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On Existence of an Energy Function for Ω-Stable Surface Diffeomorphisms

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Abstract—If the chain recurrent set of a diffeomorphism f given on a closed n-manifold M^n is hyperbolic (equivalently, f is an Ω-stable) then it coincides with the closure of the periodic points set Per_f and its chain recurrent components coincide with the basic sets. Due to C. Conley for such a diffeomorphism there is a Lyapunov function which is a continuous function $\varphi : M^n \to \mathbb{R}$ increasing out of the chain recurrent set and a constant on the chain components. But in general a Lyapunov function has critical points out of the chain recurrent set, that is it is not an energy function. In this paper we investigate the problem of the existence of an energy function for diffeomorphisms of a surface. D. Pixton constructed a Morse energy function for Morse-Smale 2-diffeomorphisms (all basic sets are trivial). It was proved by M. Barinova, V. Grines and O. Pochinka that every Ω-stable diffeomorphism $f : M^2 \to M^2$, whose all non-trivial basic sets are attractors or repellers, possesses a smooth energy function which is a Morse function outside non-trivial basic sets. The question about an existence of an energy function for 2-diffeomorphisms with zero-dimensional basic sets was open until now. The main result of this paper is that every Ω-stable 2-diffeomorphism with a zero-dimensional non-trivial basic set without pairs of conjugated points does not possess an energy function.

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1. INTRODUCTION AND FORMULATION OF RESULTS

Let M^n be a closed orientable n-manifold M^n with a metric d and $f : M^n \to M^n$ be a diffeomorphism. Recall that for $\varepsilon > 0$ a sequence of points $x_1, x_2, \dots, x_n$ such that

$$d(f(x_i), x_{i+1}) < \varepsilon \quad \text{for} \quad 1 \le i \le n-1,$$

is called an *ε-chain* of the diffeomorphism f. A point $x \in M^n$ is called *chain recurrent* if for every $\varepsilon > 0$ there exists a natural number n (depending on ε) and an ε-chain $x_1, x_2, \dots, x_n$ such that $x_1 = x_n = x$. A set R_f of all chain recurrent points is called a *chain recurrent set* of the diffeomorphism f.

It can be directly verified that the set R_f is f-invariant and compact and the relation $\sim$, defined on R_f by the rule: $x \sim y$ for $x, y \in R_f$, if for every $\varepsilon > 0$ there exist ε-chains from x to y and from y to x, is an equivalence relation. The equivalence class of points from R_f with respect to the introduced equivalence relation $\sim$ is called a *chain component*.

Definition 1. A *Lyapunov function* for a diffeomorphism $f : M^n \to M^n$ is a continuous function $\varphi : M^n \to \mathbb{R}$ with the following properties:

1) if $x \notin R_f$ then $\varphi(f(x)) < \varphi(x)$;

2) if $x, y \in R_f$ then $\varphi(x) = \varphi(y)$ iff x and y are in the same chain component;

3) $\varphi(R_f)$ is a compact nowhere dense subset of $\mathbb{R}$.

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For any point $p \in M^n$ denote by (V_p, ϕ_p) a local map such that

$$\phi_p(y) = (x_1(y), \cdots, x_n(y)) \in \mathbb{R}^n, \quad y \in V_p, \quad x_i(p) = 0, \quad i \in \{1, \ldots, n\}.$$

For a real-valued continuous function $\varphi : M^n \to \mathbb{R}$ a point $p \in M^n$ is called a *regular point* of the function φ if at the point p there exists a local chart (V_p, ϕ_p) such that $\varphi(y) = \varphi(p) + x_n(y)$. Otherwise, p is called a *critical point*. Let us denote by Cr_φ the set of the critical points of the function φ.

Definition 2. A Lyapunov function $\varphi : M^n \to \mathbb{R}$ for a diffeomorphism $f : M^n \to M^n$ is called an *energy function* if $Cr_\varphi = R_f$.

