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An intractability result for the vertex 3-colourability problem

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Abstract
The vertex 3-colourability problem is to decide whether the vertex set of a given graph can be split into three subsets of pairwise non-adjacent vertices. This problem is known to be NP-complete in a certain class of graphs, defined by an explicit description of allowed 5-vertex induced subgraphs in them. In the present paper, we improve this result by showing that the vertex 3-colourability problem remains NP-complete for a reduced set of allowed 5-vertex induced structures. It gives a step towards obtaining a complete complexity dichotomy for the mentioned problem and all the classes, defined by 5-vertex forbidden induced prohibitions.

Keywords Vertex 3-colourability · Computational complexity · Hereditary graph class

1 Introduction

A *vertex k-colouring* of a graph G is a mapping $c:\ V(G) \longrightarrow \{1, 2, \ldots, k\}$, such that $c(v_1) \neq c(v_2)$, for any adjacent vertices v_1 and v_2 of G. All the elements of $\{1, 2, \ldots, k\}$ are called *colours*. A graph is called *vertex k-colourable* if it admits a vertex k-colouring. The *vertex k-colourability problem* (the k- VertCol problem, for short) is to recognize, for a given graph, whether it posses a vertex k-coloring or not. For any $k \geq 3$, the k- VertCol problem is a classical NP-complete graph problem [4].

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A class of graphs is called *hereditary* if it is closed under deletion of vertices. Any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{S}$, which is written as $\mathcal{X} = Free(\mathcal{S})$. If $\mathcal{S}$ is finite, then $\mathcal{X}$ is called *finitely defined*.

Many papers are devoted to clarification of the computational complexity status of the k- VERTCOL problem, i.e. proving either NP-completeness or polynomial-time solvability. Unfortunately, a complexity dichotomy for the k- VERTCOL problem is far from being completed even for small values of k and hereditary classes of the form $Free(\{P_n\})$, where P_n is the simple path on n vertices. Indeed, at the present time, the complexity status of the k- VERTCOL problem is open for $Free(\{P_8\}), k = 3$ and for $Free(\{P_7\}), k = 4$. On the other hand, the 3- VERTCOL and 4- VERTCOL problems can be solved in polynomial time for $Free(\{P_7\})$ [15] and for $Free(\{P_6\})$ [1], respectively. For any k, the k- VERTCOL problem can be solved in polynomial time for $Free(\{P_5\})$ [9]. For every $k \geq 5$, the k- VERTCOL problem is NP-complete for $Free(\{P_6\})$ and 4- VERTCOL problem is NP-complete for $Free(\{P_7\})$ [10].

The complexity of the k- VERTCOL problem has been determined for all the classes of $\{H\}$-free graphs, where $|V(H)| \leq 6, k = 3$ [2] and $|V(H)| \leq 5, k = 4$ [5]. In [11], the computational complexity of the 3- VERTCOL problem has been classified for all the hereditary classes, defined by two 5-vertex forbidden induced subgraphs. These results have been extended to complete dichotomies in [12] for triples and in [13, 14] for quadruples of 5-vertex induced prohibitions. Some classes of graphs $\mathcal{X}_1^*$–$\mathcal{X}_9^*$ have been used as a tool for classification of quadruples, which will be defined below.

The notations C_n and K_n stand, respectively, for the simple cycle and the complete graph on n vertices. By F_k and W_k, where $k \geq 3$, we denote the graphs, obtained by adding a vertex v and edges $vv_1, vv_2, \ldots, vv_k$ to $(v_1, \ldots, v_k)$, which is a simple path or a simple cycle, respectively. For graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, their disjunctive union is denoted by $G_1 + G_2$.

The graphs $bull, cricket, butterfly, crown$ are depicted in Fig. 1 1 below:

The graphs $kite, dart, banner, house$ are depicted in Fig. 2 below:

The classes $\mathcal{X}_1^*$–$\mathcal{X}_9^*$ are defined as follows:

- $\mathcal{X}_1^*$: the set of all forests,

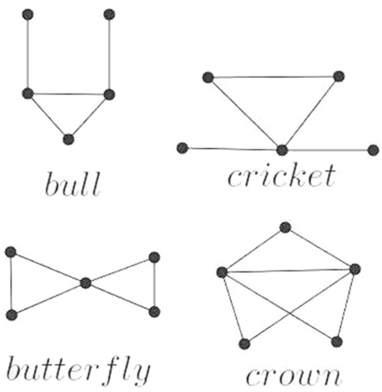

Fig. 1 The graphs $bull, cricket, butterfly, crown$

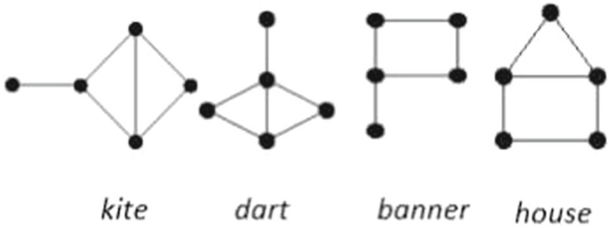

Fig. 2 The graphs $kite$, $dart$, $banner$, $house$

- $\mathcal{X}_2^*$: the set of *line graphs*, i.e. graphs of the edge adjacency, to graphs in $\mathcal{X}_1^*$,
- $\mathcal{X}_3^*$: the set of graphs, in which any 5 vertices induce a subgraph in

$$\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{cricket, kite, F_3 + K_1\},$$

- $\mathcal{X}_4^*$: the set of graphs, in which any 5 vertices induce a subgraph in

$$\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{kite, F_3 + K_1, butterfly, crown\},$$

- $\mathcal{X}_5^*$: the set of graphs, in which any 5 vertices induce a subgraph in

$$\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{kite, F_3 + K_1, house, C_4 + K_1, F_4, W_4, dart, crown\},$$

- $\mathcal{X}_6^*$: the set of graphs, in which any 5 vertices induce a subgraph in

$$\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{cricket, house, banner, C_4 + K_1, C_5\},$$

- $\mathcal{X}_7^*$: the set of graphs, in which any 5 vertices induce a subgraph in

$$\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{cricket, C_5\},$$

- $\mathcal{X}_8^*$: the set of graphs, in which any 5 vertices induce a subgraph in

$$\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{cricket, banner, house, C_4 + K_1\},$$

$\mathcal{X}_9^*$: the set of graphs, in which any 5 vertices induce a subgraph in

$$\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{kite, F_3 + K_1, dart, C_4 + K_1, banner, W_4, C_5\}.$$

Notice that $\mathcal{X}_3^* \dots \mathcal{X}_9^*$ are hereditary classes, defined by 5-vertex induced structures. The 3- VERTCOL problem is NP-complete for $\mathcal{X}_i^*$, for any $3 \leq i \leq 9$, and it is NP-complete for any finitely defined class, including $\mathcal{X}_1^*$ or $\mathcal{X}_2^*$, see [11, 12, 14]. Additionally, if a hereditary class, defined by a quadruple of 5-vertex induced subgraphs, includes none of the classes $\mathcal{X}_1^* \dots \mathcal{X}_9^*$, then the 3- VERTCOL problem is polynomial-time solvable for it, see [11–14]. These results give a complete dichotomy for the quadruples.

A step towards accomplishing the 5-vertex case is done in this paper. Namely, we strengthen NP-completeness of the 3- VertCol problem for $\mathcal{X}_9^*$ by showing that it remains NP-complete for the set $\mathcal{X}_9'$ of graphs, in which any 5 vertices induce a graph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{kite, F_3 + K_1, dart, C_4 + K_1, banner, W_4\}$.

2 The main result

The graph *necklace* is presented in Fig. 3 below.

Lemma 1 *In any vertex 3-colouring of necklace, the vertices t_1–t_4 receive the same colour.*

Proof The graph *necklace* has a vertex 3-colouring. Indeed, let us assign the colour 1 to the vertices

$$b_1, b_2, a_1, a_2, u_3, u_4, w_3, w_4, x_2, x_4, z_1, z_3, y,$$

the colour 2 to the vertices

$$a_3, a_4, b_3, b_4, u_1, u_2, w_5, w_6, x_1, x_3, y_1, y_3, z_2, z_4$$

and the colour 3 to the vertices

$$t_1, t_2, t_3, t_4, u_5, u_6, w_1, w_2, y_2, y_4, x, z.$$

Similarly, we assign colours to the remaining vertices of *necklace*. The obtained arrangement of colours is a vertex 3-colouring of *necklace*.

