Qualitative Theory of Dynamical Systems (2022) 21:47
https://doi.org/10.1007/s12346-022-00579-w

On the Solutions of Ordinary Differential Equations in the Form of Dulac Series

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Received: 18 November 2021 / Accepted: 19 February 2022

Abstract
We consider a nonlinear ordinary differential equation of arbitrary order with coefficients in the form of power series that converge in a neighborhood of the origin. The methods created in power geometry in recent years make it possible to compute formal solutions to that equation in the form of Dulac series. We describe the corresponding algorithm and prove a sufficient convergence condition for such formal solutions.

Keywords Newton polygon · Continuable solution · Formal solution · Dulac series · Convergence

Mathematics Subject Classification 34E05

1 Introduction

In this paper, we study ordinary differential equations of the form

$$\begin{aligned} &f(x, Y) = 0, \quad f(x, Y) = \sum_{|I|=0}^{M} a_I(x) Y^I, \\ &Y = (y, y', \dots, y^{(n)}), \quad y = y(x),\ x, y \in \mathbb{C}, \\ &I = (i_0, i_1, \dots, i_n),\ i_0, i_1, \dots, i_n \in \mathbb{Z}_+,\ |I| = i_0 + i_1 + \dots + i_n, \\ &Y^I = y^{i_0}(y')^{i_1} \dots (y^{(n)})^{i_n}, \\ &\mathbb{Z}_+ \text{ is the set of nonnegative integers,} \end{aligned} \tag{1}$$

and $a_I(x)$ are functions expressed as power series that converge uniformly and absolutely in a neighborhood of the point $x = 0$,

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$$a_I(x) = \sum_{m=1}^{\infty} a_{Im} x^{p_{Im}}, \quad a_{Im}, p_{Im} = \text{const}, \ a_{Im} \in \mathbb{C}, \ p_{Im} \in \mathbb{R},$$
$$p_{Im} < p_{I(m+1)}, \ \lim_{m \to \infty} p_{Im} = +\infty.. \tag{2}$$

In this paper, the exponents in the power series are supposed to be real numbers but not necessarily integers. We also assume that x is in an arbitrary open sector V with its vertex at the origin and a central angle less than 2π. The convergence of power series in a neighborhood of the point $x = 0$ is understood as their convergence in a region that is the intersection of the neighborhood and the sector V.

We will deviate from the classical Cauchy problem and search for solutions that are defined in a punctured neighborhood of the point $x = 0$ (the so-called *continuable solutions*), have asymptotic power expansions as $x \to 0$, and can be represented as convergent series in some neighborhood of the origin.

An algorithm is proposed in [1] to compute formal solutions to Eq. (1) in the form of power series or power-logarithmic series (where the coefficients of the power series are polynomials in $\ln x$). The same paper poses the question of the convergence of those series. That problem was solved for some well-known equations (e.g., Painlevé equations, Riccati equations) (see, for instance, [2, 3]). A sufficient condition was given in [1] for the convergence of power series expansions corresponding to formal solutions of an equation of the form (1). Below, we state this condition for formal solutions of Eq. (1) expressed as power-logarithmic series. For this, we need to describe the method used to compute the first approximation to the solution. The method is based on the concept of the Newton polygon. We define the Newton polygon N of Eq. (1) as the closed convex hull of some elements of the support $S(f)$ of Eq. (1), specifically, the points $Q_{Im} = (p_{Im} - |\tilde{I}|, |I|)$, $|\tilde{I}| = \sum_{k=1}^{n} k i_k$, $m \in \{1, 2, \ldots\}$. The first approximations are solutions of the so-called truncated equations, which correspond to the faces (edges and vertices) of the Newton polygon (the unknown function is $u = u(x)$),

$$\hat{f}(x, U) = 0, \ \hat{f}(x, U) = \sum_I a_I(x) U^I, \quad U = (u, u', \ldots, u^{(n)}),$$
$$U^I = u^{i_0} (u')^{i_1} \ldots (u^{(n)})^{i_n}, \tag{3}$$

where the sum extends over all sets I corresponding to the elements $Q_{I1} = (p_{I1} - |\tilde{I}|, |I|)$ of the support that belong to the considered face. To compute the first power-series approximations (asymptotic power expansions), we search for a solution to Eq. (3) in the form $u = Ax^\alpha$, $A, \alpha = \text{const}$, $A \neq 0$. In the case of an edge, if we additionally assume that it is slanted (not horizontal), i.e., $|I| \neq \text{const}$ for all sets I in the sum $\hat{f}(x, U)$, then the number α is unambiguously determined, namely $a = -(1, \alpha)$, where a is an external normal vector to the considered edge. Note also that we have $\beta = \alpha|I| + p_{I1} - |\tilde{I}| = \text{const}$ for all the points of the support that belong to the considered edge. To compute the parameter A in the case of an edge, we need to solve the algebraic equation

$$\sum_I a_{I1}\mu(\alpha, I)A^{|I|} = 0,$$
$$\mu(\alpha, I) = \alpha^{i_1}\big(\alpha(\alpha-1)\big)^{i_2}\ldots\big(\alpha(\alpha-1)\ldots(\alpha-n+1)\big)^{i_n}.$$

In the present paper, we limit ourselves to the situation when, in the case of a vertex, the numbers α are the real roots of a certain polynomial whose degree does not exceed $n|I|$. Moreover, the vector $-(1, \alpha)$ must belong to the so-called normal cone of the considered vertex, whereas the parameter A is an arbitrary nonzero number. Let us define the normal cone according to [1]. The *normal cone of a vertex* is an open sector in $\mathbb{R}^2$ with its vertex at the origin and limited by the normal cones of the edges that are incident on the considered vertex. The *normal cone of an edge* is a ray with initial point at the origin of $\mathbb{R}^2$ and directed as an external normal vector to the considered edge.

