

# Line bundles on Coulomb branches

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This is the third companion paper of [Part II]. When a gauge theory has a flavor symmetry group, we construct a partial resolution of the Coulomb branch as a variant of the definition. We identify the partial resolution with a partial resolution of a generalized slice in the affine Grassmannian, Hilbert scheme of points, and resolved Cherkis bow variety for a quiver gauge theory of type  $ADE$  or affine type  $A$ .

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## 1. Introduction

Let  $G$  be a complex reductive group and  $\mathbf{M}$  its symplectic representation of a form  $\mathbf{N} \oplus \mathbf{N}^*$ . ( $\mathbf{N}$  will be fixed hereafter.) In [Nak16, Part II] we gave a mathematically rigorous definition of the Coulomb branch of a  $3d \mathcal{N} = 4$  gauge theory associated with  $(G, \mathbf{M})$  as follows. We introduce an infinite

dimensional variety  $\mathcal{R} = \mathcal{R}_{G, \mathbf{N}}$  (the variety of triples), and define a convolution product on its  $G_{\mathcal{O}} = G[[z]]$ -equivariant homology  $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ , which is commutative. Then we define the Coulomb branch  $\mathcal{M}_C \equiv \mathcal{M}_C(G, \mathbf{N})$  as the spectrum of  $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ . It is an affine algebraic variety.

Suppose that we have a flavor symmetry, i.e.  $\mathbf{N}$  is a representation of a larger group  $\tilde{G}$  containing  $G$  as a normal subgroup. We further assume  $G_F := \tilde{G}/G$  is a torus. Then we can consider the Coulomb branch  $\mathcal{M}_C(\tilde{G}, \mathbf{N})$  for the larger group  $\tilde{G}$ . We showed that the original  $\mathcal{M}_C(G, \mathbf{N})$  is the Hamiltonian reduction  $\mathcal{M}_C(\tilde{G}, \mathbf{N}) // G_F^{\vee}$  of  $\mathcal{M}_C(\tilde{G}, \mathbf{N})$  by the dual torus  $G_F^{\vee}$ , see [Part II, Proposition 3.18]. See [Nak16, §5] for a motivation of this statement, and references in physics literature. Since  $\mathcal{M}_C(G, \mathbf{N})$  is a hamiltonian reduction by a torus, one can take the reduction at a different value of the moment map, or can consider a GIT quotient  $\mathcal{M}_C^{\varkappa}(G, \mathbf{N})$  with respect to a stability condition, which is a character  $\varkappa: G_F^{\vee} \rightarrow \mathbb{C}^{\times}$ . The former gives a deformation of  $\mathcal{M}_C(G, \mathbf{N})$  parametrized by  $\text{Spec } H_{G_F}^*(\text{pt})$ . The latter gives a quasi-projective variety  $\mathcal{M}_C^{\varkappa}(G, \mathbf{N})$  equipped with a projective morphism  $\pi: \mathcal{M}_C^{\varkappa}(G, \mathbf{N}) \rightarrow \mathcal{M}_C(G, \mathbf{N})$ . This is birational. See Remark 1.1 below.

We could understand this construction as follows. (See [Part II, §3(ix)].) Let us denote the variety of triples for the larger group  $(\tilde{G}, \mathbf{N})$  by  $\tilde{\mathcal{R}}$ . Let  $\tilde{\pi}$  be the natural projection  $\tilde{\mathcal{R}} \rightarrow \text{Gr}_{G_F}$ . We identify  $\text{Gr}_{G_F}$  with the coweight lattice of  $G_F$ , which is the weight lattice of  $G_F^{\vee}$ . For a coweight  $\varkappa$  of  $G_F$ , the inverse image  $\tilde{\pi}^{-1}(\varkappa)$  is denoted by  $\tilde{\mathcal{R}}^{\varkappa}$ . (In [Part II, §3(ix)] a coweight was denoted by  $\lambda_F$ .) Note that  $\tilde{\mathcal{R}}^0$  is nothing but the original variety of triples  $\mathcal{R}$ . The convolution product defines a multiplication

$$H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}^{\varkappa}) \otimes_{\mathbb{C}} H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}^{\varkappa'}) \rightarrow H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}^{\varkappa+\varkappa'}).$$

In particular  $H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}^{\varkappa})$  is an  $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ -module, hence defines a sheaf on  $\mathcal{M}_C(G, \mathbf{N}) = \text{Spec}(H_*^{G_{\mathcal{O}}}(\mathcal{R}))$ . We only take coweights in  $\mathbb{Z}_{\geq 0}\varkappa$  for a fixed  $\varkappa$ , and consider  $\text{Proj}(\bigoplus_{n \geq 0} H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}^{n\varkappa}))$ . This is nothing but the GIT quotient  $\mathcal{M}_C^{\varkappa}(G, \mathbf{N})$ . It is a quasi projective variety, equipped with a natural projective morphism  $\pi: \mathcal{M}_C^{\varkappa}(G, \mathbf{N}) \rightarrow \mathcal{M}_C(G, \mathbf{N})$ . We have  $H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}^{\varkappa}) = \Gamma(\mathcal{M}_C(G, \mathbf{N}), \pi_* \mathcal{O}_{\mathcal{M}_C^{\varkappa}(G, \mathbf{N})}(1))$ .

In this paper, we study  $\mathcal{M}_C^{\varkappa}(G, \mathbf{N})$  for a framed quiver gauge theory of type *ADE* or affine *A*. The original Coulomb branch  $\mathcal{M}_C(G, \mathbf{N})$  was identified with a generalized slice in the affine Grassmannian [Quiver], and a Cherkis bow variety [NT17] respectively. In both cases the variety has a natural partial resolution (actual resolution for type *A* or affine type *A*), and we identify it with the GIT quotient.

The paper is organized as follows. In §2 we show that the multiplication on  $\bigoplus H_*^{G^o}(\tilde{\mathcal{R}}^{n\kappa})$  is equal to one given by the tensor product of line bundles for a framed quiver gauge theory of type  $A_1$ . This case was studied in detail in [Ring, §4(i)]~[Ring, §4(iii)], and this section is its supplement. In §3 we show that the determinant line bundle on the Hilbert scheme of points in  $\mathbb{A}^2$  arises in our construction. In §4 we study the Coulomb branch of a framed quiver gauge theory of affine type  $A$  and identify our construction of a partial resolution with a bow variety with an appropriate stability condition. In §5 we study the Coulomb branch of a framed quiver gauge theory of type  $ADE$  and identify our construction of a partial resolution with a convolution diagram over a generalized slice in the affine Grassmannian.

**Remark 1.1.** Let us show that  $\pi$  is birational. By [Part II, §5(iv)] we can replace the representation  $\mathbf{N}$  by 0. Thus we need to compare  $\mathcal{M}_C(\tilde{G}, 0) //_{\kappa} G_F^{\vee}$  and  $\mathcal{M}_C(G, 0)$ . Note that we have a finite covering  $G'_F$  of  $G_F$  such that the corresponding covering of  $\tilde{G}$  becomes the product  $G \times G'_F$ . Moreover we can replace  $\kappa$  by its positive power, hence we may assume it lifts to  $G'_F$ . Then we get  $\mathcal{M}_C(G \times G'_F, 0) //_{\kappa} G_F^{\vee} = \mathcal{M}_C(G, 0) \times \mathcal{M}_C(G'_F, 0) //_{\kappa} G_F^{\vee}$ , which is obviously  $\mathcal{M}_C(G, 0)$ .

## Notation

We basically follow the notation in [Part II], [Quiver] and [Ring].

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## 2. Multiplication morphism

This section is a supplement to [Ring, §4(i)]~[Ring, §4(iii)].

Let  $N$  be an integer greater than 1. Let  $\mathcal{S}_N$  denote the hypersurface  $ZY = W^N$  in  $\mathbb{A}^3$ ,  $\pi: \mathcal{S}_N \rightarrow \mathcal{S}_N$  its minimal resolution, and  $\mathcal{S}_N^\circ := \mathcal{S}_N \setminus \{0\}$ . We change  $z, y, w$  to capital letters to avoid a confusion later. A weight  $\lambda$  of  $\mathrm{SL}(N)$  defines a line bundle  $\mathcal{L}_\lambda$  over  $\tilde{\mathcal{S}}_N$ . Let  $\mathcal{F}_\lambda$  denote the torsion free sheaf  $\pi_*\mathcal{L}_\lambda$  on  $\mathcal{S}_N$  for dominant  $\lambda$ . (To be consistent with other parts of this paper, we should denote a weight by  $\varkappa$ , but we keep notation in [Ring, §4].) Let us recall the notation briefly. We identify  $\mathcal{S}_N$  with  $\mathbb{A}^2//(\mathbb{Z}/N\mathbb{Z})$ , where  $\zeta \in \mathbb{Z}/N\mathbb{Z}$  acts on  $\mathbb{A}^2$  by  $\zeta \cdot (u, v) = (\zeta u, \zeta^{-1}v)$ . We have  $W = uv, Z = u^N, Y = v^N$ . The line bundle  $\mathcal{L}_{\omega_i}$  for a fundamental root  $\omega_i$  is defined so that  $\Gamma(\tilde{\mathcal{S}}_N, \mathcal{L}_{\omega_i}) = \Gamma(\mathcal{S}_N, \mathcal{F}_{\omega_i})$  is the space of the semi-invariants  $\mathbb{C}[\mathbb{A}^2]^{\chi_i}$  with  $\chi_i(\zeta) = \zeta^i$  ( $i = 1, \dots, N - 1$ ). If we identify a weight  $\lambda$  of  $\mathrm{SL}(N)$  with  $(\lambda_1 \geq \dots \geq \lambda_N)$  up to simultaneous shifts of all  $\lambda_i$ , we have  $\mathcal{L}_\lambda = \bigotimes \mathcal{L}_{\omega_i}^{\otimes (\lambda_i - \lambda_{i+1})}$ .

We realize  $\mathcal{S}_N$  and  $\tilde{\mathcal{S}}_N$  as Coulomb branches as follows:  $V$  with  $\dim V = 1, W$  with  $\dim W = N, G = \mathrm{GL}(V) = \mathbb{C}^\times, \tilde{G} = (\mathrm{GL}(V) \times T(W))/\mathbb{C}^\times$ , where  $T(W)$  is a maximal torus of  $\mathrm{GL}(W)$  consisting of diagonal matrices,  $\mathbb{C}^\times$  is the diagonal scalar subgroup,  $G_F = T(W)/\mathbb{C}^\times$ , and  $\mathbf{N} = \mathrm{Hom}(W, V)$ . Then  $\mathcal{M}_C(G, \mathbf{N})$  is  $\mathcal{S}_N$  and  $\Gamma(\mathcal{S}_N, \mathcal{F}_\lambda) \cong H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda)$ . Note that  $H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda)$  is denoted by  $i_\lambda^! \mathcal{A}^{\mathrm{for}}$  in [Ring, §4], as it is a costalk of a ring object  $\mathcal{A}^{\mathrm{for}}$  at  $\lambda$ .

We choose isomorphisms  $\Gamma(\mathcal{S}_N, \mathcal{F}_\lambda) \xrightarrow{\sim} H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda)$  for any  $\lambda$  (defined uniquely up to multiplication by a scalar).

**Lemma 2.1.** *The multiplication morphism  $\Gamma(\mathcal{S}_N, \mathcal{F}_\lambda) \otimes \Gamma(\mathcal{S}_N, \mathcal{F}_\mu) \rightarrow \Gamma(\mathcal{S}_N, \mathcal{F}_{\lambda+\mu})$  (resp.  $H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda) \otimes H_*^{G \circ}(\tilde{\mathcal{R}}^\mu) \rightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^{\lambda+\mu})$ ) is surjective for any dominant  $\lambda, \mu$ .*

*Proof.* It suffices to consider the case  $\mu = \omega_n = (1, \dots, 1, 0, \dots, 0)$  ( $n$  1's) for  $1 \leq n \leq N - 1$ . Recall that the  $\mathbb{C}^\times \times \mathbb{C}^\times$ -character of  $\Gamma(\mathcal{S}_N, \mathcal{F}_{\lambda+\mu})$  given by [Ring, Lemma 4.2] is multiplicity free. So it suffices to represent each summand  $x^{\sum_{i=1}^N ((\lambda+\omega_n)_i - m)} t^{\sum_{i=1}^N |(\lambda+\omega_n)_i - m|}$  as a product of two summands  $x^{\sum_{i=1}^N (\lambda_i - m')} t^{\sum_{i=1}^N |\lambda_i - m'|}$  and  $x^{\sum_{i=1}^N ((\omega_n)_i - m'')} t^{\sum_{i=1}^N |(\omega_n)_i - m''|}$ . Now if  $m \geq \lambda_n + 1$ , we take  $m' = m - 1, m'' = 1$ , and if  $m \leq \lambda_n$ , we take  $m' = m, m'' = 0$ . The same argument works for  $H_*^{G \circ}(\tilde{\mathcal{R}}^?)$  due to the monopole formula. Indeed, the morphism  $H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda) \otimes H_*^{G \circ}(\tilde{\mathcal{R}}^\mu) \rightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^{\lambda+\mu})$  respects the bi-grading. And the induced morphism  $H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda) \otimes_{H_*^{G \circ}(\mathcal{R})} H_*^{G \circ}(\tilde{\mathcal{R}}^\mu) \rightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^{\lambda+\mu})$  is an isomorphism generically due to the localization theorem.  $\square$

**Lemma 2.2.** *The diagram*

$$\begin{array}{ccc}
 \Gamma(\mathcal{S}_N, \mathcal{F}_\lambda) \otimes_{\mathbb{C}[\mathcal{S}_N]} \Gamma(\mathcal{S}_N, \mathcal{F}_\mu) & \xrightarrow{\sim} & H_*^{G_\circ}(\tilde{\mathcal{R}}^\lambda) \otimes_{H_*^{G_\circ}(\mathcal{R})} H_*^{G_\circ}(\tilde{\mathcal{R}}^\mu) \\
 \downarrow & & \downarrow \\
 \Gamma(\mathcal{S}_N, \mathcal{F}_{\lambda+\mu}) & \xrightarrow{\sim} & H_*^{G_\circ}(\tilde{\mathcal{R}}^{\lambda+\mu})
 \end{array}$$

*commutes up to multiplication by a scalar for any dominant  $\lambda, \mu$ .*

*Proof.* The kernels of both vertical morphisms coincide with the torsion in the upper row. Thus it suffices to check the claim generically. But generically all the four modules in question are free of rank 1. So it suffices to check the commutativity for a single  $\mathbb{C}^\times \times \mathbb{C}^\times$ -eigensection of  $\Gamma(\mathcal{S}_N, \mathcal{F}_\lambda) \otimes_{\mathbb{C}[\mathcal{S}_N]} \Gamma(\mathcal{S}_N, \mathcal{F}_\mu)$ , and this follows from the multiplicity free property of  $H_*^{G_\circ}(\tilde{\mathcal{R}}^{\lambda+\mu})$ .  $\square$

**Remark 2.3.** At the end of [Ring, §4(iii)], we wrote down an explicit isomorphism  $\Gamma(\mathcal{S}_N, \mathcal{F}_\lambda) \xrightarrow{\sim} H_*^{G_\circ}(\tilde{\mathcal{R}}^\lambda)$  when  $\lambda$  is a fundamental coweight  $\omega_i$  as

$$r^{(m, \omega_i)} \mapsto \begin{cases} v^{N-i} Y^{m-1} & \text{if } m > 0, \\ u^i Z^{-m} & \text{if } m \leq 0, \end{cases}$$

where  $r^{(m, \omega_i)}$  (denoted by  $r^m$  in [Ring, §4(iii)]) is the fundamental cycle of the fiber of  $\tilde{\mathcal{R}} \rightarrow \text{Gr}_{\tilde{G}}$  over  $(m, \underbrace{1, \dots, 1}_{i \text{ times}}, \underbrace{0, \dots, 0}_{N-i \text{ times}})$ . Thanks to Lemma 2.2 we generalize it for general dominant  $\lambda$  by products. Then Lemma 2.2 holds without ambiguity of a scalar under the generalized isomorphism. Namely it is characterized by  $\bigotimes_{i=1}^{N-1} (r^{(1, \omega_i)})^{\otimes (\lambda_i - \lambda_{i+1})} \mapsto \bigotimes_{i=1}^{N-1} (v^{N-i})^{\otimes (\lambda_i - \lambda_{i+1})}$ . By [Part II, §4] the left hand side is nothing but the fundamental class over  $(\lambda_1 - \lambda_N, \lambda_1 - \lambda_N, \lambda_2 - \lambda_N, \dots, \lambda_{N-1} - \lambda_N, 0) = (\lambda_1, \lambda_1, \lambda_2, \dots, \lambda_{N-1}, \lambda_N)$  (the first entry corresponds to  $\text{GL}(V)$  of  $\tilde{G}$  and others to  $T(W)$ ).

**Remark 2.4.** We have another way<sup>1</sup> to understand  $\mathcal{M}_C^z(G, \mathbf{N})$ . We identify  $\tilde{G} = \mathbb{C}^\times \times (\mathbb{C}^\times)^N / \mathbb{C}^\times$  with  $(\mathbb{C}^\times)^N$  by  $(r, r_1, \dots, r_N) \bmod \mathbb{C}^\times \mapsto (r_1/r, \dots, r_N/r)$ . The projection  $\tilde{G} \rightarrow G_F$  is just the quotient by the diagonal subgroup  $\mathbb{C}^\times$ . Then  $\mathbf{N} \cong \mathbb{C}^N$  is just the product of  $N$  copies of the dual of the standard representation of  $\mathbb{C}^\times$ , hence the Coulomb branch  $\mathcal{M}_C(\tilde{G}, \mathbf{N}) \cong \mathbb{C}^{2N}$ . The action of  $\pi_1(\tilde{G})^\wedge$  is the  $(\mathbb{C}^\times)^N$ -action on  $\mathbb{C}^{2N}$  given

<sup>1</sup>H.N. thanks Alexei Oblomkov for motivating him to considering this approach.

by  $(s_1, \dots, s_N) \cdot (x_1, y_1, \dots, x_N, y_N) = (s_1 x_1, s_1^{-1} y_1, \dots, s_N x_N, s_N^{-1} y_N)$ . See [Part II, §4]. We note that

$$(\mathbb{C}^\times)^{N-1} \cong \pi_1(G_F)^\wedge \rightarrow \pi_1(\tilde{G})^\wedge \cong (\mathbb{C}^\times)^N$$

is given by  $(t_1, \dots, t_{N-1}) \mapsto (t_1, t_2/t_1, t_3/t_2, \dots, t_{N-1}/t_{N-2}, 1/t_{N-1})$ . Hence  $\mathcal{M}_C(G, \mathbf{N})$  is the hamiltonian reduction of  $\mathbb{C}^{2N}$  by the action

$$(t_1 x_1, t_1^{-1} y_1, t_2/t_1 x_2, t_1/t_2 y_2, \dots, t_{N-1}^{-1} x_N, t_{N-1} y_N).$$

This is nothing but a quiver variety of type  $A_{N-1}$  with dimension vectors  $\mathbf{v} = (1, \dots, 1)$ ,  $\mathbf{w} = (1, 0, \dots, 0, 1)$ , which is known to be  $\mathcal{S}_N$ . It is also known that the GIT quotient gives a minimal resolution of  $\mathcal{S}_N$  such that the tautological line bundle for the  $i$ -th  $\mathbb{C}^\times$  is identified with  $\mathcal{L}_{\omega_i}$ .

### 3. Determinant line bundle on the Hilbert scheme

In this section we identify the determinant line bundle on the Hilbert scheme  $\text{Hilb}^n(\mathbb{A}^2)$ , or rather global sections of its pushforward to  $\text{Sym}^n \mathbb{A}^2$ , with the module over the Coulomb branch of the Jordan quiver gauge theory arising from the construction of [Part II, §3(ix)]. (See also [Ring, §2], though it is not essentially used.)

