DOI 10.1007/s10958-022-05758-3
Journal of Mathematical Sciences, Vol. 261, No. 3, February 2022

STRUCTURE OF GRAPHS OF SUSPENDED FOLIATIONS

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UDC 515.165, 514.762.5

We study the graph $G_{\mathfrak{M}}(F)$ of a foliation F with Ehresmann connection $\mathfrak{M}$. We prove that for any suspended foliation (M, F) the induced foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$ is also suspended and describe the structure of the graph $G_{\mathfrak{M}}(F)$, as well as leaves of the induced foliation. We show that the induced foliation is a suspended foliation with the same transversal manifold and the same structure group as the original foliation. Bibliography: 19 *titles. Illustrations*: 2 *figures.*

The notion of a holonomy groupoid was introduced by Ehresmann. An equivalent construction was introduced in [1]. The graph $G(F)$ or the holonomy groupoid of a foliation (M, F) is one of the main constructions in the theory of foliations. The C^*-algebras of functions for foliations (M, F) are determined on the holonomy groupoid. In the general case, the graph $G(F)$ of a foliation (M, F) of codimension q on an n-dimensional manifold M is a $(2n - q)$-dimensional manifold which is not necessarily Hausdorff.

The notion of a suspended foliation was introduced in [2]. The suspended foliations are a multi-dimensional generalization of one-dimensional foliations formed by flows given by the suspension of a diffeomorphism, well known in the theory of dynamical systems. The suspended foliations are widely used in the theory of foliations (cf. the definition in Subsection 4.1 below). First, they are perfect for constructing examples and counterexamples of foliations possessing various properties (cf., for example, [3, 4]). Second, a lot of classes of foliations turn out to be suspended. As established in [5], the transversally complete Lorentzian foliation of codimension two on a closed three-dimensional manifold is formed by trajectories of the Anosov flow and is realized either as the suspended foliation obtained by suspension of the Anosov automorphism f_A of the standard two-dimensional torus $\mathbb{T}^2$ with constant Lorentzian metric or as the folia-

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Translated from *Problemy Matematicheskogo Analiza* **114**, 2022, pp. 41-54.

tion formed by the geodesic flow on the unit tangent fibration to surfaces of constant negative curvature. The study of the structurally stability of suspended foliations on compact manifolds was started in [6] and further continued by the authors [7, 8]. Suspended foliations were used in [8] to construct a continual family of completely geodesic foliations with non-Hausdorff graphs $G(F)$ on each of the following surfaces: a torus, a cylinder, compact and open Mobius leaves. A number of examples of suspended foliations are constructed in [3] and [4]. The suspended foliations were used in [4] to prove that any countable group of conformal transformations of the standard sphere $\mathbb{S}^q$ is realized as the global holonomy group of a complete conformal foliation.

1 Statement of the Problem and the Main Results

1.1. Statement of the problem. The notion of an Ehresmann connection was proposed in [9] as a generalization of the notion of a connection in fibrations. The Ehresmann connection for a foliation (M, F) of codimension q is a smooth q-dimensional distribution $\mathfrak{M}$, transversal to leaves, which allows us to transfer curves in leaves of the foliation along any integrable curves of the distribution $\mathfrak{M}$. If, in addition, the distribution $\mathfrak{M}$ is integrable, then the Ehresmann connection is said to be *integrable*. For a foliation admitting an Ehresmann connection, the first author [10] introduced another graph $G_{\mathfrak{M}}(F)$ called the *graph of a foliation with Ehresmann connection.*

With any suspended foliation (M, F) of codimension q on an n-dimensional smooth manifold M one can associate a locally trivial fibration $p : M \to B$ over a base B; moreover, fibers of this fibration are transversal to leaves of the foliation. We emphasize that the tangent spaces to the fibers of the associated fibration $p : M \to B$ form an integrable Ehresmann connection $\mathfrak{M}$ of the foliation (M, F), called the *associated* one.

As shown in [8], nonisomorphic fibrations and, consequently, different Ehresmann connections and different graphs can be associated with the same foliation in the case of a noncompact manifold. Therefore, studying foliations with integrable Ehresmann connection, we fix not only a foliation itself, but also the associated integrable Ehresmann connection $\mathfrak{M}$ and use the notation (M, F, F^t), where $\mathfrak{M} = TF^t$.

The goal of this paper is to study the structure of the graphs $G_{\mathfrak{M}}(F)$ of suspended foliations.

1.2. The main results. First of all, we answer the question whether a foliation is suspended and prove the following criterion for a foliation to be suspended.

Theorem 1.1. *A foliation (M, F) is suspended if and only if there exists a submersion $p : M \to B$ such that the distribution $\mathfrak{M}$ formed by the tangent spaces to the leaves of the foliation is an Ehresmann connection for (M, F).*

By Theorem 1.1, the suspended foliations have the natural integrable Ehresmann connection. Using Theorem 1.1, we can prove the following assertion.

Theorem 1.2. *If (M, F) is a suspended foliation, then the induced foliation $\mathbb{F}$ on the graph $G_{\mathfrak{M}}(F)$ is also suspended; moreover, the structure group and global holonomy group of foliations (M, F) and $(G_{\mathfrak{M}}(F), \mathbb{F})$ coincide (up to an isomorphism of groups and conjugation in* Diff (T)*).*

In Section 4, we recall the notion of a suspended foliation $(M, F) = \mathrm{Sus}\,(T, B, \rho)$ and give the definition of a canonical suspended foliation. The following assertion provides a detailed description of the structure of graphs of suspended foliations and the induced foliations there.

Theorem 1.3. *Let* $(M, F) = \mathrm{Sus}\,(T, B, \rho)$ *be the foliation obtained by suspension of a homomorphism* $\rho : \pi_1(B, b) \to \mathrm{Diff}\,(T)$*, and let* $\Psi = \rho(\pi_1(B, b))$ *be its global holonomy group. Then the following assertions hold.*

(1) *The foliation* (M, F) *is isomorphic to the canonical foliation on the quotient manifold* $L_0 \times_\Psi T$*, where* $\psi(z, t) = (\psi(z), \psi(t))$ *for any* $\psi \in \Psi$*,* $(z, t) \in L_0 \times T$ *and the induced foliation on the graph* $(G_{\mathfrak{M}}(F), \mathbb{F}) = \mathrm{Sus}\,(T, \mathbb{B}, \widehat{\rho})$ *is isomorphic to the canonical foliation on the quotient manifold* $(L_0 \times L_0) \times_\Psi T$*,*

$$\psi(z_1, z_2, t) := (\psi(z_1), \psi(z_2), \psi(t)) \quad \forall\, \psi \in \Psi \quad \forall (z_1, z_2, t) \in L_0 \times L_0 \times T.$$

(2) *The quotient manifolds* $B \cong L_0/\Psi$ *and* $\mathbb{B} \cong (L_0 \times L_0)/\Psi$ *are bases of the associated fibrations* $p : M \to B$ *and* $\widehat{p} : G_{\mathfrak{M}}(F) \to \mathbb{B}$ *for the foliations* (M, F) *and* $(G_{\mathfrak{M}}(F), \mathbb{F})$ *respectively.*

(3) *The manifold* T *is the standard fiber of both associated fibrations* $p : M \to B$ *and* $\widehat{p} : G_{\mathfrak{M}}(F) \to \mathbb{B}$*;*

(4) $\widehat{\rho}(\pi_1(\mathbb{B})) = \rho(\pi_1(B)) = \Psi \subset \mathrm{Diff}\,(T)$.

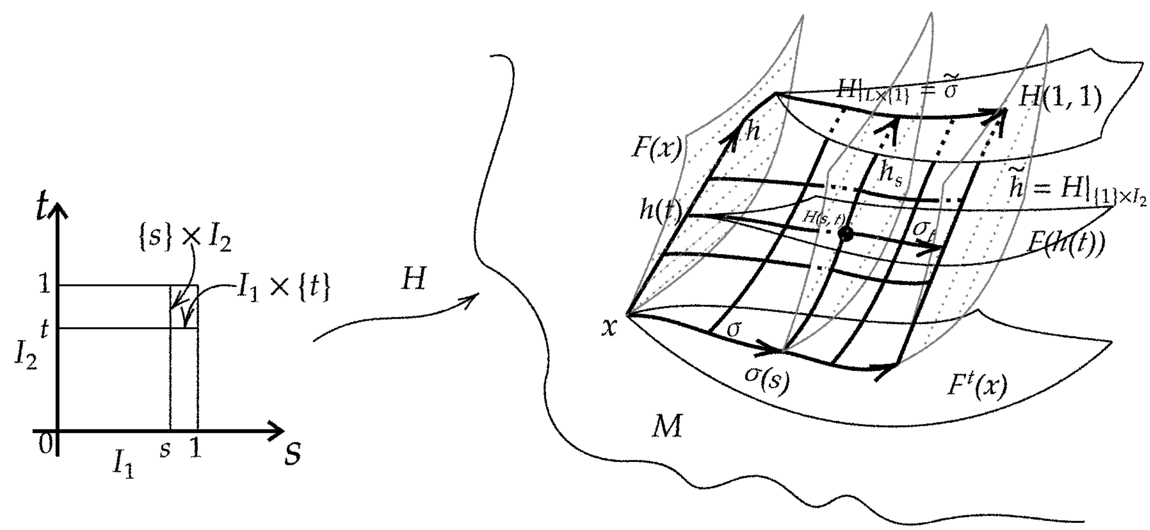

Figure 1. $\mathfrak{M} = TF^t$ is an integrable Ehresmann connection for F.