Let us now assume that truncated Eq. (3), which corresponds to the considered edge, has the nontrivial solution $u(x) = Ax^\alpha$, $A \neq 0$. If we make the substitution $y = u(x) + z$, then Eq. (1) becomes

$$\tilde{f}(x, Z) = f(x, U+Z) = \sum_{|I|=0}^{M} a_I(x)(U+Z)^I = 0,$$
$$Z = (z, z', \ldots, z^{(n)}). \tag{4}$$

Assume that the function $\tilde{f}(x, Z)$ can be expressed in the form

$$\tilde{f}(x, Z) = L(x)z + h(x, Z), \tag{5}$$

where $L(x)$ is a linear differential operator. Here, the support $S\big(L(x)z\big)$ is the point $(\gamma, 1)$, which is a vertex of the Newton polygon $N\big(\tilde{f}(x, Z)\big)$ and does not belong to the support $S\big(h(x, Z)\big)$ (see [1, sec. 1.4]). Under these conditions, the operator $L(x)$ is nontrivial, and we say that the considered face is nondegenerate. It should be noted that the nondegenerate case is one of the most common. The operator $L(x)$ in the nondegenerate case is written in the form

$$L(x) = x^\gamma\left(b_0 + b_1 x\frac{d}{dx} + \ldots + b_n x^n \frac{d^n}{dx^n}\right),$$
$$\gamma = \beta - \alpha,\ b_0, \ldots, b_n = \text{const},\ |b_0| + \ldots + |b_n| \neq 0. \tag{6}$$

If the operator $L(x)$ is identically zero (i.e., the considered face is degenerate), then we may apply the above-described algorithm to Eq. (4). The algorithm allows one to compute the following term in the expansion of the formal solution, namely $u_1(x) = A_1 x^{\alpha_1}$, where $A_1 \neq 0$ and $\alpha_1 > \alpha$. Our assumption is that, as a result of a finite number of steps of the type above described, we arrive at a nondegenerate face, and the transformed equation can be written in the form (4), which satisfies conditions (5) and (6). Also, note that if the original equation already has the required form, the first approximation to the solution may not be a power term, but a power-logarithmic one, namely $A(x)x^{\alpha}$, where $A(x)$ is a polynomial in $\ln x$ (see the next section).

The convergence condition for the formal solution of Eq. (1), whose first approximation we described above, consists in the equality of the order of the differential operator $L(x)$ and the order of the original equation, i.e., $b_n \neq 0$. This result for power expansions is given in [1, Theorem 3.4]. A proof is considered in [4] for expansions in the form of power series with constant coefficients in the case when the functions $a_I(x)$ are finite sums of monomials. A similar but somewhat more general assertion for an algebraic ODE is proven in [5] for expansions in the form of power-logarithmic series with nonnegative integer exponents. The meaning of the generalization suggested in [5] is that it covers the case when the initial face of the Newton polygon and some finite number of the subsequent ones, corresponding to the expansion terms, are degenerate (which means that the corresponding operators $L(x)$ are trivial). Furthermore, the convergence condition means that, at some step of the computation of the terms of the formal solution to the algebraic ODE, there appears a face in the Newton polygon for which the corresponding operator $L(x)$ has the form (6) and its order is equal to the order of the equation. Thus, the equality between the order of the operator $L(x)$ and that of the original equation is a basic condition for the convergence of the formal expansions. In this paper, we prove that this condition is sufficient for the convergence of the power-logarithmic expansions of solutions to Eq. (1) (expansions in the form of power-logarithmic series with arbitrary real exponents). It is not difficult to extend this result to power-logarithmic expansions of solutions with complex exponents.

Note that Dulac series appeared for the first time in the works of H. Dulac on limit cycles in the plane. The term *Dulac series* was introduced by Yu. Ilyashenko in his research on limit cycles in the plane (see, for example, [6]).

In the present paper, we use the term Dulac series to refer to power-logarithmic series.

2 Statement of the Main Result

Let us consider a nondegenerate face that belongs to the left-hand side boundary of the polygon N of Eq. (1). Assume that $u(x) = Ax^{\alpha}$, $A \neq 0$, is a solution of the truncated equation that corresponds to the considered face. After making the substitution $y = u(x) + z$ in Eq. (1) and canceling the factor x^{γ} on both sides of the obtained Eq. (see (6)), we can write the transformed equation in the form

$$L_1(x)z = F(x, Z), \tag{7}$$

where

$$Z = (z, z', \ldots, z^{(n)}),\ L_1(x) = b_0 + b_1 x \frac{d}{dx} + \ldots + b_n x^n \frac{d^n}{dx^n},$$

$$|b_0| + \ldots + |b_n| \neq 0,\ F(x, Z) = \sum_{|I|=0}^{M} f_I(x) Z^I,$$

$$f_I(x) = \sum_{m=1}^{\infty} c_{Im} x^{p_{Im}}, \; p_{Im} < p_{I(m+1)}, \; \lim_{m\to\infty} p_{Im} = +\infty.$$

Let us write the function $F_0(x) = F(x, 0) = f_{I_0}(x)$, $I_0 = (0, 0, \dots, 0)$ in the form

$$F_0(x) = \sum_{m=1}^{\infty} c_m x^{p_m}, \quad c_m = c_{I_0 m}, \; p_m = p_{I_0 m}, \; p_1 > \alpha.$$

We assume that Eq. (7) is nondegenerate, i.e., $c_1 \neq 0$ (otherwise, the function $y(x) = Ax^\alpha$ is already a solution of the original equation). The points $Q_1 = (0, 1)$ and $Q_2 = (p_1, 0)$ are the vertices of the Newton polygon of Eq. (7), and the edge $[Q_1, Q_2]$ belongs to the left boundary. The support $S\big(L_1(x)z\big)$ is the point $(0, 1)$. Moreover, the point $(0, 1)$ is not among the points of the support $S\big(F(x, Z)\big)$ in Eq. (7) (a similar condition holds in Eq. (4)). In the previous section, we noted that the original Eq. (1) in the nondegenerate case can always be reduced to the form (7). All the series in (7) converge uniformly and absolutely in some neighborhood of zero.

