8\nu^2 - 3\kappa^2)s^2 + \nu^{-1}\kappa(6\nu^2 - \kappa^2)s - 2\nu^2 + \kappa^2(a + 2)], \tag{B2b}$$

$$\frac{d^3h}{d\xi^3} = e^{-\kappa\xi}[24\nu^3 s^5 + 24\nu^2\kappa s^4 - 4\nu(10\nu^2 - 3\kappa^2)s^3 + 4\kappa(\kappa^2 - 8\nu^2)s^2 + (16\nu^3 - 12\nu\kappa^2 + \nu^{-1}\kappa^4)s + 8\nu^2\kappa - \kappa^3(a + 3)]. \tag{B2c}$$

Taking $\xi = 0$ and $s = 0$ we obtain

$$h = a,\quad \frac{dh}{d\xi} = -\kappa(a + 1),\quad \frac{d^2h}{d\xi^2} = \kappa^2(a + 2) - 2\nu^2,\quad \frac{d^3h}{d\xi^3} = 8\nu^2\kappa - \kappa^3(a + 3). \tag{B3}$$

These expressions are valid for $h_1,\ldots,h_4$ and for their derivatives. Using equations (A7) and (A8) we obtain

$$\begin{aligned}
a &= \frac{i\Omega\kappa^3}{36\nu^2K^2} - \frac{\kappa^2}{3\nu^2} - \frac{i\Omega\kappa}{9K^2} + \frac{12\nu^2K^2 - \Omega^2}{36\nu^2K^2} \equiv Q_1(\kappa),\\
-\kappa(a + 1) &= \frac{\kappa^3}{3\nu^2} - \frac{4\kappa}{3} + \frac{i\Omega}{12\nu^2} \equiv Q_2(\kappa),\\
\kappa^2(a + 2) - 2\nu^2 &= \kappa^2 + \frac{i\Omega\kappa}{4\nu^2} + \frac{K^2}{\nu^2} - 2\nu^2 \equiv Q_3(\kappa),\\
8\nu^2\kappa - \kappa^3(a + 3) &= -2\kappa^3 - \frac{i\Omega\kappa^2}{4\nu^2} + \left(8\nu^2 - \frac{K^2}{\nu^2}\right)\kappa \equiv Q_4(\kappa).
\end{aligned} \tag{B4}$$

Using equations (B3) and (B4) we obtain

$$W_0 = \begin{vmatrix} Q_1(\kappa_1) & Q_1(\kappa_2) & Q_1(\kappa_3) & Q_1(\kappa_4) \\ Q_2(\kappa_1) & Q_2(\kappa_2) & Q_2(\kappa_3) & Q_2(\kappa_4) \\ Q_3(\kappa_1) & Q_3(\kappa_2) & Q_3(\kappa_3) & Q_3(\kappa_4) \\ Q_4(\kappa_1) & Q_4(\kappa_2) & Q_4(\kappa_3) & Q_4(\kappa_4) \end{vmatrix}. \tag{B5}$$

Now we add the last row multiplied by $i\Omega(72\nu^2K^2)^{-1}$ to the first row, and the last row multiplied by $1/6\nu^2$ to the second row. This yields

$$W_0 = \begin{vmatrix} R_1(\kappa_1) & R_1(\kappa_2) & R_1(\kappa_3) & R_1(\kappa_4) \\ R_2(\kappa_1) & R_2(\kappa_2) & R_2(\kappa_3) & R_2(\kappa_4) \\ Q_3(\kappa_1) & Q_3(\kappa_2) & Q_3(\kappa_3) & Q_3(\kappa_4) \\ Q_4(\kappa_1) & Q_4(\kappa_2) & Q_4(\kappa_3) & Q_4(\kappa_4) \end{vmatrix}, \tag{B6}$$

where

$$R_1(\kappa) = \frac{\kappa^2}{3\nu^2}\left(\frac{\Omega^2}{96\nu^2K^2} - 1\right) - \frac{i\Omega\kappa}{72\nu^4} + \frac{1}{3} - \frac{\Omega^2}{36\nu^2K^2},$$
$$R_2(\kappa) = -\frac{i\Omega\kappa^2}{24\nu^4} - \frac{K^2\kappa}{6\nu^2} + \frac{i\Omega}{12\nu^2}. \tag{B7}$$