1.3. Notation and assumptions. Following [11], we denote by $P(N, H)$ the principle H-fibration over a manifold N. We denote by $\mathfrak{X}(M)$ the module of smooth vector fields on a manifold M over the algebra $\mathfrak{F}(M)$ of smooth functions. The foliation F on the manifold M is denoted by the letter F, as well as by the pair of letters (M, F). Let $\mathfrak{M}$ be a smooth distribution on the manifold M. Then

$$\mathfrak{X}_{\mathfrak{M}}(M) := \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u F \quad \forall\, u \in M\}.$$

If $\mathfrak{M}$ is integrable and $\mathfrak{M} = TF$, then $\mathfrak{X}_{\mathfrak{M}}(M)$ is denoted by $\mathfrak{X}_F(M)$. Throughout the paper, we use the notation $I = I_1 = I_2 = [0, 1]$.

The restriction of a foliation (or a metric) on a submanifold is denoted by the same letter as the original foliation (or the original metric).

The symbol $\cong$ means an isomorphism in the corresponding category and $\oplus$ denotes the direct sum of vector subspaces and distributions.

The smoothness of mappings and manifolds is understood as the smoothness of class C^r, where $r \geqslant 1$. All neighborhood are assumed to be open.

2 Integrable Ehresmann Connection for Foliation and Holonomy Groups

2.1. Integrable Ehresmann connection for a foliation. The notion of an Ehresmann connection for a foliation was introduced in [9] as a natural generalization of the notion of an Ehresmann connection for a submersion. For the sake of simplicity we give the definition only for an integrable Ehresmann connection. We use the term *vertical-horizontal homotopy* [12].

We assume that a pair of transversal foliations F and F^t of complementary dimension are given on an n-dimensional smooth manifold M, i.e., at any point $x \in M$, the tangent vector space T_xM to the manifold M is decomposed into the direct sum $T_xM = T_xF \oplus T_xF^t$, where T_xF and T_xF^t are the tangent spaces to leaves of the foliations F and F^t respectively at the point x. Piecewise smooth curves in leaves of the foliation F^t are said to be *horizontal,* whereas piecewise smooth curves in leaves of the foliation F are referred to as *vertical.* A piecewise smooth mapping H from the square $I_1 \times I_2$ to the manifold M is called a *vertical–horizontal homotopy* if the curve $H|_{\{s\}\times I_2}$ is vertical for any $s \in I_1$ and the curve $H|_{I_1\times\{t\}}$ is horizontal for any $t \in I_2$ (cf. Figure 1). In this case, the pair of paths $(H|_{I_1\times\{0\}}, H|_{\{0\}\times I_2})$ is called the *base* of H. As known, there exists at most one vertical–horizontal homotopy with a given base.

A pair of paths (σ, h) in M with the same start point $\sigma(0) = h(0)$ is said to be *admissible* if σ is a horizontal curve and h is a vertical curve.

Definition 2.1. A distribution $\mathfrak{M} = TF^t$ tangent to leaves of a foliation F^t, is called an *integrable Ehresmann connection* for the foliation F if for any admissible pair of paths (σ, h) in M there exists a vertical–horizontal homotopy H with base (σ, h).

Remark 2.1. If $\mathfrak{M} = TF^t$ is an integrable Ehresmann connection for a foliation F, then $\mathfrak{P} = TF$ is an integrable Ehresmann connection for the foliation F^t.

In what follows, for a foliation F on a manifold M with integrable Ehresmann connection TF^t we use the notation (M, F, TF^t).

2.2. Integrable Ehresmann connection for submersion. Let M and B be smooth manifolds of dimension n and m respectively, $n > m$. Let $p : M \to B$ be a surjective submersion with connected fibers. We denote by F a simple foliation formed by fibers of this submersion. We assume that an m-dimensional foliation F^t transversal to the foliation F is given on the manifold M. As above, we assume that the curves are piecewise smooth. The curves in leaves of the foliation F^t are called *horizontal,* whereas the curves in leaves of the foliation F are called *vertical.* Let $\sigma : [0,1] \to B$ be a curve with the start point $x = \sigma(0)$. If there exists a horizontal curve $\widetilde{\sigma} : [0,1] \to M$ with the start point $\widetilde{x} = \widetilde{\sigma}(0)$ such that $\sigma = p \circ \widetilde{\sigma}$, then $\widetilde{\sigma}$ is called a F^t*-lift* of the curve σ at the point $\widetilde{x}$. We say that a piecewise smooth curve $\sigma : [0,1] \to B$ of the base B possesses F^t-lifts if for any point $\widetilde{x} \in p^{-1}(x)$, where $x = \sigma(0)$, there exists an F^t*-lift* of the curve σ at the point $\widetilde{x}$.

Definition 2.2. A distribution $\mathfrak{M} = TF^t$ tangent to leaves of a foliation F^t is called an *integrable Ehresmann connection* for the foliation F if any piecewise smooth curve of a base B possesses F^t-lifts.

Using [9, Proposition 1.3], it is easy to prove the following assertion.

Proposition 2.1. *Let $p : M \to B$ be a submersion with connected fibers from an n-dimensional manifold M to an m-dimensional manifold B. Let F be a simple foliation formed by fibers of the submersion p. We assume that an m-dimensional foliation F^t transversal to the foliation F is given on the manifold M. Then the distribution $\mathfrak{M} = TF^t$ tangent to the foliation F^t is an integrable Ehresmann connection for the submersion $p : M \to B$ if and only if the distribution $\mathfrak{M}$ is an integrable Ehresmann connection for the foliation F.*

2.3. $\mathfrak{M}$-holonomy groups and their connection with germ holonomy groups. Let (M, F, F^t) be a foliation with an integrable Ehresmann connection $\mathfrak{M} = TF^t$. We consider an admissible pair of paths (δ, τ). One say that a curve $\widetilde{\delta}$ is obtained by transferring the path δ along τ relative to the Ehresmann connection $\mathfrak{M}$ if $\widetilde{\delta} := H|_{I\times\{1\}}$, which will be denoted by $\delta \overset{\tau}{\to}> \widetilde{\delta}$.

We denote by Ω_x, $x \in M$, the set of (piecewise smooth) curves with a start point x in the leaf $L^t = L^t(x)$ of the foliation F^t. We define the action of the fundamental group $\pi_1(L, x)$ from the leaf L to the set Ω_x by the rule

$$\Phi_x : \pi_1(L, x) \times \Omega_x \to \Omega_x : ([h], \sigma) \mapsto \widetilde{\sigma},$$

where $[h] \in \pi_1(L, x)$ and $\widetilde{\sigma}$ is the displacement of the curve $\sigma \in \Omega_x$ along h with respect to $\mathfrak{M}$. Let $K_{\mathfrak{M}}(L, x)$ be the kernel of the action of Φ_x, i.e.,

$$K_{\mathfrak{M}}(L, x) = \{\alpha \in \pi_1(L, x) \mid \Phi_x(\alpha, \sigma) = \sigma \quad \forall \sigma \in \Omega_x\}.$$

The quotient group $H_{\mathfrak{M}}(L, x) = \pi_1(L, x)/K_{\mathfrak{M}}(L, x)$ is called the *$\mathfrak{M}$-holonomy group* of the leaf L [9]. By the linear connections of leaves, the $\mathfrak{M}$-holonomy groups are isomorphic at different points of the same leaf.