As noted in [1], if we search for a formal solution to Eq. (7) in the form of a power series,

$$z = \sum_{k=1}^{\infty} a_k x^{s_k}, \quad a_k, s_k = \text{const}, \; s_k < s_{k+1}, \; \lim_{k\to\infty} s_k = +\infty, \tag{8}$$

then the solution only exists under certain conditions. If those conditions do not hold true, then it is necessary to extend the class of solutions of the equation, that is, we should consider solutions of the form (8), where a_k are polynomials in $\ln x$. However, if we search for such solutions, then we must in turn extend the considered class of equations. To achieve this, we below replace Eq. (7) with the following one (see [1, sec. 3.1]):

$$L_1(x)z = F(x, Z), \tag{9}$$

where

$$F(x, Z) = \sum_{|I|=0}^{M} f_I(x) Z^I, \; L_1(x) = b_0 + b_1 x \frac{d}{dx} + \ldots + b_n x^n \frac{d^n}{dx^n},$$

$$|b_0| + \ldots + |b_n| \neq 0, \; f_I(x) = \sum_{m=1}^{\infty} c_{Im} f_{Im}(\ln x) x^{p_{Im}}, \; p_{Im} < p_{I(m+1)},$$

$$\lim_{m\to\infty} p_{Im} = +\infty, \; F_0(x) = f_{I_0}(x) = \sum_{m=1}^{\infty} c_m f_m(\ln x) x^{p_m}, \; p_m < p_{m+1},$$

and f_{Im}, f_m are polynomials in $\ln x$. In this case, we assume that both the degrees of these polynomials and the absolute values of their coefficients in the equation are uniformly bounded from above by some positive number. We also assume that the series $\sum_{m=1}^{\infty} c_{Im} x^{p_{Im}}$, $0 \le |I| \le M$, converge uniformly and absolutely in some neighborhood of zero. Evidently, Eq. (7) satisfies these conditions (since all polynomials f_{Im} are identically one).

The following theorem is valid for Eq. (9).

Theorem *If the condition* $b_n \neq 0$ *is valid in Eq.* (9), *then the equation has a continuable solution represented as a Dulac series:*

$$z = \sum_{m=1}^{\infty} g_m(\ln x)x^{s_m}, \quad s_m \in \mathbb{R},\ s_m < s_{m+1},\ s_1 = p_1,\ \lim_{m\to\infty} s_m = +\infty, \tag{10}$$

where $g_m(\ln x)$ *are polynomials in* $\ln x$. *The series converges absolutely and uniformly in a neighborhood of the point* $x = 0$.

As stated above, the convergence in a neighborhood of the origin is understood as the convergence in a region that is the intersection of the said neighborhood and the sector V.

The existence of a formal solution of the form (10) for Eq. (9) is proven [1]. The theorem we stated above asserts that the series converges if $b_n \neq 0$.

Thus, if the Newton polygon of Eq. (1) has a nondegenerate face and truncated Eq. (3) corresponding to it has a nontrivial solution $u = Ax^{\alpha}$, then the theorem conditions imply that Eq. (1) has a continuable solution with the indicated asymptotic expansion and a convergent power-logarithmic expansion. It is evident from the theorem that the result can be extended to the case of the asymptotic expansion $u = A(t)x^{\alpha}$, $t = \ln x$, where $A(t)$ is a polynomial, provided that the operator $L(x)$ is of the form (6) with $b_n \neq 0$.

3 Proof of the theorem

By W we denote here a neighborhood of the point $x = 0$ in which the series (2) and, correspondingly, the series $F_0(x)$ and $F(x)$ in (7) converge absolutely and uniformly. Let us define the norm $\|v\|$ of the absolutely convergent series $v = \sum_{i=1}^{\infty} v_i$ by means of the expression $\sum_{i=1}^{\infty} |v_i|$.

The following estimates are valid for Eq. (9) in the neighborhood W:

$$\|F_0(x)\| \leq D|x|^{p_1}|\ln x|^l, \ \|f_I(x)\| \leq D|x|^{p_{I1}}|\ln x|^l, \quad D, l = \text{const} > 0.$$

As stated above, we may assume, without loss of generality, that Eq. (9) is not homogeneous, i.e., $c_1 \neq 0$, and the edge connecting the vertices $Q_1 = (0, 1)$ and $Q_2 = (p_1, 0)$ belongs to the left boundary of the Newton polygon of Eq. (9). Consequently, the inequality $p_{I1} > p_1(1 - |I|) + |\tilde{I}|$ is valid for any set $I : 1 \leq |I| \leq M$. Fix a number $\delta > 0$ such that the inequality

$$p_{I1} \geq p_1(1 - |I|) + |\tilde{I}| + \delta \tag{11}$$

is valid for all sets $I : 1 \leq |I| \leq M$.

Let us call any function $P(x)$ written in the form $P(x) = \sum_{k=1}^{K} P_k(\ln x)x^{s_k}$, where $P_k(\ln x)$ are polynomials in $\ln x$, a quasi-polynomial.