#### 3(i). Degree 2

We consider the case of the Hilbert scheme  $\text{Hilb}^2(\mathbb{A}^2)$  of two points in this subsection. We have the dilatation action of  $\mathbb{C}^\times$  on  $\mathbb{A}^2: t(u, v) = (t^{-1}u, t^{-1}v)$ . It induces a  $\mathbb{C}^\times$ -action on  $\text{Hilb}^2(\mathbb{A}^2)$ . The determinant line bundle  $\mathcal{L}$  on  $\text{Hilb}^2(\mathbb{A}^2)$  carries a natural  $\mathbb{C}^\times$ -equivariant structure. We have  $\text{Hilb}^2(\mathbb{A}^2) \simeq \tilde{\mathcal{S}}_2 \times \mathbb{A}^2$ , and  $\mathcal{L} \simeq \mathcal{O}_{\tilde{\mathcal{S}}_2}(1) \boxtimes \mathcal{O}_{\mathbb{A}^2}$ . Hence, from [Ring, Lemma 4.2], for  $l \in \mathbb{N}$ , the character of  $\Gamma(\text{Hilb}^2(\mathbb{A}^2), \mathcal{L}^l)$  equals

$$(1 - t^2)^{-1} (1 - t)^{-2} \sum_{m \in \mathbb{Z}} t^{|l-m|+|m|}.$$

On the other hand, we consider  $G = \text{GL}(V) = \text{GL}(2)$ ,  $G_F = \mathbb{C}^\times$ ,  $\tilde{G} = G \times G_F$ . The  $G = \text{GL}(V)$ -module  $\mathbf{N} = V \oplus \mathfrak{gl}(V)$  carries a commuting dilatation  $G_F$ -action; these two actions together give rise to the action of  $\tilde{G}$  on  $\mathbf{N}$ . According to [Quiver, Proposition 3.24], the Coulomb branch  $\mathcal{M}_C(G, \mathbf{N})$  is identified with  $\text{Sym}^2(\mathbb{A}^2)$ . Recall the setup of [Part II, §3(ix)]. (See also

[Ring, §2(iv)] and [Ring, §2(vi)].) We consider the variety of triple  $\tilde{\mathcal{R}}$  for the larger group  $\tilde{G}$  and  $\mathbf{N}$ , regarded as a representation of  $\tilde{G}$ . Let  $\tilde{\pi}: \tilde{\mathcal{R}} \rightarrow \text{Gr}_{G_F}$  be the projection. The affine Grassmannian  $\text{Gr}_{G_F}$  is identified with  $\mathbb{Z}$ . We denote the fiber over  $l$  by  $\tilde{\mathcal{R}}^l$ . The fiber  $\tilde{\mathcal{R}}^0$  over 0 is nothing but the original variety of triple  $\mathcal{R}$  whose equivariant Borel-Moore homology  $H_*^{G^\circ}(\mathcal{R})$  is the coordinate ring of the Coulomb branch, i.e.  $\mathbb{C}[\text{Sym}^2(\mathbb{A}^2)] = \mathbb{C}[\text{Hilb}^2(\mathbb{A}^2)]$  in this case. For  $l \in \mathbb{N} \subset \mathbb{Z} = \text{Gr}_{G_F}$ , the homology  $H_*^{G^\circ}(\tilde{\mathcal{R}}^l)$  is a module over  $H_*^{G^\circ}(\mathcal{R})$ , see [Part II, §3(ix)]. We will denote the coherent sheaf on  $\text{Sym}^2(\mathbb{A}^2)$  associated to this module by  $\mathcal{G}_l$ .

We want to identify this module with  $\Gamma(\text{Hilb}^2(\mathbb{A}^2), \mathcal{L}^l)$ . The module  $H_*^{G^\circ}(\tilde{\mathcal{R}}^l)$  is nothing but the costalk  $i_l^! \mathcal{A}^{\text{for}}$  in the setup in [Ring, §2(iv)]. By the monopole formula [Ring, (4.1)] for the character of  $H_*^{G^\circ}(\tilde{\mathcal{R}}^l)$ , we have

$$P_t^{\text{mod}} = (1 - t^2)^{-2} \sum_{\lambda_1 > \lambda_2 \in \mathbb{Z}} t^{-2|\lambda_1 - \lambda_2| + |\lambda_1 - \lambda_2 + l| + |\lambda_2 - \lambda_1 + l| + 2l + |\lambda_1 + l| + |\lambda_2 + l} + (1 - t^2)^{-1} (1 - t^4)^{-1} \sum_{\lambda \in \mathbb{Z}} t^{4l + 2|\lambda + l}.$$

**Lemma 3.1.**  $P_t^{\text{mod}} = t^{2l} (1 - t^2)^{-1} (1 - t)^{-2} \sum_{m \in \mathbb{Z}} t^{|l - m| + |m|}$ .

*Proof.* The sum in the RHS splits into 3 summands according to  $m \leq 0$ ,  $0 < m \leq l$ ,  $m > l$ , equal respectively, to  $\frac{t^l}{1 - t^2}$ ,  $lt^l$ ,  $\frac{t^{l+2}}{1 - t^2}$ . The second sum in the LHS splits into 2 summands according to  $\lambda \leq -l$ ,  $\lambda > -l$ , equal respectively, to  $\frac{t^{4l}}{1 - t^2}$ ,  $\frac{t^{4l+2}}{1 - t^2}$ . The first sum in the LHS splits into 6 summands according to  $-l \geq \lambda_1 > \lambda_2$ ,  $\lambda_1 - \lambda_2 \geq l$ , or  $-l \geq \lambda_1 > \lambda_2$ ,  $\lambda_1 - \lambda_2 < l$ , or  $\lambda_1 > \lambda_2 \geq -l$ ,  $\lambda_1 - \lambda_2 \geq l$ , or  $\lambda_1 > \lambda_2 \geq -l$ ,  $\lambda_1 - \lambda_2 < l$ , or  $\lambda_1 > -l > \lambda_2$ ,  $\lambda_1 - \lambda_2 \geq l$ , or  $\lambda_1 > -l > \lambda_2$ ,  $\lambda_1 - \lambda_2 < l$ . These summands are equal respectively, to  $\frac{t^{3l}}{(1 - t^2)(1 - t)}$ ,  $\frac{t^{3l+1}(1 - t^{l-1})}{(1 - t^2)(1 - t)}$ ,  $\frac{t^{3l}}{(1 - t^2)(1 - t)}$ ,  $\frac{t^{3l+1}(1 - t^{l-1})}{(1 - t^2)(1 - t)}$ ,  $\frac{t^{3l}}{(1 - t)^2} + \frac{(l - 2)t^{3l}}{1 - t}$ ,  $\frac{(l - 2)t^{3l+1}}{1 - t} - \frac{t^{3l+2}(1 - t^{l-1})}{(1 - t)^2}$ . Now a straightforward calculation finishes the proof.  $\square$

The evident action of  $\mathbb{G}_a^2$  on  $\mathbb{A}^2$  induces the natural free action of  $\mathbb{G}_a^2$  on  $\text{Sym}^2 \mathbb{A}^2$  such that  $\mathbb{G}_a^2 \backslash \text{Sym}^2 \mathbb{A}^2 = \mathcal{S}_2$ . Moreover, we have a projection  $\text{add}: \text{Sym} \mathbb{A}^2 \rightarrow \mathbb{A}^2$ ,  $((u_1, v_1), (u_2, v_2)) \mapsto (u_1 + u_2, v_1 + v_2)$ ; altogether we obtain an isomorphism  $\text{Sym}^2 \mathbb{A}^2 \xrightarrow{\sim} \mathcal{S}_2 \times \mathbb{A}^2$ .

**Proposition 3.2.** *Under the identification  $H_*^{G^\circ}(\mathcal{R}) \simeq \mathbb{C}[\text{Sym}^2 \mathbb{A}^2]$ , the  $H_*^{G^\circ}(\mathcal{R})$ -module  $H_*^{G^\circ}(\tilde{\mathcal{R}}^l)$  is isomorphic to the  $\mathbb{C}[\text{Sym}^2 \mathbb{A}^2]$ -module  $\Gamma(\text{Hilb}^2(\mathbb{A}^2), \mathcal{L}^l)$ . More precisely,*

- (a) *The restriction  $\mathcal{G}_l^\circ$  of  $\mathcal{G}_l$  to  $\mathcal{S}_2^\circ \times \mathbb{A}^2 \subset \mathcal{S}_2 \times \mathbb{A}^2 = \text{Sym}^2 \mathbb{A}^2$  is a line bundle isomorphic to  $\mathcal{L}^l|_{\mathcal{S}_2^\circ \times \mathbb{A}^2}$ .*

(b) *An isomorphism in (a) is defined uniquely up to multiplication by a scalar.*

(c) *An isomorphism in (a) extends to an isomorphism  $H_*^{G\circ}(\tilde{\mathcal{R}}^l) \xrightarrow{\sim} \Gamma(\text{Hilb}^2(\mathbb{A}^2), \mathcal{L}^l)$ .*

*Proof.* We consider the elements  $E_1[1]$  and  $F_1[1]$  of [Quiver, (A.7)] in  $H_*^{G\circ}(\mathcal{R}) \simeq \mathbb{C}[\text{Sym}^2 \mathbb{A}^2]$ . They have degree  $1/2$  with respect to the modified grading as in [Part II, Remark 2.8](2), see [Quiver, (A.4)]. Clearly,  $E_1[1] = u_1 + u_2$ ,  $F_1[1] = v_1 + v_2$ . The corresponding hamiltonian vector fields  $H_{E_1[1]}$  and  $H_{F_1[1]}$  on  $\mathcal{M}_C = \text{Sym}^2 \mathbb{A}^2$  commute since the Poisson bracket  $\{E_1[1], F_1[1]\}$  acts as multiplication by 2 (the number of points), and its hamiltonian vector field vanishes. The degrees of both  $H_{E_1[1]}$  and  $H_{F_1[1]}$  are  $-1/2$  since the degree of the Poisson bracket is  $-1$ . Since the degrees of  $H_*^{G\circ}(\mathcal{R})$  and  $H_*^{G\circ}(\tilde{\mathcal{R}}^l)$  are all nonnegative by the monopole formula, both  $H_{E_1[1]}$  and  $H_{F_1[1]}$  are locally nilpotent. Hence they integrate to the action of  $\mathbb{G}_a^2$  on  $H_*^{G\circ}(\mathcal{R})$  and  $H_*^{G\circ}(\tilde{\mathcal{R}}^l)$ . The action of  $\mathbb{G}_a^2$  on  $H_*^{G\circ}(\mathcal{R}) = \mathbb{C}[\text{Sym}^2 \mathbb{A}^2]$  comes from the action on  $\text{Sym}^2 \mathbb{A}^2$  discussed before the proposition. We conclude that the coherent sheaf  $\mathcal{G}_l$  on  $\text{Sym}^2 \mathbb{A}^2$  is  $\mathbb{G}_m \times \mathbb{G}_a^2$ -equivariant (the action of  $\mathbb{G}_m$  comes from the modified grading).

In particular,  $\mathcal{G}_l$  is a pullback of a  $\mathbb{G}_m$ -equivariant sheaf  $\mathcal{F}_l$  on

$$\mathbb{G}_a^2 \backslash \text{Sym}^2 \mathbb{A}^2 = \mathcal{S}_2.$$

Both  $\mathcal{G}_l$  and  $\mathcal{F}_l$  are generically of rank 1; hence both  $\mathcal{F}_l|_{\mathcal{S}_2^\circ}$  and  $\mathcal{G}_l^\circ := \mathcal{G}_l|_{\mathcal{S}_2^\circ \times \mathbb{A}^2}$  are line bundles. Recall that  $\text{Pic}(\mathcal{S}_2^\circ) = \mathbb{Z}/2\mathbb{Z}$ ; the trivial line bundle is denoted  $\mathcal{F}_{\bar{0}}$ , and the nontrivial one is denoted  $\mathcal{F}_{\bar{1}}$  in accordance with notations of [Ring, §4(ii)]. Lemma 3.1 and the argument in the proof of [Ring, Lemma 4.3] show that  $\mathcal{F}_l|_{\mathcal{S}_2^\circ} \simeq \mathcal{F}_{\bar{l}}$ , where  $\bar{l} = l \pmod{2}$ . This proves (a), and the same argument as in the proof of [Ring, Lemma 4.3] establishes (b).

For (c), we have to identify  $\mathcal{F}_l \subset j_* \mathcal{F}_{\bar{l}}$  and  $\mathcal{F}_\lambda \subset j_* \mathcal{F}_{\bar{l}}$  in notations of [Ring, §4(ii)], where  $\lambda = (l, 0)$ . We start with  $l = 1$  case. Then  $\mathcal{F}_\lambda = j_* \mathcal{F}_{\bar{1}}$ , and the character of (the global sections of)  $\mathcal{F}_1$  coincides with the character of  $j_* \mathcal{F}_{\bar{1}}$ . Hence  $\mathcal{F}_1 = j_* \mathcal{F}_{\bar{1}} = \mathcal{F}_\lambda$ .

For  $l > 1$  we have to identify  $\Gamma(\mathcal{S}_2, \mathcal{F}_l)$  inside  $\Gamma(\mathcal{S}_2^\circ, \mathcal{F}_{l \pmod{2}})$  with  $\Gamma(\tilde{\mathcal{S}}_2, \mathcal{L}_\lambda) = \Gamma(T^*\mathbb{P}^1, \mathcal{O}(l)) = \bigoplus_{k \geq 0} \Gamma(\mathbb{P}^1, \mathcal{O}(l + 2k))$ . However, the latter submodule is clearly characterized by its  $t$ -character which coincides with the  $t$ -character of  $\Gamma(\mathcal{S}_2, \mathcal{F}_l)$  by Lemma 3.1. Hence  $\Gamma(\text{Hilb}^2(\mathbb{A}^2), \mathcal{L}^l) = H_*^{G\circ}(\tilde{\mathcal{R}}^l)$ . □

Recall that the  $\tilde{G}$ -module  $\mathbf{N} = V \oplus \mathfrak{gl}(V)$  splits as a direct sum. If we set  $\mathbf{N} = \mathfrak{gl}(V)$ , then from [Part II, Remark 5.14] we obtain a homomorphism



$H_*^{G\circ}(\mathcal{R}) \hookrightarrow H_*^{G\circ}(' \mathcal{R})$  of algebras and a compatible homomorphism of modules  $H_*^{G\circ}(\tilde{\mathcal{R}}^l) \hookrightarrow H_*^{G\circ}(' \tilde{\mathcal{R}}^l)$  (where  $' \mathcal{R}, ' \tilde{\mathcal{R}}$  are varieties of triples for  $(G, \mathbf{N}), (\tilde{G}, \mathbf{N})$  respectively,  $' \tilde{\mathcal{R}}^l$  is the fiber of the projection  $' \tilde{\mathcal{R}} \rightarrow \text{Gr}_{G_F}$  over  $l$ ). According to [Quiver, Proposition 3.24], the Coulomb branch  $\mathcal{M}_C(G, \mathbf{N})$  is identified with  $\text{Sym}^2(\mathcal{S}_0)$ , and the homomorphism  $H_*^{G\circ}(\mathcal{R}) \hookrightarrow H_*^{G\circ}(' \mathcal{R})$  corresponds to the morphism  $j^2: \text{Sym}^2(\mathcal{S}_0) \hookrightarrow \text{Sym}^2(\mathbb{A}^2)$  arising from the open embedding  $j: \mathcal{S}_0 \hookrightarrow \mathbb{A}^2, (u, v) \mapsto (u, u^{-1}v), u \neq 0$ . We denote by  $' \mathcal{G}_l$  the coherent sheaf on  $\text{Sym}^2 \mathcal{S}_0$  associated to the  $H_*^{G\circ}(' \mathcal{R})$ -module  $H_*^{G\circ}(' \tilde{\mathcal{R}}^l)$ . We would like to identify the coherent sheaves  $\text{pr}_* \mathcal{L}^l$  and  $' \mathcal{G}_l$  on  $\text{Sym}^2(\mathcal{S}_0)$ , where  $\text{pr}: \text{Hilb}^2(\mathcal{S}_0) \rightarrow \text{Sym}^2(\mathcal{S}_0)$  is the Hilbert-Chow morphism. The localization of the morphism  $H_*^{G\circ}(\tilde{\mathcal{R}}^l) \hookrightarrow H_*^{G\circ}(' \tilde{\mathcal{R}}^l)$  factors through  $\mathcal{G}_l \hookrightarrow j_* j^{2*} \mathcal{G}_l = j_*^2 \text{pr}_* \mathcal{L}^l \hookrightarrow j_*^2 \mathcal{G}_l$ . The restriction of the latter morphism to  $\text{Sym}^2 \mathcal{S}_0$  is denoted by  $\theta: \text{pr}_* \mathcal{L}^l \hookrightarrow ' \mathcal{G}_l$ .

**Corollary 3.3.** *The morphism  $\theta: \text{pr}_* \mathcal{L}^l \hookrightarrow ' \mathcal{G}_l$  of coherent sheaves on  $\text{Sym}^2 \mathcal{S}_0$  is an isomorphism.*

*Proof.* Let  $T \subset \text{GL}(V) = \text{GL}(2)$  be the diagonal torus with Lie algebra  $\mathfrak{t} \subset \mathfrak{gl}(V) = \mathfrak{gl}(2)$ , with coordinates  $w_1, w_2$ . The canonical projection  $\text{Sym}^2 \mathbb{A}^2 = \mathcal{M}_C(G, \mathbf{N}) \rightarrow \mathfrak{t}/S_2 = \text{Sym}^2 \mathbb{A}^1$  is the symmetric square of the morphism  $\mathbb{A}^2 \rightarrow \mathbb{A}^1, (u, v) \mapsto uv$ . The generalized roots in  $\mathfrak{t}^\vee$  are  $w_1, w_2, w_1 - w_2$ . We change the base to  $\mathfrak{t} \rightarrow \mathfrak{t}/S_2$  and localize at a general point  $t$  of the diagonal  $w_1 - w_2 = 0$ . The corresponding fixed point sets coincide:  $(' \tilde{\mathcal{R}}^l)^t = (\tilde{\mathcal{R}}^l)^t$ ; hence  $\theta$  is an isomorphism over the general points of diagonal.