Let $\Gamma(L, x)$ be the germ holonomy group of the leaf L, commonly used in the theory of foliations [13]. Then we can define an epimorphism of groups $\chi_x : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ satisfying the commutative diagram

$$\begin{array}{ccc} & \pi_1(L, x) & \\ {\scriptstyle \beta_x} \swarrow & & \searrow {\scriptstyle \gamma_x} \\ H_{\mathfrak{M}}(L, x) & \xrightarrow{\ \chi_x\ } & \Gamma(L, x) \end{array} \tag{2.1}$$

where $\beta_x : \pi_1(L, x) \to H_{\mathfrak{M}}(L, x)$ is the quotient mapping and $\gamma_x([h]) := \langle h \rangle$ is the germ of the local holonomy diffeomorphism that is transversal to the q-dimensional disc in the leaf $L^t = L^t(x) \in F^t$ along the loop h at the point x.

A diffeomorphism $f : U \to V$ of open sets U and V is said to be *quasianalytic* if from the existence of a connected open subset $\widetilde{U} \subset U$ such that $f|_{\widetilde{U}} = id_{\widetilde{U}}$ it follows that f coincides with the identity mapping on the whole connected component of the set U containing $\widetilde{U}$.

We say that the pseudogroup of local diffeomorphisms $\mathscr{H}$ of a not necessarily connected manifold T is *quasianalytic* if every transformation $f \in \mathscr{H}$ is quasianalytic.

The following assertion follows from [14].

Theorem 2.1. *Let (M, F, F^t) be a foliation with integrable Ehresmann connection $\mathfrak{M} = TF^t$. Then an epimorphism of the holonomy groups $\chi_x\colon H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ satisfying the diagram* (2.1) *is an isomorphism for any point $x \in M$ if and only if the holonomy pseudogroup of the foliation F is quasianalitic.*

3 Graphs of Foliations with Integrable Ehresmann Connection

3.1. Definition of graphs of foliations with integrable Ehresmann connection. Let (M, F, F^t) be a foliation with integrable Ehresmann connection $\mathfrak{M} = TF^t$. We recall the construction of the graph $G_{\mathfrak{M}}(F)$ of a foliation F with integrable Ehresmann connection $\mathfrak{M}$ introduced in [10] (cf. also [15]). We consider any two points x and y in the same leaf L_α of the foliation F. We denote by $A(x, y)$ the set of piecewise smooth paths in L_α joining x and y. The paths h and g in $A(x, y)$ are said to be *equivalent* ($h \sim g$) if the loop $h \cdot g^{-1}$ equivalent to the product of paths h and g^{-1} generates the trivial element of the $\mathfrak{M}$-holonomy group $H(L_\alpha, x)$ of the leaf L_α at the point x. The equivalence class containing a path h is denoted by $\{h\}$. The set $G_{\mathfrak{M}}(F)$ of triples $(x, \{h\}, y)$, $x \in M$, $y \in L(x)$, $h \in \Phi(x, y)$ is called the *graph* of the foliation (M, F, F^t) with Ehresmann connection $\mathfrak{M} = TF^t$, whereas the mappings

$$p_1 : G_{\mathfrak{M}}(F) \to M : (x, \{h\}, y) \mapsto x,$$

$$p_2 : G_{\mathfrak{M}}(F) \to M : (x, \{h\}, y) \mapsto y$$

are referred to as the *canonical projections.*

Thus, the graph of a foliation with Ehresmann connection $G_{\mathfrak{M}}(F)$ is introduced in the same way as the classical graph $G(F)$ of a foliation [1] by replacing the germ holonomy group $\Gamma(L, x)$ of the leaf L, $x \in M$, used in the theory of foliations [13], with the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$.

3.2. Topology and smooth structure on the graph of a foliation with integrable Ehresmann connection. Hereinafter, (M, F, F^t) is a foliation of codimension q on an n-dimensional manifold M with integrable Ehresmann connection $\mathfrak{M}$. The topology in the graph $G_{\mathfrak{M}}(F)$ is introduced as follows. Let $c = (a, \{h\}, b) \in G_{\mathfrak{M}}(F)$. As known, at each point $a \in M$, there exists a contractible neighborhood $V = U \times D$ adapted to both foliations F and F^t, where U and D are contractible open neighborhoods of the point a in the leaves $L = L(a)$ and $L^t = L^t(a)$ of the foliations F and F^t respectively. This neighborhood $V = U \times D$ is called a *bifibered neighborhood* relative to (M, F, F^t). For any admissible pair of paths (σ, φ) in V started at a point a there exists a vertical–horizontal homotopy H in V with base (σ, φ) coinciding with the standard vertical–horizontal homotopy with respect to leaves of the product $U \times D$. We take any point $x \in D$ and join a with x by a smooth horizontal curve σ in D. Let $\sigma \xrightarrow{h}> \widetilde{\sigma}$. Then $\widetilde{\sigma}(0) = b$. For sufficiently small D the set of points $y = \widetilde{\sigma}(1)$ is a disc D' transversal to the foliation F at the point b. The mapping $H_h\colon D \to D'$ defined by $H_h(x) := y$ coincides with the holonomy diffeomorphism generated by the path h. We note that the family of germs $\{H_h\}_{[h] \in \pi_1(L,x)}$ of holonomy diffeomorphisms of the q-dimensional disc D at the point a forms the germ holonomy group $\Gamma(L, a)$.

We choose bifibered charts (V, φ) and (V', ψ) at points a and b respectively, so that $V = U \times D$, $V' = U' \times D'$, $\varphi(V) = \psi(V') = \mathbb{R}^k \times \mathbb{R}^q$, $\varphi(a) = \psi(b) = (0, 0)$, $\varphi(D) = \psi(D') = \{0\} \times \mathbb{R}^q$.

For any $x \in V$ we denote by x^* a point in D lying in the same local leaf L_x as the point x. We join a with x^* in D by a horizontal curve σ. Let $h \overset{\sigma}{\longrightarrow}> h^*$, $h^*(1) = y^*$. For y we can take any point in V' belonging to the same local leaf as y^*. Let s_x and s_y be arbitrary paths joining x^* with x and y^* with y in the local leaves L_x and L_y respectively. For μ we take the product of paths $s_x^{-1} \cdot h^* \cdot s_y$. By the connectedness of local leaves, the equivalence class $\{\mu\}$ is independent of the choice of the curves s_x and s_y.

By an *open neighborhood* of a point $c = (a, b, \{h\})$ we mean the set W_c of points $z = (x, y, \{\mu\})$ obtained as above. The set $\{W_c\}_{c \in G_{\mathfrak{M}}(F)}$ is the subbase of the topology for the set $G_{\mathfrak{M}}(F)$. We say that $G_{\mathfrak{M}}(F)$ equipped with this topology is the *graph* of the foliation (M, F, F^t) with Ehresmann connection.

By [10] (cf. also [15]), is known that the following assertion holds.

Theorem 3.1. *For a foliation with Ehresmann connection (M, F, F^t) the topological space of the graph $G_{\mathfrak{M}}(F)$ is Hausdorff and connected; moreover, it satisfies the second axiom of countability. On the graph $G_{\mathfrak{M}}(F)$, one can naturally introduce the structure of a Hausdorff differentiable manifold of dimension $n + k$, where $n = \dim M$, $k = dim(F)$; moreover, the canonical projections*

$$p_1 \colon G_{\mathfrak{M}}(F) \to M \colon (x, \{h\}, y) \mapsto x,$$

$$p_2 \colon G_{\mathfrak{M}}(F) \to M \colon (x, \{h\}, y) \mapsto y$$

become submersions that form locally trivial fibrations with the same standard fiber L_0.

3.3. Structure of the holonomy groupoid on the graph of a foliation. The operation $(y, \{h_2\}, z) * (x, \{h_1\}, y) := (x, \{h_1 \cdot h_2\}, z)$ transforms a graph $G_{\mathfrak{M}}(F)$ with the canonical projections p_1 and p_2 to a smooth $\mathfrak{M}$*-holonomy groupoid*. The space of left and right identities of this groupoid coincides with the *graph diagonal* $\Delta_{\mathfrak{M}} := \{(x, \{e_x\}, x) \in G_{\mathfrak{M}}(F),\ x \in M\}$, where $e_x : [0, 1] \to x$ is a constant path. The mapping $\beta : G_{\mathfrak{M}}(F) \to G(F) : (x, \{h\}, y) \mapsto (x, \langle h \rangle, y)$, where $(x, \langle h \rangle, y) \in G(F)$, is a local diffeomorphism. Both graphs $G(F)$ and $G_{\mathfrak{M}}(F)$ are equipped with the structure of a groupoid; moreover, β is an epimorphism of these groupoids, i.e., β is a surjective mapping sending the product of elements of one groupoid to the product of the corresponding elements of the other groupoid.

