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Invited Article

A survey on the modeling of hybrid behaviors: How to account for impulsive jumps properly

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ABSTRACT

We propose an overview of the modeling approaches for the mathematical description and analysis of processes that combine continuous and discontinuous behavior, namely impulsive differential equations, hybrid dynamical systems, and differential equations involving Dirac delta functions. These classes of systems are chosen due to their dominant prevalence in physics, mathematics, and control engineering research communities. A comparison of these frameworks is provided and their applicability depending on the character of the hybrid behavior is discussed. In particular, we show that special care should be taken when equations with Dirac delta function are interpreted as impulsive differential equations. We also provide insights on the stability and attractivity analysis of hybrid behaviors, highlight their essential differences to the respective stability concepts for smooth dynamical systems, and discuss specific phenomena which are peculiar for hybrid behaviors, like beating or Zeno phenomenon, modeling of multiple impulses at a single time instance, death and splitting of solutions, etc. With this, the paper attempts at bringing attention of the interested researchers to the methods available in other research communities and fostering the exchange of ideas and analysis techniques.

1. Introduction

The theory of dynamical systems knows two basic types of dynamics: continuous and discontinuous. In the first case, the system states changes continuously in time, like the position of celestial objects in astronomy. In the second case, the system state changes abruptly, like the state of an atom receiving a photon. In addition to that, in many cases dynamical systems may demonstrate mixed dynamics combining both continuous and discontinuous behavior. The state of such systems changes continuously most of the time but sometimes undergoes abrupt changes or "jumps". The focus of the present survey is on the systems with this type of dynamics.

The combination of continuous and discrete behavior arises in a variety of control-related engineering problems. In the most eloquent way it is manifested for the so-called cyber-physical systems [1], where the real physical world meets digital control mechanisms. The interaction and interplay of continuous (physical) systems and computer-based controllers, which are operated in discrete time and/or space, leads to complex dynamics, whose analysis can be challenging [2, pp. v-vi].

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The behavior that combines continuous and discontinuous parts and the corresponding modeling approaches that are capable of capturing it are generally termed *hybrid behavior* and *hybrid systems*, respectively. Further examples of hybrid behavior stem from the networked control systems [3–5], in which the continuously operating plants and continuous controllers are communicating over digital networks. This type of communication typically requires the packet-based information processing and it is additionally constrained by certain communication protocols [6,7]. It leads to a situation where a part of signals in the system is continuous and the other one is discrete or piecewise-continuous. The same happens under the event-triggered control schemes [8,9], in which the control actuation does not act continuously but only in case of the fulfillment of a certain auxiliary condition often called event-triggering mechanism [10]. Typical examples of such an approach are the event-based fault-tolerant control of aircraft engine system [11], the event-based model predictive control of a renewable hydrogen-based microgrid [13], and the pH control in microalgae raceway reactors [14], to name a few.

Hybrid behavior also arises in the modeling of the processes in which the control action may change the state of the process instantaneously and, therefore, leads to solutions with piecewise continuous trajectories. This type of control is commonly called *impulsive control* [15] and it spans a wide set of application areas ranging from secure communications [16,17], which is based on the impulsive synchronization of chaotic systems [18–20], to optimal influenza H1N1 treatment [21] and spacecraft rendezvous [22]. Additionally, we would like to mention the application of impulsive control to the observer design and state reconstruction problem under sporadic measurements [23–25]. The discontinuities of the observers' state arise at the moments of the measurements injection, which are discrete points in time. This encompasses the case of process engineering applications, e.g., for the cell population balance models in which the off-line measurements from the bioreactor are typically obtained by the operation staff [26,27]. Finally, impulsive control is used for the safety verification of continuous-time continuous-state systems whose desired set-invariance properties can be achieved by impulsive perturbations of appropriate frequency [28]. All the mentioned control tasks require proper mathematical frameworks for the modeling and analysis of the considered hybrid processes. These are mainly deployed within the frameworks of *impulsive differential equations* [29] and *hybrid dynamical systems* [30], which will be considered in details in the following sections.

Besides engineering problems, hybrid dynamics emerge in many physical and biological systems where the interactions are mediated by short pulse-like signals. For example, in biological neural networks neurons transmit information by action potentials, which are short voltage pulses [31,32]. Other examples of pulse interactions include fireflies communicating by short light pulses [33], cardiac cells [34], impacting mechanical oscillators [35], electronic oscillators [36,37], and optical systems [38–40]. Provided that the pulse duration is small and its specific shape is negligible, it is convenient to use an approximation by infinitely short pulses. Under this approximation, the system evolves autonomously except for the moments when it receives pulses, and in these moments the incoming pulses cause instant changes of the system state.

One of the topics where the assumption of infinitely short pulses allowed to obtain significant progress is mechanical systems with impacts. Starting from the pioneering works on impact dampers [41,42], further research results on the impact dynamics [43–47] have comprised a well-established mathematical foundation for the modeling and analysis of mechanical systems with interaction discontinuities. More recent results include studies of vibro-impact dynamics [48–50], dynamical behavior of oblique impact systems [51,52], and impact chattering in gear transmission systems [53,54]. A special attention has been paid to the analytic study of periodic motions [55–57] and grazing bifurcations [58,59] in discontinuous dynamical systems.

Another direction where the concept of pulse interactions is especially effective is the study of networks dynamics. The assumption of infinitely short pulses provides a simple tool to capture unit-to-unit interactions and allows the researcher to concentrate on the collective behavior of the network. In particular, this framework allowed to shed light on many important processes in neural networks, such as synchronization [60,61], asynchronous behavior [62,63], emergence of collective oscillations and complex collective dynamics [64,65], and possible mechanisms underlying cognitive function such as object working memory [66], for instance.

The concept of pulse interactions is especially effective when combined with a simple model for the local dynamics, such as phase description. Pulse-coupled phase oscillators are a popular framework for modeling biological systems [67–71], especially neural networks [72–77], locomotion of human and animals [78,79], image processing [80], as well as addressing general problems of the network dynamics [81–84]. The action of a pulse on a phase oscillator is captured by the so-called phase response curve [85], which in earlier works was also called "sensitivity function" [86] or "phase transition curve" [87]. The phase response curve (PRC) tabulates the phase shift of the oscillator perturbed by a pulse depending on the phase of the pulse reception. The PRC can be obtained numerically or even measured experimentally for oscillators of arbitrary nature. It can be useful to predict their dynamics under the action of pulse trains provided that the pulses are not too strong or too frequent [88].

Although the behavior with instantaneous impulsive jumps is typical for diverse areas spanning from cyber-physical systems to neural networks, the researchers from various areas use very different modeling approaches and rely on different analysis techniques. Our paper attempts at providing a simple introduction to several modeling frameworks and bringing attention of researchers working in one field to the methods available in other research communities. By this, we aim at fostering the exchange of ideas and methods between physicists, theoretical biologists, applied mathematicians, and control engineers who are engaged in the study of hybrid behaviors.

The paper is organized as follows. In Section 2, we present a systematic exposition of the mathematical foundations of the most popular approaches for the modeling of hybrid behavior. In particular, the main concepts of impulsive differential equations and hybrid dynamical systems are presented in subsection 2.1 and 2.2, respectively. In subsection 2.3, we recall the definition and the basic properties of the Dirac delta function and the corresponding equations which involve this formalism. Section 3 is mainly devoted to the comparison of the mentioned modeling frameworks and to the identification of their benefits with respect to different types of hybrid behaviors. Thus, we address the correspondence of the equations with Dirac delta functions and impulsive differential equations in subsection 3.1. The peculiarities of the stability and attractivity analysis of hybrid behaviors and main differences compared to the conventional Lyapunov-based approaches for smooth dynamical systems are discussed in subsections 3.2 and 3.3, respectively. Subsection 3.4 addresses the issue of multiple impulsive jumps at the same time. Finally, conclusions and a short outlook in Section 4 complete the paper.