We now consider W_0 as a function of κ_4. $Q_4(\kappa)$ is a cubic polynomial, while $R_1(\kappa)$, $R_2(\kappa)$, and $Q_3(\kappa)$ are quadratic polynomials. Consequently $W_0(\kappa_4)$ is a cubic polynomial with respect to κ_4. $W_0(\kappa_4) = 0$ when either $\kappa_4 = \kappa_1$, or $\kappa_4 = \kappa_2$, or $\kappa_4 = \kappa_3$ because in this case two columns in the determinant W_0 are identical. This implies that κ_1, κ_2, and κ_3 are the roots of cubic polynomial $W_0(\kappa_4)$. Then $W_0(\kappa_4)$ is equal to $(\kappa_4 - \kappa_1)(\kappa_4 - \kappa_2)(\kappa_4 - \kappa_3)$ times the coefficient at κ_4^3, which is equal to the coefficient at $Q_4(\kappa_4)$ in the expansion of $W_0(\kappa_4)$ with respect to the last column times (-2). As a result we obtain

$$W_0 = -2(\kappa_4 - \kappa_1)(\kappa_4 - \kappa_2)(\kappa_4 - \kappa_3)\widetilde{W}_0, \tag{B8}$$

where

$$\widetilde{W}_0 = \begin{vmatrix} R_1(\kappa_1) & R_1(\kappa_2) & R_1(\kappa_3) \\ R_2(\kappa_1) & R_2(\kappa_2) & R_2(\kappa_3) \\ Q_3(\kappa_1) & Q_3(\kappa_2) & Q_3(\kappa_3) \end{vmatrix}. \tag{B9}$$

We add the last row multiplied by $(1/3\nu^2)(1 - \Omega^2/96\nu^2K^2)$ to the first row, and the last row multiplied by $i\Omega/24\nu^4$ to the second row. This yields

$$\widetilde{W}_0 = \begin{vmatrix} S_1(\kappa_1) & S_1(\kappa_2) & S_1(\kappa_3) \\ S_2(\kappa_1) & S_2(\kappa_2) & S_2(\kappa_3) \\ Q_3(\kappa_1) & Q_3(\kappa_2) & Q_3(\kappa_3) \end{vmatrix}, \tag{B10}$$

where

$$S_1(\kappa) = \frac{i\Omega\kappa}{72\nu^4}\left(5 - \frac{\Omega^2}{16\nu^2K^2}\right) - \frac{1}{3} + \frac{K^2}{3\nu^4} - \frac{\Omega^2}{48\nu^2K^2} - \frac{\Omega^2}{288\nu^6},$$
$$S_2(\kappa) = -\frac{\kappa}{6\nu^4}\left(K^2 + \frac{\Omega^2}{16\nu^2}\right) + \frac{i\Omega K^2}{24\nu^6}. \tag{B11}$$

We now consider $\widetilde{W}_0$ as a function of κ_3. $Q_3(\kappa)$ is a quadratic polynomial, while $S_1(\kappa)$ and $S_2(\kappa)$ are linear functions of κ. Consequently $\widetilde{W}_0(\kappa_3)$ is a quadratic polynomial with respect to κ_3. $\widetilde{W}_0(\kappa_3) = 0$ when either $\kappa_3 = \kappa_1$ or $\kappa_3 = \kappa_2$ because in this case two columns in the determinant $\widetilde{W}_0$ are identical. This implies that κ_1 and κ_2 are the roots of quadratic polynomial $D(\kappa_3)$. Then $D(\kappa_3)$ is equal to $(\kappa_3 - \kappa_1)(\kappa_3 - \kappa_2)$ times the coefficient at κ_3^2, which is equal to the coefficient at $Q_3(\kappa_3)$ in the expansion of $\widetilde{W}_0(\kappa_3)$ with respect to the last column. As a result we obtain