Analogically definitions can be done for a flow f^t given on a manifold M^n. It follows from the results of C. Conley [1], that a Lyapunov function exists for any dynamical system, flow or diffeomorphism. This fact is known as "The Fundamental Theorem of Dynamical Systems". It has been observed by J. Franks [2] that applying the results of W. Wilson [3] to the construction of C. Conley gives an existence of an energy function for an arbitrary flow generated by a continuous vector field.

As it turned out, a similar fact does not hold for diffeomorphisms even in the case when their chain recurrent set is hyperbolic. The set of such diffeomorphisms coincides with the set of Ω-stable diffeomorphisms, that is, preserving the qualitative structure of the chain recurrent set under small perturbations. The chain recurrent set of such a diffeomorphism coincides with the closure of the periodic points set Per_f (see, for example, [4]) and the chain recurrent components coincide with so called *basic sets* (see [5]), the number of which is finite:

$$R_f = cl(Per_f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_m.$$

Structurally stable diffeomorphisms with a finite hyperbolic chain recurrent set are called *Morse–Smale diffeomorphisms*. All basic sets of such diffeomorphisms are *trivial*, i.e. they coincide with the periodic orbits. By analogy with Morse–Smale flows, for which K. Meyer [6] established the existence of a Morse-Bott energy function, it is natural to expect the existence of a Morse energy function for Morse–Smale diffeomorphisms. The proof of this fact is an easy exercise in a dimension $n = 1$. In a dimension $n = 2$ such a function has been constructed by D. Pixton [7], he has also constructed an example of a Morse–Smale diffeomorphism on a 3-sphere which does not possess a Morse energy function. Such an effect is associated with the possibility of a wild embedding of saddle separatrices in the ambient 3-manifold. A criterion for the existence of a Morse energy function for Morse–Smale 3-diffeomorphisms was found in [8]. Examples of Morse–Smale diffeomorphisms in a dimension $n > 3$ which do not admit Morse energy functions are also known (see, for example, [9]).

Starting from the dimension $n = 2$ a basic set of an Ω-stable diffeomorphism $f : M^n \to M^n$ can be *non-trivial*, i.e. other than a periodic orbit. It was proved in [10], that every Ω-stable diffeomorphism $f : M^2 \to M^2$, whose all non-trivial basic sets have positive dimensions, possesses a smooth energy function which is a Morse function outside non-trivial basic sets. Also a similar energy function has been constructed in [11, 12] (see also survey [13]) for some classes of 3-diffeomorphisms with non-trivial basic sets.

The question about an existence of an energy function for 2-diffeomorphisms with zero-dimensional basic sets was open until now. In papers of V. Grines, R. Plykin, H. Kalai [14–16] were considered so called *basic sets without pairs of conjugated points* (see a precise definition in the section 2) which are most closely related to the topology of the ambient manifold. The simplest example of a diffeomorphism with such a basic set is a DA-diffeomorphism obtain from an Anosov diffeomorphism by "Smale surgeries" in two directions.

The main result of the paper is a proof of the following theorem.

Theorem 1. *Every Ω-stable diffeomorphism $f : M^2 \to M^2$ with a zero-dimensional non-trivial basic set without pairs of conjugated points does not possess an energy function.*