Notation. Let $\mathbb{N}, \mathbb{R}, \mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ stand for the sets of natural, real, real non-negative, and real positive numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space for $n \in \mathbb{N}$. For a given set $\mathcal{A}$, let $\bar{\mathcal{A}}$ and $\mathtt{int}\,\mathcal{A}$ denote the closure and the interior of the set $\mathcal{A}$. Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, the distance of x to $\mathcal{A}$ is denoted by $\|x\|_{\mathcal{A}}$ and is defined by $\|x\|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} \|x - y\|$, where $|\cdot|$ denotes the Euclidean norm.

2. mathematical modeling of discontinuous behavior

In this section, we present an overview of the most common mathematical frameworks for the modeling and analysis of processes that combine continuous and discontinuous behavior. This includes (a) impulsive differential equations [89,90], which are the main approach for mathematicians, (b) hybrid dynamical systems [30,91], which are popular for control engineers and are especially beneficial for networked control systems, and finally (c) differential equations involving Dirac δ-functions [92], which are an important tool in the physicists community. Also, we would like to point out that there exists a plethora of different frameworks stemming from the computer science community, like hybrid automata [93], hybrid Petri nets [94], etc. For the interested readers, some relations between equations with Dirac/impulsive differential equations and symbolic models can be found in [95] and [96], respectively. The latter ones, however, are not in the scope of the current paper.

2.1. impulsive differential equations

Origins of the theory of impulsive differential equations date back to the papers [97,98] and the monograph [99]. In these works, a concept of a solution to the impulsive differential equation and conditions for its existence and uniqueness have been introduced. Additionally, the first assertions have been made therein regarding the stability properties of solutions. Later in 1970-th, the rigorous mathematical theory of impulsive differential equations with fixed and non-fixed moments of impulsive jumps has been developed by Samoilenko, Perestyuk, and their students. Most of these results are summarized in the monographs [29,89,100,101] and provide a classification of impulsive differential equations depending on the character of impulsive jumps [29,89], stability characterizations of solutions [102–104], extensions of Lyapunov's second method [105] and averaging theory [106–108]. More recent results on impulsive differential equations concern the extensions of dissipativity [109] and contraction [110,111] theories, applications in control [15,112], study of periodic solutions [12,129] and impulsive differential inclusions [101,113], development of the theory of global attractors [114–117].

A system of *impulsive differential equations* (or, simply, an impulsive system) is defined by three essential ingredients: (1) a differential equation $\dot{x} = f(t,x)$ with $f : \mathbb{R} \times \mathcal{M} \to \mathbb{R}^n$, $n \in \mathbb{N}$, that governs the dynamics at points x of the state space $\mathcal{M} \subseteq \mathbb{R}^n$; (2) a set $\Gamma \subset \mathbb{R} \times \mathcal{M}$ in the extended state space that triggers a jump when at time t a trajectory approaches a point $x = x(t^-) := \lim_{\varepsilon\searrow 0} x(t-\varepsilon)$, such that $(t,x) \in \Gamma$; (3) a map $g : \Gamma \to \mathbb{R}^n$ that defines the instantaneous transition of point x to a new position $x + g(t,x) \in \mathcal{M}$ for $(t,x) \in \Gamma$. This allows for a compact representation

$$\dot{x}(t) = f(t, x(t)), \qquad (t, x(t^-)) \notin \Gamma, \tag{1a}$$

$$x(t) = x(t^-) + g(t, x(t^-)), \quad (t, x(t^-)) \in \Gamma, \tag{1b}$$

where $t \geq 0$, and $x(t) \in \mathcal{M}$. The functions f, $\mathtt{id} + g$, and the set Γ are called *flow map*, *jump map*, and *impulsive set*, respectively. Here, $\mathtt{id}$ stands for the identity map, i.e., $\mathtt{id}(x) = x$ for any $x \in \mathcal{M}$. For a given impulsive set Γ, a *solution* to the impulsive system (1) corresponding to initial time $t_0 \geq 0$, and initial state $x_0 \in \mathcal{M}$ is a right-continuous function $x : [t_0, T) \to \mathcal{M}$ that satisfies the differential equation (1a) when $(t,x) \notin \Gamma$, has discontinuities of the size $g(t, x(t^-))$, when $(t, x(t^-)) \in \Gamma$, and conforms to the initial condition $x(0) = x_0$. When $T = \infty$, the corresponding solution is called *forward-complete*. Sometimes, it is convenient to rewrite equation (1b) in the form $\Delta x|_{(t,x)\in\Gamma} = g(t,x)$, where Δx denotes the difference between the values of the state after and before the jump. It is worth noting that the proposed definition of the solution is meaningful for the cases when the jump map transfers point x at time t to a new position $x + \Delta x$ so that $(t, x + \Delta x) \notin \Gamma$, i.e., after every jump the motion evolves along the continuous flow. The case of several sequential jumps will be discussed later within the framework of hybrid dynamical systems in Secs. 2.2 and 3.4.

Depending on the properties of the impulsive set Γ and the jump map g, three qualitatively different classes of impulsive differential equations can be discriminated (see also Figure 1):

(a) *Impulsive differential equations with fixed moments of jumps.* This case corresponds to $\Gamma = \mathbb{T} \times \mathcal{M}$, where $\mathbb{T} = \{t_1, t_2, \ldots\} \subset \mathbb{R}$ is a given set of jump moments. It is a convention that after every jump at time t_i the motion evolves

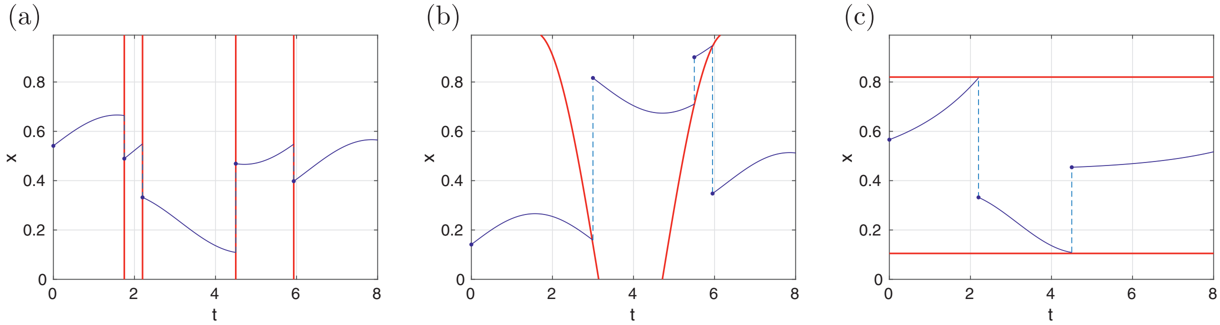

Fig. 1. Examples of typical impulsive sets $\Gamma \subset \mathbb{R} \times \mathcal{M}$ (in red) that correspond to three different classes of impulsive differential equations with one-dimensional state space $\mathcal{M} = \mathbb{R}$: (a) Impulsive differential equations with fixed moments of jumps – discontinuities occur at some predefined moments (vertical lines); (b) Impulsive differential equations with non-fixed moments of jumps – discontinuities occur when a certain relation between the time and space variable is achieved; (c) Discontinuous dynamical systems – discontinuities occur when the state reaches a certain subset of the state space (horizontal lines). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

along the continuous flow. Impulsive systems with fixed moments of jumps are widely used in many application areas, such as logistics [96,118], robotics [119], population dynamics [120–122], and medical therapeutics[123]. For example, this modeling approach was used for the analysis and design of the tumor chemotherapy strategy when the therapy is applied at some prescribed time-moments, e.g., periodically in [123]:
There, the continuous dynamics of cell populations is given as

$$\begin{aligned} \dot{C}(t) &= g_C(C(t)) - m_C(C(t), H(t)) - d_C(C(t))Z(t), \\ \dot{H}(t) &= a_H(H(t), R(t-\tau)) - m_H(C(t), H(t)) - d_H(H(t))Z(t), \\ \dot{R}(t) &= g_R(R(t)) - a_H(H(t), R(t-\tau)) - d_R(R(t))Z(t), \\ \dot{Z}(t) &= -d_Z(C(t), H(t), R(t))Z(t), \end{aligned}$$

where $C(t)$, $H(t)$, and $R(t)$ are the number of cancerous, hunting and resting cells at time t, respectively, and $Z(t)$ is the concentration of the chemotherapeutic agent at time t. Further, g_C and g_R are the reproduction rates of R and H cells, a_H is the activation of hunting cells, d_X are the losses of quantity X due to reaction of the chemotherapeutic agent Z, and m_C and m_H are the decay rates of hunting and cancerous cells independent of the presence of Z. The above continuous dynamics are in effect for times $t \neq nT$, $n \in \mathbb{N}$. At $t = nT$, an impulsive drug treatment is applied with amplitude Δ, that is,

$$C(t) = C(t^-),\; H(t) = H(t^-),\; R(t) = R(t^-),\; Z(t) = Z(t^-) + \Delta,\; \text{for } t = nT, n \in \mathbb{N}.$$

Taking the time-delay $\tau = 0$ the system's state space becomes finite-dimensional and we arrive to the impulsive differential equation model of the type (1).