$$\widetilde{W}_0 = (\kappa_3 - \kappa_1)(\kappa_3 - \kappa_2)\widetilde{W}_1, \tag{B12}$$

where

$$\widetilde{W}_1 = \begin{vmatrix} S_1(\kappa_1) & S_1(\kappa_2) \\ S_2(\kappa_1) & S_2(\kappa_2) \end{vmatrix}. \tag{B13}$$

We easily obtain

$$\widetilde{W}_1 = \frac{\kappa_2 - \kappa_1}{18\nu^8 K^2}\left[\left(\frac{\Omega^2}{16} + \nu^2 K^2\right)^2 - K^6\right]. \tag{B14}$$

Using equations (B8), (B12), and (B14) we eventually arrive at

$$W_0 = \frac{1}{9\nu^8 K^2}\left[K^6 - \left(\frac{\Omega^2}{16} + \nu^2 K^2\right)^2\right] \times \prod_{1\leqslant j<l\leqslant 4}(\kappa_l - \kappa_j). \tag{B15}$$

Appendix C. Proving that colliding *K*-roots are pinching

In this section we prove that colliding roots are always pinching. We consider $|V| < 4\nu\sqrt{3}$ and write

$$V = 4\nu\sqrt{3}\,\sin\psi, \tag{C1}$$

where $\psi \in (-\pi/2, \pi/2)$. Then we obtain

$$K_p = \frac{\nu^2}{3}(2 + 2\sin^2\psi + i\sin 2\psi), \tag{C2a}$$

$$\Omega_p = \frac{8\nu^3\sqrt{3}}{9}[-\sin\psi(3 + \sin^2\psi) + i\cos^3\psi]. \tag{C2b}$$

Next we write

$$\Omega = \Omega_p + \frac{8}{9}i\nu^3\sqrt{3}\,\vartheta, \quad K = K_p + \frac{2}{3}\nu^2\lambda, \tag{C3}$$

where ϑ varies from 0 to $3\sqrt{3}\,(8\nu^3)^{-1}[\varsigma - \Im(\Omega_p)]$ with $\varsigma > \Gamma_m = \frac{8}{9}\nu^3\sqrt{3}$. This corresponds to varying $\Im(\Omega)$ from $\Im(\Omega_p)$ to ς while keeping $\Re(\Omega) = \Re(\Omega_p)$. Substituting equation (C3) in equation (60*a*), taking into account that Ω_p and K_p satisfy the system of equations (60), and using equation (C2) yields

$$\begin{aligned} &2\lambda^3 + 3\lambda^2\cos\psi(\cos\psi + 2i\sin\psi) - 6i\lambda\vartheta\sin\psi \\ &\quad + \vartheta^2 + 2\vartheta[\cos\psi(1 + 2\sin^2\psi) - 2i\sin^3\psi] = 0. \end{aligned} \tag{C4}$$

Using the variable substitution $\lambda = z - z_0$, where

$$z_0 = \frac{1}{2}\cos\psi(\cos\psi + 2i\sin\psi), \tag{C5}$$

we reduce the cubic equation (C4) to the depressed form

$$z^3 + pz + q = 0. \tag{C6}$$

The coefficients of this equation are given by

$$p = \frac{3}{4}\cos^2\psi(4 - 5\cos^2\psi - 2i\sin 2\psi) - 3i\vartheta\sin\psi, \tag{C7}$$

$$\begin{aligned} q = &\frac{1}{4}\cos^3\psi[\cos\psi(13\cos^2\psi - 12) \\ &+ 2i\sin\psi(7\cos^2\psi - 4)] + \vartheta\left[\cos^3\psi \right.\\ &\left.+ \frac{i}{2}\sin\psi(7\cos^2\psi - 4)\right] + \frac{1}{2}\vartheta^2. \end{aligned} \tag{C8}$$