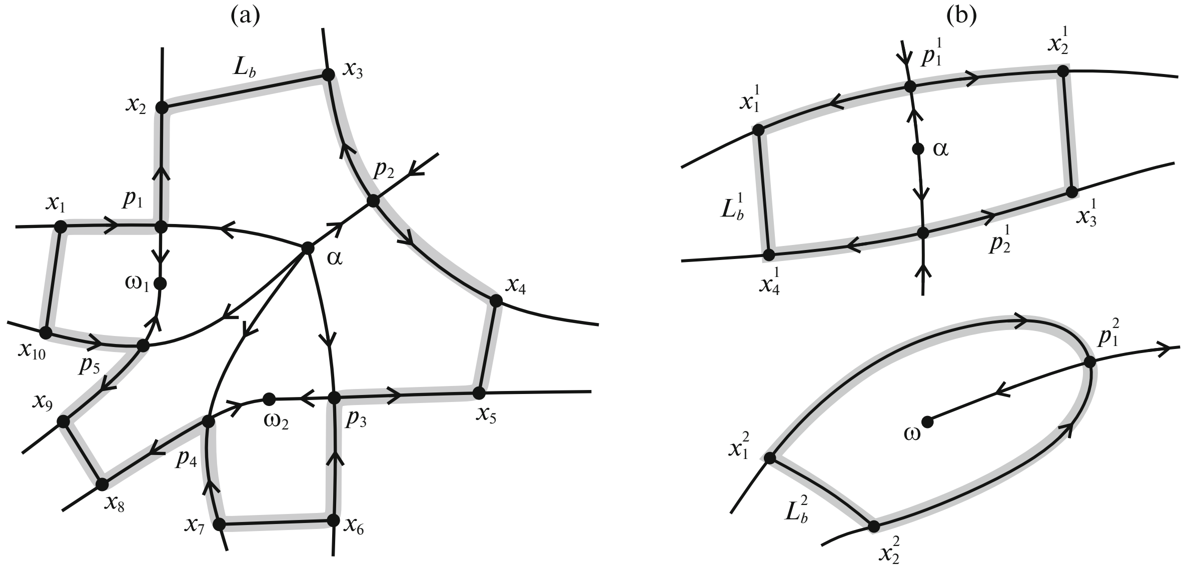

Fig. 1. Connecting curves.

2. A STRUCTURE OF A SURFACE ZERO-DIMENSIONAL NON-TRIVIAL BASIC SET WITHOUT PAIRS OF CONJUGATED POINTS

Let $f: M^2 \to M^2$ be an Ω-stable diffeomorphism whose non-wandering set contains a zero-dimensional non-trivial basic set Λ. Below we describe properties of Λ following to [16] and [17].

For every point $x \in \Lambda$ both invariant manifolds W^s_x, W^u_x have dimension one. Let $\sigma \in \{s, u\}$ and $\bar{\sigma} = s$ if $\sigma = u$, $\bar{\sigma} = u$ if $\sigma = s$. For different points $a, b \in W^\sigma_x$, $x \in \Lambda$ we denote by $[a, b]^\sigma$ a compact segment of the manifold W^σ_x bounded by the points a, b. Let $(a, b)^\sigma = [a, b]^\sigma \setminus (a \cup b)$. Two distinct points $x, y \in \Lambda$ are said to be a *pair of conjugated points* if

$$x, y \in \left(W^s_x \cap W^u_y\right), \quad (x, y)^s \cap \Lambda = (x, y)^u \cap \Lambda = \emptyset.$$

For every point $x \in \Lambda$ the set $W^\sigma_x \backslash x$ consists of two connected components, at least one of them has a non-empty intersection with the set Λ. A point $p \in \Lambda$ is called *σ-boundary* if one of the connected components of the set $W^\sigma_p \backslash p$ does not intersect Λ, let's call it *empty component*. Notice that a point can be s- and u-boundary simultaneously, in this case it is called *s, u-boundary*. The set Γ_Λ of boundary points is finite and, hence, consists of periodic points. If $p \in \Gamma_\Lambda$ and a non-empty connected component ℓ^σ_p of the set $W^\sigma_p \backslash p$ is f-invariant then

$$cl(\ell^\sigma_p \cap \Lambda) = \Lambda. \tag{1}$$

A chaplet b of a length r_b is a sequence $p_1, \ldots, p_{r_b}$, $r_b \geq 2$ of points from Γ_Λ for which there is a sequence of points $x_1, \ldots, x_{2r_b} \in (\Lambda \setminus \Gamma_\Lambda)$ such that:

1. points x_{2i-1}, x_{2i} belong to distinct separatrices of the same saddle p_i such that: both these separatrices are stable if p_i is an u-boundary point; they both are unstable if p_i is a s-boundary point; and they are of different stability if p_i is an s, u-boundary point;

2. if $x_{2i} \in W^\sigma_{p_i}$ then $x_{2i+1} \in W^{\bar{\sigma}}_{x_{2i}}$ ($x_{2r_b+1} = x_1$);

3. the set $L_b = \bigcup\limits_{i=1}^{r_b} L_{x_{2i}, x_{2i+1}}$ is a simple closed curve, where $L_{x_{2i}, x_{2i+1}} = [p_i, x_{2i}]^\sigma \cup [x_{2i}, x_{2i+1}]^{\bar{\sigma}} \cup [x_{2i+1}, p_{i+1}]^\sigma$ for $x_{2i} \in W^\sigma_{p_i}$ ($p_{r_b+1} = p_1$) (see Fig. 1).

