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Robust State Estimations in Controlled ARMA Processes with the Non-Gaussian Noises: Applications to the Delayed Dynamics

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Abstract: This paper deals with a novel theoretic approach to the robust state estimations in discrete-time dynamic systems with the non-Gaussian correlated stochastic noises. The methodology we develop is based on the so-called "worst case" robust Kalman Filter (KF) approach proposed in [2,3]. We are interested in the robust state estimation for the controlled ARMA models under assumption of the colored noises. Since an ARMA model involves the correlated noises in the equivalent Linear Model (LM) representation, the resulting dynamic system also includes the correlated stochastic variables. These two crucial properties of the ARMA models under consideration imply the impossibility of application of the classic KF-type state estimations. We use the modified "instrumental variable" method and derive an auxiliary LM with the uncorrelated noises. Application of the robust KF to this auxiliary LM makes it possible to derive a guaranteed state estimation in the initially given ARMA model. The proposed non-standard KF based state estimations are finally applied to the linear stochastic dynamic systems with the time delays.

1. INTRODUCTION

The advanced KF based state estimation methodology for diverse types of discrete and continuous dynamic systems constitutes nowadays a powerful practical tool of the modern systems engineering (see [1,7,15,17,20,29]). While the classic KF development has been around for about 55 years (see [10,18]), it has been established in many challenging engineering areas and real-world applications. Implementations of the classic and advanced KF based state estimation algorithms span from aerospace technology, to control engineering and robotics, to the signal processing, communication engineering and mathematical economics. We refer to [1,10,11,23,24,26,30]) for some important examples of application of the celebrated KF approach. Let us also mention here some successful usages of the general KF methodology in financial engineering, economy and sociology [6,25].

It is common knowledge that the Normality Hypothesis (NH) associated with the stochastic noises of dynamic processes under consideration constitutes one of the fundamental hypothesises in the classic KF theory. The same is also true for the basic Stochastic Independency Hypothesis (SIH) for the system noises. Initially developed for the LMs equipped with the linear observer, the original KF approach was next extended to some important classes of nonlinear dynamic processes (see e.g., [1,10,15,20,29]).

Our paper is devoted to a specific application of the so-called "worst-case" minimax robust KF (developed in [2,3]) to the state estimation of the controlled ARMA processes with the non-Gaussian noises. We consider some families of stochastic variables and assume that the probability distributions of these variables possess bounded second moments. The resulting robust KF state estimation involves a guaranteed state estimation for the introduced family of system noises. Recall that the generic ARMA type models imply the correlated additive noises in the equivalent LMs such that the necessary SIH does not satisfied. This fact implies that the resulting discrete dynamic model contains the non-Gaussian as well as the correlated stochastic noises. In this paper, we use a modified "instrumental variable" method and determine an auxiliary LM that contains the uncorrelated stochastic variables. This instrumental variable based approach makes it possible to apply the robust KF to the auxiliary LM associated with

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the initially given controlled ARMA model. We next study the developed non-standard state estimation methodology in the context of a linear stochastic control system with delays. The robust KF estimation strategy we propose involves a guaranteed state estimation for a consistent discrete approximation of the given delayed stochastic system.

The remainder of this paper is organized as follows: Section 2 contains the main mathematical model of a controlled ARMA process with the non-Gaussian noises. We consider a class of colored system noises characterized by a specific family of the probability distributions. Section 3 includes some basic facts related to the robust KF we developed in [2,4]. Section 4 presents a self-closed solution approach to the robust state estimation for the ARMA processes in the absence of the NH and SIH. We use here the modified instrumental variable approach. In Section 5 we apply the developed robust state estimation procedure to a linear stochastic control system with delays. Section V summarizes our paper.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section we give a formal description of the initial state-observer model. Consider the generic ARMA model of the order (l_a, l_d) (denoted as ARMA (l_a, l_d))

$$\begin{aligned} z_{t+1} = a_0 z_t + a_1 z_{t-1} + \ldots + a_{l_a} z_{t-l_a} + b u_t + \\ d_0 \xi_t + \ldots + d_{l_d} \xi_{t-l_d},\ z_0 \in \mathbb{R}, \end{aligned} \tag{1}$$