Now let  $t$  be a general point of the divisor  $w_2 = 0$ . Then the fixed point set  $(' \tilde{\mathcal{R}}^l)^t$  (resp.  $(\tilde{\mathcal{R}}^l)^t$ ) splits as a product  $\text{Gr}_{T_1} \times \text{Gr}_{T_2}$  (resp.  $\text{Gr}_{T_1} \times \mathcal{R}_{T_2, \mathbf{N}'}$ ). Here  $T_1$  (resp.  $T_2$ ) is a 1-dimensional torus with coordinate  $w_1$  (resp.  $w_2$ ) with differential  $w_1$  (resp.  $w_2$ ), and  $\mathbf{N}'$  is the 1-dimensional representation of  $T_2$  with character  $w_2$ . Note that the flavor group disappeared since its action is absorbed into the action of  $T_2$ . The morphism of localizations

$$\begin{aligned} (\mathbb{C}[\mathfrak{t}_1 \times T_1^\vee] \otimes \mathbb{C}[\mathbb{A}^2])_t &= H_*^{T\circ}((\tilde{\mathcal{R}}^l)^t)_t \rightarrow H_*^{T\circ}(' \tilde{\mathcal{R}}^l)^t)_t \\ &= (\mathbb{C}[\mathfrak{t}_1 \times T_1^\vee] \otimes \mathbb{C}[\mathcal{S}_0])_t \end{aligned}$$

at the level of spectra is nothing but  $(\text{id} \times j)_t$ . The same argument takes care of the general points of the divisor  $w_1 = 0$ . Hence the base change of  $\theta$  is an isomorphism over the general points of all the root hyperplanes. We conclude that  $\theta$  is an isomorphism. □

**3(ii). Factorization**

The projection  $\varpi_1: \mathcal{S}_1 = \mathbb{A}^2 \rightarrow \mathbb{A}^1$ ,  $(u, v) \mapsto w = uv$ , induces the projection  $\varpi_n: \text{Hilb}^n(\mathcal{S}_1) = \text{Hilb}^n(\mathbb{A}^2) \xrightarrow{\pi_n} \text{Sym}^n \mathbb{A}^2 \xrightarrow{\Pi_n} \text{Sym}^n \mathbb{A}^1 = \mathbb{A}^{(n)}$ . The embedding  $\mathbb{G}_m \subset \mathbb{A}^1$  induces the embedding  $\mathbb{G}_m^{(n)} \subset \mathbb{A}^{(n)}$ . We denote by  $\mathring{\mathbb{G}}_m^{(n)} \subset \mathbb{G}_m^{(n)}$  the open subset formed by the complement to all the diagonals; we have a Galois  $S_n$ -covering  $\mathring{\mathbb{G}}_m^n \rightarrow \mathring{\mathbb{G}}_m^{(n)}$ . We have

$$(3.4) \quad \mathring{\mathbb{G}}_m^n \times_{\mathring{\mathbb{G}}_m^{(n)}} \varpi_n^{-1}(\mathring{\mathbb{G}}_m^{(n)}) = \mathring{\mathbb{G}}_m^n \times_{\mathring{\mathbb{G}}_m^{(n)}} \Pi_n^{-1}(\mathring{\mathbb{G}}_m^{(n)}) = \mathring{\mathbb{G}}_m^n \times \mathbb{G}_m^n$$

with coordinates  $w_1, \dots, w_n$  on the first factor, and  $v_1, \dots, v_n$  on the second factor. We denote the base change  $\mathbb{A}^n \times_{\mathbb{A}^{(n)}} \text{Hilb}^n(\mathbb{A}^2)$  (resp.  $\mathbb{A}^n \times_{\mathbb{A}^{(n)}} \text{Sym}^n \mathbb{A}^2$ ) by  $\underline{\text{Hilb}}^n(\mathbb{A}^2)$  (resp.  $\underline{\text{Sym}}^n \mathbb{A}^2$ ). We have factorization isomorphisms for  $n = n' + n''$ :

$$\begin{aligned} \underline{\text{Hilb}}^n(\mathbb{A}^2)|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}} &\xrightarrow{\sim} (\underline{\text{Hilb}}^{n'}(\mathbb{A}^2) \times \underline{\text{Hilb}}^{n''}(\mathbb{A}^2))|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}}, \\ \underline{\text{Sym}}^n \mathbb{A}^2|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}} &\xrightarrow{\sim} (\underline{\text{Sym}}^{n'} \mathbb{A}^2 \times \underline{\text{Sym}}^{n''} \mathbb{A}^2)|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}} \end{aligned}$$

compatible with (3.4). By the definition of the determinant line bundle, we also have the following factorization isomorphisms:

$$(3.5) \quad \begin{aligned} & \left( \underline{\text{Hilb}}^n(\mathbb{A}^2)|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}}, \mathcal{L}^l \right) \\ & \xrightarrow{\sim} \left( (\underline{\text{Hilb}}^{n'}(\mathbb{A}^2) \times \underline{\text{Hilb}}^{n''}(\mathbb{A}^2))|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}}, \mathcal{L}^l \boxtimes \mathcal{L}^l \right), \\ & \left( \underline{\text{Sym}}^n \mathbb{A}^2|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}}, \pi_{n*} \mathcal{L}^l \right) \\ & \xrightarrow{\sim} \left( (\underline{\text{Sym}}^{n'} \mathbb{A}^2 \times \underline{\text{Sym}}^{n''} \mathbb{A}^2)|_{(\mathbb{A}^{n'} \times \mathbb{A}^{n''})_{\text{disj}}}, \pi_{n'*} \mathcal{L}^l \boxtimes \pi_{n''*} \mathcal{L}^l \right) \end{aligned}$$

compatible with the  $S_n$ -equivariant trivialization

$$(3.6) \quad \begin{aligned} \left( \mathring{\mathbb{G}}_m^n \times_{\mathring{\mathbb{G}}_m^{(n)}} \varpi_n^{-1}(\mathring{\mathbb{G}}_m^{(n)}), \mathcal{L}^l \right) &= \left( \mathring{\mathbb{G}}_m^n \times_{\mathring{\mathbb{G}}_m^{(n)}} \Pi_n^{-1}(\mathring{\mathbb{G}}_m^{(n)}), \pi_{n*} \mathcal{L}^l \right) \\ &= \left( \mathring{\mathbb{G}}_m^n \times \mathbb{G}_m^n, \mathcal{O}_{\mathring{\mathbb{G}}_m^n \times \mathbb{G}_m^n} \right) \end{aligned}$$

arising from the factorization and the identification

$$(3.7) \quad \left( \varpi_1^{-1}(\mathbb{G}_m), \mathcal{L}^l \right) = \left( \Pi_1^{-1}(\mathbb{G}_m), \pi_{1*} \mathcal{L}^l \right) = (\mathbb{G}_m \times \mathbb{G}_m, \mathcal{O}_{\mathbb{G}_m \times \mathbb{G}_m}).$$

We will need the following particular case of the above factorization isomorphisms:

$$\begin{aligned}
 & \left( (\mathbb{G}_m^{(n-1)} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{(n)}} \text{Hilb}^n(\mathbb{A}^2), \mathcal{L}^l \right) \\
 & \xrightarrow{\sim} \left( (\mathbb{G}_m^{(n-1)} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{(n-1)} \times \mathbb{A}^1} (\text{Hilb}^{n-1}(\mathbb{A}^2) \times \mathbb{A}^2), \mathcal{L}^l \boxtimes \mathcal{L}^l \right), \\
 (3.8) \quad & \left( (\mathbb{G}_m^{(n-1)} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{(n)}} \text{Sym}^n \mathbb{A}^2, \pi_{n*} \mathcal{L}^l \right) \\
 & \xrightarrow{\sim} \left( (\mathbb{G}_m^{(n-1)} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{(n-1)} \times \mathbb{A}^1} (\text{Sym}^{n-1} \mathbb{A}^2 \times \mathbb{A}^2), \pi_{n-1,*} \mathcal{L}^l \boxtimes \pi_{1*} \mathcal{L}^l \right).
 \end{aligned}$$

**3(iii). Determinant sheaves via homology groups of fibers**

We change slightly the setup of §3(i): we consider  $G = \text{GL}(V) = \text{GL}(n)$ ,  $G_F = \mathbb{C}^\times$ ,  $\tilde{G} = G \times G_F$ . The  $G = \text{GL}(V)$ -module  $\mathbf{N} = V \oplus \mathfrak{gl}(V)$  carries a commuting dilatation  $G_F$ -action; these two actions together give rise to the action of  $\tilde{G}$  on  $\mathbf{N}$ . According to [Quiver, Proposition 3.24], the Coulomb branch  $\mathcal{M}_C(G, \mathbf{N})$  is identified with  $\text{Sym}^n(\mathbb{A}^2)$ . In this case we have  $H_*^{G \circ}(\mathcal{R}) \cong \mathbb{C}[\text{Sym}^n(\mathbb{A}^2)] = \mathbb{C}[\text{Hilb}^n(\mathbb{A}^2)]$ , see [Quiver, Proposition 3.24]. For  $l \in \mathbb{N} \subset \mathbb{Z} = \text{Gr}_{G_F}$ ,  $H_*^{G \circ}(\tilde{\mathcal{R}}^l)$  forms a module over the algebra  $H_*^{G \circ}(\mathcal{R})$  as in the case  $n = 2$ , and we want to identify this module with  $\Gamma(\text{Hilb}^n(\mathbb{A}^2), \mathcal{L}^l) = \Gamma(\text{Sym}^n \mathbb{A}^2, \pi_{n*} \mathcal{L}^l)$ . Recall that  $\text{Spec } H_{G \circ}^*(\text{pt}) = \mathbb{A}^{(n)} \leftarrow \mathbb{A}^n = \text{Spec } H_{T \circ}^*(\text{pt})$ , and the base change under  $\mathbb{A}^{(n)} \leftarrow \mathbb{A}^n$  gives  $H_*^{T \circ}(\tilde{\mathcal{R}}^l)$ , where  $T$  is a Cartan torus of  $G$ . If we further localize to  $\mathring{\mathbb{G}}_m^n \subset \mathbb{A}^n$ , we have a localization isomorphism  $\mathbf{z}^* l_*^{-1}: H_*^{T \circ}(\tilde{\mathcal{R}}^l)_{\text{loc}} \xrightarrow{\sim} H_*^{T \circ}(\hat{\pi}^{-1}(l))_{\text{loc}}$  where  $\hat{\pi}: \text{Gr}_{T \times G_F} \rightarrow \text{Gr}_{G_F}$  is the obvious projection. But  $H_*^{T \circ}(\hat{\pi}^{-1}(l)) \cong H_*^{T \circ}(\text{Gr}_T) = \mathbb{C}[\mathbb{A}^n \times \mathbb{G}_m^n]$  by [Part II, Remark 3.24](2). All in all, we obtain an  $S_n$ -equivariant trivialization

$$(3.9) \quad H_*^{T \circ}(\tilde{\mathcal{R}}^l) \cong \mathcal{O}_{\mathring{\mathbb{G}}_m^n \times \mathbb{G}_m^n}.$$

Composing with the trivialization (3.6), we obtain a rational isomorphism of  $\mathbb{C}[\text{Sym}^n \mathbb{A}^2]$ -modules  $\theta: \Gamma(\text{Sym}^n \mathbb{A}^2, \pi_{n*} \mathcal{L}^l) \dashrightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^l)$ .

**Theorem 3.10.** *The rational isomorphism  $\theta: \Gamma(\text{Sym}^n \mathbb{A}^2, \pi_{n*} \mathcal{L}^l) \dashrightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^l)$  extends to the regular isomorphism of  $\mathbb{C}[\text{Sym}^n \mathbb{A}^2]$ -modules  $\theta: \Gamma(\text{Sym}^n \mathbb{A}^2, \pi_{n*} \mathcal{L}^l) \xrightarrow{\sim} H_*^{G \circ}(\tilde{\mathcal{R}}^l)$ .*

*Proof.* We follow the standard scheme, see e.g. the proof of [Quiver, Theorem 3.10]. We have to check that  $\theta$  extends through the general points of

the boundary divisor  $\mathbb{A}^n \setminus \mathbb{G}_m^n$ . If a point lies on a diagonal divisor  $w_r = w_s$ , we are reduced by localization and factorization (3.5) to Corollary 3.3. If a point lies on a coordinate hyperplane  $w_r = 0$ , we are reduced by localization and factorization (3.8), (3.7) to the evident case  $n = 1$ . We conclude by an application of [Part II, Theorem 5.26] and [Part II, Remark 5.27]. The condition  $\Pi_{n*} \pi_{n*} \mathcal{L}^l \xrightarrow{\sim} j_* \Pi_{n*} \pi_{n*} \mathcal{L}^l|_{\text{Hilb}^n(\mathbb{A}^2)^\bullet}$  of [Part II, Remark 5.27] is satisfied since the complement of  $\text{Hilb}^n(\mathbb{A}^2)^\bullet$  in  $\text{Hilb}^n(\mathbb{A}^2)$  is of codimension 2. The latter claim follows from the semismallness of  $\pi_n: \text{Hilb}^n(\mathbb{A}^2) \rightarrow \text{Sym}^n \mathbb{A}^2$ .  $\square$

### 4. Line bundles on Cherkis bow varieties

We can modify the proof of the last section to the case of quiver gauge theories of affine type  $A_{n-1}$  replacing Hilbert schemes by Cherkis bow varieties, and using results in [NT17]. We use the notation in [NT17], hence we assume the reader is familiar with it.

#### 4(i). Resolution for bow varieties

Given dimension vectors  $\underline{\mathbf{v}} = (\mathbf{v}_0, \dots, \mathbf{v}_{n-1})$ ,  $\underline{\mathbf{w}} = (\mathbf{w}_0, \dots, \mathbf{w}_{n-1})$  we consider

$$G \equiv \text{GL}(\underline{\mathbf{v}}) \stackrel{\text{def.}}{=} \prod_{i=0}^{n-1} \text{GL}(\mathbf{v}_i),$$

$$\mathbf{N} \equiv \mathbf{N}(\underline{\mathbf{v}}, \underline{\mathbf{w}}) = \bigoplus_{i=0}^{n-1} \text{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_{i+1}}) \oplus \text{Hom}(\mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{v}_i})$$

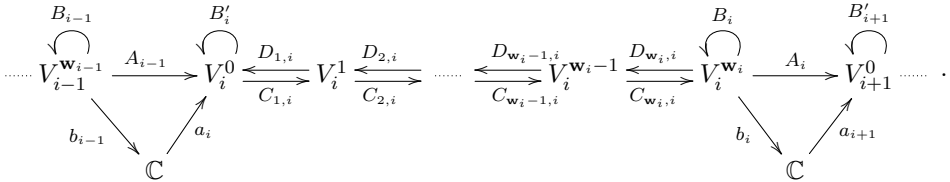
with the natural  $G$ -action on  $\mathbf{N}$ . Let  $\ell = \sum_{i=0}^{n-1} \mathbf{w}_i$ . The Coulomb branch  $\mathcal{M}_C(G, \mathbf{N})$  is isomorphic to a bow variety  $\mathcal{M}(\underline{\mathbf{v}}, \underline{\mathbf{w}})$  with a balanced condition, defined as in [NT17, §2.2]. The definition of [NT17, §2.2] is more general: we have parameters  $\varkappa_\sigma \in \mathbb{Q}$  ( $\sigma = 1, \dots, \ell$ ) of the stability condition for the GIT quotient, where  $\mathcal{M}_C(G, \mathbf{N})$  corresponds to the case  $\varkappa_\sigma = 0$  for  $\sigma = 1, \dots, \ell$ .<sup>2</sup> We have a  $\mathbb{Q}$ -line bundle from the construction, which is an actual line bundle if  $\varkappa_\sigma \in \mathbb{Z}$  for  $\sigma = 1, \dots, \ell$ . We suppose  $\varkappa_\sigma \in \mathbb{Z}$  hereafter.

---

<sup>2</sup>It was denoted by  $\nu_\sigma^{\mathbb{R}}$  in [NT17], as we also have complex parameters  $\nu^{\mathbb{C}} = (\nu_\sigma^{\mathbb{C}})_\sigma$ , which we set 0 for brevity here.

There is one more extra parameter  $\varkappa_* \in \mathbb{Z}$ , which was not explicitly explained in [NT17].<sup>3</sup> It corresponds to the quotient where either one of the stability conditions (C-S1) or (C-S2) is required in [NT17, Prop. 6.4].

Let us number vector spaces appearing in the definition of bow varieties as in [NT17, §6.1].



In particular,  $\sigma$  ( $\sigma = 1, \dots, \ell$ ) is indexed as  $(\alpha, i)$  ( $i = 0, \dots, n - 1, \alpha = 1, \dots, \mathbf{w}_i$ ). We introduce the character corresponding to parameters<sup>4</sup>  $\varkappa_*, \varkappa_{\alpha,i}$  by

$$(4.1) \quad \prod_{i=0}^{n-1} (\det V_i^1)^{-\varkappa_{1,i} + \varkappa_{2,i}} \dots (\det V_i^\alpha)^{-\varkappa_{\alpha,i} + \varkappa_{\alpha+1,i}} \dots (\det V_i^{\mathbf{w}_i-1})^{-\varkappa_{\mathbf{w}_i-1,i} + \varkappa_{\mathbf{w}_i,i}} (\det V_i^{\mathbf{w}_i})^{-\varkappa_{\mathbf{w}_i,i} + \varkappa_{1,i+1} + \delta_{i+1,0} \varkappa_*}.$$

Note that the simultaneous shift  $\varkappa_{\alpha,i} \mapsto \varkappa_{\alpha,i} + s$ , while keeping  $\varkappa_*$ , is irrelevant.

We assume

$$(4.2) \quad \varkappa_{1,i} \geq \varkappa_{2,i} \geq \dots \geq \varkappa_{\mathbf{w}_i,i}.$$

In particular, all powers appearing in (4.1) except the last one are nonpositive. This assumption is not essential, as it is satisfied if we renumber  $\varkappa_{\alpha,i}$ . Alternatively we apply reflection functors for quiver varieties [Nak03] on two way parts. Here we regard  $V_i^0$  and  $V_i^{\mathbf{w}_i}$  as framing vector spaces, and do not touch for reflection functors.

These powers, especially the last one, look slightly different from [NT17, (6.3)], where the corresponding complex parameters  $\nu_*^{\mathbb{C}}, \nu_{\alpha,i}^{\mathbb{C}}$  are put in the defining equation. But it is implicit in the proof of [Tak16, Prop. 2.9] (see

<sup>3</sup>In the original description [NT17, §2.1] of bow varieties as solutions of Nahm’s equations, parameters  $\varkappa_\sigma, \varkappa_*$  are put as the level of real part of the hyper-Kähler moment map.

<sup>4</sup>We consider the ‘corresponding’ complex parameter  $\nu_*^{\mathbb{C}}$  in [NT17, 6.2], but we put it for all  $i$ . But the sum over  $i$  only matters, so our  $\varkappa_*$  should be compared with  $n\nu_*^{\mathbb{C}}$ .

also [NT17, Prop. 3.2] and the numerical criterion [NT17, Def. 2.7]) that we have an isomorphism

$$(4.3) \quad \det V_i^{\mathbf{w}_i} \cong \det V_{i+1}^0,$$

hence the appearance of  $\varkappa_{1,i+1}$  in  $\det V_i^{\mathbf{w}_i}$  is natural. Let us denote the corresponding GIT quotient by  $\mathcal{M}_\varkappa(\mathbf{v}, \mathbf{w})$ , where  $\varkappa$  should be understood as  $\varkappa_* \in \mathbb{Z}$ ,  $(\varkappa_{\alpha,i}) \in \mathbb{Z}^\ell / \mathbb{Z}$ . Let us denote the corresponding line bundle by  $\mathcal{L}_\varkappa$ . We have the projective morphism  $\pi: \mathcal{M}_\varkappa(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_0(\mathbf{v}, \mathbf{w})$ . Let  $\mathbb{A}^{\mathbf{v}} = \prod_{i=0}^{n-1} \mathbb{A}^{\mathbf{v}_i} / \mathfrak{S}_{\mathbf{v}_i}$ . We have a factorization morphism  $\Psi: \mathcal{M}_\varkappa(\mathbf{v}, \mathbf{w}) \rightarrow \mathbb{A}^{\mathbf{v}}$ , given by eigenvalues of  $B_i$  with multiplicities, which are same as eigenvalues of  $B'_i$  thanks to the defining equation of bow varieties. We can apply [Part II, Theorem 5.26] later, as  $\mathcal{M}_\varkappa(\mathbf{v}, \mathbf{w})$  is normal ([NT17, Th. 6.15]) and all fibers of  $\Psi$  have the same dimension ([NT17, Prop. 6.13]), hence the condition of [Part II, Remark 5.27] is satisfied. Note that  $\Psi$  factors through  $\pi$ .

We have the factorization property

$$\begin{aligned} \mathcal{M}_\varkappa(\mathbf{v}, \mathbf{w}) \times_{\mathbb{A}^{\mathbf{v}}} (\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''})_{\text{disj}} \\ \cong (\mathcal{M}_\varkappa(\mathbf{v}', \mathbf{w}) \times \mathcal{M}_\varkappa(\mathbf{v}'', \mathbf{w})) \times_{\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''}} (\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''})_{\text{disj}}. \end{aligned}$$

See [NT17, Th. 6.9]. From its construction the line bundle  $\mathcal{L}_\varkappa$  is compatible with the factorization, namely  $\mathcal{L}_\varkappa$  on  $\mathcal{M}_\varkappa(\mathbf{v}, \mathbf{w})$  is sent to  $\mathcal{L}_\varkappa \boxtimes \mathcal{L}_\varkappa$  on  $\mathcal{M}_\varkappa(\mathbf{v}', \mathbf{w}) \times \mathcal{M}_\varkappa(\mathbf{v}'', \mathbf{w})$ . This is because  $\mathcal{L}_\varkappa$  is coming from the character  $\varkappa$ , given by the product of determinants of  $\text{GL}(\mathbf{v}_i)$  as in (4.1), and it factors according to a decomposition  $V_i^\alpha = (V_i^\alpha)' \oplus (V_i^\alpha)''$ . Note that this construction chooses an isomorphism between  $\mathcal{L}_\varkappa$  and  $\mathcal{L}_\varkappa \boxtimes \mathcal{L}_\varkappa$  canonically. This choice will become more explicit in the factorization formula (4.5) of a section  $y_j^\alpha$  later. This is a generalization of statements in §3(ii).