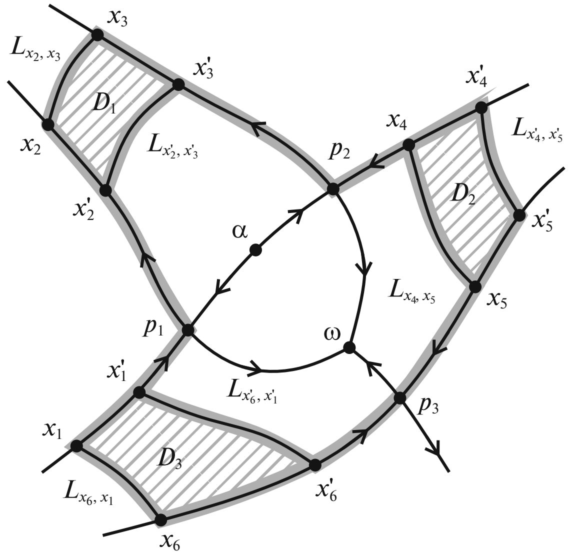

Fig. 2. Open discs bounded by different connecting curves.

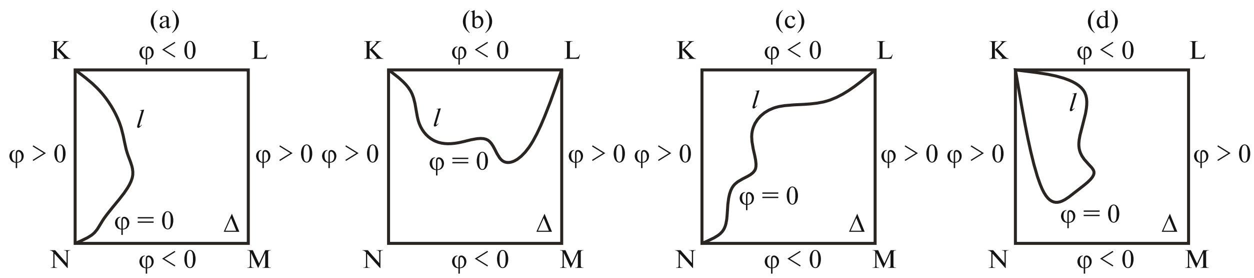

Fig. 3. Locations of the curve l: (a–c) possible and (d) impossible.

The curve L_b is called *a connecting curve of the chaplet* b.

Lemma 1. ([16]) *Let* $f: M^2 \to M^2$ *be an* Ω*-stable diffeomorphism and* Λ *be its 0-dimensional non-trivial basic set without pairs of conjugated points. Then the set* Γ_Λ *uniquely decomposes into chaplets. Moreover, for a fix chaplet* $b:\ p_1, \dots, p_{r_b}$ *there are connecting curves passing trough every point* $x_{2i} \in (W^\sigma_{p_i} \cap \Lambda)$ *and different arcs* $L_{x_{2i},x_{2i+1}}$, $L_{x'_{2i},x'_{2i+1}}$ *of distinct connecting curves bound an open 2-disc consisting of wandering points of* f *(see Fig. 2).*

3. ON CRITICAL POINTS OF A CONTINUOUS FUNCTION GIVEN ON A DISC

In this section, we prove the auxiliary lemma necessary to prove the main result of the paper.

Lemma 2. *Let* Δ *be a closed rectangle with vertices* $KLMN$ *and* $\varphi: \Delta \to \mathbb{R}$ *be a continuous function with the following properties:*

- $\varphi(K) = \varphi(L) = \varphi(M) = \varphi(N) = 0$;
- $\varphi(x) < 0$ *for every* $x \in int(KL \cup MN)$, $\varphi(x) > 0$ *for every* $x \in int(LM \cup NK)$;
- φ *does not have critical points in* $int\,\Delta$.