Unlike $G_{\mathfrak{M}}(F)$, the topological space of the graph $G(F)$, the usual graph of the foliation F constructed from the germ holonomy group is not, in general, Hausdorff. Applying the criterion for the graph $G(F)$ to be Hausdorff [1] and Theorem 2.1, we obtain the following criterion for graphs of foliations with integrable Ehresmann connection to be isomorphic.

Theorem 3.2. *Let (M, F, F^t) be a foliation with integrable Ehresmann connection $\mathfrak{M} = TF^t$. Then an epimorphism of holonomy groupoids*

$$\beta : G_{\mathfrak{M}}(F) \to G(F) : (x, \{h\}, y) \mapsto (x, \langle h \rangle, y)$$

is an isomorphism for any point $x \in M$ if and only if at least one of the following two equivalent conditions holds:

(1) *the holonomy pseudogroup of the foliation F is quasianalytic,*

(2) *the graph $G(F)$ of the foliation F is Hausdorff.*

Remark 3.1. By Theorem 3.2, the graph $G_{\mathfrak{M}}(F)$ can be regarded as the desingularization of the graph $G(F)$ provided that $G(F)$ is not a Hausdorff graph.

4 Necessary and Sufficient Condition for Foliation to Be Suspended

4.1. Suspended foliations. Let B and T be smooth manifolds of dimensions $n-q$ and q respectively. We assume that $\rho : \pi_1(B,b) \to \mathrm{Diff}\,(T)$ is a homomorphism from the fundamental group $G = \pi_1(B,b)$ to the group of diffeomorphisms of the manifold T. We consider the universal covering manifold $\widehat{B}$ for B as the right G-space. We define the action of the group G from the left on the product $\widehat{B}\times T$ of manifolds as follows: $\Phi : G\times\widehat{B}\times T \to \widehat{B}\times T$, $(g,(\widehat{b},t)) \mapsto (\widehat{b}\cdot g^{-1}, \rho(g)t)$, where $(\widehat{b},t) \in \widehat{B}\times T$. Thereby we define a smooth n-dimensional quotient manifold $M = \widehat{B}\times_G T$ with a foliation F of codimension q. The leaves of the foliation F are the images of leaves of the trivial foliation $F_{\mathrm{tr}} = \{\widehat{B}\times\{t\} | t \in T\}$ on the product $\widehat{B}\times T$ with respect to the quotient mapping $f : \widehat{B}\times T \to M$ which is a regular covering. The foliation F is said to be *suspended* and is denoted by $\mathrm{Sus}\,(T,B,\rho)$. We say that F is obtained by suspension of the homomorphism ρ.

We note that another foliation F^t is defined on the manifold M whose leaves are the images of leaves of some other trivial foliation $F^t_{\mathrm{tr}} = \{\{\widehat{b}\}\times T | \widehat{b} \in \widehat{B}\}$ on the product $\widehat{B}\times T$ with respect to the quotient mapping $f : \widehat{B}\times T \to M$. We emphasize that the distribution $\mathfrak{M} = TF^t$ is an integrable Ehresmann connection of the foliation F. Furthermore, the leaves of the foliation F^t form a locally trivial fibration $p : M \to B$ with base B.

The abstract group $\Psi = \rho(G)$ is called the *structure group* of the suspended foliation $(M,F) = \mathrm{Sus}\,(T,B,\rho)$. The group $\rho(G)$ of diffeomorphisms of the manifold T is called the *global holonomy group* of this foliation.

4.2. Proof of Theorem 1.1. *Necessity* follows from the definition of a suspended foliation.

Sufficiency. We assume that (M,F,F^t) is a foliation of codimension q on an n-dimensional manifold M with integrable connection $\mathfrak{M} = TF^t$; moreover, the foliation F^t is formed by fibers of a submersion $p : M \to B$. We consider the universal covering mapping $\varkappa : \widetilde{M} \to M$. By the Kashiwabara theorem [16] (re-opened in [9]), $\widetilde{M} = \widehat{L}\times\widehat{L}^t$, where $\widehat{L}$ is the universal covering manifold for all leaves of the foliation F and $\widehat{L}^t$ is the universal covering manifold for all leaves of the foliation F^t. Moreover, the foliations $\varkappa^*F$ and $\varkappa^*F^t$ that are liftings of F and F^t on $\widetilde{M}$ are the trivial foliations of the product $\widehat{L}\times\widehat{L}^t$. Since TF is an integrable Ehresmann connection of the foliation F^t, from Proposition 2.1 it follows that TF is an integrable Ehresmann connection for the submersion $p : M \to B$. As known, any submersion with Ehresmann connection forms a locally trivial fibration. We denote by T the standard fiber of the locally trivial fibration with projection $p : M \to B$.

Furthermore, the restriction $p|_L : L \to B$ of the projection p on an arbitrary leaf L of the foliation F is a covering mapping. Hence $\widehat{L} = \widehat{B}$ is the universal covering for the base B. Since $\widehat{L}^t$ is the universal covering for leaves of the foliation F^t, $\widehat{L}^t = \widehat{T}$ is the universal covering for the manifold T. We denote by $s : \widehat{T} \to T$ the universal covering mapping. In the above notation, $\varkappa : \widehat{B}\times\widehat{T} \to M$ is the universal covering for the manifold M.

We fix points $b \in B$, $x \in M$, and $(y,z) \in \widehat{B}\times\widehat{T}$. The fundamental group $\pi_1(M,x)$ is realized as the group K of covering transformations of the universal covering $\varkappa : \widehat{B}\times\widehat{T} \to M$; moreover, K preserves the structure of the product $\widehat{B}\times\widehat{T}$. Let pr_1 and pr_2 be the canonical projections of the product $\widehat{B}\times\widehat{T}$ on the factors. Then the mapping $g_i := pr_i \circ g$ is well defined, where $g \in K, i = 1,2$; moreover,

$$g(x_1,x_2) = (g_1(x_1), g_2(x_2)) \quad \forall (x_1,x_2) \in \widehat{B}\times\widehat{T}.$$

We assume that $K_i := \{g_i = pr_i \circ g \,|\, g \in K\}$ and $q_i \colon K \to K_i$ is the natural epimorpism of groups, $K_{ii} := q_i(q_j^{-1}(1_j))$, $i, j = 1, 2$, $i \neq j$, and 1_j is the identity of the group K_j. Moreover, K_{ii} is a normal subgroup of the group K_i and we can introduce the quotient group K_i/K_{ii}. By [17, Lemma 1], there exists an isomorphism of groups $\theta \colon K_1/K_{11} \to K_2/K_{22}$ satisfying the commutative diagram

$$\begin{array}{ccccc} K_1 & \xleftarrow{q_1} & K & \xrightarrow{q_2} & K_2 \\ {\scriptstyle h_1}\downarrow & & & & \downarrow{\scriptstyle h_2} \\ K_1/K_{11} & & \xrightarrow{\quad\theta\quad} & & K_2/K_{22} \end{array}$$

where $h_i \colon K_i \to K_i/K_{ii}$ are natural projections. We set $\Psi_i = K_i/K_{ii}$.

We take some point $\widetilde{x} = (\widetilde{y}, \widetilde{z})$ in $\varkappa^{-1}(x)$ and set $N_1(\widetilde{z}) := \{g \in K \mid g(\widehat{B} \times \{\widetilde{z}\}) = \widehat{B} \times \{\widetilde{z}\}\}$ and $N_1(\widetilde{x}) := q_1 N_1(\widetilde{z})$. Similarly, we set $N_2(\widetilde{y}) := \{g \in K \mid g(\{\widetilde{y}\} \times \widehat{T}) = \{\widetilde{y}\} \times \widehat{T}\}$ and $N_2(\widetilde{x}) := q_2 N_2(\widetilde{y})$. Since the group of covering transformations K acts on $\widehat{B} \times \widehat{T}$ freely and properly discontinuously, we conclude that the groups $N_i(\widetilde{x})$, $i = 1, 2$, $N_1(\widetilde{z})$, and $N_2(\widetilde{y})$ act on $\widehat{B} \times \widehat{T}$ and $\widehat{B}$, $\widehat{T}$ freely and properly dicontinuously respectively. The restriction $\varkappa|_{\widehat{B} \times \{\widetilde{z}\}} \colon \widehat{B} \times \{\widetilde{z}\} \to L(x)$ is the universal covering of the leaf $L(x)$ of the foliation F with the group of covering transformations $N_1(\widetilde{z})$, whereas $\varkappa|_{\{\widetilde{y}\} \times \widehat{T}} \colon \{\widetilde{y}\} \times \widehat{T} \to L^t(x) \cong T$ is the universal covering of the leaf $L^t(x)$ of the foliation F^t with the group of covering transformations $N_2(\widetilde{y})$. Since $p : M \to B$ is a fibration, $q_2|_{N_2(\widetilde{y})} : N_2(\widetilde{y}) \to N_2(\widetilde{x})$ is an isomorphism of groups. Consequently, $K_{22} = N_2(\widetilde{y}) \cong \pi_1(T)$ and the quotient mapping $s : \widehat{T} \to T = \widehat{T}/K_{22}$ is a universal covering.