Consider de auxiliary equation

$$L_1(x)u = F_{01}(x), \ F_{01}(x) = \sum_{p_1 \le p_m \le p_1+\delta} c_m f_m(\ln x)x^{p_m}.$$

It is known that the quasi-polynomial

$$u = u(x) = \sum_{p_1 \le p_m \le p_1+\delta} g_m(\ln x)x^{p_m}$$

is a solution of this equation. By means of the substitution $z = u(x) + z_1$, Eq. (9) takes the form

$$\begin{aligned} &L_1(x)z_1 = F^1(x, Z_1), \ Z_1 = (z_1, z_1', \dots, z_1^{(n)}), \\ &F^1(x, Z_1) = \sum_{|I|=0}^{M} f_I^1(x)Z_1^I, \ f_I^1(x) = \sum_{m=1}^{\infty} c_{Im}^1 f_{Im}^1(\ln x)x^{p_{Im}^1}, \\ &F_0^1(x) = F^1(x, 0) = \sum_{m=1}^{\infty} c_m^1 f_m^1(\ln x)x^{p_m^1}, \end{aligned} \tag{12}$$

which differs from (9) in that $p_1^1 \ge p_1 + \delta$. This follows from the fact that

$$F_0^1(x) = F_0(x) - F_{01}(x) + \sum_{|I|=1}^{M} f_I(x)U^I, \quad U = (u, u', \dots, u^{(n)}), \ u = u(x)$$

and from the estimate (11).

Besides, it is not difficult to prove by simple computations that the estimate

$$p_{I1}^1 \ge p_1^1(1 - |I|) + |\tilde{I}| + \delta,$$

which is similar to (11), is valid for Eq. (12).

If we apply the same reasoning to Eq. (12) and continue this process, we obtain that there exists a transformation $z = \tilde{u}(x) + \tilde{z}$, where $\tilde{u}(x)$ is a quasi-polynomial in x, that transforms Eq. (9) into a form whose only difference from (9) is that the number p_1 can now be assumed arbitrarily large. For the sake of simplicity, we assume below that the number p_1 in (9) is as large as it is necessary.

Assume that $0 < \varepsilon < \delta$. By reducing the radius of the neighborhood W of the point $x = 0$, we can achieve that the estimates

$$\|F_0(x)\| \le D|x|^{\tilde{p}_1}, \ \|f_I(x)\| \le D|x|^{\tilde{p}_{I1}}, \quad D = \text{const} > 0,$$

where $\tilde{p}_1 = p_1 - \varepsilon$ and $\tilde{p}_{I1} = p_{I1} - \varepsilon$, are valid for Eq. (9) in W. Moreover, it follows from (11) that the number ε can be chosen so small that inequalities $\tilde{p}_{I1} > \tilde{p}_1(1 - |I|) + |\tilde{I}|$ are true in (9) for all sets $I : 1 \leq |I| \leq M$. Consequently, there exists a $\tilde{\delta} > 0$ such that the estimate

$$\tilde{p}_{I1} \geq \tilde{p}_1(1 - |I|) + |\tilde{I}| + \tilde{\delta} \tag{13}$$

is valid for all sets $I : 1 \leq |I| \leq M$. To keep the proof simple, we assume that the estimates

$$\begin{aligned} &\|F_0(x)\| \leq D|x|^{p_1}, \ \|f_I(x)\| \leq D|x|^{p_{I1}}, \quad D = \text{const} > 0, \ x \in W, \\ &p_{I1} \geq p_1(1 - |I|) + |\tilde{I}| + \delta, \ 1 \leq |I| \leq M, \end{aligned} \tag{14}$$

are valid in (9). Furthermore, in what follows, we will assume, without loss of generality, that the radius of the neighborhood W is smaller than one.

Consider the differential operator $L_1(x)$. By the substitution $t = \ln x$, it takes the form of the characteristic polynomial

$$P(Q) = \sum_{j=0}^{n} \tilde{c}_j Q^j, \ Q = \frac{d}{dt}, \ Q^j = \left(\frac{d}{dt}\right)^j = \frac{d^j}{dt^j}.$$

Let $\lambda_1, \ldots, \lambda_s$ be the roots of the polynomial $P(Q)$, and let $k_1, \ldots, k_s$ be their multiplicities.

Consider the equation

$$L(x)v = h(x), \quad x \in V_1, \tag{15}$$

where the function $h(x)$ is a series of powers of x whose coefficients are polynomials in $\ln x$, and moreover, the degrees of the polynomials are jointly bounded by a certain number,

$$\|h(x)\| \leq C|x|^q, \quad C, q = \text{const}, \ C > 0, \ q > 2\left(1 + \max_{1 \leq j \leq s} |\Re\lambda_j|\right).$$

If we make the substitution $t = \ln x$, this equation takes the form

$$P(Q)\tilde{v} = h(e^t), \ \tilde{v} = \tilde{v}(t) = v(e^t). \tag{16}$$

Along with Eq. (16), we consider n equations

$$P_{jm}(Q)\tilde{v}_{jm} = h(e^t), \ P_{jm}(Q) = (Q - \lambda_j)^m, \quad m = 1, \ldots, k_j, \ j = 1, \ldots, s. \tag{17}$$

Lemma *For each $m = 1, \ldots, k_j$, there exists a solution $\tilde{v}_{jm}(t)$ of Eq. (17) such that the estimate*

$$\|v_{jm}(x)\| \leq 2Cq^{-1}|x|^q, \quad x \in W,$$

is valid for the functions $v_{jm}(x) = \tilde{v}_{jm}(\ln x)$.