Then $\varphi^{-1}(0) \cap int\,\Delta \neq \emptyset$ *and every connected component of the set* $\varphi^{-1}(0) \cap int\,\Delta$ *is a curve whose closure separates either* KL *from* MN *or* LM *from* NK *(see Fig. 3 (a), (b), (c)).*

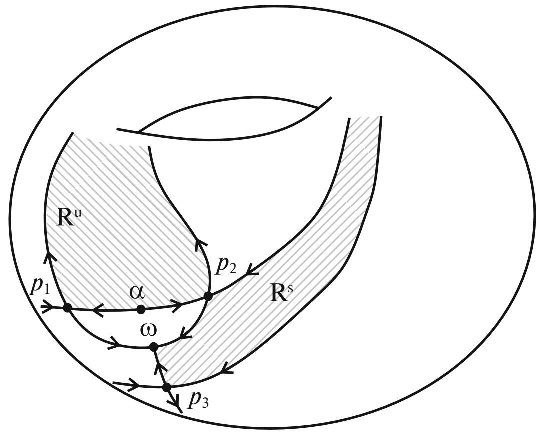

Fig. 4. Bands R^s and R^u.

Proof. Firstly notice that $\varphi^{-1}(0) \cap int\,\Delta \neq \emptyset$ as φ has values of different signs on $\partial\Delta$. Let l be one of the connected components of $\varphi^{-1}(0) \cap int\,\Delta$. As φ does not have critical points in $int\,\Delta$ then l is a 1-dimensional submanifold without boundary and, hence, l is homeomorphic either to $\mathbb{S}^1$ or $\mathbb{R}$. In the first case l bounds a 2-disc in $int\,\Delta$ whose necessary contains a critical point of φ, that contradicts the conditions of the lemma. Thus, l is a closed subset of $int\,\Delta$ which is homeomorphic to $\mathbb{R}$.

By Jordan curve theorem (see, for example, [18]) $int\,\Delta \setminus l$ is a disjoint union of two 2-discs D_1 and D_2 and l is the boundary of each in $int\,\Delta$. As $\partial\Delta = KLMN$ then boundary of D_i, $i = 1, 2$ consists of l and a part of the curve $KLMN$. Since φ is continuous, equals 0 on l and is non-zero on the intervals KL, LM, MN, NM then each interval is completely contained in one of the sets ∂D_1 or ∂D_2. If every set contains at least one interval (see Fig. 3 (a), (b), (c)) then lemma is proved. In the opposite case φ is zero on $cl(l)$ and $cl(l)$ bounds a 2-disc in $int\,\Delta$ (see Fig. 3 (d)) whose necessary contains a critical point of φ, that contradicts the conditions of the lemma. □

4. PROOF OF THE MAIN THEOREM

In this section we prove that *every Ω-stable diffeomorphism $f : M^2 \to M^2$ with a zero-dimensional non-trivial basic set without pairs of conjugated points does not possess an energy function.*

Proof. Let $f : M^2 \to M^2$ be an Ω-stable diffeomorphism and Λ be its 0-dimensional non-trivial basic set without pairs of conjugated points. Let us conduct a proof from the opposite, that is suppose that f possess an energy function $\varphi : M^2 \to \mathbb{R}$. Then φ is an energy function for f^k, for every $k \in \mathbb{N}$ and without loss of generality we can assume that all boundary points from Γ_Λ and their separatrices are fixed. Moreover, let us suppose that $\varphi(x) = 0$ for every point $x \in \Lambda$.