We notice that K_p, z_0, p, and q are functions of ψ. We have the following relations: $K_p(-\psi) = \overline{K_p(\psi)}$, $z_0(-\psi) = \overline{z_0(\psi)}$, $p(-\psi) = \overline{p(\psi)}$ and $q(-\psi) = \overline{q(\psi)}$. It follows from these relations and equation (C6) that $z(-\psi) = \overline{z(\psi)}$ and, consequently, $\lambda(-\psi) = \overline{\lambda(\psi)}$. Let us now assume that the colliding roots are pinching for $\psi = \tilde{\psi}$. Then it follows from that the trajectories of colliding roots for $\psi = -\tilde{\psi}$ are symmetric to those for $\psi = \tilde{\psi}$ and, consequently, the roots are pinching for $\psi = -\tilde{\psi}$. This observation enables us to carry out the analysis only for $\psi \geqslant 0$.

Below the two roots of equation (C6) corresponding to the two colliding *K*-roots of the dispersion equation (60*a*) will be called the colliding *z*-roots. We denote them as z_1 and z_2. The third root of equation (C6) is z_3. When $K = K_p$ we have $\lambda = 0$ and, consequently, $z = z_0$. Then it follows that $z_1 = z_2 = z_0$ when $\vartheta = 0$, and, in accordance with equation (C5), the trajectories of the colliding *z*-roots defined by equations $z = z_1(\vartheta)$ and $z = z_2(\vartheta)$ start in the upper part of the complex *z*-plane. Since $z_1 + z_2 + z_3 = 0$ it follows that $z_3 = -2z_0$ when $\vartheta = 0$, and the trajectory of the third root defined by equations $z = z_3(\vartheta)$ starts in the lower part of the complex *z*-plane.

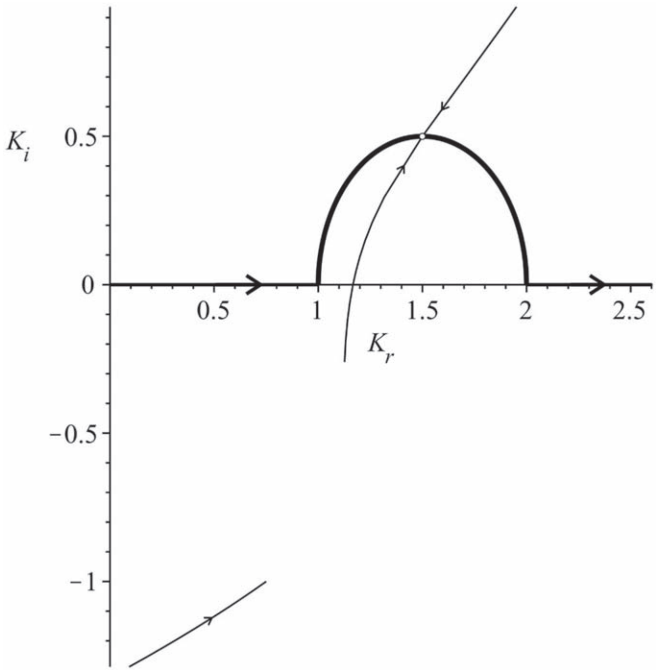

Figure 3. The thick line shows the deformed integration contour. The thin lines with arrows show the trajectories of K-roots in the complex K-plane. The scale in the axes is arbitrary.