Let  $\mathring{\mathbb{A}}^{|\mathbf{v}|}$  denote the open subset of  $\mathbb{A}^{|\mathbf{v}|}$  consisting of  $w_i^k \neq w_i^l$  ( $k \neq l$ ),  $w_i^k \neq w_{i+1}^l$ ,  $w_i^k \neq 0$  (for  $i$  with  $\mathbf{w}_i \neq 0$ ). Let  $\mathring{\mathbb{A}}^{\mathbf{v}} = \mathring{\mathbb{A}}^{|\mathbf{v}|} / \prod \mathfrak{S}_{\mathbf{v}_i}$ . It is the complement of union of all generalized root hyperplanes of  $(G, \mathbf{N})$  in the sense of [Part II, §5(i)].

We order eigenvalues of  $B_i$  (which are also eigenvalues of  $B'_i$ ) as  $w_{i,1}, \dots, w_{i,\mathbf{v}_i}$ . We consider them as coordinates of  $\mathbb{A}^{\mathbf{v}_i}$ , and functions on  $\mathcal{M}_\varkappa(\mathbf{v}, \mathbf{w}) \times_{\mathbb{A}^{\mathbf{v}}} \mathring{\mathbb{A}}^{|\mathbf{v}|}$ . (Here  $|\mathbf{v}| = \sum \mathbf{v}_i$ .) Define a section  $y_{i,k}^\alpha$  of the vector bundle  $(V_i^\alpha)^*$  by

$$y_{i,k}^\alpha \stackrel{\text{def.}}{=} b_i \prod_{\substack{1 \leq l \leq \mathbf{v}_i \\ l \neq k}} (B_i - w_{i,l}) C_{\mathbf{w}_i,i} \cdots C_{\alpha+1,i}$$

and a rational section  $y_i^\alpha$  of the line bundle  $(\det V_i^\alpha)^*$  defined over  $\Psi^{-1}(\mathbb{A}^{\mathbf{v}})$  by

$$(4.4) \quad y_i^\alpha \stackrel{\text{def.}}{=} y_{i,1}^\alpha \wedge y_{i,2}^\alpha \wedge \cdots \wedge y_{i,\mathbf{v}_i}^\alpha \prod_{k>l} (w_{i,k} - w_{i,l})^{-1}.$$

Note that this is  $\mathfrak{S}_{\mathbf{v}_i}$ -invariant, as signs from  $y_{i,1}^\alpha \wedge y_{i,2}^\alpha \wedge \cdots \wedge y_{i,\mathbf{v}_i}^\alpha$  and  $\prod_{k>l} (w_{i,k} - w_{i,l})$  cancel.

We also define sections  $z_{i+1,k}^0$  ( $k = 1, \dots, \mathbf{v}_{i+1}$ ) of  $V_{i+1}^0$  and  $z_{i+1}^0$  of  $\det V_{i+1}^0$  by

$$z_{i+1,k}^0 \stackrel{\text{def.}}{=} \prod_{\substack{1 \leq l \leq \mathbf{v}_{i+1} \\ l \neq k}} (B'_{i+1} - w'_{i+1,l}) a_{i+1},$$

$$z_{i+1}^0 \stackrel{\text{def.}}{=} z_{i+1,1}^0 \wedge z_{i+1,2}^0 \wedge \cdots \wedge z_{i+1,\mathbf{v}_{i+1}}^0 \prod_{k>l} (w'_{i+1,k} - w'_{i+1,l})^{-1}.$$

We regard  $z_{i+1}^0$  as a section of  $\det V_i^{\mathbf{w}_i}$  via (4.3).

They are compatible with the factorization as follows. Let  $y_{i,k}^{\prime\alpha}$  ( $1 \leq k \leq \mathbf{v}'_i$ ),  $y_i^{\prime\alpha}$ ,  $y_{i,k}^{\prime\prime\alpha}$  ( $\mathbf{v}'_i + 1 \leq k \leq \mathbf{v}_i$ ),  $y_i^{\prime\prime\alpha}$  be defined for  $\mathcal{M}_{\mathcal{X}}(\mathbf{v}', \mathbf{w})$ ,  $\mathcal{M}_{\mathcal{X}}(\mathbf{v}'', \mathbf{w})$  respectively. As in [NT17, Lem. 6.11], we have

$$y_{i,k}^\alpha = \begin{cases} y_{i,k}^{\prime\alpha} \prod_{l=\mathbf{v}'_i+1}^{\mathbf{v}_i} (w_{i,k} - w_{i,l}) & \text{if } 1 \leq k \leq \mathbf{v}'_i, \\ y_{i,k}^{\prime\prime\alpha} \prod_{l=1}^{\mathbf{v}'_i} (w_{i,k} - w_{i,l}) & \text{if } \mathbf{v}'_i + 1 \leq k \leq \mathbf{v}_i, \end{cases}$$

and hence

$$(4.5) \quad y_i^\alpha = y_i^{\prime\alpha} \wedge y_i^{\prime\prime\alpha} \prod_{k=1}^{\mathbf{v}'_i} \prod_{l=\mathbf{v}'_i+1}^{\mathbf{v}_i} (w_{i,k} - w_{i,l}).$$

We have a similar formula for  $z_{i+1}^0$ .

Let  $y^{\mathcal{X}}$  be a section of  $\mathcal{L}_{\mathcal{X}}$  given by

$$(4.6) \quad y^{\mathcal{X}} \stackrel{\text{def.}}{=} \prod_{i=0}^{n-1} (y_i^1)^{\mathcal{X}_{1,i} - \mathcal{X}_{2,i}} \cdots (y_i^\alpha)^{\mathcal{X}_{\alpha,i} - \mathcal{X}_{\alpha+1,i}} \cdots (y_i^{\mathbf{w}_i-1})^{\mathcal{X}_{\mathbf{w}_i-1,i} - \mathcal{X}_{\mathbf{w}_i,i}}$$

$$\times \begin{cases} (y_i^{\mathbf{w}_i})^{\mathcal{X}_{\mathbf{w}_i,i} - \mathcal{X}_{1,i+1} - \delta_{i+1,0} \mathcal{X}_*} & \text{if } \mathcal{X}_{\mathbf{w}_i,i} - \mathcal{X}_{1,i+1} - \delta_{i+1,0} \mathcal{X}_* \geq 0, \\ (z_{i+1}^0)^{-\mathcal{X}_{\mathbf{w}_i,i} + \mathcal{X}_{1,i+1} + \delta_{i+1,0} \mathcal{X}_*} & \text{otherwise.} \end{cases}$$

Note that powers in the first line are nonnegative by the assumption (4.2). The power in the second line is nonnegative by the definition.

By factorization  $\mathcal{M}_{\varkappa}(\underline{\mathbf{v}}, \underline{\mathbf{w}})$  is isomorphic to product of bow varieties with  $\dim V_i^0 = \dim V_i^1 = \dots = \dim V_i^{\mathbf{w}_i} = 1$ ,  $\dim V_j^\alpha = 0$  ( $j \neq i$ ) over  $\mathbb{A}^{|\underline{\mathbf{v}}|}$ . Those bow varieties are [NT17, 6.5.1] ( $n = 1$ ) and [NT17, 6.5.3] ( $n > 1$ ). In either cases, they are locally isomorphic to  $\mathbb{C} \times \mathbb{C}^\times$ , as we exclude  $w_i^k = 0$ . We also see that  $y_i^\alpha, z_{i+1}^0$  are nonvanishing over  $\mathbb{A}^{|\underline{\mathbf{v}}|}$ , hence  $y^\varkappa$  also.

Let us turn to the gauge theory side. We define the flavor symmetry as follows: We consider the action of  $T(\underline{\mathbf{w}}) = \prod_i T^{\mathbf{w}_i}$  on  $\mathbf{N}$  induced from the standard action of  $T^{\mathbf{w}_i}$  on  $\mathbb{C}^{\mathbf{w}_i}$ . Together with  $G$ , we have an action of  $(G \times T(\underline{\mathbf{w}}))/\mathbb{C}^\times$ , where  $\mathbb{C}^\times$  is embedded in  $G \times \prod_i T^{\mathbf{w}_i}$  as the diagonal scalars. We have an extra  $\mathbb{C}_{\text{dil}}^\times$  acting on  $\mathbf{N}$  by scaling on the component  $\text{Hom}(\mathbb{C}^{\mathbf{v}_{n-1}}, \mathbb{C}^{\mathbf{v}_0})$ . Let  $\tilde{G} = \mathbb{C}_{\text{dil}}^\times \times (G \times T(\underline{\mathbf{w}}))/\mathbb{C}^\times$ ,  $G_F = \tilde{G}/G = \mathbb{C}_{\text{dil}}^\times \times T(\underline{\mathbf{w}})/\mathbb{C}^\times$ . Then  $H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$  is a module over  $H_*^{G \circ}(\mathcal{R}) = \mathbb{C}[\mathcal{M}_C(G, \mathbf{N})]$  by the construction in [Part II, §3(ix)]. Here  $\tilde{\pi}: \tilde{\mathcal{R}} = \mathcal{R}_{\tilde{G}, \mathbf{N}} \rightarrow \text{Gr}_{G_F}$  and  $\mathcal{R}^\varkappa = \tilde{\pi}^{-1}(\varkappa)$  as before, and  $\varkappa = (\varkappa_*, \varkappa_{\alpha, i})$  is a coweight of  $G_F$ , regarded as a point in  $\text{Gr}_{G_F}$ . We can also consider  $\mathcal{M}_C^\varkappa(G, \mathbf{N}) = \text{Proj}(\bigoplus_{n \geq 0} H_*^{G \circ}(\tilde{\mathcal{R}}^{n\varkappa}))$ , which is endowed with a projective morphism  $\mathcal{M}_C^\varkappa(G, \mathbf{N}) \rightarrow \mathcal{M}_C(G, \mathbf{N})$ . Let us use the standard basis of  $\mathbb{C}^{\mathbf{v}_i}$  to take a maximal torus  $T$  of  $G$  consisting of diagonal matrices. We identify  $\mathbb{A}^{\underline{\mathbf{v}}}$  with the spectrum of  $H_G^*(\text{pt}) = H_T^*(\text{pt})^{\Pi \mathfrak{S}_{\mathbf{v}_i}}$ . We have  $\varpi: \mathcal{M}_C(G, \mathbf{N}) \rightarrow \mathbb{A}^{\underline{\mathbf{v}}}$  given by the structural homomorphism  $H_G^*(\text{pt}) \rightarrow H_*^{G \circ}(\mathcal{R})$  when  $\varkappa = 0$ . We compose  $\mathcal{M}_C^\varkappa(G, \mathbf{N}) \rightarrow \mathcal{M}_C(G, \mathbf{N})$  with  $\varpi$  to apply [Part II, Theorem 5.26] to  $\mathcal{M}_C^\varkappa(G, \mathbf{N})$  later.

Let  $\mathbf{N}_T$  denote  $\mathbf{N}$  regarded as a  $T$ -module. We have the pushforward homomorphism  $\iota_*: H_*^{T \circ}(\mathcal{R}_{T, \mathbf{N}_T}) \rightarrow H_*^{T \circ}(\mathcal{R}_{G, \mathbf{N}}) = H_*^{G \circ}(\mathcal{R}_{G, \mathbf{N}}) \otimes_{H_G^*(\text{pt})} H_T^*(\text{pt})$  of the inclusion  $\mathcal{R}_{T, \mathbf{N}_T} \rightarrow \mathcal{R}_{G, \mathbf{N}}$  (see [Part II, §5(iii)]). We put the flavor symmetry as above for  $T$ , i.e.,  $\tilde{T} \stackrel{\text{def.}}{=} \mathbb{C}_{\text{dil}}^\times \times (T \times T(\underline{\mathbf{w}}))/\mathbb{C}^\times$ . We have  $\tilde{T}/T = \mathbb{C}_{\text{dil}}^\times \times T(\underline{\mathbf{w}})/\mathbb{C}^\times = G_F$ . We consider  $\tilde{\pi}_T: \mathcal{R}_{\tilde{T}, \mathbf{N}_T} \rightarrow \text{Gr}_{G_F}$  as above, and  $\tilde{\pi}_T^{-1}(\varkappa)$ . We have a natural inclusion  $\tilde{\pi}_T^{-1}(\varkappa) \rightarrow \tilde{\mathcal{R}}^\varkappa = \tilde{\pi}^{-1}(\varkappa)$ , denoted again by  $\iota$ , and the pushforward homomorphism

$$\iota_*: H_*^{T \circ}(\tilde{\pi}_T^{-1}(\varkappa)) \rightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa) \otimes_{H_G^*(\text{pt})} H_T^*(\text{pt}).$$

Let  $\pi_T: \mathcal{R}_{\tilde{T}, \mathbf{N}_T} \rightarrow \text{Gr}_{\tilde{T}}$  be the projection.

Next we introduce a class  $y^\varkappa \in H_*^{T \circ}(\tilde{\pi}_T^{-1}(\varkappa))$ , whose image under  $\iota_*$  will be identified with  $y^\varkappa$ . We begin with  $y_i^\alpha, z_{i+1}^0$ , which will be identified with  $y_i^\alpha, z_{i+1}^0$  respectively. For  $y_i^\alpha$  we choose  $\varkappa$  so that

$$(4.7) \quad \begin{aligned} & \text{the corresponding component } -\varkappa_{\alpha, i} + \varkappa_{\alpha+1, i} \text{ (or } -\varkappa_{\mathbf{w}_i, i} + \\ & \varkappa_{1, i+1} + \delta_{i+1, 0} \varkappa_* \text{ if } \alpha = \mathbf{w}_i) \text{ is } -1, \text{ and all others appearing} \\ & \text{in powers of (4.1) are zero.} \end{aligned}$$



For  $\mathbf{z}_i^0$  we choose  $\varkappa$  so that  $-\varkappa_{\mathbf{w}_i,i} + \varkappa_{1,i+1} + \delta_{i+1,0}\varkappa_*$  is 1 instead. A choice of  $\varkappa$  is unique up to overall shifts of  $T(\mathbf{w})$ -component. We lift this coweight  $\varkappa$  of  $G_F$  to  $\tilde{T}$  by setting the  $T$ -component as

$$(4.8) \quad \underbrace{(\varkappa_{1,i}, \dots, \varkappa_{1,i})}_{\mathbf{v}_i \text{ times}}_{i=0}^{n-1}.$$

Let us denote the lift by  $\tilde{\varkappa}$ . We define  $y_i^\alpha$  and  $\mathbf{z}_i^0$  as the fundamental class of  $\pi_T^{-1}(\tilde{\varkappa})$  according to the choice of  $\tilde{\varkappa}$ . This is an analog of  $y_{i,k}$  considered in [NT17, §6.8.1]. By the localization theorem, it is nonvanishing over  $\mathbb{A}^{|\mathbf{v}|}$ . For general  $\varkappa$  we define  $y^\varkappa$  as the product in (4.6) with  $y_i^\alpha, \mathbf{z}_{i+1}^0$  replaced by  $y_i^\alpha, \mathbf{z}_{i+1}^0$ .

We define a rational isomorphism

$$\theta : \Gamma(\mathcal{M}_0(\mathbf{v}, \mathbf{w}), \pi_*(\mathcal{L}_\varkappa)) \dashrightarrow H_*^{G^\circ}(\tilde{\pi}^{-1}(\varkappa))$$

by sending  $y^\varkappa$  to  $\iota_* y^\varkappa$ . It is  $\coprod \mathfrak{S}_{\mathbf{v}_i}$ -equivariant, hence it is indeed an isomorphism as above.

**Theorem 4.9.** *Under the assumption (4.2)  $\theta$  extends to an isomorphism  $\Gamma(\mathcal{M}_0(\mathbf{v}, \mathbf{w}), \pi_*(\mathcal{L}_\varkappa)) \xrightarrow{\sim} H_*^{G^\circ}(\tilde{\mathcal{R}}^\varkappa)$  of  $\mathbb{C}[\mathcal{M}_0(\mathbf{v}, \mathbf{w})] = H_*^{G^\circ}(\mathcal{R})$ -modules.*

*Proof.* As in the proofs of [Quiver, Theorem 3.10], [NT17, Th. 6.18], we need to study how the Coulomb branch and the bow variety look like around the general points  $t$  of the boundary divisor in  $\mathbb{A}^{|\mathbf{v}|}$ . In our case,

- (a)  $w_{i-1,k}(t) = w_{i,l}(t)$  for some  $i, k, l$ , but all others are distinct. Moreover  $w_{j,r}(t) \neq 0$  if  $\mathbf{w}_j \neq 0$ . (We understand  $i \neq i - 1$ , hence  $n \geq 2$ .)
- (b)  $w_{i,k}(t) = w_{i,l}(t)$  for distinct  $k, l$  and some  $i$ , but all others are distinct. Moreover  $w_{j,r}(t) \neq 0$  if  $\mathbf{w}_j \neq 0$ .
- (c) All pairs like in (a),(b) are distinct, but  $w_{i,k}(t) = 0$  for  $i$  with  $\mathbf{w}_i \neq 0$ .

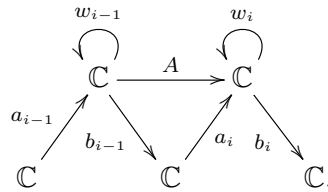
See the proof of [NT17, Th. 6.18]. The gauge theory  $(G, \mathbf{N}, \tilde{G})$  with the flavor symmetry group  $\tilde{G}$  is replaced by  $(Z_G(t), \mathbf{N}^t, Z_{\tilde{G}}(t))$ . In our case,  $Z_{\tilde{G}}(t) = \mathbb{C}_{\text{dil}}^\times \times (Z_G(t) \times T(\mathbf{w}))/\mathbb{C}^\times$ , and  $(Z_G(t), \mathbf{N}^t) = (\text{GL}(\mathbf{v}') \times T^{|\mathbf{v}''|}, \mathbf{N}(\mathbf{v}', \mathbf{w}'))$ , where  $\mathbf{v}', \mathbf{w}'$  are given below,  $\mathbf{v}'' = \mathbf{v} - \mathbf{v}'$  and  $T^{|\mathbf{v}''|}$  acts trivially on  $\mathbf{N}(\mathbf{v}', \mathbf{w}')$ :

- (a)  $\mathbf{w}' = 0, \mathbf{v}'_i = 1 = \mathbf{v}'_{i-1}$  and other entries are 0.
- (b)  $\mathbf{w}' = 0, \mathbf{v}'_i = 2$  and other entries are 0.
- (c)  $\mathbf{v}'_i = 1, \mathbf{w}'_i = \mathbf{w}_i$  and other entries are 0.

The extra factor  $T(\underline{\mathbf{w}})$  acts trivially in (a),(b), while it acts through  $T(\underline{\mathbf{w}}) \rightarrow T^{\mathbf{w}_i}$  in (c). On the other hand  $\mathbb{C}_{\text{dil}}^\times$  acts trivially in (b),(c) and (a) with  $i \neq 0$ .

By the same argument as in the proofs of [Quiver, Theorem 3.10], [NT17, Th. 6.18] both  $y^\varkappa$  and  $y^\varkappa$  are related to  $y'^\varkappa, y'^\varkappa$  by nonvanishing regular functions defined on a neighborhood of  $t$  in  $\mathbb{A}^{|\mathbf{v}|}$  under the factorization. Therefore it is enough to check that the isomorphism  $\theta$  extends for the local models (a),(b),(c) above.