Proof of the lemma The function

$$\tilde{v}_{jm}(t) = \int_{\infty}^{t} \frac{(t-\tau)^{m-1}}{(m-1)!} e^{-\lambda_j(\tau-t)} h(e^{\tau})\, d\tau \tag{18}$$

is a solution of Eq. (17) (see, e.g., [7, chapter V]). Choose the integration path $\Im\tau = \Im t$ and make the substitution $\tau = t - \xi$, $0 \le \xi < +\infty$. Taking into account that $\|e^{-\lambda_j(\tau-t)} h(e^{\tau})\| = \|e^{\lambda_j \xi} h(e^{t-\xi})\| \le C e^{\xi \Re\lambda_j + q\Re(t-\xi)}$, we obtain

$$\|v_{jm}(x)\| = \|\tilde{v}_{jm}(t)\| \le C e^{q\Re t} \int_0^{+\infty} \frac{\xi^{m-1}}{(m-1)!} e^{(\Re\lambda_j - q)\xi}\, d\xi \le$$
$$\le C e^{q\Re t} \int_0^{+\infty} \frac{\xi^{m-1}}{(m-1)!} e^{-q\xi/2}\, d\xi = C(2q^{-1})^m e^{q\Re t} \le 2Cq^{-1}|x|^q$$

for $q > 2(1 + |\Re\lambda_j|)$. This proves the lemma. □

Equation (16) has a solution $\tilde{v} = \tilde{v}(t)$ that can be expressed as a sum of solutions of Eq. (17), namely

$$\tilde{v} = \sum_{j=1}^{s} \sum_{m=1}^{k_j} A_{jm} \tilde{v}_{jm},$$

where A_{jm} are the coefficients in the partial-fraction decomposition of $1/P(Q)$ (see [7, chapter V]),

$$\frac{1}{P(Q)} = \sum_{j=1}^{s} \sum_{m=1}^{k_j} \frac{A_{jm}}{(Q-\lambda_j)^m}. \tag{19}$$

Hence, it follows from the lemma that Eq. (15) has a solution $v = v(x)$ that satisfies the inequality

$$\|v(x)\| \le 2CA_0 q^{-1} |x|^q \tag{20}$$

in the neighborhood W of the point $x = 0$, with $A_0 = \sum_{j=1}^{s} \sum_{m=1}^{k_j} |A_{jm}|$.

Note that if the function $\tilde{v}(t)$ is a solution of Eq. (16), then the function $\tilde{v}_1 = \tilde{v}_1(t) = \tilde{v}'(t)$ is a solution of the equation

$$P_1(Q)\tilde{v}_1 = h_1(t),\ P_1(Q) = Q^{-1}\big(P(Q) - P(0)\big),\ h_1(t) = h(e^t) - P(0)\tilde{v}(t).$$

We have $\|h_1(t)\| \le C(1 + 2A_0P_0q^{-1})e^{q\Re t}$, $P_0 = |P(0)|$, and if $2A_0P_0q^{-1} \le 1$, then

$$\|h_1(t)\| \le 2Ce^{q\Re t}. \tag{21}$$

Let $\lambda_1^1, \ldots, \lambda_{s_1}^1$ be the roots of the polynomial $P_1(Q)$. Thus, if the inequalities $q > 2(1 + \max_{1\le j\le s_1} |\Re\lambda_j^1|)$, $q > 2(1 + \max_{1\le j\le s} |\Re\lambda_j|)$, and (21) are valid, then we obtain the estimate

$$\|\tilde{v}_1(t)\| \leq 4CA_1q^{-1}e^{q\Re t},$$

where A_1 is a constant equal to the sum of the absolute values of the coefficients in the partial-fraction decomposition of $1/P_1(Q)$. Therefore, an estimate similar to (20) is valid for the function $v_1(x) = v'(x)$, namely

$$\|v_1(x)\| \leq 4CA_1q^{-1}|x|^{q-1}.$$

Reasoning in a similar manner, we obtain the following equations for the functions $\tilde{v}_j(t) = \tilde{v}^{(j)}(t)$, $j = 1, \ldots, n$:

$$\begin{aligned} &P_j(Q)\tilde{v}_j = h_j(t), \quad h_j(t) = h_{j-1}(t) - P_{j-1}(0)\tilde{v}_{j-1}(t), \\ &P_j(Q) = Q^{-1}\big(P_{j-1}(Q) - P_{j-1}(0)\big), \quad P_0(Q) = P(Q),\ \tilde{v}_0(t) = \tilde{v}(t), \\ &h_0(t) = h(e^t),\ j = 1, \ldots, n-1. \end{aligned}$$

If $q > n$ and $q > 2(1 + |\Re\lambda|)$ for all numbers λ that are roots of the polynomials $P_j(Q)$, and $q \geq 2A_j|P_j(0)|$, $j = 0, \ldots, n-1$, where A_j are constants equal to the sum of absolute values of the coefficients in the partial-fraction decompositions of $1/P_j(Q)$, then we obtain the following estimates:

$$\|\tilde{v}_j(t)\| \leq 2^{j+1}CA_jq^{-1}e^{q\Re t}, \quad \|h_j(t)\| \leq 2^jCe^{q\Re t}, \quad j = 0, \ldots, n-1.$$

For the function $\tilde{v}_n(t) = \tilde{v}^{(n)}(t)$, we have the equality $b_n\tilde{v}_n(t) = h_{n-1}(t) - P_{n-1}(0)\tilde{v}_{n-1}(t)$, from which we deduce the estimate

$$\|\tilde{v}_n(t)\| \leq |b_n^{-1}|\,2^{n-1}Ce^{q\Re t}(1 + 2A_{n-1}|P_{n-1}(0)|q^{-1})e^{q\Re t} \leq |b_n^{-1}|\,2^nCe^{q\Re t}.$$

The estimates we have obtained imply the validity of the estimate

$$\|v^{(j)}(x)\| \leq BC|x|^{q-j}, \quad j = 0, \ldots, n,\ x \in W, \tag{22}$$

for a sufficiently large number q and a constant $B \geq 1$ that depends only on n and b_n.