(b) *Impulsive differential equations with non-fixed moments of jumps*. For this case, $\Gamma = \{(t,x) \in \mathbb{R} \times \mathcal{M} : \Phi(t,x) = 0\}$ for some function $\Phi : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$. That is, the occurrence of impulses depends on both time t and state x. An example of such process is, for instance, the Lotka-Volterra model of interaction of two biological species with external impulsive regulation that depends on species' population and accounts for seasonal factors [124], or Hopfield neural networks with state-dependent impulses [125]:

$$\begin{aligned} \dot{x}(t) &= -Cx(t) + Af(x(t)), t \in (\theta_i, \theta_{i+1}] \quad \text{and} \quad t \neq \theta_i + \tau_i(x(t^-)), \\ x(t) &= x(t^-) + J_i(x(t^-)), t = \theta_i + \tau_i(x(t^-)), \end{aligned}$$

where the state variable $x(t) \in \mathcal{M} \subset \mathbb{R}^n$, A, C, and J_i are given matrices of appropriate dimension and $f : \mathcal{M} \to \mathbb{R}^n$ is a nonlinear activation function. The strictly increasing sequence of jump moments $\{\theta_i\}_{i=0}^{\infty}$ has no finite accumulation points. The impulsive set Γ is a union of hypersurfaces $\Gamma_i = \{(t,x) \in \mathbb{R}_+ \times \mathcal{M} : t = \theta_i + \tau_i(x)\}$ defined by continuous nonlinear functions $\tau_i : \mathcal{M} \to \mathbb{R}_+$. Impulsive jumps occur when the integral curve intersects any hypersurface Γ_i, $i = 1, \ldots, \infty$.

(c) *Discontinuous dynamical systems*. Here, the impulsive set and the jump map do not depend on time t, i.e., $\Gamma = \mathbb{R} \times \hat{\Gamma}$, with $\hat{\Gamma} \subset \mathcal{M}$, and the jump map has the form $g(t,x) = \hat{g}(x)$ with $\hat{g} : \hat{\Gamma} \to \mathcal{M}$. Typical examples of processes that can be modeled by discontinuous dynamical systems are pulse-coupled oscillators in which impulses occur when the state reaches a certain threshold. For example, a network of N pulse-coupled phase oscillators can be modeled as [70]

$$\begin{aligned} \dot{\bar{\theta}}(t) &= (\omega, \omega, \ldots, \omega), I(\bar{\theta}(t^-)) = \emptyset, \\ \bar{\theta}(t) &= \left(\Psi^{k_1}(\theta_1(t^-)), \Psi^{k_2}(\theta_2(t^-)), \ldots, \Psi^{k_N}(\theta_N(t^-))\right) \mod 2\pi, I(\bar{\theta}(t^-)) \neq \emptyset. \end{aligned}$$

Here, $\bar{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)$ are the phases of the oscillators, $\theta_j \in \mathbb{R}$ (mod2π), and $I(\bar{\theta}(t^-)) = \{j | \theta_j(t^-) = 2\pi\}$ is the set of oscillators whose phases reach the threshold 2π at time t. Such oscillators are said to emit pulses, and these pulses are received by the other oscillators at the same time. Each oscillator j receives pulses from a certain set N_j of other oscillators. When all the oscillators are below the threshold their phases grow uniformly with the speed ω. When some oscillators reach the threshold, they emit pulses and send them to their peers. When an oscillator receives a pulse, its phase changes according to the map $\Psi(\theta) = \theta + c\Phi(\theta)$, where $\Phi(\theta)$ is the phase response curve and c is the scalar coupling gain. Note that the phase change is given by the k_j-fold superposition of the map $\Psi(\theta)$, where k_j is the number of pulses received by the j-th oscillator simultaneously: $k_j = |I(\bar{\theta}) \cap N_j|$, where $|\cdot|$ denotes the cardinality of a set.

Discontinuous dynamical systems may also arise in control regulation problems, e.g, in the "predator-prey" model subjected the external influences in form of the removal or addition of certain percentage Δ of biomass both for the prey and predator [126]:

$$\begin{aligned}
\dot{m}(t) &= m(t)(r_1 - q_1 M(t)),\ (m, M) \notin \Gamma,\\
\dot{M}(t) &= -M(t)(r_2 - q_2 m(t)),\ (m, M) \notin \Gamma,\\
m(t) &= (1 - \Delta)m(t^-),\ (m, M) \in \Gamma,\\
M(t) &= (1 - \Delta)M(t^-),\ (m, M) \in \Gamma,
\end{aligned}$$

where $m(t) > 0$ and $M(t) > 0$ denote the amount of biomass constituting the prey and predator populations at time $t \geq 0$; constant coefficients $r_1, r_2, q_1, q_2 > 0$ characterize the relative growth, decay, and predatory interaction between the populations. For a given parameter $k > 0$, the impulsive set $\hat{\Gamma} = \{(m, M) \in \mathbb{R}_+ \times \mathbb{R}_+ : M = km\}$ corresponds to the ray with slope k. The impulsive moments correspond to the instances when the ratio of the quantities of biomasses of the predator and prey reaches the value k.

The essential property of both impulsive differential equations with non-fixed moments of jumps (b) and discontinuous dynamical systems (c) is that each solution has its distinct moments of discontinuities, whereas all solutions share the same moments of jumps in the case (a). Also, in the cases (b) and (c), a solution may exhibit the so-called "beating" phenomenon, which is characterized by an infinitely many impulsive jumps within a finite interval of time. This situation occurs when the solution meets the set Γ more often and often, so that the time between two consecutive intersections of the solution and the impulsive set converges to zero. This dynamical phenomenon is also termed *Zeno behavior* and it will be discussed in more details in the following Sec. 2.2.

As highlighted in [89], the representation (1) allows for a variety of extensions, which can be used for the modeling of complex processes and phenomena. For example, if the jump map is allowed to be multi-valued then the corresponding solution undergoes instantaneous splitting into several solutions when it meets the jump set. That is, $g(t, x) \in \mathcal{P}(\mathcal{M})$, where $\mathcal{P}(\mathcal{M})$ is the powerset over the state space $\mathcal{M}$. A particularly peculiar situation emerges if we assume that $g(t, x) = \emptyset$ is the empty set for all $x \in \mathcal{R}$ with some $\mathcal{R} \subset \mathcal{M}$. Such systems are called "mortal" [127]. Their solution "dies" when the trajectory meets the "death" set $\mathcal{R}$. A typical problem studied in the context of these systems, is finding the average trajectory lifetime, or the probability of its death within a given time. Finally, a number of extensions of the framework of impulsive differential equations have been developed that combine instantaneous impulsive jumps with other dynamical effects like, time delays [128,130–135], switching of the flow dynamics [136–141], stochastic perturbations [142–144], and extend the framework to handle infinite-dimensional state spaces [145–149], non-instantaneous jumps [150–152], and fractional order derivatives [153–156].