Consider the case (a) with  $n \geq 3$ . Let us consider the local model for the bow variety side. It is [NT17, 6.5.6]:



Since we assume  $w_{i-1}, w_i \neq 0$ , the relevant  $C_{\alpha,i-1}, D_{\alpha,i-1}, C_{\beta,i}, D_{\beta,i}$  ( $\alpha = 1, \dots, \mathbf{w}_{i-1}, \beta = 1, \dots, \mathbf{w}_i$ ) are isomorphisms, hence can be normalized by the group action and defining equations. Thus they are omitted. It is also clear that the  $\varkappa$ -stability condition is automatically satisfied, hence

$$\mathcal{M}_\varkappa(\underline{\mathbf{v}}, \underline{\mathbf{w}}') \cong \mathcal{M}_0(\underline{\mathbf{v}}, \underline{\mathbf{w}}').$$

We normalize  $a_{i-1} = 1, b_i = 1$  thanks to the conditions (S1),(S2). The defining equation is  $(w_i - w_{i-1})A = a_i b_{i-1}$ . On the other hand, we have introduced functions  $y_{i-1}, y_i, y_{i-1,i}$  in [NT17, 6.5.6], which are  $y_{i-1} = b_{i-1} a_{i-1} = b_{i-1}, y_i = b_i a_i = a_i, y_{i-1,i} = A$ . (We change  $y_{i-1,i}$  in [NT17] by its inverse.) The variety  $\mathcal{M}_0(\underline{\mathbf{v}}, \underline{\mathbf{w}}')$  is

$$\{(w_{i-1}, w_i, y_{i-1}, y_i, y_{i-1,i}^{\pm 1}) \mid y_{i-1} y_i = y_{i-1,i} (w_i - w_{i-1})\}.$$

In this case, line bundles  $\det V_{i-1}^\alpha, (\det V_i^\beta)^*$  are trivialized by their non-vanishing sections  $C_{\alpha,i-1} \cdots C_{1,i-1} a_{i-1} = y_{i-1} / y_{i-1}^\alpha, b_i C_{\mathbf{w}_i,i} \cdots C_{\beta+1,i} = y_i^\beta$ , and sections  $y_{i-1}^\alpha, z_i^0, y_i^\beta, z_{i+1}^0$  are identified with  $y_{i-1}, y_i, 1, 1$  respectively. Therefore

$$y^\varkappa = \begin{cases} y_{i-1}^{\varkappa_{1,i-1} - \varkappa_{1,i} - \delta_{i,0} \varkappa_*} & \text{if } \varkappa_{\mathbf{w}_{i-1},i-1} - \varkappa_{1,i} - \delta_{i,0} \varkappa_* \geq 0, \\ y_{i-1}^{\varkappa_{1,i-1} - \varkappa_{\mathbf{w}_{i-1},i-1}} y_i^{-\varkappa_{\mathbf{w}_{i-1},i-1} + \varkappa_{1,i} + \delta_{i,0} \varkappa_*} & \text{otherwise.} \end{cases}$$

Next let us consider the local model in the Coulomb branch side. The group  $T(\underline{\mathbf{w}})$  acts trivially on  $\mathbf{N}(\underline{\mathbf{v}}, \underline{\mathbf{w}}')$ . The extra  $\mathbb{C}_{\text{dil}}^\times$ -action appears when

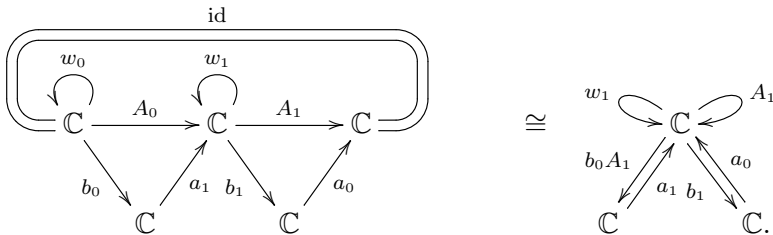
$i = 0$ , but it can be absorbed to the  $\mathrm{GL}(\mathbf{v}_{i-1})$ -action, as we assume  $n \geq 3$ . We take an isomorphism  $Z_{\tilde{G}}(t) \cong \mathbb{C}^\times \times \mathbb{C}^\times \times T^{|\mathbf{v}'|} \times G_F$ , then  $H_*^{G \circ}(\mathcal{R}^\varkappa) \cong H_*^{G \circ}(\mathcal{R})$ . It means that the line bundle is trivialized. Then  $y_{i-1}^\alpha, z_i^0, y_i^\beta$  or  $z_{i+1}^0$  is the fundamental class of the fiber over the coweight  $(\varkappa_{1,i-1} - \delta_{i0}\varkappa_*, \varkappa_{1,i})$  of  $\mathrm{GL}(\mathbf{v}_{i-1}) \times \mathrm{GL}(\mathbf{v}_i)$  according to a suitable choice of  $\varkappa$  as in (4.7). (The ambiguity of shifts does not matter, as it only gives an invertible function.) Now recall  $y_{i-1}, y_i, y_{i-1,i}$  are fundamental classes of fibers over  $(1, 0), (0, 1), (1, 1)$  respectively under  $\mathcal{M}_C(\underline{\mathbf{v}}', \underline{\mathbf{w}}') \cong \{y_{i-1}y_i = y_{i-1,i}(w_i - w_{i-1})\}$  by [Part II, Theorem 4.1]. Thus  $y_{i-1}^\alpha, z_i^0, y_i^\beta, z_{i+1}^0$  are equal to  $y_{i-1}, y_i, 1, 1$  up to invertible functions respectively. Since both  $y^\varkappa$  and  $z^\varkappa$  are defined as products, they are equal up to an invertible function. Therefore the isomorphism of line bundles extends over  $\mathcal{M}_0(\underline{\mathbf{v}}', \underline{\mathbf{w}}')$ .

For (a) with  $n = 2$ , the gauge theory side is reduced to the case  $(\mathrm{GL}(\underline{\mathbf{v}}'), \mathbf{N}(\underline{\mathbf{v}}', 0)) = (\mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C} \oplus \mathbb{C})$  with the  $\mathbb{C}^\times \times \mathbb{C}^\times$ -action  $(t_0, t_1) \cdot (x, y) = (t_1 t_0^{-1}x, t_0 t_1^{-1}y)$  and the flavor group  $G_F$  remains only as the  $\mathbb{C}_{\mathrm{dil}}^\times$ -action by  $t_* \cdot (x, y) = (x, t_*y)$  for  $t_* \in \mathbb{C}_{\mathrm{dil}}^\times$ . Since the diagonal subgroup  $\mathbb{C}^\times \subset \mathbb{C}^\times \times \mathbb{C}^\times$  acts trivially on  $\mathbb{C} \oplus \mathbb{C}$ , the action factors through the quotient  $\mathbb{C}^\times \times \mathbb{C}^\times \rightarrow \mathbb{C}^\times; (t_0, t_1) \mapsto t_0 t_1^{-1}$ . The Coulomb branch has the corresponding factor  $\mathbb{C} \times \mathbb{C}^\times = \mathcal{M}_C(\mathbb{C}^\times, 0)$ . We can change the second summand  $\mathbb{C}$  of  $\mathbb{C} \oplus \mathbb{C}$  by its dual thanks to [Part II, §4(v)]. Hence we are reduced to the situation in [Ring, §4(iii)] with  $V = \mathbb{C}, W = \mathbb{C}^2$ . In particular,

$$\mathcal{M}_C(\mathrm{GL}(\underline{\mathbf{v}}'), \mathbf{N}(\underline{\mathbf{v}}', 0)) = \mathbb{C} \times \mathbb{C}^\times \times \mathcal{M}_C(\mathbb{C}^\times, \mathbb{C}^2) = \mathbb{C} \times \mathbb{C}^\times \times \mathcal{S}_2,$$

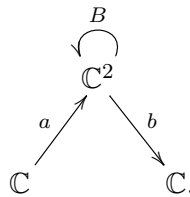
and the corresponding  $\mathcal{M}_{\tilde{G}}^\varkappa(\mathrm{GL}(\underline{\mathbf{v}}'), \mathbf{N}(\underline{\mathbf{v}}', 0))$  is  $\mathbb{C} \times \mathbb{C}^\times \times T^*\mathbb{P}^1$ . According to the choice of  $\varkappa$  as in (4.7), the section  $y_0^\alpha, y_1^\beta, z_0^0$  or  $z_1^0$  is the fundamental class of fiber over  $(\varkappa_{1,0} - \varkappa_{1,1}, -\varkappa_*, 0) \in \mathrm{Gr}_{\tilde{G}}$  if we identify  $\mathrm{Gr}_{\tilde{G}}$  with the coweight lattice of  $\tilde{G} = \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times / \mathbb{C}^\times$ , and also with  $\mathbb{Z}^3 / \mathbb{Z}$ . Concretely  $(\varkappa_{1,0} - \varkappa_{1,1}, -\varkappa_*, 0)$  is  $(1, 1, 0), (0, 1, 0), (-1, -1, 0), (0, -1, 0)$  for  $y_0^\alpha, y_1^\beta, z_1^0, z_0^0$  respectively.

On the other hand, the local model of the bow variety is given in [NT17, 6.5.4] with  $\mathbf{w}_1 = \mathbf{w}_2 = 0$ . Since  $A_0$  is an isomorphism by the conditions (S1),(S2), we can normalize it to 1. Then we can factor out  $(w_1, A_1) \in \mathbb{C} \times \mathbb{C}^\times$ , and the remaining factor is  $\mathcal{S}_2$  and its resolution  $T^*\mathbb{P}^1$ . Line bundles are given by characters of  $\mathbb{C}^\times$  acting on  $\mathbb{C}$  on the right side:



Moreover  $y_0^\alpha$  (resp.  $y_1^\beta$ ) is identified with  $b_0A_1$  (resp.  $b_1$ ). Since we identify the fundamental class of fiber over  $(1, 1, 0)$  (resp.  $(0, 1, 0)$ ) with  $b_0A_1$  (resp.  $b_1$ ) as in the end of [Ring, §4(iii)], we conclude that the isomorphism of line bundles extends. For  $\varkappa_* = 1$ , corresponding to the case  $z_0^0$  or  $z_1^0$ , we need to use the opposite stability condition, and  $\mathbb{P}^1$  is replaced by the dual  $\mathbb{P}^1$ . We replace linear maps above by its transpose to apply [Ring, §4(iii)]. Then the fundamental class for  $(-1, -1, 0)$  (resp.  $(0, -1, 0)$ ) are identified with  $a_1$  (resp.  $a_0$ ), and hence  $z_1^0$  (resp.  $z_0^0$ ). Therefore the assertion is true also in this case. It is also true for general  $\varkappa$  thanks to Lemma 2.2.

Next consider the case (b). First suppose  $n \geq 2$ . The local model for the bow variety is [NT17, 6.5.5]:

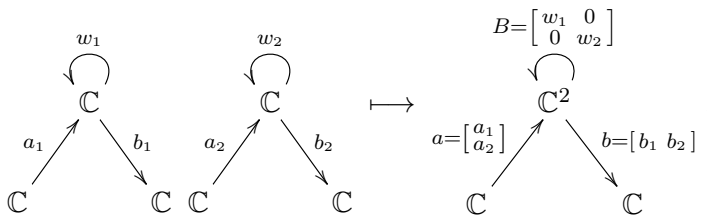


where we drop subscripts  $i$ . Linear maps  $C_{\alpha,i}, D_{\alpha,i}$  ( $\alpha = 1, \dots, \mathbf{w}_i$ ) are isomorphisms thanks to the assumption that eigenvalues of  $B$  are nonzero. Therefore they are normalized by the group action and defining equations, and omitted. We have  $\mathcal{M}_\varkappa(\underline{\mathbf{v}}', \underline{\mathbf{w}}') \cong \mathcal{M}_0(\underline{\mathbf{v}}', \underline{\mathbf{w}}')$  as before.

Let  $w_1, w_2$  be eigenvalues of  $B$ . Then  $\mathbb{C}[\mathcal{M}_0(\underline{\mathbf{v}}', \underline{\mathbf{w}}') \times_{\mathbb{A}^2} \mathbb{A}^2]$  is

$$\mathbb{C}[w_1, w_2, 'y_1^\pm, 'y_2^\pm, \xi] / ('y_1 - 'y_2 = \xi(w_1 - w_2))$$

where  $'y_1 = b(B - w_2)a, 'y_2 = b(B - w_1)a, \xi = ba$ . Thanks to the conditions (S1),(S2) we trivialize the dual of the vector bundle associated with  $V = \mathbb{C}^2$  by a frame  $\{b, bB\}$ . The factorization morphism is given by



Hence the trivialization  $b \wedge bB$  of  $\det V^*$  is  $b_1b_2(w_1 - w_2)$  over the open subset  $w_1 \neq w_2$ . On the other hand the section  $y^\alpha$  of (4.4) is  $b_1b_2(w_1 - w_2)$ .

(cf. (4.5).) Therefore  $y^\alpha = b \wedge bB$ . Thus  $y^\alpha$  extends to a nonvanishing section over  $\mathcal{M}_0(\mathbf{v}', \mathbf{w}')$ . The same is true for  $z^0$ .

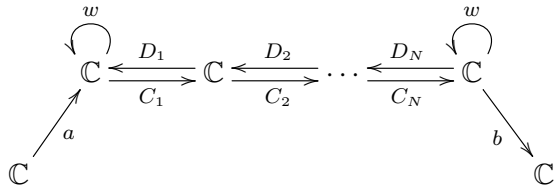
On the other hand, we have an isomorphism

$$H_*^{\text{GL}(2)^\circ}(\tilde{\mathcal{R}}^\varkappa) \cong H_*^{\text{GL}(2)^\circ}(\text{Gr}_{\text{GL}(2)})$$

if we choose an isomorphism  $Z_{\tilde{G}}(t) \cong \text{GL}(2) \times T^{|\mathbf{v}''|} \times G_F$ . The homology class  $y^\varkappa$  is identified with a power of  $'y_1, 'y_2$ , which is an invertible function. Therefore the isomorphism of line bundles extends over  $\mathcal{M}_0(\mathbf{v}', \mathbf{w}')$ .

For (b) with  $n = 1$ , we are reduced to the situation of Corollary 3.3 if  $\nu_* < 0$ . Thus the local model  $\mathcal{M}_{\tilde{\mathcal{C}}}(\text{GL}(2), \mathfrak{gl}(2))$  is  $\text{Hilb}^2(\mathcal{S}_0)$ , and the line bundle is a power of the determinant line bundle. On the other hand, the local model of the bow variety is given in [NT17, 6.5.2] with  $\mathbf{w} = 0$ . It coincides with the description in [Nak99, §1] with constraint  $A$  being invertible. It is nothing but  $\text{Hilb}^2(\mathcal{S}_0)$  and the relevant line bundles coincide. Moreover our definition of the section  $y^\varkappa$  is compatible with the open embedding  $\text{Sym}^2(\mathcal{S}_0) \setminus \Delta_{\mathcal{S}_0} \hookrightarrow \text{Sym}^2(\mathbb{A}^2) \setminus \Delta_{\mathbb{A}^2}$  ( $\Delta_?$  denotes the diagonal) as in Corollary 3.3. And the isomorphism is unique up to a multiplicative scalar on  $\text{Sym}^2(\mathbb{A}^2) \setminus \Delta_{\mathbb{A}^2}$  by Proposition 3.2(b). Therefore our isomorphism coincides with one in Corollary 3.3, hence extends over  $\Delta_{\mathcal{S}_0}$ . If  $\nu_* > 0$ , we take transposes of linear maps to deduce the assertion from the  $\nu_* < 0$  case.

Let us consider the case (c). First suppose  $n > 1$ . The local model for the bow variety side is [NT17, 6.5.3]:



where we set  $N = \mathbf{w}_i$  and drop subscripts  $i$ . We have  $\mathcal{M}_0(\mathbf{v}', \mathbf{w}')$   $\cong \mathcal{S}_N = \{YZ = W^N\}$ . Here  $a$  and  $b$  are normalized to 1 thanks to the conditions (S1),(S2), and we set  $y = C_N \cdots C_1, z = D_1 \cdots D_N$ . The section  $y^\alpha$  ( $\alpha = 1, \dots, N$ ) of the line bundle  $(\det V^\alpha)^*$  is  $bC_N \cdots C_{\alpha+1}$ . Sections  $y^N, z^0$  are nowhere vanishing, as well as the corresponding  $y^N, z^0$ . So let us ignore  $y^N, y^N, z^0, z^0$  hereafter. In particular, we omit the second line in (4.6) for the definition of  $y^\varkappa$ .

After the normalization  $a = b = 1$ , it becomes a quiver variety of type  $A_{N-1}$ . When  $\varkappa_{1,i} > \varkappa_{2,i} > \cdots > \varkappa_{\mathbf{w}_i,i}$ , it is easy to see that  $\mathcal{M}_{\varkappa}(\mathbf{v}', \mathbf{w}')$  is the minimal resolution  $\tilde{\mathcal{S}}_N$  of  $yz = w^N$  so that  $(\det V^1)^*, \dots, (\det V^{N-1})^*$

correspond to line bundles  $\mathcal{L}_{\omega_1}, \dots, \mathcal{L}_{\omega_{N-1}}$ , corresponding to weights  $\omega_1, \dots, \omega_{N-1}$  in [Ring, §4(i)]. On the other hand,  $(\det V^N)^*$  is the trivial line bundle  $\mathcal{O}_{\tilde{\mathcal{S}}_N}$ . (The  $\varkappa$ -stability under the assumption  $\varkappa_{1,i} > \varkappa_{2,i} > \dots > \varkappa_{\mathbf{w}_i,i}$  coincides with the stability used in [Nak98].) Moreover the section  $y^\alpha$  is  $v^{N-\alpha}$  under the isomorphism  $\Gamma(\tilde{\mathcal{S}}_N, \mathcal{L}_{\omega_\alpha}) \cong \mathbb{C}[\mathbb{A}^2]^{\chi_{\omega_\alpha}}$ . (This holds even for  $\alpha = N$ .) This remains true if  $\varkappa_{\alpha-1,i} > \varkappa_{\alpha,i}$ , and other inequalities may *not* be strict if we replace  $\tilde{\mathcal{S}}_N$  by a partial resolution of  $\mathcal{S}_N$ . Thus  $y^\varkappa$  is a section of the line bundle  $\mathcal{L}_\varkappa = \bigotimes_{\alpha=1}^{N-1} \mathcal{L}_{\omega_\alpha}^{\otimes(\varkappa_{\alpha,i} - \varkappa_{\alpha+1,i})}$ , given by the product  $\bigotimes_{\alpha=1}^{N-1} (v^{N-\alpha})^{\otimes(\varkappa_{\alpha,i} - \varkappa_{\alpha+1,i})}$ .