Since $K_{11} \subset N_1(\widetilde{z})$, it follows that K_{11} freely and properly discontinuously acts on $\widehat{B}$. Thereby we can define the quotient manifold $L_0 := \widehat{B}/K_{11}$. Let $\tau : \widehat{B} \to L_0$ be a quotient mapping. Moreover, the quotient group $\Psi_1 = K_1/K_{11}$ freely and properly dicontinuously acts on L_0 and the quotient manifold L_0/Ψ_1 is diffeomorphic to the base B. If $k : \widehat{B} \to B$ is a universal covering mapping, then

$$k = s \circ \tau. \tag{4.1}$$

The direct product $K_{11} \times K_{22}$ of normal subgroups K_{ii} of the groups K_i is a normal subgroup of the group K, and, consequently, we can define the smooth quotient manifold $(\widehat{B} \times \widehat{T})/(K_{11} \times K_{22})$ diffeomorphic to the product of manifolds $(\widehat{B}/K_{11}) \times (\widehat{T}/K_{22})$. The group Ψ freely and properly continuously acting on $L_0 \times T$ is isomorphic to the quotient group $K/(K_{11} \times K_{22})$; moreover, there exists a diffeomorphism $\Theta \colon M \to (L_0 \times T)/\Psi$ satisfying the commutative diagram

$$\begin{array}{ccc} \widehat{B} \times \widehat{T} & \xrightarrow{\tau \times s} & L_0 \times T \\ {\scriptstyle \varkappa}\downarrow & & \downarrow{\scriptstyle f} \\ (\widehat{B} \times \widehat{T})/K = M & \xrightarrow{\Theta} & (L_0 \times T)/\Psi \end{array} \tag{4.2}$$

where $f \colon L_0 \times T \to (L_0 \times T)/\Psi$ is the quotient mapping on the space of orbits. From [17, Lemma 1] it follows that the group Ψ is isomorphic to each of the quotient groups $K_1/K_{11} = \Psi_1$ and $K_2/K_{22} = \Psi_2$. Since the group Ψ preserves the structure of product, on the quotient manifold $(L_0 \times T)/\Psi$ we can define a couple of transversal foliations $(\mathscr{F}, \mathscr{F}^t)$ that are the images of the trivial foliations on the product $L_0 \times T$ under the quotient mapping f. Furthermore, Θ is an isomorphism of each pair of foliations F, $\mathscr{F}$ and F^t, $\mathscr{F}^t$. We assume that the group $\Psi \cong \Psi_1$ acts on L_0 from the right by the formula

$$y.\psi = \psi^{-1}(y) \;\Leftrightarrow\; \psi(y) = y.\psi^{-1}, \quad \psi \in \Psi,$$

whereas the group $\Psi \cong \Psi_2$ acts on T from the left by means of the group Ψ_2. In what follows, the quotient manifold $(L_0 \times T)/\Psi$ is denoted by $L_0 \times_\Psi T$, as in [15].

Since $p : M \to B$ is a fibration, the fundamental group $G = \pi_1(B, b)$ of the base B is isomorphic to the group K_1. Consequently, there exists a group homomorphism $\rho = \theta \circ h_1 : G \cong K_1 \to \Psi_2 \subset \mathrm{Diff}\,(T)$. It is easy to verify that $(M, F) = \mathrm{Sus}\,(T, B, \rho)$ is the foliation obtained by suspension of the homomorphism $\rho : G = \pi_1(B, b) \to \mathrm{Diff}\,(T)$. The theorem is proved.

Corollary 4.1. *The structure group Ψ of a suspended foliation $(M, F) = \mathrm{Sus}\,(T, B, \rho)$ is isomorphic to the quotient group $\pi_1(M, x)/(K_{11} \times K_{22})$ of the fundamental group $\pi_1(M, x)$ along the direct product of normal subgroups isomorphic to the fundamental groups $\pi_1(L_0)$ and $\pi_1(T)$, where L_0 is the space of regular covering for any leaf L of the foliation (M, F) whose group of covering transformations is isomorphic to the holonomy group $H_{\mathfrak{M}}(L)$.*

The manifold L_0 is diffeomorphic to the leaf L_α of the foliation (M, F) with the trivial holonomy group $H_{\mathfrak{M}}(L_\alpha)$, if it exists.

Definition 4.1. The suspended foliation $(L_0 \times_\Psi T, \mathscr{F})$ constructed in the proof of Theorem 1.1 is said to be *canonical*, and the diffeomorphism Θ satisfying the commutative diagram (4.2) is referred to as the *representation* of the suspended foliation.

From the proof of Theorem 1.1 we obtain the following assertion.

Theorem 4.1. *Let $(M, F) = \mathrm{Sus}\,(T, B, \rho)$ be the foliation obtained by suspension of the homomorphism $\rho : G = \pi_1(B, b) \to \mathrm{Diff}\,(T)$. Then*

(1) *the foliation (M, F) is isomorphic to the canonical suspended foliation $(L_0 \times_\Psi T, \mathscr{F})$,*

(2) *the manifold T is diffeomorphic to the standard fiber of the transversal fibration $p : M \to B$,*

(3) *the manifold B is diffeomorphic to the quotient manifold L_0/Ψ,*

(4) *the restriction $p|_L : L \to B$ is the covering mapping for any leaf L of the foliation (M, F).*

4.3. Interpretation of the global holonomy group of a suspended foliation. Let $(M, F) = \mathrm{Sus}\,(T, B, \rho)$ be the foliation obtained by suspension of the homomorphism $\rho : G \to \mathrm{Diff}\,(T)$, where $G = \pi(B, b)$, and let $\Psi = \rho(G)$ be the global holonomy group of this foliation. Since each suspended foliation is isomorphic to the canonical suspended foliation, we can interpret the global holonomy group Ψ as follows. We fix a point $b \in B$ and identify the standard fiber T of the fibration $p : M \to B$ with the fiber $p^{-1}(b)$. Let $\psi = \rho([h])$, where $[h] \in G = \pi(B, b)$, is an arbitrary transformation in the group Ψ. As known, in each homotopy class of paths of the manifold there exists a piecewise smooth path. Therefore, without loss of generality we can assume here and in what follows that $h : I \to B$ is a piecewise smooth loop at the point b. We consider an arbitrary point $x \in T$. Since $\mathfrak{M}$ is an integrable Ehresmann connection for the submersion $p : M \to B$, there exists an $\mathfrak{M}$-lift $\widetilde{h}$ of the path h at the point x. Then $\psi(x) = \widetilde{h}(1)$. Since $p|_{L(x)} : L \to B$ is the covering mapping of the leaf $L(x)$ on B, $\widetilde{h}(1)$ is independent of the choice of h in the homotopy class of loops $[h]$.