2.2. hybrid dynamical systems

Hybrid dynamical systems, which were proposed in [30,91], constitute a convenient mathematical framework for the modeling and analysis of processes with state-dependent moments of jumps. Paradigmatically, they are capable of capturing a similar dynamic behavior as discontinuous dynamical systems, but the mathematical treatment is somewhat different. In particular, hybrid dynamical systems use a specific compartmentalized concept of time, which is described by a discrete and a continuous component in this context. Here, the continuous component describes the continuous time, which parametrizes the episodes of continuous evolution of the system, during which it is governed by a differential equation. The discrete component corresponds to the number of encountered jumps and separates the individual continuous episodes. This implies a distinct notion of a solution as a map from the two-dimensional time-space to the state space, that differs from the one used in impulsive differential equations. This approach allowed to develop a wide range of analysis methods, which nowadays constitute a well-established branch of hybrid systems and which are actively used for control design purposes in many engineering applications, e.g., in networked control systems [157] and for the event-triggered control of nonlinear systems [158].

In the next paragraphs, we introduce the notion of a hybrid dynamical system following [30].

Definition 1 (Hybrid time domain). A subset $E = \bigcup_j ([t_j, t_{j+1}], j) \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_0$ is called a *hybrid time domain* if it is a union of a finite or infinite sequence of indexed intervals $[t_j, t_{j+1}] \times \{j\}$, $j = 0, 1, 2, \ldots$, for some ordered sequence of time points $0 \leq t_0 \leq t_1 \leq t_2 \leq \ldots$ in $\mathbb{R}$. In case of a finite number m of intervals, the last one is allowed to be half-open of the form $[t_{m-1}, T)$ with T finite or $T = \infty$.

Definition 2 (Hybrid arc). A function $x : E \to \mathbb{R}^n$ is called a *hybrid arc* if E is a hybrid time domain and if for each $j = 0, 1, 2, \ldots$, the function $t \to x(t, j)$ is locally absolutely continuous on the interval $I^j = \{t : (t, j) \in E\}$.

Equipped with these definitions, we are set to define the notion of a hybrid dynamical system. Given a hybrid arc x, the notation $\mathtt{dom}\,x$ represents its domain, which is a hybrid time domain.

Definition 3 (Solution to a hybrid system)**.** Let C, D be subsets of $\mathbb{R}^n$. A hybrid arc x is a *solution to the hybrid dynamical system*

$$\dot{x} = f(x), \quad x \in C, \tag{2a}$$

$$x^+ = g(x), \quad x \in D. \tag{2b}$$

if $x(0,0) \in \bar{C} \cup D$ and

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t : (t,j) \in \mathtt{dom}\,x\}$ has nonempty interior

$$\begin{array}{lll} x(t,j) \in C & \text{for all} & t \in \mathtt{int} I^j, \\ \dot{x}(t,j) = f(x(t,j)) & \text{for almost all} & t \in I^j; \end{array}$$

(S2) for all $(t,j) \in \mathtt{dom}\,x$ such that $(t, j+1) \in \mathtt{dom}\,x$,

$$x(t,j) \in D, \quad x(t,j+1) = g(x(t,j)).$$

Thus, the state of the hybrid system, represented by x, can change according to the differential equation (2a) while $x(t,j) \in C$, and it can change according to the difference equation (2b) while $x(t,j) \in D$. The sets C and D are called the *flow set* and the *jump set*, respectively. The functions $f : C \to \mathbb{R}^n$ and $g : D \to C \cup D$ are called the *flow map* and *jump map*. It is worth noting that the flow and jump maps are allowed to be empty or to coincide with $\mathbb{R}^n$. Moreover, these sets may have intersection points: if x is in both sets simultaneously, it may either continue its evolution along the continuous trajectory of (2a), or it may be instantaneously transferred to a new position according to (2b), i.e., this situation gives birth to two different solutions. This peculiarity of hybrid dynamical systems enables them to model even more complex dynamic behaviors than discontinuous dynamical systems.

To illustrate the above definitions, we consider a mathematical model of a bouncing ball given by hybrid dynamical system of the form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g \end{cases}, \qquad (x_1, x_2)^\top \in C, \tag{3a}$$

$$\begin{cases} x_1^+ = x_1 \\ x_2^+ = -\lambda x_2 \end{cases}, \quad (x_1, x_2)^\top \in D, \tag{3b}$$

where $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}$ denote the position and the velocity of the ball at time $t \in [0, \infty)$, respectively. The gravity constant g and restitution coefficient λ are positive scalar parameters. The flow set C and jump set D are defined as follows:

$$C = \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 \geq 0\},$$

$$D = \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 \leq 0 \text{ and } x_2 \leq 0\}.$$

Fig. 2 shows the graph of the positional component x_1 of a solution over the hybrid time domain for the initial conditions $x_1(0) = 10$, $x_2(0) = 0$. The ball will make an infinite number of jumps before comes to rest after a finite time due to frictional losses ($\lambda < 1$) each time it touches the ground at $x_1 = 0$. Therefore, the number of jumps j goes to infinity within finite time t.

This phenomenon is called *Zeno behavior* (corresponds to the "beating" phenomenon in impulsive differential equations) and it can be formalized as a property of hybrid arcs (or, the respective hybrid time domains): A hybrid arc $x : E \to \mathbb{R}^n$ is called *complete* if $\mathtt{dom}\,x$ is unbounded, i.e., if

$$\sup_{(t,j) \in \mathtt{dom}\,x} (t + j) = \infty. \tag{4}$$

A hybrid arc $x : E \to \mathbb{R}^n$ is called *Zeno arc* if it is complete and $\sup_{\mathtt{dom}\,x} t < \infty$. This implies that, for a Zeno arc, an infinite number of jumps occurs during a finite time. In this case, the time $T_{\text{Zeno}} = \sup_{\mathtt{dom}\,x} t$ is called a Zeno time. Solutions for $t \geq T_{\text{Zeno}}$ are not defined, which may cause considerable problems for appropriate modeling and analysis of real systems with hybrid behavior and their interconnections [160].

More complex rules for the construction of hybrid time domains were proposed in [161–163]. In [161], the concept of generalized hybrid time domain has been introduced where a discrete-time axis was generalized to a countable ordinal that can have infinitely many accumulation points, which correspond to Zeno occurrences. In [162,163], a notion of the three-dimensional extended hybrid time domain has been proposed, in which the third component tracks the number of encountered Zeno behaviors. Both approaches allow to prolong solutions beyond the Zeno time T_{Zeno}.

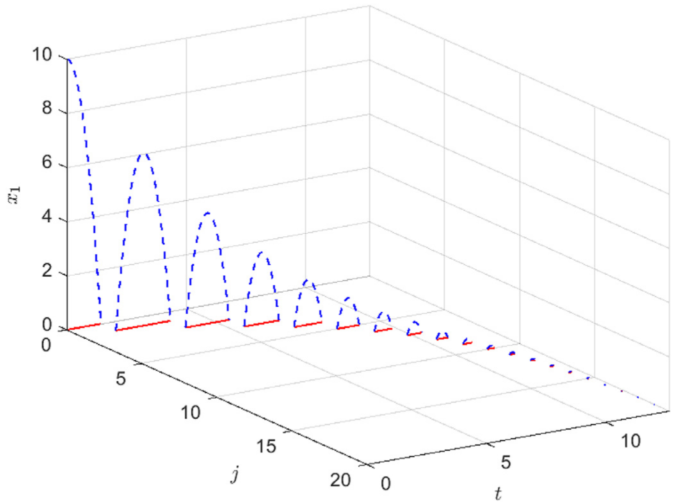

Fig. 2. A plot of a hybrid time domain (solid red line) and the corresponding hybrid arc (dashed blue line) representing the evolution of the ball's height starting from $x_1(0) = 10$ with zero velocity. The parameter values are $g = 9.81$ and $\lambda = 0.8$. The simulation was performed using Hybrid Equations (HyEQ) Toolbox [159] for MATLAB/SIMULINK. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