in the absence of the classic Gaussian hypothesis for the stochastic variable $\xi_t,\ t \in \mathbb{N}$. Here $l_a,\ l_d \in \mathbb{N}$ and z_0 denotes an initial value of the state variable. The original discrete-time model (1) with the known coefficients describes a controlled dynamics with the control input u_t and $b \in \mathbb{R}$. Assume $u_t \in U$, where $U \subset \mathbb{R}$ is a compact set of admissible control inputs. We next assume that the controlled ARMA model (1) is equipped with the linear observer

$$y_t = c_t z_t + \zeta_t,\ t \in \mathbb{N} \tag{2}$$

with the known coefficients $c_t,\ t \in \mathbb{N}$. We also assume the non-Gaussian character of the stochastic variable $\zeta_t,\ t \in \mathbb{N}$ in (2). The concrete probabilistic characterization of the stochastic variables ξ_t and ζ_t in (1)-(2) is given as follows:

$$\begin{aligned} \xi_t \sim G_\xi \in \mathcal{P}(0, r_1), \\ \zeta_t \sim G_\zeta \in \mathcal{P}(0, r_2), \end{aligned} \tag{3}$$

where $\mathcal{P}(0, r_1)$ and $\mathcal{P}(0, r_2)$ in (3) are classes of probability distribution functions (pdf's) defined as follows

$$\mathcal{P}(0, r_1) := \{G(\cdot)|\ \sum_{t=0}^{\infty} \xi_t G_\xi = 0,\ \sum_{t=0}^{\infty} \xi_t^2 G_\xi \le r_1\},$$

$$\mathcal{P}(0, r_2) := \{G(\cdot)|\ \sum_{t=0}^{\infty} \zeta_t G_\zeta = 0,\ \sum_{t=0}^{\infty} \zeta_t^2 G_\zeta \le r_2\}.$$

Here $r_1 > 0$ and $r_2 > 0$ are known positive numbers. We next assume that the typical stochastic independency hypothesis for variable ξ_t in ARMA (l_a, l_d) model (1) are satisfied:

$$\sum_{t=0}^{\infty} \xi_t \xi_s G_\xi = 0\ \forall s \ne t,\ s, t \in \mathbb{N}.$$

The probabilistic characterization (3) of the uncertainties ξ and ζ in the controlled ARMA (l_a, l_d) model describes a very wide class of possible stochastic variables. Let us note that the possible Gaussian (discrete-time) stochastic processes also belong to the given families $\mathcal{P}(0, r_1)$ and $\mathcal{P}(0, r_2)$. Therefore, the ARMA involved dynamic state-observer model (1)-(2) generalizes the classic ARMA models with the standard Gaussian noises. In parallel to the concept of a "white noise" we call a stochastic variable in (3) a "colored noise". We also refer to [2] for the rigorous mathematical description of the classes $\mathcal{P}(0, r_1)$ and $\mathcal{P}(0, r_2)$ of pdf's.

It is well known that the controlled ARMA (l_a, l_d) model of the type (1) can be rewritten as a linear type state-space system (LM)

$$x_{t+1} = A x_t + B u_t + D w_t \tag{4}$$

We refer to [14,22] for the necessary technical details. Note that

$$x_t := (z_t, z_{t-1}, ..., z_{t-l_a})^T \in \mathbb{R}^{l_a+1},$$

and moreover, the corresponding system and control matrices in (4) can be specified as follows:

$$A := \begin{pmatrix} a_0 & a_1 & ... & a_{l_a-1} & a_{l_a} \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 1 & 0 \end{pmatrix},\ B := \begin{pmatrix} b \\ ... \\ ... \\ 0 \end{pmatrix}.$$

Observe, that $A \in \mathbb{R}^{(l_a+1)\times(l_a+1)}$ and $B \in \mathbb{R}^{(l_a+1)\times 1}$ and the control input u_t in (4) is a scalar for every $t \in \mathbb{N}$. Moreover, we have

$$D := \begin{pmatrix} d_0 & d_1 & ... & d_{l_d-1} & d_{l_d} \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(l_a+1)\times(l_d+1)}.$$