5 Structure of Induced Foliation

5.1. Induced foliation and its Ehresmann connection. Let $F = \{L_\alpha \,|\, \alpha \in J\}$ be a foliation with integrable Ehresmann connection $\mathfrak{M}$. By Theorem 3.1, the canonical projections

$p_i : G_{\mathfrak{M}}(F) \to M$, $i = 1, 2$, are submersions forming locally trivial fibrations with connected fibers. Consequently, on the graph $G_{\mathfrak{M}}(F)$, we can define the foliation

$$\mathbb{F} = \{p_1^{-1}(L_\alpha) \,|\, L_\alpha \in F\} = \{p_2^{-1}(L_\alpha) \,|\, L_\alpha \in F\},$$

called the *induced foliation.*

Theorem 5.1. *We assume that (M, F, F^t) is a foliation with integrable Ehresmann connection $\mathfrak{M}$, $G_{\mathfrak{M}}(F)$ is the graph of (M, F, F^t), and $p_i : G_{\mathfrak{M}}(F) \to M$, $i = 1, 2$, are the canonical projections. Then the q-dimensional distribution*

$$\mathfrak{N} := (p_{1*})^{-1}\mathfrak{M} \cap (p_{2*})^{-1}\mathfrak{M} \tag{5.1}$$

is an integrable Ehresmann connection for the induced foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$, and any $\mathfrak{M}$-curve in M possesses $\mathfrak{N}$-lifts relative to the canonical projections; *moreover,*

1) *for any $\mathfrak{M}$-curve $\sigma : I \to M$ with the start point $x = \sigma(0)$ and any point $z = (x, \{h\}, y) \in p_1^{-1}(x)$, the $\mathfrak{N}$-lift at the point x is a curve $\delta(s) = (x(s), \{h_s\}, y(s))$, $s \in I$, where $x(s) = \sigma(s)$, $h \xrightarrow{\sigma|_{[0,s]}}> h_s$, and $y(s) = h_s(1)$,*
2) *if $\gamma(s) = (x(s), \{h_s\}, y(s))$, $s \in I$, is an arbitrary $\mathfrak{N}$-curve started at the point $\gamma(0) = z = (x, \{h\}, y)$ in $G_{\mathfrak{M}}(\mathscr{F})$, then $\sigma(s) = x(s)$ is an $\mathfrak{M}$-curve*; *moreover, $h \xrightarrow{\sigma|_{[0,s]}}> h_s$ and $y(s) = h_s(1)$.*

Proof. We first prove the validity of assertions 1) and 2).

1. Let $\sigma : I \to M$ be an arbitrary $\mathfrak{M}$-curve started at the point $x = \sigma(0)$, and let $z \in p_1^{-1}(x)$. Then $z = (x, \{h\}, y)$. Since the pair of paths (σ, h) is admissible for the vertical–horizontal homotopy, there exists $h \xrightarrow{\sigma|_{[0,s]}}> h_s$ for any $s \in I$. Let $x(s) = \sigma(s)$, $y(s) = h_s(1)$. Then the curve $\delta(s) = (x(s), \{h_s\}, y(s))$, $s \in I$, belongs to the graph $G_{\mathfrak{M}}(F)$ and $(p_1 \circ \delta)(s) = x(s) = \sigma(s)$. Since

$$p_1\Big(\frac{d\delta(s)}{ds}\Big) = \frac{dx(s)}{ds} \in \mathfrak{M}_{\sigma(s)}, \quad p_2\Big(\frac{d\delta(s)}{ds}\Big) = \frac{dy(s)}{ds} \in \mathfrak{M}_{h_s(1)} \quad \forall\, s \in I,$$

$\delta(s)$ is an $\mathfrak{N}$-curve with the start point $\delta(0) = (x, \{h\}, y) = z$, i.e., δ is an $\mathfrak{N}$-lift of the $\mathfrak{M}$-curve σ at the point z with respect to $p_1 : G_{\mathfrak{M}}(F) \to M$.

Similarly, any $\mathfrak{M}$-curve σ possesses $\mathfrak{N}$-lifts with respect to the canonical projection $p_2 : G_{\mathfrak{M}}(F) \to M$.

2. We consider an arbitrary $\mathfrak{N}$-curve $\delta(s) = (x(s), \{h_s\}, y(s))$, $s \in I$, started at the point $\delta(0) = z = (x, \{h\}, y)$. We set $\sigma(s) := x(s)$, $h \xrightarrow{\sigma|_{[0,s]}}> \widetilde{h}_s$ and $\widetilde{y}(s) := \widetilde{h}_s(1)$. By assertion 1), the curve $\gamma(s) := (\sigma(s), \{\widetilde{h}_s\}, \widetilde{y}(s))$ is an $\mathfrak{N}$-lift of the curve $\sigma(s)$ at the point z. By the existence and uniqueness theorem for the Cauchy problem in the theory of ordinary differential equations, there exists a unique $\mathfrak{N}$-lift of an $\mathfrak{M}$-curve σ at a point z. By the uniqueness of an $\mathfrak{N}$-lift of the curve σ at the point z, we necessarily have $\delta(s) = \gamma(s)$ for any $s \in I$.

Thus, assertions 1) and 2) are proved. Consequently, the distribution $\mathfrak{N}$ is smooth. We verify the integrability of the distribution $\mathfrak{N}$. For any $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}_{\mathfrak{N}}(G_{\mathfrak{M}}(F))$ the differential p_{i*} of the smooth mapping $p_i : G_{\mathfrak{M}} \to M$ possesses the property $p_{i*}[\mathbb{X}, \mathbb{Y}] = [p_{i*}\mathbb{X}, p_{i*}\mathbb{Y}]$. By definition, $\mathfrak{N}$, $p_{i*}\mathbb{X}, p_{i*}\mathbb{Y} \in \mathfrak{X}_{\mathfrak{M}}(M)$. Since $\mathfrak{M} = TF^t$ is an integrable distribution, $[p_{i*}\mathbb{X}, p_{i*}\mathbb{Y}] \in \mathfrak{X}_{\mathfrak{M}}(M)$.

Consequently, $[\mathbb{X}, \mathbb{Y}] \in p_{1*}^{-1}(\mathfrak{M}) \cap p_{2*}^{-1}(\mathfrak{M}) \subset \mathfrak{X}_{\mathfrak{N}}(G_{\mathfrak{M}}(F))$. By the Frobenius theorem, $\mathfrak{N}$ is an integrable distribution. Consequently, there exists a foliation $\mathbb{F}^t$ such that $T\mathbb{F}^t = \mathfrak{N}$.

Let us show that $\mathfrak{N}$ is an Ehresmann connection for the foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$. Let (δ, k) be an admissible pair of paths, i.e., $\delta : I_1 \to G_{\mathfrak{M}}(F)$ is an integrable curve of the distribution $\mathfrak{N}$ started at an arbitrary point z, whereas $k : I_2 \to \mathbb{L}$ is a path in the leaf $\mathbb{L}$ of the foliation $\mathbb{F}$ passing through $z = k(0)$. We set $\sigma = p_1 \circ \delta$ and $h = p_1 \circ k$. Then (σ, h) is an admissible pair of paths started at the point $x = p_1(z)$ in M. Consequently, there exists a vertical–horizontal homotopy H with base (σ, h). Moreover, we introduce the displacement $\sigma \xrightarrow{h|_{[0,t]}}> \sigma_t$ of the $\mathfrak{M}$-horizontal curve σ along the vertical path $h|_{[0,t]}$ relative to the integrable Ehresmann connection $\mathfrak{M}$. Consequently, σ_t is an $\mathfrak{M}$-horizontal curve. By the above, there exists an $\mathfrak{N}$-lift δ_t of the curve σ_t at a point $k(t) \in p_1^{-1}(h(t))$. By definition,

$$\widetilde{H}(s,t) = \sigma_t(s) \quad \forall (s,t) \in I_1 \times I_2.$$

Then $p_1 \circ \widetilde{H} = H$. By this equality, it is easy to verify that $\widetilde{H}$ is a vertical–horizontal homotopy with base (δ, k). Hence $(G_{\mathfrak{M}}(F), \mathbb{F}, \mathbb{F}^t)$ is a foliation with integrable Ehresmann connection. □

Definition 5.1. A distribution $\mathfrak{N} = T\mathbb{F}^t$ tangent to leaves of a foliation $\mathbb{F}^t$, is called the *induced integrable Ehresmann connection* for the foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$.

5.2. Structure of leaves of the induced foliation. We consider only smooth coverings for smooth manifolds. We recall some definitions and facts concerning regular coverings [18]. A covering mapping $f : N \to B$ is *regular* if for any closed path h in B all its lifts $\widehat{h}$ to B are simultaneously closed or not closed.

According to [18], a covering mapping $f : N \to B$ is regular if and only if for some point $y \in N$ the induced monomorphism of the fundamental group $f_{\#} : \pi_1(N, y) \to \pi_1(B, b)$, where $b = f(y)$, maps $\pi_1(N, y)$ onto the normal subgroup $f_{\#}(\pi_1(N, y))$ in $\pi_1(B, b)$.

Furthermore, a covering mapping $f : N \to B$ is regular if and only if the group of covering transformations $\Psi := \{\psi \in \mathrm{Diff}\,(N) \mid f = f \circ \psi\}$ transitively acts on each leaf $f^{-1}(b)$, $b \in B$.

As known, for any smooth regular covering mapping $f : N \to B$ the group of covering transformations Ψ is isomorphic to the quotient group $\pi_1(B, b)/f_{\#}(\pi_1(N, y))$, where $y \in N$ and $b = f(y)$. We note that for fixed points $y \in N$ and $b = f(y)$ the group of covering transformations Ψ is uniquely defined.