2.3. equations with dirac delta function

The Dirac delta function was introduced by Paul Dirac [92] as a function modeling the density distribution of an idealized point mass. Mathematically, the Dirac delta function is defined by the action of an associated distribution [164]. This distribution, D_δ acts as a linear functional on a set of appropriate test functions $f : \mathbb{R} \to \mathbb{R}$ (e.g., smooth functions with compact support) and maps every function to its value at zero:

$$D_\delta[f] = f(0). \tag{5}$$

A heuristic characterization of the delta function could be given as a function of a real variable which equals zero everywhere except the origin where it equals infinity:

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \tag{6}$$

with the additional restriction

$$\int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{7}$$

From a physical perspective, one can also think of a delta function as a limit of a sequence of pulse-like functions δ_n, whose support tends to zero while the integral remains equal to unity. Such an understanding makes delta functions ideal for modeling signals in the form of short pulses. For example, consider an integrate-and-fire neuron governed by the equation [63]

$$\tau \dot{V}(t) = -V(t) + RI(t), \tag{8}$$

where τ is the membrane time constant, R is the membrane resistivity, and $I(t)$ is the synaptic current arriving at the soma. This current is the sum of the contributions of signals arriving at different synapses. As those signals take the form of short action potentials, i.e., brief spikes, they can be modeled as delta functions. Thus, the total current is given by

$$RI(t) = \tau \sum_j J_j \sum_k \delta(t - t_j^k), \tag{9}$$

where J_j is the amplitudes of the postsynaptic potential of the j-th synapse, and t_j^k are the spike arrival times at that synapse.

The presence of a delta function on the right-hand side of an ODE implies the discontinuity of its solutions. To show that consider a system

$$\frac{dx}{dt} = f(x) + \delta(t), \tag{10}$$

where $x \in \mathbb{R}$. The impact of the delta function is negligible at all times except $t = 0$. At the latter point it is possible to integrate the equation as follows:

$$x(0^+) = x(0^-) + \int_{-0}^{+0} (x(t) + \delta(t))dt = x(0^-) + 1. \tag{11}$$

Thus, Eq. (10) is equivalent to an impulsive differential equation

$$\dot{x}(t) = f(x), \qquad t \neq 0, \tag{12a}$$

$$x(t) = x(t^-) + 1, \qquad t = 0. \tag{12b}$$

In general, every ODE with a delta function on the right-hand side can be reformulated as an impulsive differential equation. However, the calculation of the jumps in ODEs with delta- functions in the right-hand side requires special care in many cases as we discuss in Section 3.1.

3. comparison between the frameworks

3.1. state-dependent jumps in equations with dirac delta functions

In the previous section we have shown that the dynamics of equation (10) is equivalent to that of the impulsive differential equation (12). Similarly, an ODE with a Dirac delta function on the right-hand side can be reformulated as an IDE. Both modeling approaches are often used interchangeably to describe dynamical systems with jumps [61,73,82], and equations with Dirac delta functions are frequently considered to be a more compact way of writing the corresponding impulsive differential equations. However, special care should be taken when the magnitude of the jump depends on the system state. This dependence can be expressed as a multiplication of the delta function by a function $g(x)$ of the state variable, such that the equation takes form

$$\frac{dx}{dt} = f(x) + g(x)\delta(t). \tag{13}$$

Frequently, authors assume that the magnitude of the discontinuity in the solution of (13) is given by the value $g(x(0^-))$ evaluated for the system's state *just before* the jump [73,76,77,81,83,84,165,166]. This means, that (13) is interpreted to be equivalent to the impulsive differential equation

$$\dot{x}(t) = f(x(t)), \qquad t \neq 0, \tag{14a}$$

$$x(t^+) = x(t^-) + g(x(t^-)), \quad t = 0. \tag{14b}$$

For example, Izhikevich [167] formulates a general model of a network with pulse coupling in the form

$$\frac{dx_i}{dt} = f_i(x_i) + \varepsilon \sum_{j=1}^{n} g_{ij}(x_i)\delta(t - t_j^* - \eta_{ij}), \tag{15}$$

where f_i describes the local dynamics of the i-th node (neuron) of the network, while g_{ij} describes the effect of a pulse generated by the j-th neuron on the i-th. The moment when the j-th neuron emits a pulse is denoted by t_j^*, and η_{ij} is the coupling delay between the j-th and the i-th neurons. When the i-th neuron receives a pulse, its state changes immediately. The magnitude of this change is assumed to be the following: $x_i(t+0) = x_i(-0) + \varepsilon g_{ij}(x_i(t-0))$.

The following simple example illustrates why this approach may lead to inconsistent results:

Let $\mathbb{T} = \{t_1, t_2, t_3, \ldots\}$ be a strictly increasing sequence of impulse times in $(0, \infty)$ with no finite accumulation point, i.e., $\lim_{i\to\infty} t_i = \infty$. Consider a linear impulsive differential equation

$$\dot{x} = x, \qquad t \notin \mathbb{T}, \tag{16a}$$

$$\Delta x = -\frac{x}{2}, \quad t \in \mathbb{T}, \tag{16b}$$

where $x(t) \in \mathbb{R}_{>0}$. Let us evaluate the hypothesis that the behavior defined by the impulsive differential equation (16) is equivalent to the following equation with a Dirac delta function:

$$\dot{x} = x + \sum_{t_i \in \mathbb{T}} \left(-\frac{x}{2}\right)\delta(t - t_i). \tag{17}$$

For this sake, we introduce a nonlinear strictly monotonic transformation of variable $z = \ln x$, $x > 0$. The application of this transformation to equations (16) and (17) leads to two *different* systems with different solutions. Indeed,

$$z = \ln x \quad \Rightarrow \quad \dot{z} = \frac{1}{x}\dot{x} \quad \Rightarrow \quad \dot{x} = x\dot{z} = e^z\dot{z}. \tag{18}$$

That is, equation (17) can be represented as

$$\dot{z} = 1 + \sum_{t_i \in \mathbb{T}} \left(-\frac{1}{2}\right)\delta(t - t_i). \tag{19}$$

The representation (19) corresponds to the impulsive differential equation (cf. (11))

$$\dot{z} = 1, \qquad t \notin \mathbb{T}, \tag{20a}$$

$$\Delta z = -\frac{1}{2}, \qquad t \in \mathbb{T}. \tag{20b}$$

Let us now perform the change of variable in (16). Following (18), the differential equation (16a) reads as $\dot{z} = 1$. Evaluating the value of the state variable x after a jump at $t \in \mathbb{T}$, we obtain:

$$x + \Delta x = x - \frac{x}{2} = e^{z+\Delta z}.$$

Thus, $e^z = 2e^{z+\Delta z}$, which yields $\Delta z = -\ln 2$. Finally, the impulsive differential equation in z-coordinates derived from (16) has the following form

$$\dot{z} = 1, \qquad t \notin \mathbb{T}, \tag{21a}$$

$$\Delta z = -\ln 2, \quad t \in \mathbb{T}. \tag{21b}$$

Clearly, if (16) and (17) were equivalent, the impulsive equations (20) and (21) for z would coincide. Thus, (16) and (17) are *not* equivalent for the assumed interpretation of the delta function. While the continuous dynamics are the same, the magnitudes of the jumps are different. For a more detailed discussion of this issue, see [168].

The described difference in the magnitude of the jumps becomes even more dramatic for nonlinear systems, e.g, for oscillation systems with impulses [29, Section 7.1]. Summarizing, one should take special care about the calculation of the jump map when using an ODE with delta functions, for example by employing canonical graph completion techniques [169] or by approximating the delta-function by a sequence of continuous functions [170]. The explicit notation (1) for impulsive differential equations avoids this issue. For a more detailed exposition, see [168].

The nonlinear transformation of variables is a key element in stability analysis via the Lyapunov function method. As, for scalar equations, the Lyapunov function actually defines some (usually nonlinear) transformation of the system's state. Following the arguments used in the above calculations, the application of the Lyapunov method to impulsive differential equations and to their counterparts with Dirac delta functions may lead to qualitatively different outcomes in the case that the jump map is not carefully calculated. Further peculiarities of the stability analysis of hybrid processes are discussed in subsections 3.2 and 3.3.