Let us now assume that the trajectory of one of the K-roots crosses the real axis in the complex K-plane when $\vartheta = \vartheta_c$. This implies that $\Im(K(\vartheta_c)) = 0$. It is straightforward to check that for the corresponding z-root $\Im(z(\vartheta_c)) = 0$. We introduce the notation $z_c = z(\vartheta_c)$. Substituting z_c for z and ϑ_c for ϑ in equation (C6), separating the real an imaginary parts of the obtained equation, and using equations (C7) and (C8) we obtain the system of two equations,

$$4z_c^3 + 3z_c\cos^2\psi(4 - 5\cos^2\psi) + 2\vartheta_c^2 + 4\vartheta_c\cos^3\psi + \cos^4\psi(13\cos^2\psi - 12) = 0, \quad (C9)$$

$$(\cos^3\psi + \vartheta_c)(6z_c - 7\cos^2\psi + 4) = 0. \quad (C10)$$

Since $\vartheta_c > 0$ it follows from equation (C10) that

$$z_c = \frac{1}{6}(7\cos^2\psi - 4). \quad (C11)$$

The corresponding K-root crosses the real axis at $K_p + \frac{2}{3}\nu^2(z - z_0) = \frac{1}{9}\nu^2(8 + \cos^2\psi) > 0$, that is it crosses the positive part of the real axis. Substituting equation (C11) in equation (C9) yields

$$\vartheta_c^2 + 2\vartheta_c\cos^2\psi - \frac{\sin^2\psi(4 + 5\cos^2\psi)^2}{108} = 0. \quad (C12)$$

The positive root to this equation is

$$\vartheta_c = \sqrt{\cos^6\psi + \frac{\sin^2\psi(4 + 5\cos^2\psi)^2}{108}} - \cos^3\psi. \quad (C13)$$

Introducing $\varsigma = \frac{8}{9}\nu^3\sqrt{3}\,\xi$ we obtain that $\vartheta \in [0, \xi - \cos^3\psi]$, where now ξ is any quantity larger than 1. It is not difficult to show that $\vartheta_c < 1 - \cos^3\psi$. This implies that $\vartheta_c \in (0, \xi - \cos^3\psi)$.

Differentiating equation (C6) with respect to ϑ we can calculate $dz/d\vartheta$. Substituting $z = z_c$ and $\vartheta = \vartheta_c$ in the obtained formula and taking its imaginary part yields

$$\frac{d\Im(z)}{d\vartheta} = \frac{-27\sin\psi(\cos^3\psi + \vartheta_c)^2}{\sin^4\psi(3 + \sin^2\psi)^2 + 81\sin^2\psi(\cos^3\psi + \vartheta_c)^2}. \quad (C14)$$

We see that $d\Im(z)/d\vartheta < 0$. This result implies that a root trajectory can cross the real axis only moving from the upper to the lower complex z-plane. Hence, only the trajectories of colliding roots can cross the real axis, and only once. If we assume that both trajectories cross the real axis, then, for some value of ϑ they will be in the lower part of the complex z-plane. As a result, we would obtain $\Im(z_1 + z_2 + z_3) < 0$, which contradicts to the condition $z_1 + z_2 + z_3 = 0$. Hence, we conclude that only one trajectory of the two colliding roots can cross the real axis. On the other hand, one trajectory must cross the real axis because equation (C6) has a real root when $\vartheta = \vartheta_c$. Therefore, the trajectory of one of the colliding roots crosses the real axis and the trajectory of the other colliding root does not cross it. We see that the union trajectories of two colliding z-roots crosses the real axis just once, that is the odd number of times, and the same is true for the trajectories of the two K-colliding roots. In addition, the crossing occurs at the positive real axis. This implies that the colliding K-roots are pinching.

The analytical study presented in this section was supported by the direct numerical solution of the dispersion equation. The numerical results confirmed that the trajectory of one colliding roots crosses the positive real axis once, while the trajectory of the second root does not cross it. The picture of root trajectories and pinched integration contour are shown in figure 3 for $\psi = 45°$. The picture is similar for other values of ψ with the only difference that the double root is in the lower complex K-plane when $\psi < 0$.

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