Let us proceed to the main part of the theorem proof.

Proof of the theorem According to the lemma and the reasoning given above, if condition (14) holds true and the number p_1 is sufficiently large, then the equation

$$L_1(x)u = F_0(x) \tag{23}$$

has a solution $u = u_1(x)$ that can be expressed as a series of powers of x, with coefficients that are polynomials in $\ln x$, and the estimates

$$\|u_1^{(j)}(x)\| \leq DB|x|^{p_1-j}, \quad j = 0, 1, \ldots, n, \tag{24}$$

are valid for this solution in the neighborhood W of the point $x = 0$.

Consider the sequence of functions ($k = 1, 2, \dots$)

$$h_k = h_k(x) = \sum_{m=1}^{k} u_m(x), \; h_0(x) = 0, \tag{25}$$

where $u_1(x)$ is a solution of Eq. (23) and $u_{m+1}(x)$ is the solution described in the lemma for the equation

$$\begin{aligned} &L_1(x)u_{m+1} = F_{0m}(x), \; F_{0m}(x) = F\big(x, H_m(x)\big) - F\big(x, H_{m-1}(x)\big), \\ &H_0 = (0, 0, \dots, 0), \; H_m = (h_m, h_m', \dots, h_m^{(n)}), \; m = 1, 2, \dots. \end{aligned} \tag{26}$$

We want to prove that the estimates

$$\begin{aligned} &\|u_k^{(j)}(x)\| \le DB|x|^{p_1 - j + (k-1)\delta_1}, \quad \|h_k^{(j)}(x)\| \le 2DB|x|^{p_1 - j}, \\ &\quad j = 0, \dots, n, \; \delta_1 = \delta/2, \end{aligned} \tag{27}$$

are valid for any integer $k \ge 1$ in some neighborhood $W_1 \subset W$ of the point $x = 0$. The radius r of the neighborhood W_1 is to be determined later.

This was already proven for $k = 1$ (see (24), (25)). Suppose that (27) is true for $1 \le k \le K$. Then, it follows from (26) that

$$F_{0K}(x) = F\big(x, H_K(x)\big) - F\big(x, H_{K-1}(x)\big) = \sum_{|I|=1}^{M} f_I(x) \left(H_K^I(x) - H_{K-1}^I(x)\right).$$

We will use the notations $b_l = \big(h_K^{(l)}(x)\big)^{i_l}$, $a_l = \big(h_{K-1}^{(l)}(x)\big)^{i_l}$, $l = 0, 1, \dots, n$. We can now write that

$$F_{0K}(x) = \sum_{|I|=1}^{M} f_I(x) \left(\prod_{l=0}^{n} b_l - \prod_{l=0}^{n} a_l\right).$$

Express $\prod_{l=0}^{n} b_l - \prod_{l=0}^{n} a_l$ in the form

$$\prod_{l=0}^{n} b_l - \prod_{l=0}^{n} a_l = \sum_{0 \le j \le n} a_0 \dots a_{j-1}(b_j - a_j)b_{j+1} \dots b_n.$$

After some simple computations, we find that

$$\begin{aligned} &\|b_j - a_j\| \le (4DB)^{i_j} |x|^{(p_1 - j)i_j + \delta_1(K-1)}, \\ &\|a_0 \dots a_{j-1} b_{j+1} \dots b_n\| \le (2DB)^{|I| - i_j} |x|^{p_1(|I| - i_j) - |\tilde{I}| + j i_j}, \end{aligned}$$

for $r^{\delta_1} \leq 1/2$. From this, we deduce the estimate

$$\left\| \prod_{l=0}^{n} b_l - \prod_{l=0}^{n} a_l \right\| \leq (4DB)^{|I|} |x|^{p_1|I|-|\tilde{I}|+\delta_1(K-1)}.$$

Hence, taking (14) into account, we get the estimate

$$\|F_{0K}(x)\| \leq D\tilde{M}(4DB)^M |x|^{p_1+\delta_1(K-1)+\delta},$$

where $\tilde{M}$ is the number of sets I that fulfills the condition $1 \leq |I| \leq M$.

Assume now that the radius r of the neighborhood W_1 satisfies the condition $\tilde{M}(4BD)^M r^{\delta_1} \leq 1$, $r^{\delta_1} \leq 1/2$. According to the above, Eq. (26) for $m = K$ has a solution $u_{K+1}(x)$ for which the estimates

$$\|u_{K+1}^{(j)}(x)\| \leq DB|x|^{p_1-j+K\delta_1}, \quad \|h_{K+1}^{(j)}(x)\| \leq 2DB|x|^{p_1-j},$$
$$j = 0, \ldots, n, \; x \in V_r,$$

are valid. Hence, estimates (27) are proved for all integers $k \geq 1$.

It follows from the proven estimates that the series

$$h^{(j)}(x) = \sum_{k=1}^{\infty} u_k^{(j)}(x), \quad j = 0, \ldots, n,$$

converge absolutely and uniformly in the neighborhood W_1 of the point $x = 0$ (we mean the convergence in the intersection of W_1 with the sector V).