3.2. stability analysis for systems with jumps

One of the peculiarities of dynamical systems with jumps emerges when their stability is addressed. For continuous dynamical systems, the Lyapunov stability of a solution $x(t)$ is characterized by the property that every perturbed solution $y(t)$ starting at $y_0 = y(t_0)$ close enough to $x_0 = x(t_0)$ remains close to $x(t)$ as time progresses. Formally,

$$\forall \varepsilon > 0: \ \exists \delta = \delta(\varepsilon) > 0: \ \forall y_0 \in O_\delta(x_0): \ \forall t > t_0: \ \|x(t) - y(t)\| < \varepsilon, \tag{22}$$

where $O_\delta(x_0)$ denotes the open δ-neighborhood of x_0.

Directly applied, this definition turns out to be too restrictive for many systems with jumps. For example, consider a population of oscillators described by their phases ϕ_j which grow uniformly with $d\phi_j/dt = \omega_j$. Further, assume that the phase passing through zero corresponds to an observable event, for example, an action potential. When one oscillator "fires" (i.e., reaches $\phi_j = 0$), the others receive an impulse, so that their phases jump according to the map

$$\phi_k \mapsto \phi_k + Z(\phi_k), \ \ k \neq j, \tag{23}$$

where $Z(\cdot)$ is the phase resetting curve. It was shown [171,172] that in the case of two oscillators such a system may display mutual synchronization, where both oscillators fire periodically with the same period. If the oscillators are not identical ($\omega_1 \neq \omega_2$), the firing occurs with a time lag. For $Z(\phi) = -\kappa \cdot \sin(2\pi\phi)$, this regime is stable for a range of values for κ in the sense that the system returns back to firing with the same period and the same time lag after a small perturbation. However, even for an arbitrarily small perturbation, the firing events of the periodic and the perturbed solution occur at slightly

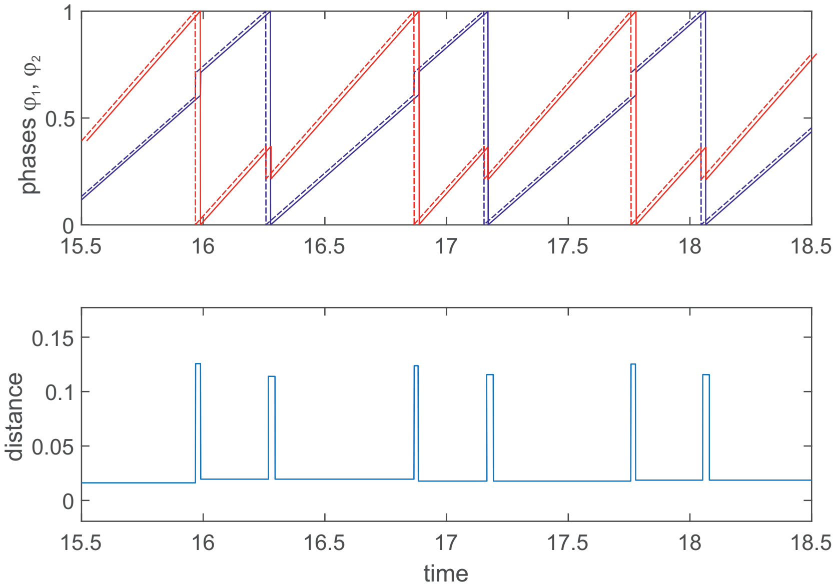

Fig. 3. Top panel: the periodic (solid lines) and the perturbed (dashed lines) solutions of the two oscillators with pulse coupling. Blue (red) lines correspond to the phase of the first (second) oscillator. Bottom panel: the distance between the two solutions. The frequencies $\omega_1 = 1$,m $\omega_2 = 1.4$, the phase resetting curve $Z(\phi) = -0.15 \sin 2\pi\phi$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

different time moments, see Fig. 3. Between these moments, the difference between the two solutions is of constant order corresponding to the jump magnitude. Applying the definition (22) of stability to this situation, would therefore characterize the solution as unstable.

This example motivates the introduction of a modified definition of stability, more adequate for systems with jumps. One suitable notion is given by Samoilenko and Perestyuk [89], which introduces intervals of temporal forbearance around the moments of discontinuity. During those intervals a significant difference between the original and perturbed solution is allowed. This means that one solution is considered to be close to another if the difference between both is small outside of small neighborhoods around its discontinuities. In particular, the time lag between two corresponding jumps of the perturbed and unperturbed system has to be small, as well. More formally, we have the following definition for the stability of solutions to impulsive differential equations:

Definition 4. Let $x(t)$ denote a solution to the impulsive system (1) defined for all $t \geq t_0$ for some initial time $t_0 \in \mathbb{R}$. The solution $x(t)$ is called stable if for any $\varepsilon > 0$ and $\eta > 0$ there exists $\delta = \delta(\varepsilon, \eta) > 0$ such that for any other solution $y(t)$ to (1) with $\|x(t_0) - y(t_0)\| < \delta$ it holds that $\|x(t) - y(t)\| < \varepsilon$ for all $t \geq t_0$ such that $|t - t_i| > \eta$, where t_i, $i \in \mathbb{N}$ are the moments of impulsive jumps of $x(t)$, which can be also state-dependent.

The introduction of η in Def. 4 in comparison to (22) resolves the issue of momentary, large state deviations due to small jump time perturbations by allowing arbitrary deviations at times close to the solution's discontinuities. Consequently, the neighborhood $O_\delta(\varepsilon, \eta)$ of the admissible initial values for y depends also on the value of η, which defines the size of the time-interval around jumps that is disregarded during the evaluation of ε-closeness. Hence, a smaller η requires a smaller δ, if the latter is assumed to be chosen as a maximal size of the neighborhood.

The asymptotic stability property of solutions to impulsive differential equations can be defined as follows [89]:

Definition 5. A solution $x(t)$ is called asymptotically stable if it is stable in the sense of Def. 4 and there exists a number $\delta_0 > 0$ such that for any other solution $y(t)$ with $\|x(t_0) - y(t_0)\| < \delta_0$, the following holds:

$$\lim_{t\to\infty} \|x(t) - y(t)\| = 0.$$

Note that the solutions depicted in Fig. 3 are not asymptotically stable in the sense of Def. 5, since the corresponding distances between the original and the perturbed solutions do not converge to zero. This is a consequence of the persistent phase shift induced by the perturbation. However, the trajectory of the perturbed solution converges towards the periodic orbit described by the original solution, when it is considered as a set. To capture this behavior, the notions of asymptotically attractive sets is important for the long-term characterization of hybrid behaviors. This will be discussed in Section 3.3.

3.3. stability with respect to a set and attractivity concepts

In many cases, stability analysis of a given solution to the system with jumps can be reduced to the stability analysis of the trivial solution (zero equilibrium point) of the auxiliary system, which can be derived from the original one by the appropriate change of variables[1] However, the asymptotic stability of a closed set, rather than of an equilibrium point, is of interest for many application areas. We illustrate this by the example of a sample-and-hold control system from [30]:

Consider a continuous-time control system $\dot{z} = \tilde{f}(z, u)$ with the state $z(t) \in \mathbb{R}^n$, $n \in \mathbb{N}$, control input $u(t) \in \mathbb{R}^m$, $m \in \mathbb{N}$, and $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and a state-feedback controller $u = \kappa(z)$. A *sample-and-hold* implementation of the feedback comprises in iterative repetition of two steps:

- *sample*: measure the state of the system, and use the feedback controller to obtain the control value based on the measurements;
- *hold*: apply the computed constant control value for certain amount of time T.