The gauge theory  $(\mathrm{GL}(\underline{\mathbf{v}}'), \mathbf{N}(\underline{\mathbf{v}}', \underline{\mathbf{w}}'))$  is one studied in [Ring, §4(iii)] with  $N = \mathbf{w}'_i$ . We have an extra  $\mathbb{C}_{\mathrm{dil}}^\times$  in the flavor symmetry group, but it acts trivially on  $\mathbf{N}(\underline{\mathbf{v}}', \underline{\mathbf{w}}')$ . Let us ignore  $\mathbb{C}_{\mathrm{dil}}^\times$  from now on. Recall  $y^\varkappa$  is the fundamental class of  $\pi_T^{-1}(\tilde{\varkappa})$  where  $\tilde{\varkappa} = (\varkappa_{1,i}, \varkappa_{1,i}, \varkappa_{2,i}, \dots, \varkappa_{N,i})$  is a coweight of  $(\mathbb{C}^\times \times T^N)/\mathbb{C}^\times = (\mathrm{GL}(\underline{\mathbf{v}}'_i) \times T^{\mathbf{w}'_i})/\mathbb{C}^\times$ . This is so for the lift  $\tilde{\varkappa}$  of a particular  $\varkappa$  as in (4.7), but remains to be true for  $\tilde{\varkappa}$  of arbitrary  $\varkappa$  with (4.2) if we ignore the second line in (4.7). See [Part II, Theorem 4.1]. On the other hand, the fundamental class of  $\pi_T^{-1}(\tilde{\omega}_\alpha)$  corresponds to  $v^{N-\alpha}$  by the computation in [Ring, §4(iii)], where  $\tilde{\omega}_\alpha = (1, \underbrace{1, \dots, 1}_{\alpha \text{ times}}, \underbrace{0, \dots, 0}_{N-\alpha \text{ times}})$  is

also a coweight of  $(\mathbb{C}^\times \times T^N)/\mathbb{C}^\times$ . Since

$$\sum_{\alpha=1}^{N-1} (\varkappa_{\alpha,i} - \varkappa_{\alpha+1,i}) \tilde{\omega}_\alpha = \varkappa$$

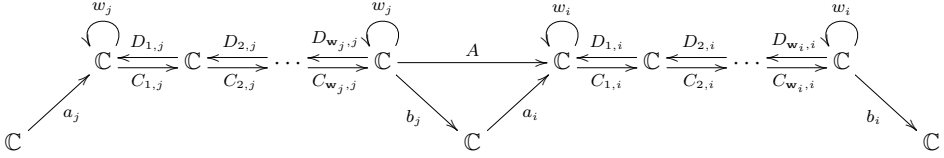
holds (up to shift), the class  $y^\varkappa$  is equal to  $\bigotimes_{\alpha=1}^{N-1} (v^{N-\alpha})^{\otimes(\varkappa_{\alpha,i} - \varkappa_{\alpha+1,i})}$ , which is nothing but  $y^\varkappa$ . This is nothing but the isomorphism normalized as in Remark 2.3. Thus the isomorphism extends over  $\mathcal{M}_\varkappa(\underline{\mathbf{v}}', \underline{\mathbf{w}}')$ .

If  $n = 1$ , we have  $\mathbf{N}(\underline{\mathbf{v}}', \underline{\mathbf{w}}') = \mathrm{End}(\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C}^{\mathbf{w}'_i}, \mathbb{C})$  and  $\mathrm{GL}(\underline{\mathbf{v}}') = \mathbb{C}^\times$  acts trivially on the summand  $\mathrm{End}(\mathbb{C})$ . On the other hand  $\mathbb{C}_{\mathrm{dil}}^\times$  acts on  $\mathrm{End}(\mathbb{C})$  by scaling and trivially on  $\mathrm{Hom}(\mathbb{C}^{\mathbf{w}'_i}, \mathbb{C})$ . Then we can separate  $\mathrm{End}(\mathbb{C})$  and  $\mathrm{Hom}(\mathbb{C}^{\mathbf{w}'_i}, \mathbb{C})$ , and both are already treated. □

### 4(ii). Computation

For a later purpose we compute the case (a) with  $n \geq 3$  in more detail. Let us drop the assumption  $\underline{\mathbf{w}}' = 0$  and study general cases with  $\mathbf{w}'_i, \mathbf{w}'_{i-1}$ . Let us also write  $j$  instead of  $i - 1$ . Let us suppose  $i \neq 0$  for brevity. Therefore we ignore  $\varkappa_*$ . Let us also drop ‘ $\prime$ ’ from dimension vectors.

Let us consider the local model for the bow variety side. It is [NT17, 6.5.6]:



Note that  $\mathbb{C}[\mathcal{M}_0(\underline{\mathbf{v}}, \underline{\mathbf{w}})]$  written in [NT17, 6.5.6] is wrong, hence we will give a detail.

We normalize  $a_j = 1, b_i = 1$  thanks to the conditions (S1),(S2). We also know that  $A \neq 0$  thanks to (S1),(S2). The defining equation for the middle triangle is  $(w_i - w_j)A = a_i b_j$ .

We introduce functions

$$\begin{aligned} z_j &= D_{1,j} \cdots D_{\mathbf{w}_j,j} A^{-1} a_i, & z_i &= b_j A^{-1} D_{1,i} \cdots D_{\mathbf{w}_i,i}, \\ z_{j,i} &= D_{1,j} \cdots D_{\mathbf{w}_j,j} A^{-1} D_{1,i} \cdots D_{\mathbf{w}_i,i}, \\ y_j &= b_j C_{\mathbf{w}_j,j} \cdots C_{1,j}, & y_i &= C_{\mathbf{w}_i,i} \cdots C_{1,i} a_i, \\ y_{j,i} &= C_{\mathbf{w}_i,i} \cdots C_{1,i} A C_{\mathbf{w}_j,j} \cdots C_{1,j}. \end{aligned}$$

Then

$$\begin{aligned} z_j z_i &= (w_i - w_j) z_{j,i}, \\ y_j y_i &= (w_i - w_j) y_{j,i}, & z_{j,i} y_{j,i} &= w_i^{\mathbf{w}_i} w_j^{\mathbf{w}_j}, \\ z_i y_i &= (w_i - w_j) w_i^{\mathbf{w}_i}, \\ z_j y_j &= (w_i - w_j) w_j^{\mathbf{w}_j} \\ z_i y_{j,i} &= w_i^{\mathbf{w}_i} y_j, & z_j y_{j,i} &= w_j^{\mathbf{w}_j} y_i, & y_i z_{j,i} &= w_i^{\mathbf{w}_i} z_j, & y_j z_{j,i} &= w_j^{\mathbf{w}_j} z_i. \end{aligned}$$

We have  $\mathcal{M}_0(\underline{\mathbf{v}}, \underline{\mathbf{w}}) \cong \{(w_j, w_i, y_j, y_i, y_{j,i}, z_j, z_i, z_{j,i}) \mid \text{above equations}\}$ . On the other hand, this is isomorphic to the Coulomb branch, where  $y_j, y_i, y_{j,i}$  are fundamental classes of fibers over  $(1, 0), (0, 1), (1, 1)$ , and  $z_j, z_i, z_{j,i}$  are those over  $(-1, 0), (0, -1), (-1, -1)$ .

Let us suppose  $w_j, w_i \neq 0$ . Then all  $C_{\alpha,j}, D_{\alpha,j}, C_{\beta,i}, D_{\beta,i}$  become isomorphisms. Since  $z_{j,i} y_{j,i} = w_i^{\mathbf{w}_i} w_j^{\mathbf{w}_j}, z_{j,i}$  and  $y_{j,i}$  are invertible. We can eliminate  $z_{j,i}, z_i = y_{j,i}^{-1} w_i^{\mathbf{w}_i} y_j, z_j = y_{j,i}^{-1} w_j^{\mathbf{w}_j} y_i$ . Hence

$$\mathcal{M}_0(\underline{\mathbf{v}}, \underline{\mathbf{w}})|_{w_j, w_i \neq 0} \cong \{(w_j^{\pm 1}, w_i^{\pm 1}, y_j, y_i, y_{j,i}^{\pm 1}) \mid y_j y_i = y_{j,i} (w_i - w_j)\}.$$

On the other hand when  $w_j \neq w_i$ , we can eliminate  $y_{j,i} = (w_i - w_j)^{-1}y_j y_i$ ,  $z_{j,i} = (w_i - w_j)^{-1}z_j z_i$ . Hence

$$\begin{aligned} \mathcal{M}_0(\mathbf{v}, \mathbf{w})|_{w_j \neq w_i} &\cong \{(w_j, w_i, y_j, y_i, z_j, z_i) \mid y_j z_j \\ &= (w_i - w_j)w_j^{\mathbf{w}_j}, y_i z_i = (w_i - w_j)w_i^{\mathbf{w}_i}\}_{|w_i \neq w_j}. \end{aligned}$$

This is an open subset in the product of type  $A_{\mathbf{w}_j-1}$  and  $A_{\mathbf{w}_i-1}$  simple singularities.

Let us recall sections  $y_j^\alpha = b_j C_{\mathbf{w}_j,j} \cdots C_{\alpha+1,j}$ ,  $y_i^\beta = b_i C_{\mathbf{w}_i,i} \cdots C_{\beta+1,i}$  of  $(\det V_j^\alpha)^*$ ,  $(\det V_i^\beta)^*$  respectively. We consider other sections

$$\begin{aligned} {}'y_j^\alpha &= C_{\mathbf{w}_i,i} \cdots C_{1,i} A C_{\mathbf{w}_j,j} \cdots C_{\alpha+1,j}, & z_j^\alpha &\stackrel{\text{def.}}{=} D_{1,j} \cdots D_{\alpha,j}, \\ z_i^\beta &\stackrel{\text{def.}}{=} b_j A^{-1} D_{1,i} \cdots D_{\beta,i}, & {}'z_i^\beta &\stackrel{\text{def.}}{=} D_{1,j} \cdots D_{\mathbf{w}_j,j} A^{-1} D_{1,i} \cdots D_{\beta,i}. \end{aligned}$$

We have

$$\begin{aligned} y_j {}'y_j^\alpha &= y_{j,i} y_j^\alpha \\ z_j y_j^\alpha &= (w_i - w_j) w_j^{\mathbf{w}_j - \alpha} z_j^\alpha, \\ y_i y_j^\alpha &= (w_i - w_j) {}'y_j^\alpha, \\ z_{j,i} y_j^\alpha &= w_j^{\mathbf{w}_j - \alpha} z_i z_j^\alpha. \end{aligned}$$

Note  $z_i z_j^\alpha = b_j A^{-1} D_{1,i} \cdots D_{\mathbf{w}_i,i} D_{1,j} \cdots D_{\alpha,j}$ . Similarly we have

$$\begin{aligned} y_j z_j^\alpha &= w_j^\alpha y_j^\alpha, \\ y_{j,i} z_j^\alpha &= w_j^\alpha {}'y_j^\alpha. \end{aligned}$$

Let us consider the local model in the Coulomb branch side. Let us take a coweight  $(m, 1^\alpha, 0^{\mathbf{w}_j - \alpha}, n, 0^{\mathbf{w}_i})$  of  $(\text{GL}(V_j) \times T^{\mathbf{w}_j} \times \text{GL}(V_i) \times T^{\mathbf{w}_i})/\mathbb{C}^\times$ . Let  $\alpha_{r,m,n}$  denote the fundamental class of the fiber for the projection  $\tilde{\mathcal{R}} \rightarrow \text{Gr}_{\tilde{G}}$ . We can compute products of  $\alpha_{r,m,n}$  with  $y_i, y_j, y_{j,i}, z_i, z_j, z_{j,i}$  by the formula in [Part II, §4]. A calculation shows that

$$(4.10) \quad \alpha_{r,m,n} = \begin{cases} y_j^{m-n-1} y_{j,i}^n y_j^\alpha & \text{if } m > n \geq 0, \\ z_i^{-n} y_j^{m-1} y_j^\alpha & \text{if } m > 0 \geq n, \\ y_i^{n-m} y_{j,i}^{m-1} {}'y_j^\alpha & \text{if } n \geq m > 0, \\ y_i^n z_j^{-m} z_j^\alpha & \text{if } n \geq 0 \geq m, \\ z_i^{m-n} z_{j,i}^{-m} z_j^\alpha & \text{if } 0 \geq m \geq n, \\ z_j^{n-m} z_{j,i}^{-n} z_j^\alpha & \text{if } 0 \geq n \geq m \end{cases}$$

gives an isomorphism of  $\mathbb{C}[\mathcal{M}_0(\mathbf{v}, \mathbf{w})]$ -modules.



### 5. Determinant line bundles on convolution diagram over the affine Grassmannian

In this section we identify the determinant line bundles on the convolution diagrams over slices in the affine Grassmannian, or rather global sections of their pushforwards to the slices, with the modules over the Coulomb branches of the corresponding quiver gauge theories arising from the construction of [Part II, §3(ix)].

#### 5(i). Slices revisited

Recall the setup and notations of [Quiver, §2(x)]. We define the iterated convolution diagram  $\widetilde{\mathcal{W}}_\mu^\lambda$  as the moduli space of the following data:

- (a) a collection of  $G$ -bundles  $\mathcal{P}_{\text{triv}} = \mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_N$  on  $\mathbb{P}^1$ ;
- (b) a collection of rational isomorphisms  $\sigma_s: \mathcal{P}_{s-1} \rightarrow \mathcal{P}_s$ ,  $1 \leq s \leq N$ , regular over  $\mathbb{P}^1 \setminus \{0\}$ , with a pole of degree  $\leq \omega_{i_s}$  at 0;
- (c) a  $B$ -structure  $\phi$  on  $\mathcal{P}_N$  of degree  $w_0\mu$  having fiber  $B_- \subset G$  at  $\infty \in \mathbb{P}^1$  (with respect to the trivialization  $\sigma := \sigma_N \circ \dots \circ \sigma_1$ ).

We have an evident proper birational projection  $\pi: \widetilde{\mathcal{W}}_\mu^\lambda \rightarrow \overline{\mathcal{W}}_\mu^\lambda$  (where  $\lambda = \sum_{s=1}^N \omega_{i_s}$ ), sending  $(\mathcal{P}_0, \dots, \mathcal{P}_N, \sigma_1, \dots, \sigma_N, \phi)$  to  $(\mathcal{P}_N, \sigma, \phi)$ .

More generally, we will need an evident generalization  $\pi: \widetilde{\mathcal{W}}_\mu^\lambda \rightarrow \overline{\mathcal{W}}_\mu^\lambda$  for an arbitrary sequence of dominant coweights  $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ ,  $\sum_{s=1}^n \lambda_s = \lambda$ , in place of  $(\omega_{i_1}, \dots, \omega_{i_N})$ .

Now recall the setup and notations of [Quiver, §2(ix)]; in particular, we set  $\alpha = \lambda - \mu$ . We pick  $\mathbb{N}[Q_0] \ni \gamma \leq \alpha$ , and set  $\beta = \alpha - \gamma$ .

**Proposition 5.1.** *We have a factorization isomorphism of the varieties over  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^{\gamma^*})_{\text{disj}}$ :*

$$(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^{\gamma^*})_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda \xrightarrow{\sim} (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^{\gamma^*})_{\text{disj}} \times_{\mathbb{A}^{\beta^*} \times \mathbb{A}^{\gamma^*}} (\mathbb{Z}^{\beta^*} \times \widetilde{\mathcal{W}}_{\lambda-\gamma}^\lambda).$$

*It is compatible with the factorization isomorphism of zastava (see [Quiver, §2(i)]) under projection  $s_\mu^\lambda \circ \pi$ .*

*Proof.* The same argument as in the proof of [BFGM02, Proposition 2.4].  $\square$

We fix  $i \in Q_0$ ; recall that  $\alpha_i$  is the corresponding simple coroot. In what follows we will use a particular case of Proposition 5.1 similar to [Quiver, Proposition 2.9], where  $\gamma = \alpha_i$  and  $\beta = \alpha - \alpha_i$ . Here we are additionally able to identify  $\widetilde{\mathcal{W}}_\mu^\lambda$  with the minimal resolution of the Kleinian surface  $\mathcal{S}_{\langle \lambda, \alpha_i \rangle}$ .

Recall the birational isomorphism of [Quiver, §2(ix)]

$$\varphi: (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda \dashrightarrow (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\beta^*} \times \mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \mathcal{S}_{\langle \lambda, \alpha_i^\vee \rangle}).$$

**Proposition 5.2.** *The birational isomorphism  $\varphi$  extends to a regular isomorphism of the varieties over  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}}$ :*

$$(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda \xrightarrow{\sim} (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\beta^*} \times \mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \widetilde{\mathcal{S}}_{\langle \lambda, \alpha_i^\vee \rangle}).$$

*Proof.* Like in the proof of [Quiver, Proposition 2.9], it suffices to prove the claim over  $\mathring{Z}^{\alpha^*}$ . So we restrict to this open subset without further mentioning this and introducing new notations for the corresponding open subsets in the convolution diagrams over slices. Like in [Quiver, Proposition 2.10], we will identify  $\widetilde{\mathcal{W}}_\mu^\lambda$  with a certain blowup of  $\widetilde{\mathcal{W}}_\mu^\lambda$ . To this end we consider a convolution diagram  $\overline{\text{Gr}}_G^{\lambda_1} \widetilde{\times} \cdots \widetilde{\times} \overline{\text{Gr}}_G^{\lambda_n} \rightarrow \overline{\text{Gr}}_G^\lambda$ ,  $\sum_{s=1}^n \lambda_s = \lambda$ , and denote it by  $\pi: \widetilde{\text{Gr}}_G^\lambda \rightarrow \overline{\text{Gr}}_G^\lambda$ . Then just as in [Quiver, §2(ii)], we have  $\widetilde{\mathcal{W}}_\mu^\lambda = \widetilde{\text{Gr}}_G^\lambda \times_{\text{Bun}_G(\mathbb{P}^1)} \text{Bun}_B^{w_0\mu}(\mathbb{P}^1)$ . The sequences  $\underline{\lambda}$  we need will have at most one term not equal to a fundamental coweight, so that

$$\underline{\lambda} = (\omega_{j_1}, \dots, \omega_{j_{d-1}}, \lambda_d, \omega_{j_{d+1}}, \dots, \omega_{j_n}).$$

In fact, we can choose a collection of sequences  $(\lambda) = ({}^{(0)}\underline{\lambda}, {}^{(1)}\underline{\lambda}, \dots, {}^{(a)}\underline{\lambda}) = \underline{\lambda} = (\omega_{i_1}, \dots, \omega_{i_N})$  such that for any  $b < a$  the sequence  ${}^{(b+1)}\underline{\lambda}$  is obtained from the sequence  ${}^{(b)}\underline{\lambda}$  by the procedure  ${}^{(b)}\underline{\lambda} \rightsquigarrow {}^{(b)}\underline{\lambda}' =: {}^{(b+1)}\underline{\lambda}$  described in three cases (i–iii) below.

(i) In case  $\lambda_d$  is not a fundamental coweight, but  $\langle \lambda_d, \alpha_j^\vee \rangle = 1$  for certain vertex  $j$  (which may or may not happen to coincide with our chosen vertex  $i$ ), we set

$$n' = n + 1, \lambda'_d = \lambda_d - \omega_j, \underline{\lambda}' = (\omega_{j_1}, \dots, \omega_{j_{d-1}}, \lambda'_d, \omega_j, \omega_{j_{d+1}}, \dots, \omega_{j_n}).$$

Then the convolution morphism  $\varpi: \widetilde{\text{Gr}}_G^{\underline{\lambda}'} \rightarrow \widetilde{\text{Gr}}_G^{\underline{\lambda}}$  is an isomorphism up to codimension 2, and hence the convolution morphism  $\varpi: \widetilde{\mathcal{W}}_\mu^{\underline{\lambda}'} \rightarrow \widetilde{\mathcal{W}}_\mu^{\underline{\lambda}}$  is an isomorphism (recall that we restricted ourselves to the open subset over  $\mathring{Z}^{\alpha^*}$ ).

(ii) If  $\underline{\lambda} = (\omega_{j_1}, \dots, \omega_{j_e}, \omega_{j_{e+1}}, \dots, \omega_{j_n})$ , we set

$$n' = n, \underline{\lambda} = (\omega_{j_1}, \dots, \omega_{j_{e+1}}, \omega_{j_e}, \dots, \omega_{j_n}),$$

i.e. we just swap two neighbouring fundamental coweights. It follows from

(i) above that  $\widetilde{\mathcal{W}}_\mu^{\underline{\lambda}'} = \widetilde{\mathcal{W}}_\mu^{\underline{\lambda}}$  (over  $\mathring{Z}^{\alpha^*}$ ).

(iii) In case  $\langle \lambda_d, \alpha_j^\vee \rangle \geq 2$ , we set

$$\begin{aligned} n' &= n + 2, \quad d' = d + 1, \quad \lambda'_{d'} = \lambda_d - 2\omega_j, \\ \underline{\lambda}' &= (\omega_{j_1}, \dots, \omega_{j_{d-1}}, \omega_j, \lambda'_{d'}, \omega_j, \omega_{j_{d+1}}, \dots, \omega_{j_n}). \end{aligned}$$

We also set  $n'' = n$ ,  $\lambda''_d = \lambda_d - \alpha_j$ ,  $\underline{\lambda}'' = (\omega_{j_1}, \dots, \omega_{j_{d-1}}, \lambda''_d, \omega_{j_{d+1}}, \dots, \omega_{j_n})$ .