Since (26) implies the equality

$$L_1 h_k(x) = F_0(x) + F\big(x, h_k^I(x)\big) - L_1 u_{k+1}(x),$$

the function $z = h(x) = \sum_{k=1}^{\infty} u_k(x)$ satisfies Eq. (9) in the neighborhood W_1. This concludes the proof of the theorem. □

4 Examples

4.1. Let us consider the Abel equation of the first kind in normal form:

$$y' + y^3 + cx^p = 0. \tag{28}$$

The support of the left side of the equation consists of three points: $Q = (-1, 1)$, $Q_1 = (p, 0)$, and $Q_2 = (0, 3)$. If $p > -\frac{3}{2}$, the convex hull of these points (Newton polygon) is a triangle with vertices Q, Q_1, and Q_2, and edges $\Gamma_1 = [Q, Q_1]$, $\Gamma_2 = [Q, Q_2]$, and $\Gamma_3 = [Q_1, Q_2]$ (see Fig. 1). The left boundary of the triangle is formed by the three vertices indicated above and two edges, Γ_1 and Γ_2.

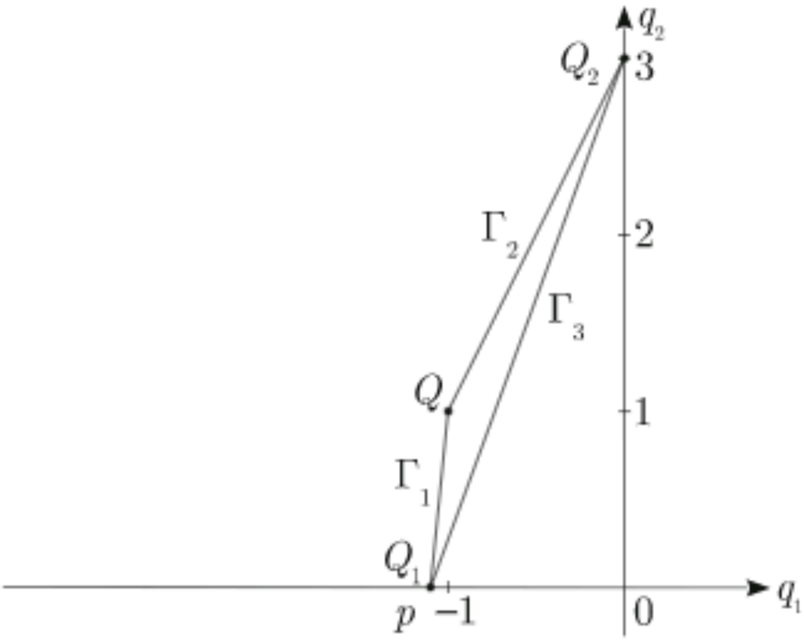

Fig. 1 .

Let us consider the edge Γ_1. Here the conditions of the theorem hold. If $p > -1$, it is possible to prove that Eq. (28) has a solution expressed as a power series that converges absolutely and uniformly in some neighborhood of the point $x = 0$, namely

$$y = \sum_{k=1}^{\infty} a_k x^{s_k}, \quad a_k,\ s_k = \text{const},\ s_1 = p+1, a_1 = \frac{-c}{p+1},$$
$$s_k < s_{k+1},\ \lim_{k\to\infty} s_k = +\infty.$$

If $p = -1$, the equation has a solution that can be expressed as a power-logarithmic series (Dulac series) that converges absolutely and uniformly in some neighborhood of the point $x = 0$, namely

$$y = -c\ln x + c^3 x(\ln^3 x - 3\ln^2 x + 6\ln x - 6) + \sum_{k=3}^{\infty} a_k(\ln x)x^{s_k},$$
$$s_k = \text{const},\ 1 < s_k < s_{k+1},\ \lim_{k\to\infty} s_k = +\infty,$$

where $a_k(t)$ are polynomials.

If $-3/2 < p < -1$, the equation has a solution expressed as a series that converges absolutely and uniformly in some neighborhood of the point $x = 0$, namely

$$y = \frac{-c}{p+1}x^{p+1} + \sum_{k=2}^{\infty} b_k(\ln x)x^{s_k},$$
$$s_k = \text{const},\ p+1 < s_k < s_{k+1},\ \lim_{k\to\infty} s_k = +\infty,$$

where $b_k(t)$ are either polynomials or constants (depending on p).

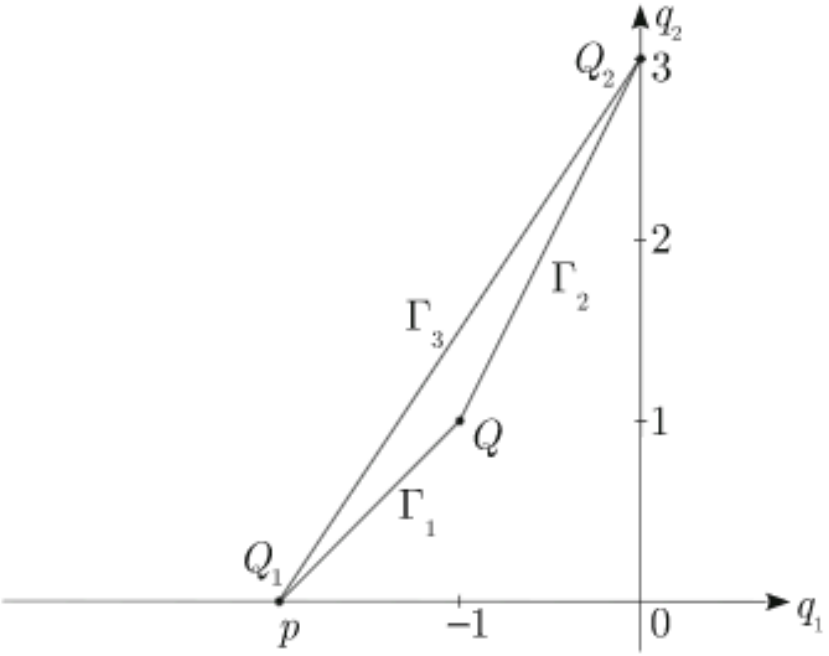

Fig. 2 .