For every chaplet b of the basic set Λ consider a connecting curve L_b. By statement 1 the curves L_b and $f(L_b)$ bound a set of r_b open 2-discs consisting of wandering points of f. Denote by D_b the closure of them. Then $R_b = \bigcup\limits_{n\in\mathbb{Z}} f^n(D_b)$ consists of r_b bands. As Λ is 0-dimensional then among all such bands for all chaplets there is a band R^s whose boundary contains two stable separatrices ℓ_1^s, ℓ_2^s of some u or s, u-boundary points and there is a band R^u whose boundary contains two unstable separatrices ℓ_1^u, ℓ_2^u of some s- or s, u-boundary points (see Fig. 4). It follows from equality (1) and hyperbolicity of the basic set Λ that $R^s \cap R^u \neq \emptyset$ and by statement 1 every connected component of this intersection is a rectangle bounded by segments of the separatrices ℓ_1^u, ℓ_1^s, ℓ_2^u, ℓ_2^s. Let Δ be one of them, bounded by segments KL, LM, MN, NK of the separatrices ℓ_1^u, ℓ_1^s, ℓ_2^u, ℓ_2^s, accordingly (see Fig. 5).

Due to the definition of the energy function and stable and unstable manifolds, the function $\varphi|_\Delta$ satisfies all conditions of Lemma 2. Then there is a curve l_0 with $\varphi(x) = 0$ for $x \in l_0$, whose closure separates either KL from MN or LM from NK. The same situation is with every rectangle $\Delta_n = f^n(\Delta)$, $n \in \mathbb{Z}$, that is there is a curve l_n with $\varphi(x) = 0$ for $x \in l_n$, whose closure separates either $f^n(KL)$

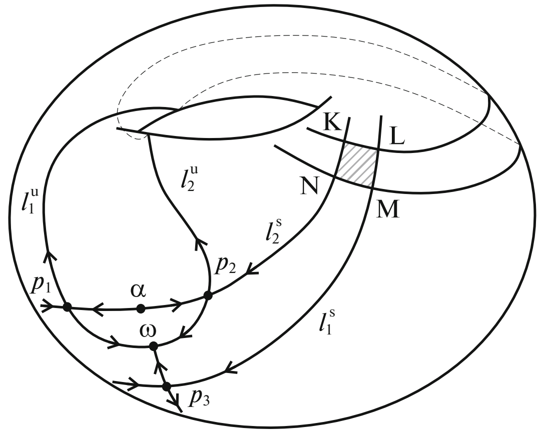

Fig. 5. Disc bounded by separatrices.

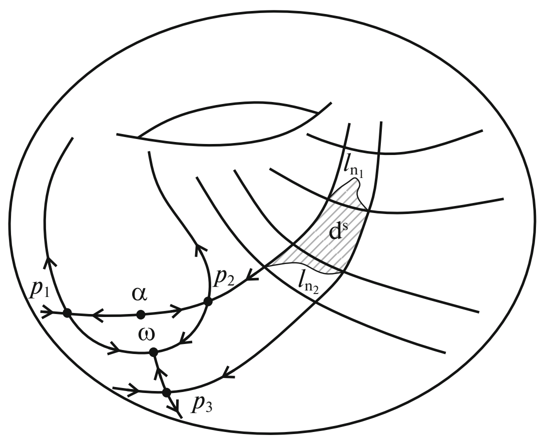

Fig. 6. The disc d^s.

from $f^n(MN)$ or $f^n(LM)$ from $f^n(NK)$. Thus, we find two curves $l_{n_1}, l_{n_2}, n_1 \neq n_2$ such that exactly one connected component of either the set $R^s \setminus cl(l_{n_1} \cup l_{n_2})$ or the set $R^u \setminus cl(l_{n_1} \cup l_{n_2})$ is a 2-disc.

Let us assume for definiteness that we find such a disc d^s as a connected component of the set $R^s \setminus cl(l_{n_1} \cup l_{n_2})$ (see Fig. 6). Then $\varphi(x) \geqslant 0$ for $x \in \partial d^s$. From other side by the construction $int(d^s) \cap (\ell_1^u \cup \ell_2^u) \neq \emptyset$ and, hence, the minimum of the function $\varphi|_{cl(d^s)}$ is situated in $int(d^s)$. But it is contradict the fact that φ has no critical points in $int(R^s)$ as, due to the statement 1, $int(R^s)$ consists of wandering points of f. □

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