The following assertion is obtained from [15] and provides a description of the structure of leaves of the induced foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$.

Theorem 5.2. *Let (M, F, F^t) be a foliation with integrable Ehresmann connection $\mathfrak{M} = TF^t$. Let L_0 be the standard fiber of the canonical fibrations $p_i : G_{\mathfrak{M}}(F) \to M$, $i = 1, 2$. Then the induced foliation $\mathbb{F}$ on the graph $G_{\mathfrak{M}}(F)$ possesses the following properties.*

1. *Any leaf $\mathbb{L} \in \mathbb{F}$ is diffeomorphic to the quotient manifold $(L_0 \times L_0)/\Phi$, where L_0 is the standard fiber of the fibration $p_i : G_{\mathfrak{M}}(F) \to M$, $i = 1, 2$, and the group Φ is isomorphic to the holonomy group $H_{\mathfrak{M}}(L)$ of the leaf $L = p_i(\mathbb{L})$.*
2. *For any leaf L of the foliation F there exists a regular covering mapping $L_0 \to L$ with the group of covering transformations that is isomorphic to the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L)$.*

3. *For any point $z = (x, \{h\}, y)$ of the graph $G_{\mathfrak{M}}(F)$ the intersection $p_1^{-1}(x) \cap p_2^{-1}(y)$ is bijective to the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L)$ of the leaf $L = L(x)$).*
4. *The group $H_{\mathfrak{M}}(L)$ is isomorphic to the quotient group $\pi_1(\mathbb{L})/(\pi_1(L_0) \times \pi_1(L_0))$ that is the fundamental group $\pi_1(\mathbb{L})$ for the direct product of normal subgroups which is isomorphic to the fundamental group $\pi_1(L_0)$.*
5. *The holonomy groups $H_{\mathfrak{M}}(L)$ and $H_{\mathfrak{N}}(\mathbb{L})$, where $L = \pi_1(\mathbb{L})$ and $\mathfrak{N}$ is the Ehresmann connection for the induced foliation given by (5.1), are naturally isomorphic.*

6 Feature of the Induced Foliation of Graph of Suspended Foliation

Proof of Theorem 1.2. Let (M, F, F^t) be a suspended foliation with natural integrable Ehresmann connection, and let $(G_{\mathfrak{M}}(F), \mathbb{F})$ be the induced foliation. By Theorem 5.1, on the graph $G_{\mathfrak{M}}(F)$, there exists a foliation $\mathbb{F}^t$ such that $\mathfrak{N} = T\mathbb{F}^t$ is an integrable Ehresmann connection for the foliation $\mathbb{F}$. Let us show that $\mathbb{F}$ is a suspended foliation. By Theorem 1.1, it suffices to prove that the foliation $\mathbb{F}^t$ is formed by fibers of some submersion $r : G_{\mathfrak{M}}(F) \to \mathbb{B}$.

Let us show that the restriction $p_1|_{\mathbb{L}^t} : \mathbb{L}^t \to L^t$ on an arbitrary leaf $\mathbb{L}^t$ of the foliation $\mathbb{F}^t$ is a diffeomorphism on the corresponding leaf L^t of the foliation F^t. Let $z = (x, \{h\}, y) \in \mathbb{L}^t$. Then $x, y \in L^t$ and $p_1(\mathbb{L}^t) \subset L^t$, where $L^t = L^t(x)$. Since $p_1|_{\mathbb{L}^t}$ is a submersion of manifolds of the same dimension, from the definition of a smooth structure on a graph it follows that $p_1|_{\mathbb{L}^t}$ is a local diffeomorphism. By assertion 1) of Theorem 5.1, for any piecewise smooth curve $\sigma : I \to L^t$ started at the point $x' = \sigma(0)$ and any point $z' \in (p_1|_{\mathbb{L}^t})^{-1}(x')$ there exists a piecewise smooth curve $\delta : I \to \mathbb{L}^t$ started at the point $z' = \delta(0)$ such that $p_1 \circ \delta = \sigma$. Since the leaves L^t and $\mathbb{L}^t$ are linearly connected, the restriction $p_1|_{\mathbb{L}^t} : \mathbb{L}^t \to L^t$ is surjective. Consequently, $p_1|_{\mathbb{L}^t} : \mathbb{L}^t \to L^t$ is a covering mapping.

Let us show that $p_1|_{\mathbb{L}^t} : \mathbb{L}^t \to L^t$ is bijective. Assume the contrary. Let there exist points $z = (x, \{h\}, y)$ and $z' = (x, \{h'\}, y')$ in the leaf $\mathbb{L}^t$ that are projected to $x \in L^t$. We join z with z' by a piecewise smooth path $\delta : [0, 1] \to \mathbb{L}^t$, $\delta(0) = z$, $\delta(1) = z'$. Then $\sigma = p_1 \circ \delta : [0, 1] \to L^t$ is a loop at the point x. From assertions 1) and 2) of Theorem 5.1 it follows that $\delta(s) = (x(s), \{h_s\}, y(s))$, where $x(s) = \sigma(s)$, $h \xrightarrow{\sigma|_{[0,s]}} h_s$, $y(s) = h_s(1)$. Consequently, $z' = (x, \{h'\}, y') = (x, \{h_1\}, h_s(1))$. We emphasize that $\mathfrak{P} = TF$ is an integrable Ehresmann connection of the foliation (M, F^t) (cf. Remark 2.1). Since the foliation (M, F^t) is formed by fibres of a locally trivial fibration, all its $\mathfrak{P}$-holonomy groups are trivial. Therefore, $h = h_1$ and $y' = h_1(1) = h(1) = y$. Consequently, $z' = (x, \{h\}, h) = z$. Hence $p_1|_{\mathbb{L}^t} : \mathbb{L}^t \to L^t$ is a bijection that is a local diffeomorphism, i.e., a diffeomorphism. Thus, each leaf of the foliation $\mathbb{F}^t$ is diffeomorphic to the standard fiber T of the fibration $p : M \to B$.

Since the foliation (M, F^t) is formed by fibers of a locally trivial fibration with projection $p : M \to B$, at any point $x \in M$ there exists a coordinate neighborhood such that each leaf of this foliation intersects this neighborhood along at most one local leaf. Consequently, at the point $z = (x, \{h\}, y)$ of the graph $G_{\mathfrak{M}}(F)$ there exists a bifibered coordinate neighborhood $V = U \times D$, defined in Subsection 3.2, such that each leaf of the foliation $\mathbb{F}^t$ intersects this neighborhood along one local leaf of the form $\{c\} \times D$, where $c \in U$. This means that $\mathbb{F}^t$ is a regular foliation in the sense of Palais [19]. As proved in [19], the space of leaves of any regular foliation is equipped with the structure of a smooth, not necessarily Hausdorff manifold.

Let us show that the topological space of leaves $\mathbb{B} = G_{\mathfrak{M}}(F)/\mathbb{F}^t$ of a foliation $\mathbb{F}^t$ is always Hausdorff. We denote by $\xi : M \to B = M/F^t$ and $\eta : G_{\mathfrak{M}}(F) \to \mathbb{B} = G_{\mathfrak{M}}(F)/\mathbb{F}^t$ the projections onto the leaf space. By the definition of an integrable Ehresmann connection $T\mathbb{F}^t = \mathfrak{N}$, there exists a mapping $\mu : \mathbb{B} \to B$ satisfying the commutative diagram

$$\begin{array}{ccc} G_{\mathfrak{M}}(F) & \xrightarrow{p_1} & M \\ \eta \big\downarrow & & \big\downarrow \xi \\ G_{\mathfrak{M}}(F)/\mathbb{F}^t = \mathbb{B} & \xrightarrow{\mu} & M/F^t = B \end{array} \tag{6.1}$$

We note that the mappings p_1, η, and ξ are simultaneously continuous and open. Since the diagram (6.1) is commutative, the mapping $\mu : \mathbb{B} \to B$ is also continuous and open. Let $\mathbb{L}_1^t$ and $\mathbb{L}_2^t$ be any different leaves of the foliation $\mathbb{F}^t$. Then $w_1 = [\mathbb{L}_1^t] = \eta(\mathbb{L}_1^t)$ and $w_2 = [\mathbb{L}_2^t] = \eta(\mathbb{L}_2^t)$ are any two distinct points in $\mathbb{B}$. We set $v_1 = \mu(w_1)$ and $v_2 = \mu(w_2)$. Since B is Hausdorff, for $v_1 \neq v_2$ there exist disjoint neighborhoods $V_1 = V_1(v_1)$ and $V_2 = V_2(v_2)$ of the points v_1 and v_2 respectively. Therefore, $W_1 = \mu^{-1}(V_1)$ and $W_2 = \mu^{-1}(V_2)$ are disjoint neighborhoods of the points w_1 and w_2, in $\mathbb{B}$ respectively.