The colored noise $w_t,\ t \in \mathbb{N}$ in (4) is in fact determined by the delayed components of stochastic variables ξ_t from the originally given system (1):

$$w_t := (\xi_t, \xi_{t-1}, ..., \xi_{t-l_d})^T \in \mathbb{R}^{(l_d+1)}.$$

Taking into consideration the non-Gaussian nature of the stochastic variables ξ_t in the originally given controlled ARMA (l_a, l_d) (1), we conclude that

$$w_t \sim G_w \in \mathcal{P}^{(l_d+1)}(0, r_1). \tag{5}$$

The power symbol in (5) is understood here as a Cartesian power. We next deduce

$$\mathcal{P}^{(l_d+1)}(0, r_1) = \{G(\cdot)|\ \sum_{t=0}^{\infty} w_t G_w = 0,\ \sum_{t=0}^{\infty} w_t w_t^T G_w \le R\},$$

where $R \in \mathbb{R}^{(l_d+1)\times(l_d+1)}$ is a limiting covariance matrix for the newly defined stochastic vector w_t in the state equation (4). Using the above definition of the class $\mathcal{P}(0, r_1)$, we finally obtain

$$\mathcal{R}_1 := \mathrm{diag}(r_1).$$

Note that the probabilistic characterization (5) for the stochastic variable w_t is a direct consequence of the non-Gaussian assumption (3). However, the timely different uncertainties in model (4) constitute the correlated stochastic variables. Evidently, w_t and w_s are correlated for all indexes $t,\ s \in \mathbb{N}$ such that

$$t + (l_d + 1) > s > t.$$

For the above indexes $t,\ s$ the covariance $\text{Cov}(w_t, w_s)$ between w_t and w_s is non-zero and can be easily calculated:

$$\text{Cov}(w_t, w_s) := \sum_{t=0}^{\infty} w_t w_s^T G_w =$$
$$\begin{pmatrix} 0 & 0 & \ldots & 0 & \sigma^2_{t-s} & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \sigma^2_{t-s-1} & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \sigma^2_{t-l_d} \end{pmatrix} \neq 0.$$

The above important fact, namely, the evident stochastic dependence (correlation) of the noises w_t and w_s for

$$t + (l_d + 1) > s > t$$

is a direct consequence of the specific definition of the uncertainty vector w_t in LM (4) in the auxiliary LM (4).

The observation process (2) associated with the originally given ARMA (l_a, l_d) model (1) can also be rewritten

$$y_t = Hx_t + \zeta_t,\ t \in \mathbb{N} \tag{6}$$

with the observer matrix

$$H := (\ c_t\ 0\ \ldots\ 0\ ) \in \mathbb{R}^{1\times(l_a+1)}.$$

Let us recall that the distribution of ζ_t in (6) is given by (3).

We next consider the resulting state-observer LM given by (4) and(6). Note that this model involves the correlated, non-Gaussian system and observer noises such that the generic NH and SIH are not valid. This conclusion makes it impossible a direct application of the classic KF and extended to the obtained LM (4)-(6).

3. FOUNDATIONS OF THE ROBUST KALMAN FILTER

We now present the main mathematical facts related to the robust Kalman Filter introduced in [2]. This advanced version of the KF will next be applied to the controlled ARMA model in the presence of observations, namely, to (4)-(6) for $t = 1, ..., N \in \mathbb{N}$.

Consider the main state-observer model (4)-(6) and assume for the moment that the SIH is formally satisfied. The covariance matrix $\text{Cov}[(x_t^T, y_t^T)^T]$ of the joint vector $(x_t^T, y_t^T)^T$ can be expressed as follows:

$$\text{Cov}[(x_t^T, y_t^T)^T] = \begin{bmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1}H^T \\ H\Sigma_{t|t-1} & H\Sigma_{t|t-1}H^T + \text{Cov}[\zeta_t] \end{bmatrix}.$$