We have an open subvariety  ${}^{\circlearrowleft}\overline{\mathrm{Gr}}_G^{\lambda_d} := \mathrm{Gr}_G^{\lambda_d} \sqcup \mathrm{Gr}_G^{\lambda_d - \alpha_j} \subset \overline{\mathrm{Gr}}_G^{\lambda_d}$ , and also an open subvariety  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda} := \mathrm{Gr}_G^{\omega_{j_1}} \widetilde{\times} \dots \widetilde{\times} {}^{\circlearrowleft}\overline{\mathrm{Gr}}_G^{\lambda_d} \widetilde{\times} \dots \widetilde{\times} \mathrm{Gr}_G^{\omega_{j_n}} \subset \widetilde{\mathrm{Gr}}_G^{\lambda}$ . We have a closed subvariety  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda''} := \mathrm{Gr}_G^{\omega_{j_1}} \widetilde{\times} \dots \widetilde{\times} \mathrm{Gr}_G^{\lambda''_d} \widetilde{\times} \dots \widetilde{\times} \mathrm{Gr}_G^{\omega_{j_n}} \subset {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$ . We will denote the restriction of the convolution morphism  $\varpi: \widetilde{\mathrm{Gr}}_G^{\lambda'} \rightarrow \widetilde{\mathrm{Gr}}_G^{\lambda}$  to  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda} \subset \widetilde{\mathrm{Gr}}_G^{\lambda}$  by  $\varpi: {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$ . Similarly, if  $j \neq i$  but  $\lambda_d - \alpha_i$  is dominant, we define the open subsets  ${}^{\circlearrowleft}\overline{\mathrm{Gr}}_G^{\lambda_d} := \mathrm{Gr}_G^{\lambda_d} \sqcup \mathrm{Gr}_G^{\lambda_d - \alpha_i} \subset \overline{\mathrm{Gr}}_G^{\lambda_d}$  and  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda} \subset \widetilde{\mathrm{Gr}}_G^{\lambda}$ . Then (if  $j \neq i$ ) the convolution morphism  $\varpi: {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$  is an isomorphism, while  $\varpi: {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$  is the blowup of  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$  along the closed subvariety  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda''} \subset {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$ .

Indeed, étale-locally,  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$  splits as a product  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda''} \times \mathcal{S}_{N_j}$  where  $N_j := \langle \lambda_d, \alpha_j^\vee \rangle$ , and  $\varpi$  splits as a product  $\mathrm{Id} \times \overline{\varpi}$  where  $\overline{\varpi}: \mathcal{S}'_{N_j} \rightarrow \mathcal{S}_{N_j}$  is the restriction of  $\varpi$  to any slice  $\mathcal{S}_{N_j}$ . Now  $\mathcal{S}'_{N_j}$  is a normal surface, smooth if  $N_j = 2$ , and the fiber of  $\overline{\varpi}$  over  $0 \in \mathcal{S}_{N_j}$  is the projective line if  $N_j = 2$ . Furthermore, if  $N_j > 2$ , then the fiber of  $\overline{\varpi}$  over  $0 \in \mathcal{S}_{N_j}$  is a union of two projective lines intersecting at a point; this point in  $\mathcal{S}'_{N_j}$  has Kleinian  $A_{N_j-3}$ -singularity (in particular, it is smooth if  $N_j = 3$ ). The check reduces to the case of rank 1 by the argument of [MOV05, Section 3]. In rank 1 it follows e.g. from [MV03]. We conclude that  $\overline{\varpi}: \mathcal{S}'_{N_j} \rightarrow \mathcal{S}_{N_j}$  is the blowup of  $\mathcal{S}_{N_j}$  at  $0 \in \mathcal{S}_{N_j}$  (in effect, the minimal resolution  $\widetilde{\mathcal{S}}'_{N_j}$  of  $\mathcal{S}'_{N_j}$  must coincide with the minimal resolution  $\widetilde{\mathcal{S}}_{N_j}$  of  $\mathcal{S}_{N_j}$ , hence  $\mathcal{S}'_{N_j}$  must be obtained from  $\widetilde{\mathcal{S}}_{N_j}$  by blowing down all the exceptional divisor components except for the two outermost ones), and hence  $\varpi: {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$  is the blowup of  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$  along the closed subvariety  ${}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda''} \subset {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda}$ .

We define  ${}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda} := {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda} \times_{\mathrm{Bun}_G(\mathbb{P}^1)} \mathrm{Bun}_B^{w_0\mu}(\mathbb{P}^1)$ ,  ${}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda'} := {}^{\circlearrowleft}\widetilde{\mathrm{Gr}}_G^{\lambda'} \times_{\mathrm{Bun}_G(\mathbb{P}^1)} \mathrm{Bun}_B^{w_0\mu}(\mathbb{P}^1)$ , and we define  $\varpi: {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda} \supset {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda''}$ ,  $\varpi: {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda}$  similarly. By the argument used in the proof of [Quiver, Lemma 2.16], the morphisms  $\overline{\mathrm{Gr}}_G^{\lambda} \xrightarrow{p \circ \pi} {}^{\circlearrowleft}\mathrm{Bun}_G(\mathbb{P}^1) \leftarrow \mathrm{Bun}_B^{w_0\mu}(\mathbb{P}^1)$  are Tor-independent, hence  $\varpi: {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda}$  is the blowup of  ${}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda}$  along the closed subvariety  ${}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda''} \subset {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda}$ , while  $\varpi: {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda'} \rightarrow {}^{\circlearrowleft}\widetilde{\mathcal{W}}_\mu^{\lambda}$  is an isomorphism (if  $j \neq i$ ).

In case  $j \neq i$ , the open subvariety  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \mathfrak{W}_\mu^\lambda \subset (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$  coincides with the whole of  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$ . Hence the convolution morphism  $\varpi: (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda \rightarrow (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$  is an isomorphism.

In case  $j = i$ , the open subvariety  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \mathfrak{W}_\mu^\lambda \subset (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$  coincides with the whole of  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$ . Furthermore, the closed subvariety  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^{\lambda''} \subset (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$  coincides with the singular locus (with its reduced scheme structure) of  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$ . Arguing by induction, we conclude that  $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda$  coincides with  $\text{Bl}_{\lfloor \frac{\langle \lambda, \alpha_i^\vee \rangle}{2} \rfloor}$ , where  $\text{Bl}_0 := (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_\mu^\lambda \cong (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\beta^*} \times \mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \mathcal{S}_{\langle \lambda, \alpha_i^\vee \rangle})$ , and  $\text{Bl}_b$  is the result of blowup of  $\text{Bl}_{b-1}$  at its singular locus,  $b = 1, \dots, \lfloor \frac{\langle \lambda, \alpha_i^\vee \rangle}{2} \rfloor$ . Hence,  $\text{Bl}_{\lfloor \frac{\langle \lambda, \alpha_i^\vee \rangle}{2} \rfloor} \cong (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\beta^*} \times \mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \widetilde{\mathcal{S}}_{\langle \lambda, \alpha_i^\vee \rangle})$ .

The proposition is proved.  $\square$

## 5(ii). Determinant line bundles

Note that we have a whole collection of morphisms from  $\widetilde{\mathcal{W}}_\mu^\lambda$  to  $\text{Gr}_G$ : for  $1 \leq s \leq N$  we set  $p_s(\mathcal{P}_0, \dots, \mathcal{P}_N, \sigma_1, \dots, \sigma_N, \phi) := (\mathcal{P}_s, \sigma_s \circ \dots \circ \sigma_1)$ . Recall the determinant line bundle  $\mathcal{L}$  on  $\text{Gr}_G$  (see e.g. [Quiver, §2(iii)]). For  $1 \leq s \leq N$  we define the relative determinant line bundle  $\mathcal{D}_s$  on  $\widetilde{\mathcal{W}}_\mu^\lambda$  as  $\mathcal{D}_s := p_s^* \mathcal{L} \otimes p_{s-1}^* \mathcal{L}^{-1}$  (where  $p_s^* \mathcal{L}$  is understood as a trivial line bundle). For a collection of integers  $\varkappa = (k_1, \dots, k_N) \in \mathbb{Z}^N$ , we define a line bundle  $\mathcal{D}^\varkappa$  on  $\widetilde{\mathcal{W}}_\mu^\lambda$  as  $\bigotimes_{s=1}^N \mathcal{D}_s^{\otimes k_s}$ . In other words, for the obvious projection  $\mathbf{p}: \widetilde{\mathcal{W}}_\mu^\lambda \rightarrow \widetilde{\text{Gr}}_G^\lambda$  and similarly defined line bundle  $\mathcal{D}_{\text{Gr}}^\varkappa$  on the Grassmannian convolution diagram  $\widetilde{\text{Gr}}_G^\lambda$ , we have  $\mathcal{D}^\varkappa = \mathbf{p}^* \mathcal{D}_{\text{Gr}}^\varkappa$ . In particular,  $\mathcal{D}^{(1,1,\dots,1)} = p_N^* \mathcal{L}$  is trivial.

For  $i \in Q_0$ , we set  $N_i = \langle \lambda, \alpha_i^\vee \rangle = \#\{s : \omega_{i_s} = \omega_i\}$ . We order the set of indices  $s$  such that  $\omega_{i_s} = \omega_i$ :  $s_1^{(i)} < \dots < s_{N_i}^{(i)}$ . We associate to  $\varkappa \in \mathbb{Z}^N$  a collection of coweights  $\varkappa^{(i)} = \sum_{n=1}^{N_i-1} (k_{s_n^{(i)}} - k_{s_{n+1}^{(i)}}) \omega_n$ ,  $i \in Q_0$ , of  $\text{PGL}(W_i)$ . We will denote by  $\Lambda_F^+ \subset \mathbb{Z}^N$  the set of all  $\varkappa$  such that  $k_{s_1^{(i)}} \geq k_{s_2^{(i)}} \geq \dots \geq k_{s_{N_i}^{(i)}}$  for any  $i \in Q_0$ . We will denote by  $\Lambda_F^{++} \subset \Lambda_F^+$  the set of all  $\varkappa$  such that  $k_{s_1} \geq k_{s_2}$  for any  $1 \leq s_1 < s_2 \leq N$ .

**Proposition 5.3.** *The factorization isomorphism of Proposition 5.1 lifts to a canonical (in the sense explained during the proof) isomorphism of line*

bundles

$$\left( (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^{\gamma^*})_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_{\mu}^{\lambda}, \mathcal{O}_{(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^{\gamma^*})_{\text{disj}}} \otimes \mathcal{D}^{\mathcal{Z}} \right) \xrightarrow{\sim} \left( (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^{\gamma^*})_{\text{disj}} \times_{\mathbb{A}^{\beta^*} \times \mathbb{A}^{\gamma^*}} (\mathring{Z}^{\beta^*} \times \widetilde{\mathcal{W}}_{\lambda-\gamma}^{\lambda}), \mathcal{O}_{(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^{\gamma^*})_{\text{disj}}} \otimes \mathcal{O}_{\mathring{Z}^{\beta^*}} \boxtimes \mathcal{D}^{\mathcal{Z}} \right).$$

*Proof.* The factorization isomorphism of Proposition 5.1 associates to the data of  $(\mathcal{P}_0, \dots, \mathcal{P}_N, \sigma_1, \dots, \sigma_N, \phi)$  the data of  $(\mathcal{P}_0^{(1)} = \dots = \mathcal{P}_N^{(1)}, \sigma_1^{(1)} = \dots = \sigma_N^{(1)} = \text{id}, \phi^{(1)})$  and  $(\mathcal{P}_0^{(2)}, \dots, \mathcal{P}_N^{(2)}, \sigma_1^{(2)}, \dots, \sigma_N^{(2)}, \phi^{(2)})$ . By construction, the relative determinant of  $\mathcal{P}_s$  and  $\mathcal{P}_{s-1}$  coincides with the relative determinant of  $\mathcal{P}_s^{(2)}$  and  $\mathcal{P}_{s-1}^{(2)}$ .  $\square$

We consider the Kleinian surface resolution  $\widetilde{\mathcal{S}}_{N_i} \xrightarrow{\pi} \mathcal{S}_{N_i} \xrightarrow{H} \mathbb{A}^1$  with a line bundle  $\mathcal{L}_{\mathcal{Z}^{(i)}}$ . See [Ring, §4(i)].

**Corollary 5.4.** *The factorization isomorphism of Proposition 5.2 lifts to a canonical (in the sense explained during the proof) isomorphism of line bundles*

$$\left( (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\alpha^*}} \widetilde{\mathcal{W}}_{\mu}^{\lambda}, \mathcal{O}_{(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}}} \otimes \mathcal{D}^{\mathcal{Z}} \right) \xrightarrow{\sim} \left( (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}} \times_{\mathbb{A}^{\beta^*} \times \mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \widetilde{\mathcal{S}}_{N_i}), \mathcal{O}_{(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\text{disj}}} \otimes \mathcal{O}_{\mathring{Z}^{\beta^*}} \boxtimes \mathcal{L}_{\mathcal{Z}^{(i)}} \right).$$

*Proof.* Due to Proposition 5.3, it suffices to construct an isomorphism  $(\widetilde{\mathcal{W}}_{\lambda-\alpha_i}^{\lambda}, \mathcal{D}^{\mathcal{Z}}) \xrightarrow{\sim} (\widetilde{\mathcal{S}}_{N_i}, \mathcal{L}_{\mathcal{Z}^{(i)}})$ . This reduces to the case of rank 1 by the argument of [MOV05, Section 3]. In rank 1 we compare the weights of the Cartan torus in the fixed points.

Namely,  $G = \text{GL}(2)$ ,  $\omega$  is the fundamental weight  $(1, 0)$ ,  $\lambda$  is a sequence  $(\omega, \dots, \omega)$  ( $N$  times),  $\alpha = (1, -1)$  is the simple root,  $\lambda = N\omega = (N, 0)$ ,  $\lambda - \alpha = (N - 1, 1)$ , and we will write  $\widetilde{\mathcal{W}}$  for  $\widetilde{\mathcal{W}}_{\lambda-\alpha}^{\lambda}$ . Then  $\widetilde{\mathcal{W}}$  is a locally closed subvariety of the convolution diagram  $\text{Gr}_G^{\omega} \times \dots \times \widetilde{\text{Gr}}_G^{\omega}$  ( $N$  times). The latter convolution diagram is the moduli space of flags of lattices  $L_0 \supset L_1 \supset \dots \supset L_N$  where  $L_0 = V \otimes \mathbb{C}[[z]]$ ,  $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ , and  $\dim L_n/L_{n+1} = 1$  for any  $n = 0, \dots, N - 1$ . The fixed points  $\widetilde{\mathcal{W}}^T = \{p_0, \dots, p_{N-1}\}$  (where  $T \subset \text{GL}(2) = \text{GL}(V)$  is the diagonal torus) are as follows:  $p_r = (L_0^{(r)} \supset \dots \supset L_N^{(r)})$  where  $L_n^{(r)}$  is spanned by  $z^n e_1, e_2$  for  $0 \leq n < r$ , and by  $z^{n-1} e_1, z e_2$  for  $r \leq n \leq N - 1$ . In particular,  $L_0^{(r)} = L_0$ , and  $L_N^{(r)} = z^{N-1} \mathbb{C}[[z]]e_1 \oplus z \mathbb{C}[[z]]e_2$ . The fiber of  $\mathcal{D}_s$  at  $p_r$  is  $\mathbb{C}z^{s-1}e_1$  for  $1 \leq s \leq r$ , and  $\mathbb{C}e_2$  for  $s = r + 1$ , and  $\mathbb{C}z^{s-2}e_1$  for  $r + 1 < s \leq N$ . Let  $T_1$  be the image of  $T \subset \text{GL}(2)$  in  $\text{PGL}(2)$ . The natural action of  $T$  on the convolution diagram factors through  $T_1$ , and the

action of  $T_1$  lifts to an action on  $\mathcal{D}_s$ : the character of the fiber (at a fixed point) isomorphic to  $\mathbb{C}z^l e_1$  (resp.  $\mathbb{C}z^l e_2$ ) is 1 (resp.  $x_1^{-1}$ ). Here  $x_1$  is the generator of  $X^*(T_1)$ . Recall the action of  $\mathbb{C}^\times \times \mathbb{C}^\times$  on  $\widetilde{\mathcal{S}}_N$  in [Ring, §4(i)]. We will be interested in the action of the first copy of  $\mathbb{C}^\times$ . It factors through the quotient modulo the subgroup of  $N$ -th roots of unity:  $\mathbb{C}^\times \rightarrow \mathbb{C}^\times / \sqrt[N]{1}$ . We identify  $\mathbb{C}^\times / \sqrt[N]{1}$  with  $T_1$  so that the pullback of  $x_1 \in X^*(T_1)$  to  $\mathbb{C}^\times$  coincides with  $x^N$ . Then the identification  $\widetilde{\mathcal{S}}_N \simeq \widetilde{\mathcal{W}}$  is  $\mathbb{C}^\times \rightarrow T_1$ -equivariant, it takes  $p_r \in \widetilde{\mathcal{S}}_N$  to  $p_r \in \widetilde{\mathcal{W}}$ , and the characters of  $\mathbb{C}^\times$  in the fibers of  $\mathcal{L}_{\omega_s - \omega_{s-1}}$  and  $\mathcal{D}_s$  at the respective fixed points in  $\widetilde{\mathcal{S}}_N$  and  $\widetilde{\mathcal{W}}$  match up to an overall twist (independent of a fixed point) by the character  $x$  of  $\mathbb{C}^\times$ .

This defines the desired isomorphism  $(\widetilde{\mathcal{W}}, \mathcal{D}_s) \xrightarrow{\sim} (\widetilde{\mathcal{S}}_N, \mathcal{L}_{\omega_s - \omega_{s-1}})$  up to multiplication by an invertible constant, and hence  $(\widetilde{\mathcal{W}}, \mathcal{D}^\varkappa) \xrightarrow{\sim} (\widetilde{\mathcal{S}}_N, \mathcal{L}^\varkappa)$  (also up to multiplication by an invertible constant). This is the only ambiguity in the choice of isomorphism of corollary. □

**5(iii). Sections of determinant line bundles**

For  $1 \leq s \leq N$ , we set  $\lambda_s := \omega_{i_1} + \dots + \omega_{i_s}$ . Then the projection  $p_s: \widetilde{\mathcal{W}}_\mu^\lambda \rightarrow \text{Gr}_G^{\lambda_s}$  lands into  $\overline{\text{Gr}}_G^{\lambda_s}$ . The determinant line bundle

$$\mathcal{L}|_{\overline{\text{Gr}}_G^{\lambda_s}} \simeq \mathcal{O}_{\overline{\text{Gr}}_G^{\lambda_s}} \left( \sum_{i \in Q_0} \langle \lambda_s, \alpha_i^\vee \rangle S_{\lambda_s - \alpha_i} \cap \overline{\text{Gr}}_G^{\lambda_s} \right)$$

has a canonical section  $z_{\lambda_s}$  vanishing to the order  $\langle \lambda_s, \alpha_i^\vee \rangle$  at the semiinfinite orbit  $S_{\lambda_s - \alpha_i}$  intersecting  $\overline{\text{Gr}}_G^{\lambda_s}$  in codimension 1. For  $\varkappa = (k_1 \geq \dots \geq k_N) \in \Lambda_F^{++}$ , the line bundle  $\mathcal{D}^\varkappa = \bigotimes_{s=1}^N p_s^* \mathcal{L}^{\otimes(k_s - k_{s+1})}$  (we set  $k_{N+1} = 0$ ) has a section  $z^\varkappa := \bigotimes_{s=1}^N p_s^* z_{\lambda_s}^{k_s - k_{s+1}}$ . In particular, recall that  $\mathcal{D}^{(1,1,\dots,1)} = p_N^* \mathcal{L}$  is trivial, but the section  $z^{(1,1,\dots,1)} = p_N^* z_{\lambda_N} \neq 1$  since it vanishes along some semiinfinite orbits.