Let c be an arbitrary 3-colouring of *necklace*. Notice that in any 3-colouring of W_4 pairs of its non-adjacent degree 3 vertices are both monochromatic. Hence,

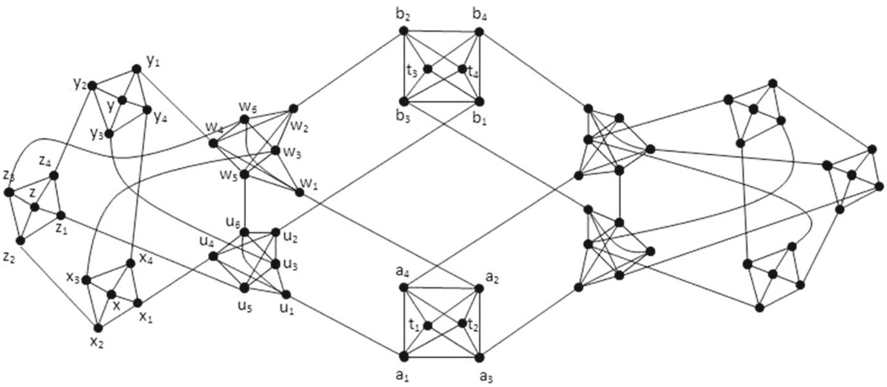

Fig. 3 The graph *necklace*

$$c(t_1) = c(t_2), c(t_3) = c(t_4), c(a_1) = c(a_2), c(a_3) = c(a_4), c(b_1) = c(b_2),$$

$$c(b_3) = c(b_4), c(u_1) = c(u_2), c(w_1) = c(w_2), c(u_3) = c(u_4), c(w_3) = c(w_4),$$

$$c(u_5) = c(u_6), c(w_5) = c(w_6), c(x_1) = c(x_3), c(x_2) = c(x_4), c(y_1) = c(y_3),$$

$$c(y_2) = c(y_4), c(z_1) = c(z_3), c(z_2) = c(z_4).$$

Let us show that $c(a_1) = c(b_1)$. Assume the opposite, i.e. that $c(a_1) = 1$ and $c(b_1) = 2$. Then,

$$c(a_2) = 1, c(b_2) = 2, c(u_1) = c(u_2) = c(w_1) = c(w_2) = 3.$$

Without loss of generality, let $c(u_6) = 1$ and $c(w_5) = 2$. Then,

$$c(u_5) = c(w_3) = c(w_4) = 1 \text{ and } c(w_6) = c(u_3) = c(u_4) = 2.$$

Hence,

$$c(x_1) = c(x_3) = c(y_1) = c(y_3) = c(z_1) = c(z_3) = 3.$$

As

$$c(x_2) = c(x_4), c(y_2) = c(y_4), c(z_2) = c(z_4)$$

and because of the existence of the edges x_2z_2, y_2z_4, x_4y_4, the vertices x_2, y_2, z_2 must have pairwise distinct colours. But, the colour 3 is forbidden for them. We have a contradiction. Therefore, $c(a_1) = c(a_2) = c(b_1) = c(b_2)$. Similarly,

$$c(a_3) = c(a_4) = c(b_3) = c(b_4) \text{ and, hence, } c(t_1) = c(t_2) = c(t_3) = c(t_4).$$

□

Theorem 1 *The* 3- VERTCOL *problem is NP-complete for graphs in* $\mathcal{X}'_9$.

Proof The *implantation of necklace* is an operation, applied to a vertex x of a graph G with the neighbourhood $\{x_1, x_2, x_3, x_4\}$ as follows. The vertex x is deleted, *necklace* is added to the resultant graph, and then the edges $x_1t_1, x_2t_2, x_3t_3, x_4t_4$ are added. Lemma 1 guarantees that the graph, obtained after the implantation of *necklace*, is vertex 3-colourable iff G is vertex 3-colourable. Let G be a 4-*regular graph*, i.e. a graph, whose all vertices have degree equal 4. We sequentially apply the implantation of $necklace$ to each vertex of G. The resultant graph G^* is vertex 3-colourable iff G is vertex 3-colourable. As $necklace \in \mathcal{X}_9'$, then it is not hard to see that $G^* \in \mathcal{X}_9'$. The 3- VertCol problem is known to be NP-complete for 4-regular graphs [3]. Hence, the 3- VertCol problem for 4-regular graphs can be reduced in polynomial time to the same problem for graphs in $\mathcal{X}_9'$. Therefore, it is NP-complete for $\mathcal{X}_9'$. □

3 Conclusions and future work

The vertex 3-colourability problem for sets of 5-vertex induced subgraphs is considered in this paper. We prove here that it is NP-complete for some class of this type, improving a previously known result for a broader set of allowed 5-vertex induced fragments. It gives a step to a complete complexity dichotomy for the considered problem and all the hereditary classes, defined by 5-vertex induced prohibitions, which is now known only for quadruples of such forbidden elements. Obtaining this dichotomy is a challenging research problem for future work.

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