Here

$$\Sigma_{t|t-1} := \text{Cov}[x_t | y^*_{t-1}]$$

is a (conditional) covariance matrix of the state x_t and $\text{Cov}[\zeta_t]$ is a covariance matrix of the vector ζ_t. By y_t^* we denote here the following "observation history" vector:

$$y_t^* := (y_1^T, ..., y_t^T)^T.$$

Let now

$$x_{t|t} := E(x_t | y_t^*)$$

be the a posteriori state estimation of the state vector x_t in (4) and

$$x_{t|t-1} := E(x_t | y^*_{t-1})$$

be a consistent KF state prediction. The "worst-case" robust KF from [2] can now be expressed as follows:

<u>state estimation step</u>

$$\begin{aligned} & x_{0|0} = x_0, \\ & x_{t|t-1} = Ax_{t-1|t-1} + Bu_t, \\ & x_{t|t} = x_{t|t-1} + \psi^{opt}(\delta y_t)\delta y_t,\ t = 1, ..., N, \\ & \psi^{opt}(\delta y_t) := S_{12}S_{22}^{-1}, \end{aligned} \tag{7}$$

and

<u>moments estimation step</u>

$$\begin{aligned} & \Sigma_{0|0} = 0, \\ & \Sigma_{t|t-1} = A\Sigma_{t-1|t-1}A^T + D\text{Cov}[\xi_t]D^T\ \forall t = 1, ..., N, \\ & \Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}\mathcal{Q}_t^T - \mathcal{Q}_t\Sigma_{t|t-1} + \\ & \mathcal{Q}_t\Sigma_{t|t-1}\mathcal{Q}_t^T + S_{12}S_{22}^{-1}\text{Cov}[\zeta_t]S_{22}^{-1}S_{12}^T. \end{aligned} \tag{8}$$

Here

$$\delta y_t := y_t - E(y_t | y^*_{t-1}) = H\delta x_t + \zeta_t$$

with

$$\delta x_t := x_t - x_{t|t-1}.$$

Moreover, $\Sigma_{t|t} := \text{Cov}[x_t]$ is a covariance matrix of the state vector x_t in (4). Note that matrices S_{12} and S_{22} are defined as proposed in [2].

The obtained relations (7)-(8) also include the additional notation:

$$\mathcal{Q}_t := S_{12}S_{22}^{-1}H,$$

and

$$\mathcal{R}_2 := \text{diag}(r_2, 0).$$

The robust KF (7)-(8) guarantees a "worst-case" robust state estimation and constitutes a specific robustness framework for the wide classes of pdf's under consideration. The main optimization problem that implies (7)-(8) has the following formal structure:

$$\begin{aligned} & \text{minimize} \sup_{\mathcal{P}(0,S)} E||\delta x_t - \psi(\delta y_t)\delta y_t||^2 \\ & t = 1, ..., N \in \mathbb{N} \\ & \text{subject to } \psi(\cdot) \in \mathbb{L}_2. \end{aligned} \tag{9}$$

By $\mathbb{L}_2$ we denote here the classic Lebesgue space of all square integrable functions. Note that the main minimization problem (9) is considered on a finite time interval determined by a (large) natural number N. By

$$S := \text{diag}(\mathcal{R}_1, \mathcal{R}_2)$$

we denote here the block-diagonal matrix S, namely, the upper bound of the covariance matrix $\text{Cov}[(\xi_t^T, \zeta_t^T)^T]$ of the joint noise vector (ξ_t^T, ζ_t^T) in the state-observer model under consideration. A further useful generalization of this approach can be found in [2].

Let us note that the presented generalization (7)-(8) of the classic KF was developed in the absence of the NH. However, it is based on the fundamental SIH. As we can see the equivalent LM (4) additionally involves the correlated additive system noises w_t. This fact implies some further necessary formal transformations of the state space model in order to guarantee the necessary SIH. We next propose such a mathematically rigorous transformation and consider the celebrated "instrumental variable" approach.

4. ROBUST STATE ESTIMATION IN THE PRESENCE OF CORRELATED NON-GAUSSIAN NOISES

As shown in the previous section, the specific structure of the non-Gaussian stochastic noise w_t in the resulting state equation (4) evidently implies the strong correlation between w_t and w_s for all indexes $t, s \in \mathbb{N}$ that satisfy

$$t + (l_d + 1) > s > t.$$

On the other hand, the "worst-case" robust KF from Section 3 was developed under the main assumption on the timely uncorrelated (non-autocorrelated) noises. In this section we apply the celebrated "instrumental variable" method to (4) and construct an auxiliary (with respect to (4)-(6)) uncorrelated state-observer mode. Next one can directly apply the proposed ("worst-case") robust KF to this auxiliary state-observer model with the resulting timely uncorrelated stochastic noises.