**5(iv). Example**

We consider  $G = \text{SL}(3)$ ,  $\mu = 0$ ,  $\underline{\lambda} = (\omega_j, \omega_i)$ ,  $\lambda = \omega_i + \omega_j = \alpha_i + \alpha_j$ . The slice  $\overline{\mathcal{W}}_\mu^\lambda$  is the closure of the minimal nilpotent orbit in  $\mathfrak{sl}_3$ , and  $\widetilde{\mathcal{W}}_\mu^\lambda$  is the cotangent bundle  $T^*\mathbb{P}^2$  where  $\mathbb{P}^2 = \mathbb{P}(V)$ , and  $V$  has a basis  $b_1, b_2, b_3$ , and  $V^*$  has the dual basis  $a_1, a_2, a_3$ . We assume that these bases are eigenbases for a Cartan torus  $T$ , and the weight of  $a_1$  equals  $\omega_i$ ,  $\text{wt}(a_2) = \omega_i - \alpha_i$ ,  $\text{wt}(a_3) = -\omega_j$ . The zastava  $Z^\lambda$  is given by equation  $y_i y_j = (w_i - w_j) y_{j,i}$ , and the open

zastava  $\mathring{Z}^\lambda \subset Z^\lambda$  is given by  $y_{j,i} \neq 0$ . The weights  $\text{wt}(y_i) = \alpha_i$ ,  $\text{wt}(y_j) = \alpha_j$ ,  $\text{wt}(w_i) = \text{wt}(w_j) = 0$ ,  $\text{wt}(y_{j,i}) = \lambda$ .

We have the canonical projections  $\widetilde{\mathcal{W}}_\mu^\lambda \rightarrow \overline{\mathcal{W}}_\mu^\lambda \rightarrow Z^\lambda$ , and a section  $\mathring{Z}^\lambda \hookrightarrow \overline{\mathcal{W}}_\mu^\lambda$ . We consider the incidence quadric  $Q \subset V \times V^*$  given by  $a_1b_1 + a_2b_2 + a_3b_3 = 0$ . Its categorical quotient modulo the ‘hyperbolic’  $\mathbb{C}^\times$ -action is  $\overline{\mathcal{W}}_\mu^\lambda$ , and the composed projection  $Q \rightarrow Z^\lambda$  acts as

$$y_i = a_1b_2, \quad y_j = a_2b_3, \quad y_{j,i} = a_1b_3, \quad w_i = -a_1b_1, \quad w_j = a_3b_3, \quad w_i - w_j = a_2b_2.$$

The preimage of the open zastava  $\mathring{Z}^\lambda \subset Z^\lambda$  is given by  $a_1 \neq 0 \neq b_3$ . The composition  $\mathring{Z}^\lambda \hookrightarrow \mathcal{W}_\mu^\lambda \hookrightarrow \text{Gr}_{\text{SL}(3)}^\lambda \rightarrow \mathcal{B}$  (the flag variety of  $\text{SL}(3)$ ) is nothing but the evaluation at  $0 \in \mathbb{P}^1$  morphism (viewing  $\mathring{Z}^\lambda$  as based maps from  $\mathbb{P}^1$  to  $\mathcal{B}$ ).

The Picard group  $\text{Pic}(\widetilde{\mathcal{W}}_\mu^\lambda) \simeq \mathbb{Z}$ , generated by the first determinant bundle  $\mathcal{L}_1 = \mathcal{D}_1$  that coincides with the pullback of  $\mathcal{O}(1)$  from  $\mathbb{P}^2$ . The global sections  $\Gamma(\widetilde{\mathcal{W}}_\mu^\lambda, \mathcal{D}_1)$  are the functions on the incidence quadric  $Q$  having weight 1 with respect to the hyperbolic  $\mathbb{C}^\times$ . In particular, this line bundle has  $T$ -eigensections  $a_1, a_2, a_3$ . The restriction of  $a_1$  to  $\mathring{Z}^\lambda \subset \widetilde{\mathcal{W}}_\mu^\lambda$  is nowhere vanishing. The restriction of  $a_2$  vanishes along the divisor  $\text{div}(y_j) \subset \mathring{Z}^\lambda$ , and the restriction of  $a_3$  vanishes along the divisor  $\text{div}(w_j) \subset \mathring{Z}^\lambda$ . Note that  $a_3 = p_1^*z_{\lambda_1} \in \Gamma(\widetilde{\mathcal{W}}_\mu^\lambda, \mathcal{D}_1)$ . Furthermore, the section of the trivial line bundle (i.e. a function)  $p_2^*z_{\lambda_2} = -a_3b_1$ .

Comparing with §4(ii), we conclude that (in our situation  $\alpha = 1$ )

$$z_i = -a_2b_1, \quad z_j = a_3b_2, \quad z_{j,i} = -a_3b_1, \quad y_j^1 = a_2, \quad 'y_j^1 = a_1, \quad z_j^1 = a_3.$$

From (4.10) we conclude that  ${}^1r^{0,0} = z_j^1 = a_3$  (the fundamental class of the preimage of the cocharacter  $(0, 1, 0, 0) \in X_*(\text{GL}(V_j) \times T^{\mathbf{w}_j} \times \text{GL}(V_i) \times T^{\mathbf{w}_i})$ ).

Similarly, the fundamental class of the preimage of the cocharacter  $(0, 0, 0, 1) \in X_*(\text{GL}(V_j) \times T^{\mathbf{w}_j} \times \text{GL}(V_i) \times T^{\mathbf{w}_i})$  is the section  $b_1$  of the pullback of  $\mathcal{O}(1)$  from  $\mathbb{P}(V^*)$  to  $T^*\mathbb{P}(V^*) = \widetilde{\mathcal{W}}_\mu^{\lambda'}$  where  $\lambda' = (\omega_i, \omega_j)$ .

More generally, the fundamental class of the preimage of the cocharacter  $(\min(k_1, k_2), k_1,$

$\min(k_1, k_2), k_2)$  restricted to  $\mathring{Z}^\lambda$  vanishes to the order  $k_1 - k_2$  at the divisor  $w_j = 0$  if  $k_1 \geq k_2$ , and to the order  $k_2 - k_1$  at the divisor  $w_i = 0$  if  $k_1 \leq k_2$ , and is invertible elsewhere, in particular at  $w_i = w_j$ . Hence for  $k \geq 0$  the fundamental class of the preimage of the cocharacter  $(\min(k_1, k_2) - k, k_1, \min(k_1, k_2) - k, k_2)$  restricted to  $\mathring{Z}^\lambda$  is invertible off the zero divisors

of  $w_i$  and  $w_j$ . This follows from (4.10) (note that  $z_{j,i}$  is invertible at the generic point of the divisor  $w_i = w_j$ ).

**5(v). Determinant sheaves on slices via homology groups of fibers**

We recall the setup of [Quiver, §3(iii)] and [Quiver, §3(v)]. We set  $G = \text{GL}(V)$ ,  $G_F = T(W)$ ,  $\tilde{G} = G \times G_F$ . The group  $\tilde{G}$  acts on  $\mathbf{N}_\mu^\lambda$ . According to [Quiver, Theorem 3.10], the Coulomb branch  $\mathcal{M}_C(G, \mathbf{N})$  is identified with  $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$ . Our choice of basis of the character lattice of  $T(W)$  defines a cone of dominant coweights of  $\text{GL}(W) \supset T(W)$ . It is nothing but  $\Lambda_F^+$  introduced in §5(ii). For  $\varkappa \in \Lambda_F^+$ , the homology  $H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$  forms a module over the algebra  $H_*^{G \circ}(\mathcal{R})$ , and for  $\varkappa \in \Lambda_F^{++}$  we want to identify this module with  $\Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \mathcal{D}^\varkappa) = \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_* \mathcal{D}^\varkappa)$ . Here the assumption  $\varkappa \in \Lambda_F^{++}$  is *not* essential, as we can renumber  $i_1, \dots, i_N$  so that  $k_1 \geq k_2 \geq \dots \geq k_N$ .

First we consider the case  $\varkappa = (1, \dots, 1, 0, \dots, 0)$ , i.e.  $\mathcal{D}^\varkappa = p_s^* \mathcal{L}$ . We allow  $s = N$ , so that  $\varkappa = (1, \dots, 1)$ . Let  $\mathbf{N}_T$  denote  $\mathbf{N}$  regarded as a  $T$ -module. We have the pushforward homomorphism  $\iota_*: H_*^{T \circ}(\mathcal{R}_{T, \mathbf{N}_T}) \rightarrow H_*^{T \circ}(\mathcal{R}) = H_*^{G \circ}(\mathcal{R}) \otimes_{H_G^*(\text{pt})} H_T^*(\text{pt})$  of the inclusion  $\mathcal{R}_{T, \mathbf{N}_T} \rightarrow \mathcal{R}$  (see [Part II, §5(iii)]). We set  $\tilde{T} := T \times G_F = T \times T(W)$ . We consider  $\tilde{\pi}_T: \mathcal{R}_{\tilde{T}, \mathbf{N}_T} \rightarrow \text{Gr}_{G_F}$ , and the fiber  $\tilde{\pi}_T^{-1}(\varkappa)$ . We have a natural inclusion  $\tilde{\pi}_T^{-1}(\varkappa) \rightarrow \tilde{\pi}^{-1}(\varkappa) = \tilde{\mathcal{R}}^\varkappa$ , denoted again by  $\iota$ , and the pushforward homomorphism

$$\iota_*: H_*^{T \circ}(\tilde{\pi}_T^{-1}(\varkappa)) \rightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa) \otimes_{H_G^*(\text{pt})} H_T^*(\text{pt}).$$

Let  $\pi_T: \mathcal{R}_{\tilde{T}, \mathbf{N}_T} \rightarrow \text{Gr}_{\tilde{T}}$  be the projection. We lift the coweight  $\varkappa$  of  $G_F$  to  $\tilde{T}$  by setting the  $w_{i,r}$ -coordinate of the  $T$ -component to be 0 for any  $w_{i,r}$ ,  $i \in Q_0$ ,  $1 \leq r \leq a_i$ . Let us denote it by  $\tilde{\varkappa}$ . We consider the fundamental class of  $\pi_T^{-1}(\tilde{\varkappa})$  and denote it by  $\mathbf{z}^\varkappa$ . By the localization theorem, it is nonvanishing over  $\mathbb{A}^{|\alpha|}$ . Note that the lift  $\tilde{\varkappa}$  has different  $T$  component from one in (4.8). The class  $\mathbf{z}^\varkappa$  is different from  $\mathbf{y}^\varkappa$  used in §4.

We define a rational isomorphism  $\theta: \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_* \mathcal{D}^\varkappa) \dashrightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$  by sending  $\mathbf{z}^\varkappa$  to  $\iota_* \mathbf{z}^\varkappa$ . It is  $S_\alpha$ -equivariant, hence it is indeed an isomorphism as above.

**Theorem 5.5.** *The rational isomorphism  $\theta: \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_* \mathcal{D}^\varkappa) \dashrightarrow H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$  extends to the regular isomorphism of  $\mathbb{C}[\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}]$ -modules  $\theta: \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_* \mathcal{D}^\varkappa) \xrightarrow{\sim} H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$ .*



*Proof.* We follow the standard scheme, see e.g. the proof of [Quiver, Theorem 3.10]. We have to check that  $\theta$  extends through the general points of the boundary divisor  $\mathbb{A}^{|\alpha|} \setminus \mathring{\mathbb{G}}_m^{|\alpha|}$ . Namely,

- (a)  $w_{j,s}(t) = w_{i,r}(t)$  for some  $i \neq j$  connected by an edge,  $r, s$ , but all others are distinct. Moreover  $w_{k,p}(t) \neq 0$  if  $N_k \neq 0$ .
- (b)  $w_{i,r}(t) = w_{i,s}(t)$  for distinct  $r, s$  and some  $i$ , but all others are distinct. Moreover  $w_{j,p}(t) \neq 0$  if  $N_j \neq 0$ .
- (c) All pairs like in (a),(b) are distinct, but  $w_{i,r}(t) = 0$  for  $i$  with  $N_i \neq 0$ .

The gauge theory  $(G, \mathbf{N}, \tilde{G})$  with the flavor symmetry group  $\tilde{G}$  is replaced by  $(Z_G(t), \mathbf{N}^t, Z_{\tilde{G}}(t))$ . In our case,  $Z_{\tilde{G}}(t) = Z_G(t) \times T(W)$ , and  $(Z_G(t), \mathbf{N}^t) = (\mathrm{GL}(V') \times T'', \mathbf{N}(V', W'))$ , where  $V', W'$  are given below,  $V = V' \oplus V''$  and  $T''$  acts trivially on  $\mathbf{N}(V', W')$ :

- (a)  $W' = 0, V'_j = \mathbb{C} = V'_i$  and other entries are 0.
- (b)  $W' = 0, V' = \mathbb{C}^2$  and other entries are 0.
- (c)  $V'_i = 1, W'_i = \mathbb{C}^{N_i}$  and other entries are 0.

The extra factor  $T(W)$  acts trivially in (a),(b), while it acts through  $T(W) \rightarrow T(W_i)$  in (c).

By the same argument as in the proofs of [Quiver, Theorem 3.10], both  $z^{\varkappa}$  and  $z^{\varkappa}$  are related to  $z'^{\varkappa}, z'^{\varkappa}$  by nonvanishing regular functions defined on a neighborhood of  $t$  in  $\mathbb{A}^{|\alpha|}$  under the factorization. Therefore it is enough to check that the isomorphism  $\theta$  extends for the local models (a),(b),(c) above.

(a) According to §5(iv), both  $z'^{\varkappa}$  and  $z'^{\varkappa}$  are invertible at the general points of the divisor  $w_i = w_j$  (recall that we assume  $w_i \neq 0 \neq w_j$ ).

(b) The zero divisor of  $z'^{\varkappa}$  is the union of the zero divisors of  $w_{i,1}$  and  $w_{i,2}$ ; in particular,  $z'^{\varkappa}$  is invertible at the general points of the divisor  $w_{i,1} = w_{i,2}$  (recall that we assume  $w_{i,1} \neq 0 \neq w_{i,2}$ ). The homology class  $z'^{\varkappa}$  is invertible as well.

(c) We make use of the  $\mathbb{C}^\times \times \mathbb{C}^\times$ -action on  $\tilde{\mathcal{S}}_{N_i}$  of [Ring, §4(i)]. A dominant weight  $\lambda$  of [Ring, §4(i)] is now  $\varkappa^{(i)} = \omega_n = (1, \dots, 1, 0, \dots, 0)$  ( $n$  1's, and we allow the possibility  $n = N_i$ , when  $\varkappa^{(i)} = (1, \dots, 1)$ ). The fundamental class  $z'^{\varkappa^{(i)}}$  is an eigenvector of  $\mathbb{C}^\times \times \mathbb{C}^\times$  with the eigencharacter  $x^n t^n$ . Since all the eigenspaces are 1-dimensional, it suffices to check that  $z'^{\varkappa^{(i)}}$  has the same eigencharacter. Now the  $x$ -character of  $z'^{\omega_n}$  is  $x^n$  since  $z'^{\omega_n}$  is a highest vector of the irreducible  $\mathrm{GL}(2)$ -module with highest weight  $(n, 0)$ .

The exponent of the  $t$ -character of  $z^{l\omega_n}$  is minimal among all such exponents with the fixed  $x$ -character. Hence the  $t$ -character of  $z^{l\omega_n}$  is  $t^n$ .

For the sake of completeness, note that the divisor of  $z^{l\omega_n}$  is the union of  $E_1, \dots, E_{n-1}$  and the strict transform of  $\{Z = 0\}$  (notation of the beginning of §2 and [Ring, §4(i)]).

We conclude by an application of [Part II, Theorem 5.26] and [Part II, Remark 5.27]. The condition  $\Pi_*\pi_*\mathcal{D}^\varkappa \xrightarrow{\sim} j_*\Pi_*\pi_*\mathcal{D}^\varkappa|_{(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*})^\bullet}$  of [Part II, Remark 5.27] is satisfied since  $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$  is Cohen-Macaulay, and the complement of  $(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*})^\bullet$  in  $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$  is of codimension 2. The latter claim follows from the semismallness of  $\pi: \overline{\mathcal{W}}_{\mu^*}^{\lambda^*} \rightarrow \overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$  as in the proof of [Quiver, Lemma 2.7], and the Cohen-Macaulay property is proved the same way as in [Quiver, Lemma 2.6] and [Quiver, Lemma 2.16].  $\square$

Now we construct an isomorphism  $\theta_\varkappa: \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_*\mathcal{D}^\varkappa) \xrightarrow{\sim} H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$  for arbitrary  $\varkappa \in \Lambda^{++}$  inductively, with Theorem 5.5 as the base of induction. More precisely, we write  $\varkappa = \sum_l \varkappa_l$ , where each  $\varkappa_l$  is of the form  $(1, \dots, 1, 0, \dots, 0)$  considered in Theorem 5.5.

**Theorem 5.6.** *There is a unique isomorphism  $\theta_\varkappa: \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_*\mathcal{D}^\varkappa) \xrightarrow{\sim} H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$  making the following diagram commutative:*

$$\begin{array}{ccc} \bigotimes_{\mathbb{C}[\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}]}^l \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_*\mathcal{D}^{\varkappa_l}) & \xrightarrow[\bigotimes_{\theta_{\varkappa_l}}]{\sim} & \bigotimes_{H_*^{G \circ}(\mathcal{R})}^l H_*^{G \circ}(\tilde{\mathcal{R}}^{\varkappa_l}) \\ \downarrow & & \downarrow \\ \Gamma(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}, \pi_*\mathcal{D}^\varkappa) & \xrightarrow{\sim} & H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa). \end{array}$$

*Proof.* Assume  $\varkappa = \varkappa' + \varkappa''$ , and  $\theta_{\varkappa'}, \theta_{\varkappa''}$  are already constructed. Then we restrict to  $(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*})^\bullet \xrightarrow{j} \overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$ , and note that  $j^*\pi_*\mathcal{D}^\varkappa$  is the quotient of  $j^*\pi_*\mathcal{D}^{\varkappa'} \otimes j^*\pi_*\mathcal{D}^{\varkappa''}$  modulo torsion, due to factorization and Lemma 2.1. Similarly,  $j^*H_*^{G \circ}(\tilde{\mathcal{R}}^\varkappa)$  is the quotient of  $j^*H_*^{G \circ}(\tilde{\mathcal{R}}^{\varkappa'}) \otimes j^*H_*^{G \circ}(\tilde{\mathcal{R}}^{\varkappa''})$  modulo torsion. So we define  $j^*\theta_\varkappa$  as the quotient of  $j^*\theta_{\varkappa'} \otimes j^*\theta_{\varkappa''}$  modulo torsion. Finally, we define  $\theta_\varkappa$  as  $j_*j^*\theta_\varkappa$ . The resulting diagram commutes thanks to Lemma 2.2.  $\square$

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### Erratum to [Ring]

As pointed out by Bielawski [Bie21], the statement of Proposition 5.20 is not correct. The action of  $G^\vee$  on the second factor should be twisted by the automorphism of  $G^\vee$  interchanging conjugacy classes of  $g$  and  $g^{-1}$ . We use the automorphism in the proof of Proposition 5.20, as  $\mathfrak{C}_{G^\vee}$  is induced from that. The same correction applies to the last sentence of Remark 5.22.

We also need to twist the first factor of the diagonal embedding  $\mathrm{Gr}_G \rightarrow (\mathrm{Gr}_G)^2$  in Proposition 5.23 by the automorphism induced by  $g \mapsto g^{-1}$ ,  $G((z)) \rightarrow G((z))$ . The induced automorphism  $\mathcal{C}_G$  on  $D_G(\mathrm{Gr}_G)$  corresponds to  $\mathfrak{C}_{G^\vee}$  under the derived Satake equivalence  $\Psi$ . The latter appeared in line –6 at page 318.

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