If $p = -3/2$, then the equation has the solution $y = \dfrac{a}{\sqrt{x}}$, where a is any of the roots of the equation $2a^3 - a + 2c = 0$.

Let us consider the edge Γ_2 when $p > -3/2$. The functions $u_{1,2}(x) = \dfrac{\pm 1}{\sqrt{2x}}$ are solutions of the truncated equation $u' + u^3 = 0$, which corresponds to the considered edge. By the substitution $y = u_{1,2}(x) + z$, we can write Eq. (28) in the form (see (7))

$$L_1(x)z = -3u_{1,2}(x)xz^2 - xz^3 - cx^{p+1},\ L_1(x) = x\frac{d}{dx} + \frac{3}{2}. \tag{29}$$

Since the inequality $p + 1 > -\dfrac{3}{2}$ holds, we can prove that, for each of the functions $u_{1,2}(x) = \pm(2x)^{-1/2}$, Eq. (29) has a solution in the form of a power series that converges absolutely and uniformly in some neighborhood of the point $x = 0$, namely

$$z = z_{1,2}(x) = \frac{-cx^{p+1}}{p + 5/2} + \dots .$$

Thus, Eq. (28) has the solutions

$$y = y_{1,2}(x) = \pm\frac{1}{\sqrt{2x}} + z_{1,2}(x).$$

If $p < -\dfrac{3}{2}$, the left boundary of the Newton polygon of Eq. (28) consists of the vertices $Q_1 = (p, 0)$ and $Q_2 = (0, 3)$, and the edge $\Gamma_3 = [Q_1, Q_2]$ (see Fig. 2).

The function $u_1(x) = -(cx^p)^{1/3}$ is a solution of the truncated equation $u^3 + cx^p = 0$, which corresponds to the edge Γ_3. By means of the substitution $y = u_1(x) + z$, we can write Eq. (28) in the form (see (5))

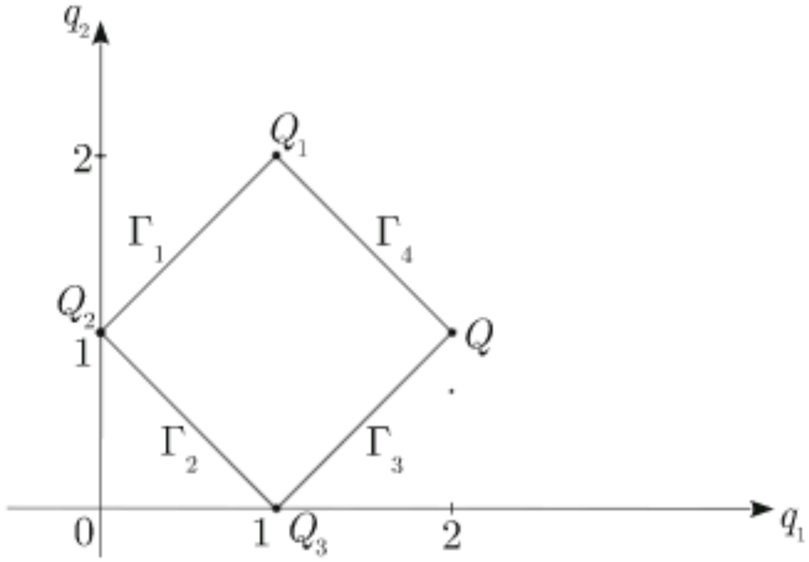

Fig. 3 .

$$L(x)z = -z' - 3u_1(x)z^2 - z^3 - u_1'(x), \quad L(x) = 3u_1^2(x).$$

The operator $L(x)$ is of order zero, whereas Eq. (28) is of order one. Therefore, the conditions of the theorem do not hold in this case.

4.2. In this example, we show that the process of construction of a formal solution may lead to a divergent series if the conditions of the theorem do not hold.

Consider the Riccati equation

$$x^3y' - xy^2 + y - x = 0. \tag{30}$$

The Newton polygon of the equation is shown in Fig. 3.

The function $u_1(x) = x^{-1}$ is a solution of the truncated equation $-xu^2 + u = 0$, which corresponds to the edge Γ_1 of the Newton polygon of Eq. (30). The substitution $y = x^{-1} + z$ transforms Eq. (30) into

$$x^3z' - xz^2 - z - 2x = 0. \tag{31}$$

The Newton polygons of Eqs. (30) and (31) coincide. But now we should consider the edge Γ_2. Here, the operator $L_1(x) = -1$ is of order zero and the sufficient condition of convergence does not hold. The current approximation $u_2(x) = -2x$ is a solution of the truncated equation $-u - 2x = 0$, which corresponds to the lower left edge of the Newton polygon of Eq. (31). Following the method described in the theorem to construct a formal solution, we compute the terms of the expansion $y = u_1(x) + u_2(x) + \dots$ of such a solution and obtain that $u_k(x) = a_k x^{2k-3}$, $a_k \leq -k!$, $k \geq 2$. Thus, the series $u_1(x) + u_2(x) + \dots$, which is a formal solution of Eq. (30), diverges in any neighborhood of the origin.

Note that it is not difficult to obtain an integral of Eq. (30) in a finite form. However, we were only interested here in showing that the sufficient condition of convergence for the power expansion of the solution of the equation is essential.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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