Let us start by introduction to the "instrumental variable" method for a state-observer model. We refer to [22] for the further mathematical details. For the variable z_t from (1) consider the so-called "lag-operator"

$$L(z_t) := z_{t-1}$$

associated with the controlled ARMA (l_a, l_d) (1). It is common knowledge that the above operator L make it possible to represent the given ARMA (l_a, l_d) process in the algebraic form:

$$z_t = \frac{d(L)}{a(L)}\xi_t,$$

where

$$a(L) := 1 - a_0 L - ... - a_{l_a} L^{l_a},$$

and

$$d(L) := 1 - d_1 L - ... - d_{l_d} L^{l_d}$$

are operator-polynomials with respect to the operator L. We now are ready to define formally the general "instrumental variable" for ARMA (l_a, l_d) (1)

$$\tilde{z}_t := z_{t-l_d} \equiv L^{l_d}(z_t). \tag{10}$$

Recall that the basic definition (10) of the "instrumental variable" $\tilde{z}_t$ involves a specific "forming filter" (see e.g., [22]). As we can see, the above concept of the "instrumental variable" concept makes it possible to obtain the state dynamics with the stochastically independent variables. From (10), we deduce

$$\mathrm{Cov}(z_{t+1}, \tilde{z}_t) = \mathrm{Cov}(z_{t+1}, z_{t-l_d}) = 0 \tag{11}$$

and the basic SIH in the context of the instrumental variable (10) is reestablished.

The above fact, namely, the obtained stochastic independency of z_{t+1} and $\tilde{z}_t$ (expressed by (11)) motivates the re-definition of the ARMA involved LM (4)

$$\begin{aligned}
&x_{t+1} = A^{l_d+1}x_{t-l_d-1} + A^{l_d}Bu_{t-l_d-1} + A^{l_d}Dw_{t-l_d} + ...+ \\
&ABu_{t-1} + ADw_{t-1} + Bu_t + Dw_t = \\
&A^{l_d+1}x_{t-l_d-1} + (A^{l_d}Bu_{t-l_d-1} + ... + ABu_{t-1} + Bu_t)+ \\
&(A^{l_d}Dw_{t-l_d} + ... + ADw_{t-1} + Dw_t).
\end{aligned} \tag{12}$$

Evidently, the dynamics (12) constitutes an equivalent rewriting of the original LM (4). We now redefine the originally given discrete time and introduce the new time-step

$$\tau := l_d + 2.$$

Then the discrete-time dynamic system (12) has the following equivalent LM representation:

$$x_{t+\tau} = A^{l_d+1}x_t + \tilde{B}u_t + \epsilon_t \tag{13}$$

The control input in (12) and (13) is given by

$$\tilde{B}u_t := (A^{l_d}Bu_{t-l_d-1} + ... + ABu_{t-1} + Bu_t),$$

and the stochastic system noise ϵ_t has the following expression:

$$\epsilon_t := (A^{l_d}Dw_{t-l_d-1} + ... + ADw_{t-1} + Dw_t).$$

Evidently, the new stochastic variables ϵ_t in the resulting LM (13) satisfy the basic SIH. Moreover, the corresponding probability distributions of ϵ_t belong to the previously defined pdf's functional class $\mathcal{P}^{(l_d+1)}(0, r_1)$. This fact motivates a direct application of the proposed robust KF (7)-(8) to the delayed LM (13) with uncorrelated noises. The complete recursive robust KF for LM (13) can now be written as follows:

$$\begin{aligned}
&\underline{\text{state estimation step}} \\
&x_{0|0} = x_0, \\
&x_{t|t-\tau} = A^{l_d+1}x_{t-\tau|t-\tau} + \tilde{B}u_t, \\
&x_{t|t} = x_{t|t-\tau} + \psi^{opt}(\delta y_t)\delta y_t,\ t = \tau, ..., N + \tau, \\
&\psi^{opt}(\delta y_t) := S_{12}S_{22}^{-1}, \\
&\underline{\text{moments estimation step}} \\
&\Sigma_{0|0} = 0, \\
&\Sigma_{t|t-\tau} = A^{l_d+1}\Sigma_{t-\tau|t-\tau}(A^{l_d+1})^T + D\mathrm{Cov}[\xi_t]D^T \\
&\forall t = \tau, ..., N + \tau, \\
&\Sigma_{t|t} = \Sigma_{t|t-\tau} - \Sigma_{t|t-\tau}\mathcal{Q}_t^T - \mathcal{Q}_t\Sigma_{t|t-\tau}+ \\
&\mathcal{Q}_t\Sigma_{t|t-\tau}\mathcal{Q}_t^T + S_{12}S_{22}^{-1}\mathrm{Cov}[\zeta_t]S_{22}^{-1}S_{12}^T.
\end{aligned} \tag{14}$$

Here we re-define the components of (7)-(8) using the new step-size τ in (13)

$$\begin{aligned}
&\delta y_t := y_t - E(y_t|y^*_{t-\tau}) = H\delta x_t + \zeta_t, \\
&\delta x_t := x_t - x_{t|t-\tau}.
\end{aligned}$$

The resulting robust KF (14) is now written in conformity with the basic SIH for the delayed LM (13).

Using the closed form of the instrumental variable robust KF (13), we now propose a state estimation based control design for the resulting ARMA model (13). Let $g : \mathbb{R}^{l_a+1} \to U$ be a Lipschitz function and

$$\begin{aligned}
&u_{t-1} = ... = u_{t-l_d-1} = 0, \\
&u_t := g(x_{t|t}).
\end{aligned}$$

In a simple (but useful) case of the proportional control scheme we have

$$\begin{aligned}
&u_{t-1} = ... = u_{t-l_d-1} = 0, \\
&u_t := k^T x_{t|t},
\end{aligned} \tag{15}$$

where $k \in \mathbb{R}^{l_a+1}$ is a gain vector. The corresponding closed-loop ARMA model (13) involving the proportional state estimation based control design (15) has the following form

$$x_{t+\tau} = A^{l_d+1}x_t + Bk^T x_{t|t} + \epsilon_\tau, \tag{16}$$

Note that in this case the state prediction step in (14) can be rewritten as follows:

$$x_{t+\tau|t} = (A^{l_d+1} + Bk^T)x_{t|t}. \tag{17}$$

Finally note that the state estimation based control strategy (15)-(17) constitutes an adequate control design approach to the linear dynamic systems with stochastic uncertainties.

5. APPLICATION OF THE ROBUST STATE ESTIMATIONS TO THE LINEAR STOCHASTIC CONTROL SYSTEM WITH DELAYS

In this section we apply the developed robust KF methodology to the state estimation of the following delayed integral dynamics

$$X(t) = \int_0^\infty X(t-s)\mu(ds)+bu(t)+\int_0^t \theta(t-s)dL(s). \tag{18}$$

Here μ is a probability measure on $[0, \infty)$, $L(\cdot)$ is a Levy stochastic process with $E(L(t)) = 0$ and $\theta(\cdot)$ is a scalar measurable function. The control input $u(\cdot)$ and $b \in \mathbb{R}$ are defined similar to (1). We assume that $u(t) \in U \subset \mathbb{R}$ and that U is a compact.

The resulting stochastic process $X(\cdot)$ can often be considered as a suitable approximation of an initially given controlled Stochastic Delayed Differential Equation (SDDE). The delayed stochastic models (18) and the original SDDEs constitute a useful modelling framework for many real-world dynamic systems with random uncertainties. We refer to [4] for the delayed robot dynamics and to [9,12,27] for the general theory of the delayed systems and SDDEs. For the basic model (18) we additionally assume that

$$E[L^2(t)] \leq \infty,$$

where $r_1 \in \mathbb{R}_+$ is defined in (3).