Such the implementation can be modeled using an additional timer variable τ that tracks the elapsed time since the last sampling. Jumps occur when the timer variable reaches T, i.e.,

$$\begin{aligned} \dot{z} &= \tilde{f}(z, u) \\ \dot{u} &= 0 \qquad , \quad \text{when} \quad x := (z, u, \tau)^\top \in \mathbb{R}^n \times \mathbb{R}^m \times [0, T) \\ \dot{\tau} &= 1 \end{aligned}$$

and

$$\begin{aligned} z^+ &= z \\ u^+ &= \kappa(z), \quad \text{when} \quad x := (z, u, \tau)^\top \in \mathbb{R}^n \times \mathbb{R}^m \times \{T\}. \\ \tau^+ &= 0 \end{aligned}$$

The goal of the asymptotically stabilizing feedback control is then the uniform global asymptotic stability of the set $\mathcal{A} = \{0\} \times \mathbb{R}^m \times [0, T]$. This property does not impose any restrictions of the components u and τ, but ensures the convergence of component z to zero. As highlighted in [30, Example 3.1], if the control design enforces boundedness of u, which is a property guaranteeing that the implementation of the controller is feasible, then the set $\mathcal{A}$ can be chosen to be bounded. For example, if u is picked from a compact set of controls $\mathcal{U}$, then it is possible to consider $\mathcal{A} = \{0\} \times \mathcal{U} \times [0, T]$. Asymptotic stability of an equilibrium point is a special case of asymptotic stability of a closed set, since an equilibrium point is a closed set containing a single point.

In sequel, we recall the existing definitions for the uniform global asymptotic stability of closed sets for impulsive differential equations and hybrid dynamical systems. For this purpose, we employ the so-called comparison functions, which are widely in use for the global analysis of solutions. A special attention will be paid to the peculiarities of the attractivity concepts in different modeling frameworks.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}_\infty$ function, also written $\alpha \in \mathcal{K}_\infty$, if α is zero at zero, continuous, strictly increasing, and unbounded. Function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ function ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for all $t \geq 0$, and $\beta(r, \cdot)$ is strictly decreasing with $\lim_{t\to\infty} \beta(r, t) = 0$ for all $r \geq 0$.

Definition 6. Let $x(t)$ denote a solution to the impulsive system (1) defined for all $t \geq t_0$ for some initial time $t_0 \in \mathbb{R}$, and $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be *globally asymptotically stable* (GAS) if there exists $\beta \in \mathcal{KL}$ such that for all $x_0 \in \mathbb{R}^n$, it holds that

$$\|x(t)\|_{\mathcal{A}} \leq \beta(\|x_0\|_{\mathcal{A}}, t - t_0) \quad \text{for all} \quad t \geq t_0. \tag{24}$$

Properties of $\mathcal{KL}$-functions immediately suggest that

$$\lim_{t\to\infty} \|x(t)\|_{\mathcal{A}} = 0 \quad \text{for any} \quad x_0 \in \mathbb{R}^n, \tag{25}$$

i.e., every solution eventually converge towards the set $\mathcal{A}$ as time t goes to infinity.

One of the most powerful tools to verify stability in sense of Def. 6 is based on the extension of Lyapunov's second method firstly proposed in [105] and summarized in [29]. These results provide sufficient conditions for the asymptotic stability of the equilibrium in terms of properties of the auxiliary Lyapunov-like scalar-valued function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. The derivative of V along the trajectories of impulsive system between jump moments characterizes the flow dynamics, the size of jumps of V at trajectories' discontinuities characterizes the jump dynamics of the system.

Based on this characterization, the corresponding flow dynamics and jumps dynamics can contribute or play against the desired stability property, e.g., if the derivative of V is negative-definite, then the flows are stabilizing; if the jump dynamics are expanding, i.e., the value of V evaluated after the jump is larger than before the jump, then the discrete dynamics is destabilizing. Additional constraints (called *dwell-time conditions*), which restrict the number/frequency of impulsive jumps and balance continuous dynamics and discontinuous dynamics of the system, are required to guarantee the desired stability

[1] The derivation of this transformation can be sometimes more complicated compared to a similar procedure for ODEs, e.g., in the case of state-dependent moments of jumps. We refer the interested reader to [89, §17] for a detailed exposition of this issue.

property. The described approach is applicable to impulsive systems with fixed moments of jumps (see, e.g., [173]) or to systems with variable jump moments if apriori estimates of the inter-impulse intervals can be made (see e.g., [89,174,175]). Clearly, if both, continuous flows and discrete jumps contribute towards stability, no such restrictions are needed (see, e.g., [176,177]).

Different types of dwell-time conditions have been employed for the stability analysis of equilibria and closed sets of impulsive systems, e.g., fixed dwell-time [89], (reverse) average dwell-time [177,178], non-fixed impulse-time moments within predefined time-windows [179–181], eventually uniformly convergent impulse frequency [182–184], and eventually uniformly bounded impulse frequency [185]. The mentioned dwell-times provide different levels of robustness of the GAS with respect to the perturbations of the moments of jumps and different conservatism of the resulting sufficient stability conditions. We refer the reader to [185] for a detailed comparison of the corresponding dwell-time conditions.

In general, the attractivity concept in the sense of (25) cannot be applied in case of solutions that have a bounded, i.e., not complete, domain of existence since (25) requires the existence of a limit at infinity. Typical examples of such systems are discontinuous dynamical systems or impulsive differential equation with non-fixed moments of jumps involving beating phenomenon, e.g., [101, Example 1],

$$\dot{x}(t) = 0, \qquad (t, x(t^-)) \notin \Gamma,$$
$$x(t) = x^2(t^-)\mathtt{sign}(x(t^-)), \quad (t, x(t^-)) \in \Gamma$$

with $\Gamma = \{(t,x) \in \mathbb{R}^2 : x = \arctan(\tan(t))\}$. The motions with initial condition $x(0) \in (-1, 0)$ are subjected to countably many impulses on the time-interval $\left(\frac{3\pi}{4}, \pi\right)$. The sequence of times at which the motion is subjected to impulsive jumps has the limit point $t = \pi$. Hence, the solution that corresponds to this motion cannot be extended to the interval $t \geq \pi$.

On the contrary, for hybrid dynamical systems, a solution with Zeno behavior has a complete domain of definition in the sense of (4). In the theory of hybrid dynamical systems, the corresponding concepts of *pre-asymptotic stability* and *pre-attractivity* is used instead of asymptotic stability and attractivity according to (24), (25) for the long-term characterization of solutions:

Definition 7. Consider a hybrid system (2) on $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be

(a) *uniformly globally stable* if there exists $\alpha \in \mathcal{K}_\infty$ such that any solution x to (2) satisfies $\|x(t,j)\|_{\mathcal{A}} \leq \alpha(\|x(0,0)\|_{\mathcal{A}})$ for all $(t,j) \in \mathtt{dom}\, x$;
(b) *uniformly globally pre-attractive* if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution x to (2) with $\|x(0,0)\|_{\mathcal{A}} \leq r$, $(t,j) \in \mathtt{dom}\, x$ and $t + j \geq T$ imply $\|x(0,0)\|_{\mathcal{A}} \leq \varepsilon$;
(c) *uniformly globally pre-asymptotically stable* (UGpAS) if it is both uniformly globally stable and uniformly globally pre-attractive.

In the framework of hybrid dynamical systems, a time point (t,j) is 'large' if the sum of the continuous time t and the number j of jumps occurred is 'large'. Such an approach together with the introduction of the hybrid time domain (Definition 1) and double parametrization of solutions (Definition 3) allowed to develop a variety of novel Lyapunov-based stability analysis results, including the converse Lyapunov theorem, which are summarized in [30]. In particular, the pre-attractiveness of the origin for the bouncing ball model (3) can be concluded [91], whilst the origin is not attractive in the sense of (25) for the corresponding model in the form of impulsive differential equations.

Finally, it has been shown recently in [186] that the standard notion of asymptotic stability (Def. 6) for impulsive systems, whereby the state is ensured to approach $\mathcal{A}$ only as continuous time elapses, is too weak to allow for any meaningful type of robustness with respect to external inputs in a time-varying impulsive system setting. By strengthening the inequality (24) to

$$\|x(t)\|_{\mathcal{A}} \leq \beta(\|x_0\|_{\mathcal{A}}, t - t_0 + n_{(t_0,t]}) \quad \text{for all} \quad t \geq t_0, \tag{26}$$

where $n_{(t_0,t]}$ denotes the number of impulse-time instants contained in $(t_0, t]$, some well-established robustness results for time-invariant non-impulsive systems have been transferred to impulsive systems in [187]. Inequality (26) requires that the convergence to the set $\mathcal{A}$ occurs not only as time elapses but also as the number of jumps increases, and, therefore, the set $\mathcal{A}$ is sometimes termed *strongly GAS* in this case.