If $p_2(\mathbb{L}_1^t) \neq p_2(\mathbb{L}_2^t)$, then the separation of the points $w_1 = [\mathbb{L}_1^t] = \eta(\mathbb{L}_1^t)$ and $w_2 = [\mathbb{L}_2^t] = \eta(\mathbb{L}_2^t)$ in $\mathbb{B}$ is proved in a similar way.

It remains to consider the case $w_1 = w_2$. In this case, $p_i(\mathbb{L}_1^t) = p_i(\mathbb{L}_2^t) = L^t$, $i = 1, 2$. We consider an arbitrary point $z = (x, \{h\}, y) \in \mathbb{L}_1^t$. Then there is a point $z' = (x, \{h'\}, y) \in \mathbb{L}_2^t$, where the paths h and h' join $x = h(0) = h'(0)$ with $y = h(1) = h'(1)$ in the leaf L^t of the foliation F^t. Since all holonomy groups of the foliation F^t are trivial, the loop $h \cdot h'$ determines the trivial element of the $\mathfrak{P}$-holonomy group of the leaf L^t. Therefore, $\{h\} = \{h'\}$. Consequently, $z = z'$ and the leaves $\mathbb{L}_1^t$ and $\mathbb{L}_2^t$ coincide, which contradicts our assumption. In other words, this case is impossible.

Thus, we have proved that $\mathbb{B} = G_{\mathfrak{M}}(F)/\mathbb{F}^t$ is a smooth Hausdorff manifold. Since the universal covering manifold for the graph $G_{\mathfrak{M}}(F)$ is diffeomorphic to the product of manifolds $(\widehat{B} \times \widehat{B}) \times \widehat{T}$, the distribution $\mathfrak{B} = T\mathbb{F}$ is an integrable Ehresmann connection for the foliation $\mathbb{F}^t$. Hence the foliation $\mathbb{F}^t$ is formed by fibers of the submersion $\eta : G_{\mathfrak{M}}(F) \to \mathbb{B}$ with integrable Ehresmann connection $\mathfrak{B}$. Consequently, from Theorem 1.1 it follows that the induced foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$ is a suspended foliation. Let $(G_{\mathfrak{M}}(F), \mathbb{F}) = \mathrm{Sus}(T, \mathbb{B}, \widehat{\rho})$.

By the above, for suspended foliations F and $\mathbb{F}$ the associated fibrations $p : M \to B$ and $\widehat{p} : G_{\mathfrak{M}}(F) \to \mathbb{B}$ have the same standard fiber T. Let $\widehat{\Psi} = \widehat{\rho}(\pi_1(\mathbb{B}))$ be the global holonomy group of the foliation $\mathbb{F}$. We fix a point $\widehat{b} \in \mathbb{B}$. Let $b = \mu(\widehat{b})$, where μ satisfies the commutative diagram (6.1). We identify the standard fiber T of the fibration $\widehat{p} : G_{\mathfrak{M}}(F) \to \mathbb{B}$ with $\widehat{p}^{-1}(\widehat{b})$ and the standard fiber T of the fibration $p : M \to B$ with $p^{-1}(b)$. Let $\widehat{\psi} = \widehat{\rho}([f])$, where $[f] \in \pi(\mathbb{B}, \widehat{b})$, be an arbitrary transformation in the group $\widehat{\Psi}$. We consider an arbitrary point $z = (x, \{k\}, y) \in T = \widehat{p}^{-1}(\widehat{b})$. According to the interpretation of the global holonomy group $\widehat{\Psi}$ on T (cf. Subsection 4.3), we have $\widehat{\psi}(z) = \widetilde{f}(1)$, where $\widetilde{f}$ is an $\mathfrak{N}$-lift of the loop f at the point z. Since $p_{1*}(\mathfrak{N}) = \mathfrak{M}$ and the diagram (6.1) is commutative, we have $\psi(x) = \widetilde{h}(1)$, where $\widetilde{h} = p_1 \circ \widetilde{f}$; moreover, $\widetilde{h}$ is an $\mathfrak{M}$-lift of the loop h at the point x, where $[h] = \mu_*([f])$ (cf. Figure 2). Consequently, $p_1 \circ \widehat{\psi} = \psi \circ p_1$.

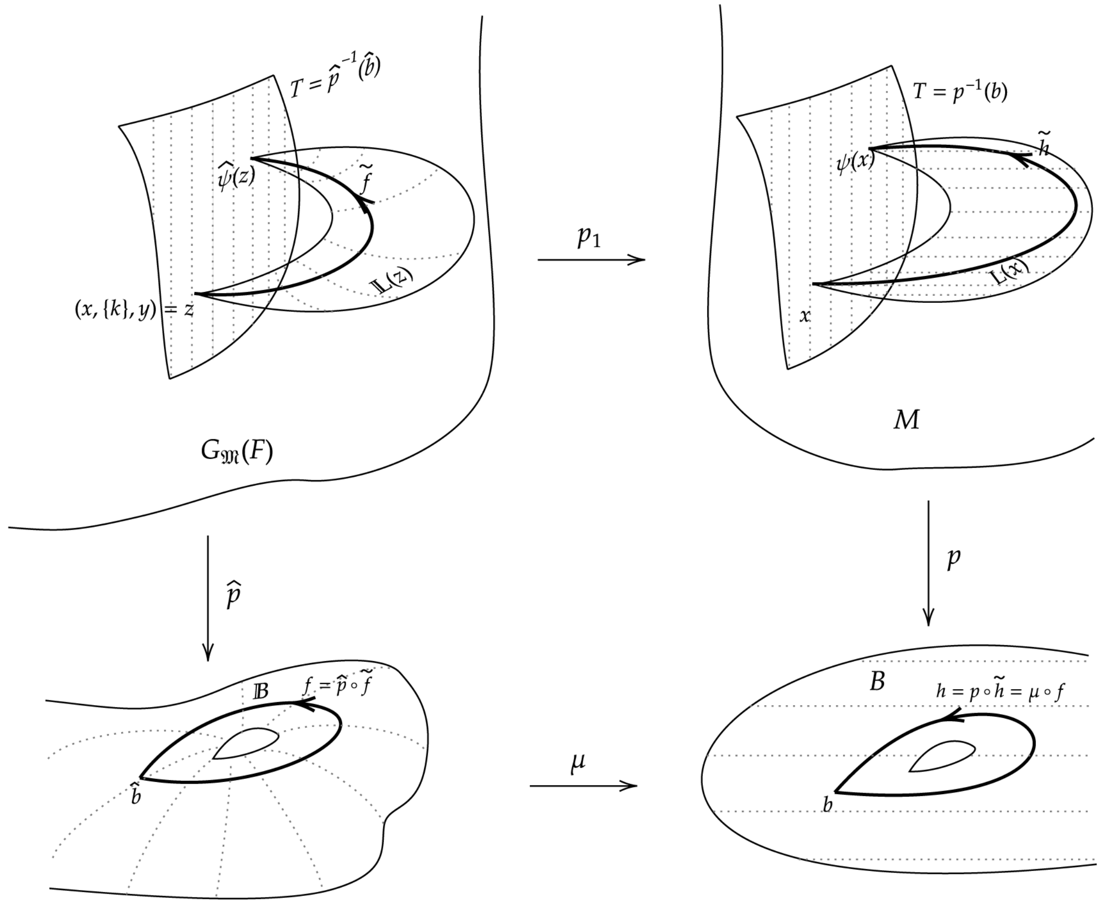

Figure 2.

Thus, from the interpretation of the global holonomy group of suspended foliations and commutativity of the diagram (6.1) it follows that the diffeomorphism $\widehat{p}_1|_T : T \cong \widehat{p}_1^{-1}(\widehat{b}) \to T \cong p_1^{-1}(b)$ is the conjugation of the groups $\widehat{\Psi}$ and Ψ.

From Theorems 1.2 and 4.1 we obtain Theorem 1.3. $\square$

Acknowledgments

The work is supported by the Laboratory of Dynamical Systems and Applications of NRU HSE, the Ministry on Science and Higher Education of the Russian Federation (grant No. 075-15-2019-1931).

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Submitted on January 5, 2022