It is well known that a probability measure μ can be approximated by a convex combination of the Dirac measures (see e.g., (19)). Let $\delta_l,\ l = 0, ..., l_a$ be Dirac measures. Assume that

$$\mu(ds) = \sum_{l=0}^{l_a} a_l \delta_l(ds), \tag{19}$$

where

$$\sum_{l=0}^{l_a} a_l = 1,\ a_l \geq 0\ \forall\ l = 0, ..., l_a. \tag{20}$$

Here l_a is a sufficiently big natural number. Moreover, let

$$\begin{aligned} \theta(s) = d_0\chi_{[0,1)}(s) + d_1\chi_{[2,3)}(s) + \\ ... + d_{l_d}\chi_{[2l_d, 2l_d+1]}(s) \end{aligned} \tag{21}$$

for a given natural number l_d and coefficients $\{d_1, ..., d_{l_d-2}\}$. By $\chi_I(\cdot)$ we denote here the usual characteristic function of an interval I. We now consider the dynamic equation (18) taking into consideration the specific approximation (19) of μ. Moreover, the parameter function $\theta(\cdot)$ in (18) is assumed to be given by (21). For the selected parameters the general dynamic equation (18) can be rewritten as follows:

$$\begin{aligned} X(t) = \sum_{l=0}^{l_a} a_l X(t-l) + bu(t) + \\ \sum_{l=0}^{l_d} d_l(L(t-2l) - L(t-(2l+1))) = \\ \sum_{l=0}^{l_a} a_l X(t-l) + bu(t) + \sum_{l=0}^{l_d} d_l Y_{t-l}, \end{aligned} \tag{22}$$

where

$$Y(t-l) := L(t-2l) - L(t-2l-1).$$

Evidently,

$$Y(t-l),\ l = 0, ..., l_d$$

in (22) are stochastically independent variables. The resulting expression (22) evidently constitutes a special case of the controlled ARMA model (1) with the barycentric coefficients $a_l,\ l = 0, ..., l_a$ determined in (19). Let us additionally assume that

$$E[(L(t-2l) - L(t-(2l+1)))^2] \leq r_1.$$

In that case the stochastic variables $Y(t-l)$ in the resulting ARMA model (22) have a pdf from the generic class $\mathcal{P}(0, r_1)$ introduced in Section 2. We now can use the instrumental variable based robust KF (14) for the state estimation in the specific ARMA model (22).

Using the proposed instrumental variable based approach, we can rewrite the obtained ARMA model (22) in the form of the equivalent LM (13). The state estimation based control design scheme (15)-(17) can now be applied the resulting equivalent LM associated with the specific ARMA model (22).

6. CONCLUDING REMARKS

In this contribution, we proposed a new conceptual KF based solution approach to the robust state estimation for the general controlled ARMA processes with the specific colored noises. We consider a wide family of the probability distributions associated with the system noises and replace the classic NH in the conventional KF by an alternative assumption involving a class of non-Gaussian systems disturbances. The non-standard noise probability distributions under consideration possess the bounded second moments. The minimax based KF approach we propose finally leads to a numerically consistent recursive computational scheme and makes it possible a self-closed numerical treatment of the optimal state estimation problem for the ARMA processes under consideration.

Since the consideration of a general ARMA processes implies the violation of the basic SIH, we consider an further necessary modification of the classic method of "instrumental variable" in order to reestablish the necessary SIH in an equivalent LM. This approach makes it possible to use the developed robust KF also in the case of the ARMA processes with the colored noises. The resulting robust feedback-type control design for the originally given ARMA model can now involve the obtained specific state estimation.

Analytic approaches we propose in this paper are developed in the context of a conceptually new approach to the robust state estimation in the controlled ARMA models.

Our paper presents some necessary conceptual aspects of the newly developed robust state estimation methodology for LMs with the colored noises. The theoretical solution approaches we elaborated need the further comprehensively studies and numerical simulations. Note that the proposed robust state estimation methodology can also incorporate the heavy-tailed probability distributions associated with the corresponding outliers involved system noises. The developed robust KF based state estimation in general controlled ARMA models can be used in several real-world data driven modelling frameworks for the real-world dynamic systems. It provides a mathematically rigorous optimization based methodology and extends the conventional approaches to the state estimation and state based control of the ARMA processes.

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