3.4. treatment of several jumps at the same time

An important situation that may happen in all different classes of hybrid systems, and requires careful consideration of the effect it entails, is the simultaneous occurrence of several jumps. For example, in a network with pulse coupling, several pulses can arrive to one unit at the same time. If the system is defined with Dirac delta functions on the right-hand side, this may result in the emergence of a delta function with a larger amplitude. For instance, if in Eq. (9) some impact times t_k^j coincide, this results in the instantaneous change of the voltage by the total amount $\sum J_{ij}$, where the sum runs over all j corresponding to the coinciding t_k^j. If the system is defined in terms of an impulsive differential equation, the case of several jumps at the same time must be separately defined. For example, if two pulses arrive simultaneously at a phase oscillator at the same time, the map (23) has to be applied twice to ensure a continuous dependence of trajectories on initial conditions [70,82].

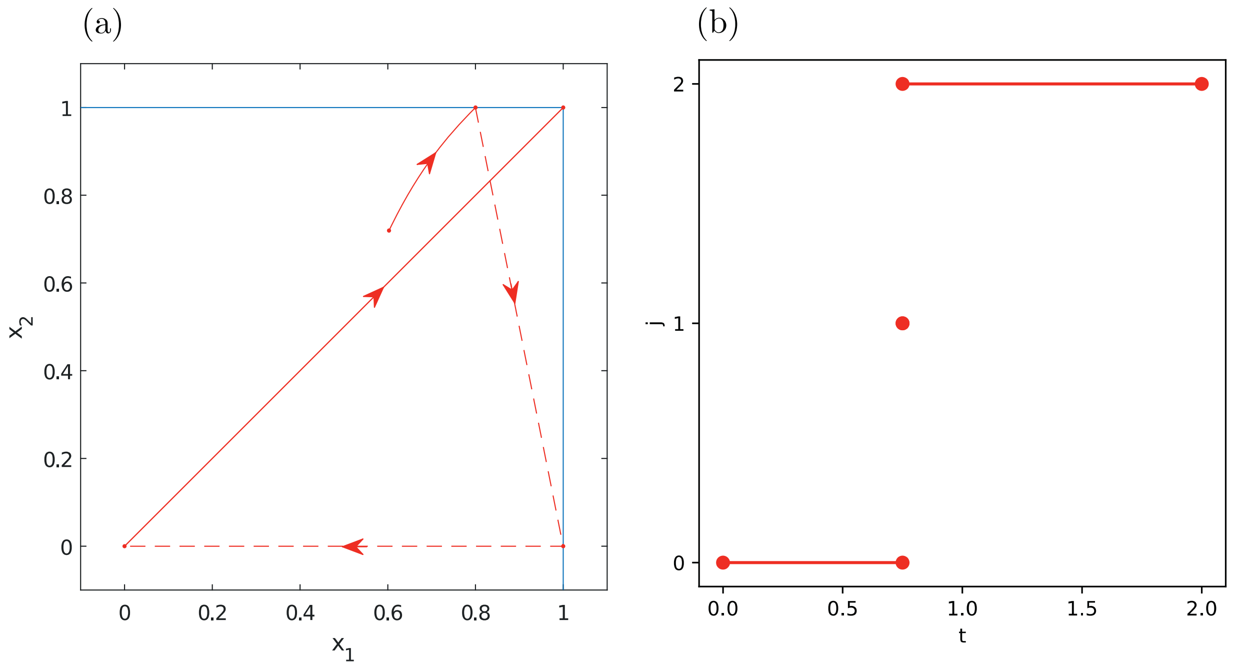

Fig. 4. An "avalanche" in a population of two integrate-and-fire oscillators. (a): The continuous dynamics is plotted by red solid lines, jumps are plotted by red dashed lines. The jumps set is depicted by blue lines. (b): A plot of the hybrid time domain (in solid red) for the corresponding hybrid trajectory. It is possible to access the state of the system before the first jump, after the first jumps, and after the second jump at times $(0.75, 0)$, $(0.75, 1)$, and $(0.75, 2)$, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

More complex situations might arise when one jump immediately induces one or several subsequent jumps. Such a mechanism causes for instance so-called avalanches in neural networks [188]. If the coupling between the neurons is modeled as instantaneous interaction, the firing of some neurons can lead to the excitation of their projection targets if these are close enough to their threshold at that time. The induced firing may again cause other neurons to fire, and so on, releasing an avalanche of firing events over several neuronal layers.

The most convenient way to formally define such the dynamics is to use hybrid dynamical systems. For example, consider a population of N integrate-and-fire oscillators [60], each described by the voltage-like state variable x_i subject to the continuous dynamics

$$\dot{x}_i = S_0 - \gamma x_i. \tag{27}$$

When $x_i = 1$, the oscillator "fires" and x_i resets to zero. At this moment the oscillator emits a pulse which is immediately received by all the other oscillators and causes an instant change of their states. Each oscillator is pulled up by an amount of ε, but not above unity, so that

$$x_i^+ = \min\{1, x_i + \varepsilon\}. \tag{28}$$

The dynamics of such the system can be defined in terms of the hybrid system (2a)-(2b) with the state $x = \{x_1, x_2, \ldots, x_N\}$, the flow set $C = \bigcap_{i=1}^N \{x | x_i < 1\}$ and the jump set $D = \bigcup_{i=1}^N \{x | x_i = 1\}$. The continuous dynamics is given by $f = (f_1, \ldots, f_N)$ with $f_j(x) = S_0 - \gamma x_i$, $i = 1, \ldots, N$, while the jump map is given by $g = (g_1, \ldots, g_N)$ with

$$g_i(x) = \begin{cases} 0, & \text{if } x_i \in \{0, 1\}, \\ \min\{1, x_i + M\varepsilon\}, & \text{if } x_i < 1, \end{cases} \tag{29}$$

where $M \geq 1$ is the number of components x_i equal to 1. Then, as M units approach the threshold $x_i = 1$ simultaneously, the system reaches the jump set C. That is, a jump occurs and the units which fire reset to zero. All other units receive an input of strength $M\varepsilon$. If this input drives some other units to the threshold, the system remains in the jump set, and a second jump occurs, which may again entail further jumps, etc. An illustration of such an "avalanche" in a an ensemble of two neurons is given in Fig. 4 (a).

It is clear that this type of behavior cannot be completely described neither by equations with Dirac delta functions, nor by impulsive differential equations. Both these frameworks cannot represent a sequence of jumps occurring simultaneously, since they rely on solutions, which are piecewise continuous functions of time t. In particular, this means that at most two different states can be associated with a time moment t^*, namely $\lim_{\varepsilon \searrow 0} x(t^* - \varepsilon)$ and $\lim_{\varepsilon \searrow 0} x(t^* + \varepsilon)$. If x is discontinuous at t^*, these correspond to the system's states before and after a jump. In contrast, the hybrid dynamical systems approach allows to capture an arbitrary number of different states at a given time moment t^* as follows: $x(t^*, j), x(t^*, j+1), x(t^*, j+2), \ldots$. The hybrid time domain that corresponds to the "avalanche" in the preceding example is given in Fig. 4 ((b)).

4. conclusions and outlook

The present paper provides an overview of the most common modeling approaches for the mathematical description and analysis of processes that combine continuous and discontinuous behavior. In particular, we give a comparison between impulsive differential equations, hybrid dynamical systems, and differential equation with Dirac delta function from the viewpoint of their modeling capabilities and the respective analysis techniques. We discuss a comprehensive list of application areas from various research domains and typical dynamical processes, whose modeling requires accounting for instantaneous impulsive jumps. With this, we attempt to raise the attention of the interested researchers to the methods available in other research communities and to foster the exchange of ideas and analysis techniques. An exploration of the connections of the frameworks considered in the paper and symbolic models and methods stemming from the computer science community are of a great interest for the future study.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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